FAST LOOPS ON SEMI-WEIGHTED HOMOGENEOUS HYPERSURFACE SINGULARITIES

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Abstract. We show the existence of essential fast loops on semi-weighted homogeneous hypersurface singularities with weights $w_1 \geq w_2 > w_3$. In particular we show that semi-weighted homogeneous hypersurface singularities are metrically conical only if their two lowest weights are equal.

1. Introduction

Let $X \subset \mathbb{R}^n$ be a subanalytic set with a singularity at $x$. It is well-known for small real numbers $\epsilon > 0$ that there exists a homeomorphism from the Euclidean ball $B(x, \epsilon)$ to itself which maps $X \cap B(x, \epsilon)$ onto the straight cone over $X \cap S(x, \epsilon)$ with vertex at $x$. The homeomorphism $h$ is called a topologically conical structure of $X$ at $x$ and, since John Milnor proved the existence of topologically conical structure for algebraic complex hypersurfaces with an isolated singularity \cite{9}, some authors say $\epsilon$ is a Milnor radius of $X$ at $x$. Some developments of the Lipschitz geometry of complex algebraic singularities come from the following question: given an algebraic subset $X \subset \mathbb{C}^n$ with an isolated singularity at $x$, is there $\epsilon > 0$ such that $X \cap B(x, \epsilon)$ is bi-Lipschitz homeomorphic to the cone over $X \cap S(x, \epsilon)$ with vertex at $x$? When we have a positive answer for this question we say that $(X, x)$ is metrically conical. Some motivations for this question were given in \cite{3}, \cite{6} and, in the same papers, the above question was answered negatively. The strategy used in \cite{6} to show that some examples of complex weighted homogeneous surface singularities $(X, x)$ are not metrically conical was to exhibit nontrivial loops on $X \cap S(x, \epsilon)$ which the diameter goes to 0 faster than linearly as $\epsilon \to 0$. In this paper we analyze semi-weighted homogeneous hypersurface singularities under the same

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point of view above and, in particular, we show that semi-weighted homogeneous hypersurface singularities are metrically conical only if its two lowest weights are equal.

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2. Preliminaries

2.1. Inner metric. Given an arc $\gamma: [0,1] \to \mathbb{R}^n$, we remember that the length of $\gamma$ is defined by

$$l(\gamma) = \inf \{ \sum_{i=1}^{m} |\gamma(t_i) - \gamma(t_{i-1})| : 0 = t_0 < t_1 < \cdots < t_{m-1} < t_m = 1 \}.$$

Let $X \subset \mathbb{R}^n$ be a subanalytic connected subset. It is well-know that the function $d_X: X \times X \to [0, +\infty)$

$$d_X(x,y) = \inf \{ l(\gamma) : \gamma: [0,1] \to X; \gamma(0) = x, \gamma(1) = y \}$$

is a metric on $X$, so-called inner metric on $X$.

Theorem 2.1 (Pancake Decomposition [8]). Let $X \subset \mathbb{R}^n$ be a subanalytic connected subset. Then, there exist $\lambda > 0$ and $X_1, \ldots, X_m$ subanalytic subsets such that:

a. $X = \bigcup_{i=1}^{m} X_i$,

b. $d_X(x,y) \leq \lambda|x - y|$ for any $x, y \in X_i, i = 1, \ldots, m$.

2.2. Horn exponents. Let $\beta \geq 1$ be a rational number. The germ of

$$H_\beta = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 = z^\beta, \ z \geq 0 \}$$

at $0 \in \mathbb{R}^3$ is called a $\beta$-horn.

By results of [1], we know that a $\beta_1$-horn is bi-Lipschitz equivalent, with respect to the inner metric, to a $\beta_2$-horn if, and only if $\beta_1 = \beta_2$. Let $\Omega \subset \mathbb{R}^n$ be a 2-dimensional subanalytic set. Let $x_0 \in \Omega$ be a point such that $\Omega$ is a topological 2-dimensional manifold without boundary near $x_0$.

Theorem 2.2. [1] There exists a unique rational number $\beta \geq 1$ such that the germ of $\Omega$ at $x_0$ is bi-Lipschitz equivalent, with respect to the inner metric, to a $\beta$-horn.
The number $\beta$ is called the horn exponent of $\Omega$ at $x_0$. We use the notation $\beta(\Omega, x_0)$. By Theorem 2.2 $\beta(\Omega, x_0)$ is a complete intrinsic bi-Lipschitz invariant of germs of subanalytic sets which are topological 2-dimensional manifold without boundary. In the following, we show a way to compute horn exponents.

According to [2], $\beta(\Omega, x_0) + 1$ is the volume growth number of $\Omega$ at $x_0$, i.e.

$$\beta(\Omega, x_0) + 1 = \lim_{r \to 0^+} \frac{\log H^2[\Omega \cap B(x_0, r)]}{\log r}$$

where $H^2$ denotes the 2-dimensional Hausdorff measure with respect to Euclidean metric on $\mathbb{R}^n$.

2.3. Order of contact of arcs. Let $\gamma_1: [0, \epsilon) \to \Omega$ and $\gamma_2: [0, \epsilon) \to \Omega$ be two continuous semianalytic arcs with $\gamma_1(0) = \gamma_2(0) = x_0$ and not identically equal to $x_0$. We suppose that the arcs are parameterized in the following way:

$$\|\gamma_i(t) - x_0\| = t, \ i = 1, 2.$$ 

Let $\rho(t)$ be a function defined as follows: $\rho(t) = \|\gamma_1(t) - \gamma_2(t)\|$. Since $\rho$ is a subanalytic function there exist numbers $\lambda \in \mathbb{Q}$ and $a \in \mathbb{R}$, $a \neq 0$, such that

$$\rho(t) = at^\lambda + o(t^\lambda).$$

The number $\lambda$ is called an order of contact of $\gamma_1$ and $\gamma_2$. We use the notation $\lambda(\gamma_1, \gamma_2)$ (see [4]).

Let $K$ be the field of germs of subanalytic functions $f: (0, \epsilon) \to \mathbb{R}$. Let $\nu: K \to \mathbb{R}$ be a canonical valuation on $K$. Namely, if $f(t) = at^\beta + o(t^\beta)$ with $\alpha \neq 0$ we put $ord_t(f(t)) = \beta$.

Lemma 2.3. Let $\gamma_1, \gamma_2$ be a pair of semianalytic continuous arcs such that $\gamma_1(0) = \gamma_2(0) = x_0$ and $\gamma_i \neq x_0$ ($i = 1, 2$). Let $\tilde{\gamma}_1(\tau)$ and $\tilde{\gamma}_2(\tau)$ be semianalytic parameterizations of $\gamma_1$ and $\gamma_2$ such that $\|\tilde{\gamma}_i(\tau) - x_0\| = \tau + o(\tau)$, $i = 1, 2$. Then $ord_\tau\|\tilde{\gamma}_1(\tau) - \tilde{\gamma}_2(\tau)\| \leq \lambda(\gamma_1, \gamma_2)$.

The following result is an alternative way to compute horn exponents of germs of subanalytic sets which are topological 2-dimensional manifold without boundary.
Theorem 2.4. Let $\Omega \subset \mathbb{R}^n$ be a 2-dimensional subanalytic set. Let $x_0 \in \Omega$ be a point such that $\Omega$ is a topological 2-dimensional manifold without boundary near $x_0$. Then $\beta(\Omega, x_0) = \min \{ \lambda(\gamma_1, \gamma_2) : \gamma_1, \gamma_2$ are semianalytic arcs on $\Omega$ with $\gamma_1(0) = \gamma_2(0) = x_0 \}$. 

Lemma 2.3 and Theorem 2.4 were proved in [5].

3. Fast loops

Let $X \subset \mathbb{R}^n$ be a subanalytic set with a singularity at $x$. Let $\epsilon > 0$ be a Milnor radius of $X$ at $x$ and let us denote by $X^*$ the set $X \cap B(x, \epsilon) \setminus \{x\}$. Given a positive real number $\alpha$, a continuous map $\gamma : S^1 \to X^*$ is called an $\alpha$-fast loop if there exists a homotopy $H : S^1 \times [0, 1] \to X \cap B(x, \epsilon)$ such that

1. $H(\theta, 0) = x$ and $H(\theta, 1) = \gamma(\theta)$, $\forall \theta \in S^1$,
2. $\lim_{r \to 0^+} \frac{1}{r^a} \mathcal{H}^2(\text{Im}(H) \cap B(x, r)) = 0$ for each $0 < a < \alpha$,

where $\text{Im}(H)$ denotes the image of $H$.

Given a subanalytic set $X$ and a singular point $x \in X$, according to [2], there exists a positive number $c$ such that any $\alpha$-fast loop $\gamma : S^1 \to X^*$ with $\alpha > c$ is necessarily homotopically trivial. Such a number $c$ is called distinguished for $(X, x)$. We define the $\upsilon$ invariant in the following way:

$$\upsilon(X, x) = \inf \{ c : c$ is distinguished for $(X, x) \}.$$ 

The number $\upsilon(X, x)$ defined above is inspired by the first characteristic exponent for the local metric homology presented in [2].

Example 3.1. Let $K \subset \mathbb{R}^n$ be a straight cone over a Nash submanifold $N \subset \mathbb{R}^n$, with vertex at $p$. Then every loop $\gamma : S^1 \to K^*$ is a 2-fast loop. Moreover, if $\alpha > 2$, then each $\alpha$-fast loop $\gamma : S^1 \to K^*$ is homotopically trivial. We can sum up it saying $\upsilon(K, p) = 2$.

Proposition 3.2. Let $(X, x)$ and $(Y, y)$ be subanalytic germs. If there exists a germ of a bi-Lipschitz homeomorphism, with respect to inner metric, between $(X, x)$ and $(Y, y)$, then $\upsilon(X, x) = \upsilon(Y, y)$.
Proof. Let \( f: (X, x) \to (Y, y) \) be a bi-Lipschitz homeomorphism, with respect to the inner metric. Given \( A \subset X \), let us denote \( \tilde{A} = f(A) \). In this case, \( A = f^{-1}(\tilde{A}) \), where \( f^{-1} \) denotes the inverse map of \( f: (X, x) \to (Y, y) \).

Claim. There are positive constants \( k_1, k_2, \lambda_1, \lambda_2 \) such that

\[
\frac{1}{k_1} \mathcal{H}^2(\tilde{A} \cap B(y, \frac{r}{\lambda_2})) \leq \mathcal{H}^2(A \cap B(x, r)) \leq k_2 \mathcal{H}^2(\tilde{A} \cap B(y, \lambda_1 r)).
\]

In fact, using Pancake Decomposition Theorem (see Subsection 2.1) and using that \( f \) and \( f^{-1} \) are Lipschitz maps, we obtain positive constants \( \lambda_1, \lambda_2 \) such that

\[
f(A \cap B(x, r)) \subset (\tilde{A} \cap B(y, \lambda_1 r)) \quad \text{and} \quad f(\tilde{A} \cap B(y, r)) \subset (A \cap B(x, \lambda_2 r))
\]

and we also obtain positive constants \( k_1, k_2 \) such that

\[
\mathcal{H}^2(f(A \cap B(x, r))) \leq k_1 \mathcal{H}^2(A \cap B(x, r)) \quad \text{and} \quad \mathcal{H}^2(f^{-1}(\tilde{A} \cap B(y, r))) \leq k_2 \mathcal{H}^2(\tilde{A} \cap B(y, r)).
\]

Our claim follows from these two inequalities and the two inclusions above.

Now, we use this claim to show that given \( \alpha > 0 \), a loop \( \gamma: S^1 \to X \setminus \{x\} \) is an \( \alpha \)-fast loop if, and only if, \( f \circ \gamma: S^1 \to Y \setminus \{y\} \) is an \( \alpha \)-fast loop. In fact, let \( \gamma: S^1 \to X \setminus \{x\} \) be a loop and \( H: S^1 \times [0, 1] \to X \) a homotopy such that \( H(\theta, 0) = x \) and \( H(\theta, 1) = \gamma(\theta) \), \( \forall \theta \in S^1 \). Thus, \( f \circ H: S^1 \times [0, 1] \to Y \) is a loop and \( f \circ H(\theta, 0) = x \) and \( f \circ H(\theta, 1) = f \circ \gamma(\theta) \), \( \forall \theta \in S^1 \). Let us denote \( A = \text{Im}(H) \) and \( \tilde{A} = \text{Im}(f \circ H) \), i.e., \( \tilde{A} = f(A) \). Given \( 0 < a < \alpha \), by the above claim, we have that

\[
\lim_{r \to 0^+} \frac{1}{r^a} \mathcal{H}^2(A \cap B(x, r)) = 0
\]

if, and only if,

\[
\lim_{r \to 0^+} \frac{1}{r^a} \mathcal{H}^2(\tilde{A} \cap B(y, r)) = 0.
\]

In other words, it was shown that \( \gamma: S^1 \to X \setminus \{x\} \) is an \( \alpha \)-fast loop if, and only if, \( f \circ \gamma: S^1 \to Y \setminus \{y\} \) is an \( \alpha \)-fast loop, hence \( \nu(X, x) = \nu(Y, y) \). \( \square \)

**Corollary 3.3.** Let \( X \subset \mathbb{R}^n \) be a subanalytic set and \( x \in X \) an isolated singular point. If \( \nu(X, x) > 2 \), then \( (X, x) \) is not metrically conical.

**Proof.** Let \( N \) be the intersection \( X \cap S(x, \epsilon) \) where \( \epsilon > 0 \) is chosen sufficiently small. Since \( x \) is an isolated singular point of \( X \), we have \( N \subset \mathbb{R}^n \) is a Nash submanifold. If \( (X, x) \) is metrically conical, \( X \cap B(x, \epsilon) \) must be bi-Lipschitz homeomorphic (with
respect to the inner metric) to the straight cone over \(N\) with vertex at \(x\). Thus, it follows from Proposition 3.2 that \(v(X, x) = 2\). \(\square\)

4. SEMI-WIGHTED HOMOGENEOUS HYPERSURFACE SINGULARITIES

Remind that a polynomial function \(f: \mathbb{C}^3 \rightarrow \mathbb{C}\) is called \textit{semi-weighted homogeneous} of degree \(d \in \mathbb{N}\) with respect to the weights \(w_1, w_2, w_3 \in \mathbb{N}\) if \(f\) can be presented in the following form: \(f = h + \theta\) where \(h\) is a weighted homogeneous polynomial of degree \(d\) with respect to the weights \(w_1, w_2, w_3\), the origin is an isolated singularity of \(h\) and \(\theta\) contains only monomials \(x_1^{m_1}x_2^{m_2}x_3^{m_3}\) such that \(w_1m_1 + w_2m_2 + w_3m_3 > d\).

An algebraic surface \(S \subset \mathbb{C}^3\) is called \textit{semi-weighted homogeneous} if there exists a semi-weighted homogeneous polynomial \(f = h + \theta\) such that

\[
S = \{(x_1, x_2, x_3) \in \mathbb{C}^3 : f(x_1, x_2, x_3) = 0\}.
\]

The set

\[
S_0 = \{(x_1, x_2, x_3) \in \mathbb{C}^3 : f(x_1, x_2, x_3) = 0\}
\]

is called a \textit{weighted approximation} of \(S\).

**Theorem 4.1.** Let \(S \subset \mathbb{C}^3\) be a semi-weighted homogeneous algebraic surface with an isolated singularity at origin \(0 \in \mathbb{C}^3\). If the weights of \(S\) satisfy \(w_1 \geq w_2 > w_3\), then \(v(S, 0) > 2\). In particular, \((S, 0)\) is not metrically conical.

**Proof.** Let \(S \subset \mathbb{C}^3\) be defined by the semi-weighted polynomial \(f = h + \theta\) of degree \(d\) and let \(S_0\) be the following weighted homogeneous approximation of \(S\):

\[
S_0 = \{(x_1, x_2, x_3) \in \mathbb{C}^3 : h(x_1, x_2, x_3) = 0\}.
\]

Let us consider a family of functions defined as follows:

\[
F(X, u) = h(X) + u\theta(X),
\]

where \(u \in [0, 1]\), \(X = (x_1, x_2, x_3)\). Let \(V(X, u)\) be the vector field defined by:

\[
V(X, u) = -\sum_{i=1}^{3} Q_i(X, u) \frac{\partial}{\partial x_i} + \frac{\partial}{\partial u}
\]

where

\[
P(X, u) = \sum_{i=1}^{3} \left| \frac{\partial F}{\partial x_i}(X, u) \right|^{2\alpha_i} \quad \text{and} \quad Q_i(X, u) = \theta(X) \left| \frac{\partial F}{\partial x_i}(X, u) \right|^{2\alpha_i - 2} \frac{\partial F}{\partial x_i}(X, u)
\]
and \( \alpha_i = \frac{(d-w_1)(d-w_2)(d-w_3)}{d-w_i} \), \( i = 1, 2, 3 \).

It was shown, by L. Fukui and L. Paunescu (see [7] p. 445), that the flow of this vector field gives a modified analytic trivialization [7] of the family \( F^{-1}(0) \).

In particular, we obtain a homeomorphism \( \Phi : (S_0, 0) \rightarrow (S, 0) \) which defines a correspondence of subanalytic continuous arcs. Moreover, \( \Phi \) satisfies the following equation

\[
\Phi(X) = X + \int_0^1 W(\Phi(X), u)du
\]

where \( W(X, u) = V(X, u) - \frac{\partial}{\partial u} \).

**Proposition 4.2.** Let \( \gamma(t) = (t^{w_1}x_1(t), t^{w_2}x_2(t), t^{w_3}x_3(t)) \) be such that \( x_1(t), x_2(t) \) and \( x_3(t) \) are subanalytic continuous functions, \( 0 \leq t < \epsilon \), with \( (x_1(0), x_2(0), x_3(0)) \neq (0, 0, 0) \). If

\[
\eta(t) = \int_0^1 W(\gamma(t), u)du
\]

with \( \eta(t) = (\eta_1(t), \eta_2(t), \eta_3(t)) \), then \( \text{ord}_t|\eta_i(t)| > w_i \) for all \( i = 1, 2, 3 \).

**Proof of the proposition.** Let \( m = (d-w_1)(d-w_2)(d-w_3) \). Since \( h \) has isolated singularity at \( 0 \in \mathbb{C}^3 \), \( \exists \lambda_1 > 0 \) such that

\[
P(\gamma(t), u) \geq \lambda_1 \sum_{i=1}^{3} \left( \frac{\partial h}{\partial x_i}(\gamma(t)) \right)^{2\alpha_i}.
\]

Moreover, since each \( \frac{\partial h}{\partial x_i} \) is weighted homogeneous of degree \( d-w_i \), \( \exists \lambda_2 > 0 \) such that

\[
\left| \frac{\partial h}{\partial x_i}(\gamma(t)) \right|^{2\alpha_i} \geq \lambda_2 t^{2m}.
\]

Hence, \( \text{ord}_t P(\gamma(t), u) \leq 2m \). By hypothesis,

\[
\text{ord}_t|\theta(\gamma(t))| > d \quad \text{and} \quad \text{ord}_t\left| \frac{\partial F}{\partial x_i}(\gamma(t), u) \right|^{2\alpha_i-1} \geq (2\alpha_i - 1)(d - w_i).
\]

Now, we can conclude that \( \text{ord}_t \) of \( \frac{\theta}{P}|\frac{\partial F}{\partial x_i}|^{2\alpha_i-1} \) on \( \gamma(t) \) is bigger than

\[
d + (2\alpha_i - 1)(d - w_i) - 2m = w_i.
\]

Finally, since

\[
\eta_i(t) = \int_0^1 \frac{\theta(\gamma(t))}{P(\gamma(t), u)} \left| \frac{\partial F}{\partial x_i}(\gamma(t), u) \right|^{2\alpha_i-2} \frac{\partial F}{\partial x_i}(\gamma(t), u)du,
\]

we have proved the proposition. \( \square \)
According to Lemma 1 of [6], we can take an essential loop \( \Gamma \) from \( S^1 \) to the link of the weighted homogeneous approximation \( S_0 \) of \( S \) of the form:

\[
\Gamma(\theta) = (x_1(\theta), x_2(\theta), 1).
\]

Let \( H_0 : [0,1] \times S^1 \rightarrow S_0 \) be defined by

\[
H_0(r, \theta) = (r^{w_1/w_3} x_1(\theta), r^{w_2/w_3} x_2(\theta), r).
\]

Then, \( H : [0,1] \times S^1 \rightarrow S \) defined by

\[
H(r, \theta) = \Phi \circ H_0(r, \theta)
\]

is a subanalytic homotopy satisfying: \( H(0, \theta) = x \) and \( H(1, \theta) = \Phi \circ \Gamma(\theta) \). We are going to show the image of \( H \) (im\( (H) = \Omega \)) has volume growth number at origin bigger than 2. Actually, since the volume growth number of \( \Omega \) at 0 is \( 1 + \beta(\Omega, 0) \), we are going to show that \( \beta(\Omega, 0) \) is bigger than 1. So, let us consider two arcs \( \gamma_1 \) and \( \gamma_2 \) on \( (\Omega, 0) \).

Claim. Each \( \gamma_i \) can be parameterized in the following form:

\[
\gamma_i(s) = (s^{w_1/w_3} x_{i1}(s), s^{w_2/w_3} x_{i2}(s), s x_{i3}(s))
\]

where \( x_{i1}(s), x_{i2}(s) \) and \( x_{i3}(s) \) are subanalytic continuous functions and \( x_{i3}(0) = 1 \), \( (i = 1, 2) \).

In fact, first of all, let us fix \( i \) and denote \( \gamma = \gamma_i \). For each \( s > 0 \), let \( \gamma(s) \) be the point on the arc \( \gamma \) such that \( \rho(\gamma(s)) = s^{w_1/w_3} \), where

\[
\rho(x_1, x_2, x_3) := |x_1|^{w_2/w_3} + |x_2|^{w_1/w_3} + |x_3|^{w_1/w_2} \frac{1}{w_1 w_2 w_3}.
\]

In particular, \( \gamma(s) = (s^{w_1/w_3} x_1(s), s^{w_2/w_3} x_2(s), s x_3(s)) \) where \( x_1(s), x_2(s) \) and \( x_3(s) \) are subanalytic continuous functions, with \( (x_1(0), x_2(0), x_3(0)) \neq (0, 0, 0) \). For each \( s > 0 \), let \( \xi(s) \) be the point on the image \( \text{im}(H_0) \subset S_0 \) such that \( \Phi(\xi(s)) = \gamma(s) \). It follows from eq. (4.1) that

\[
\gamma(s) = \xi(s) + \eta(s)
\]

where \( \eta(s) = \int_0^1 W(\gamma(s), u) du \). By Proposition 4.2 (taking \( s = t^{w_3} \)), it follows that

\[
\xi(s) = (s^{w_1/w_3} z_1(s), s^{w_2/w_3} z_2(s), s z_3(s)) \quad \text{with} \quad (z_1(0), z_2(0), z_3(0)) = (x_1(0), x_2(0), x_3(0)).
\]
Since the image $\text{Im}(H_0)$ is invariant by the $\mathbb{R}_+^3$-action
\[ s \cdot (x_1, x_2, x_3) = (s^{-\frac{1}{2}}x_1, s^{-\frac{2}{3}}x_2, sx_3) \]
and $\xi(s) \in \text{Im}(H_0)$ for all $s > 0$, we have that $(z_1(0), z_2(0), z_3(0)) \in \text{Im}(H_0)$, hence $z_3(0) = x_3(0) = a$ is a positive real number. Finally, via the simple change $s \mapsto a^{-1}s$, we show what was claimed above.

In order to finalize the proof of Theorem 4.1 by Lemma 2.3 we have
\[ \lambda(\gamma_1, \gamma_2) \geq \text{ord}_s(|\gamma_1(s) - \gamma_2(s)|) \]
and, since
\[ \text{ord}_s(|\gamma_1(s) - \gamma_2(s)|) > 1, \]
\[ \lambda(\gamma_1, \gamma_2) > 1. \] Therefore, we can use Theorem 2.4 to get $\beta(\Omega, 0) > 1$. □

References