Journal *of* Singularities
Volume 1 (2010), 134-145

received 9 November 2009
in revised form 20 October 2010
DOI: 10.5427/jsing.2010.1i

ON BORDISM AND COBORDISM GROUPS OF MORIN MAPS

ENDRE SZABÓ, ANDRÁS SZŰCS, AND TAMÁS TERPAI

Abstract. We prove that the unoriented cobordism groups of Morin maps are 2-primary in nearly all cases. In the second part we define and compute a ring structure on the rational cobordism group of oriented fold maps.

1. Introduction

We will consider Morin maps of n-manifolds into $(n+k)$-manifolds, $k > 0$. That is, all maps will be assumed to be locally generic and have differentials of rank n or $n-1$ everywhere. It is known [5] that at each point a Morin map is either regular or has a singularity of type Σ^{1_r} for some $r \geq 1$, $r \leq \frac{n}{k+1}$, in which case it has the local form

$$(t_1, \dots, t_{n-1}, x) \mapsto (t_1, \dots, t_{n-1},\ t_1 x + \dots + t_r x^r, \dots, t_{(k-1)r+1} x + \dots + t_{kr} x^r,$$
$$t_{kr+1} x + \dots + t_{kr+r-1} x^{r-1} + x^{r+1}).$$

Definition 1. *A Morin map will be called Σ^{1_r}-map if it has no singularities Σ^{1_s} for $s > r$. The cobordism group of Σ^{1_r}-maps of n-manifolds into $\mathbb{R}^{n+k}$ can be defined in a natural way (see [11]). Let us denote this group by $\Sigma^{1_r}(n,k)$ (no orientability is required). This group for $r = 1$ will be denoted by $Fold(n,k)$, and its oriented version will be $Fold^{SO}(n,k)$. For $r = 0$ (i.e., for the cobordism groups of immersions), the notation will be $Imm(n,k)$ and $Imm^{SO}(n,k)$, respectively.*

Our goal in the first part is to evaluate the cobordism groups $\Sigma^{1_r}(n,k)$ modulo finite 2-primary groups. In the second part we will define and compute the ring structure on the bigraded group

$$\bigoplus_{n,k} Fold^{SO}(n,k) \otimes \mathbb{Q}.$$

2. The unoriented cobordism groups of Morin maps

Theorem 1.

1. *The cobordism group $\Sigma^{1_r}(n,k)$ of Morin maps without singularities Σ^{1_s} for $s > r$ is a finite 2-primary group if*
    - *$r = \infty$ (i.e., we allow all Morin maps),*
    - *r is arbitrary and k is odd,*
    - *$r \not\equiv 0 \bmod 4$ and k is even.*
2. *For $r \equiv 0 \bmod 4$ and k even the rank of the free part of the group $\Sigma^{1_r}(n,k)$ is equal to that of $H^{n-r(k+1)-k}(BO(k);\mathbb{Z})$. (Recall that the latter is equal to the number of partitions of $\frac{n-r(k+1)-k}{4}$ with entries not greater than $k/2$, in particular it is zero when $\frac{n-r(k+1)-k}{4}$ is not an integer).*

This theorem improves [10, Theorem 1]. We do not know whether the groups $\Sigma^{1_r}(n,k)$ have odd torsion in the last case above or not.

The proof will be based on the so called Kazarian conjecture proved by the second author in [11]. In order to formulate this conjecture we recall the so called Kazarian space (considered already by R. Thom). For a given list τ of allowed singularities the Kazarian space $\mathcal{K}_\tau$ is the subset of the universal jet bundle corresponding to the allowed singularities τ. (This space $\mathcal{K}_\tau$ is very useful in computing the so called Thom polynomials giving the homology classes of different singularity strata.) On the other hand the second author constructed a space X_τ, whose homotopy groups give the cobordism groups of the so called τ-maps, i.e., maps with singularities only from the list τ (see [11] and the references there). For τ-maps a universal (virtual) normal bundle can be defined, it is a virtual bundle over the Kazarian space $\mathcal{K}_\tau$ and it will be denoted by ν. The (strengthened version of) Kazarian conjecture says that

$$X_\tau \cong \Omega^\infty S^\infty T\nu$$

where $T\nu$ is the "Thom space" of the virtual bundle ν. Note that although the Thom space of a virtual bundle is not defined, the space $\Omega^\infty S^\infty T\nu$ is well-defined (see [11]).

Recall that the Kazarian space is glued together from "blocks", one for each allowed monosingularity η. The block for η is the total space of the vector bundle over BG_η associated to the source representation λ_η, where G_η is the maximal compact subgroup of the symmetry group of η. We shall denote this vector bundle by ξ_η. Recall from [7],[11] that this bundle is the universal normal bundle of the η-stratum in the source manifold. Analogously, the vector bundle over BG_η associated to the target representation $\tilde\lambda_\eta$ is the universal normal bundle of the η-stratum in the target manifold, and we shall denote this vector bundle by $\tilde\xi_\eta$. For $\eta = \Sigma^{1_r}$ we abbreviate $\xi_{\Sigma^{1_r}}$ to ξ_r and $\tilde\xi_{\Sigma^{1_r}}$ to $\tilde\xi_r$. For Morin maps, these data have been calculated in [9], [10], [7] (more necessary information on the Kazarian spaces $\mathcal{K}_\tau$ will be given in the proof of Theorem 6).

We will first investigate the cohomology of the (stable) Thom space $T\nu$ of the virtual normal bundle ν over the Kazarian space. To obtain that, we use twisted Thom isomorphism to reduce the question to the determination of the cohomology groups of the Kazarian space with coefficients twisted by ν.

Along with the classes of maps mentioned above, we will need to use prim (i.e., *pr*ojected *im*mersion) maps. Recall that a Morin map f is called a *prim map,* if the one dimensional vector bundle $\ker df$ over the set of singular points is trivial, and moreover it is trivialised. The name prim comes from the property that being a prim map is equivalent to being the *pr*ojection of an *im*mersion into the product of the original target manifold with the real line. The analogue of the bundle ξ_r for prim maps will be denoted by $\bar\xi_r$.

3. Calculation

We will consider coefficients from a ring R where there is division by 2, and we will denote the local system twisted by the determinant bundle of some vector bundle ζ by R_ζ. We shall say also that R_ζ is twisted by the class $w_1(\zeta)$.

We will need to compute $H^*(T\nu; R)$, where ν is the virtual normal bundle over the Kazarian space $\mathcal{K}_\tau$. Using the twisted Thom isomorphism we have: $H^*(T\nu; R) \cong H^{*-k}(\mathcal{K}_\tau; R_\nu)$.

We shall consider the Kazarian spectral sequence [11] for prim maps and then for arbitrary Morin maps. Recall that the Kazarian spectral sequence starts from the cohomologies of BG_η and converges to those of $\mathcal{K}_\tau$. This time we need the spectral sequence that converges to the cohomologies of the Kazarian space with coefficients in R_ν.

3.1. **Case of even k ($k =$ codimension of the map).**

First, consider the prim case. We know that in this case the Kazarian spectral sequence converges to the cohomology of the Kazarian space for $(k+1)$-codimensional immersions, which is $BO(k+1)$. This is so because a codimension k prim map can be identified with its codimension $k+1$ lift to an immersion. The virtual normal bundle is hence stably the same as the canonical bundle over $BO(k+1)$. We claim that for k even these cohomology groups vanish.

Lemma 1. *The twisted cohomology $H^*(BO(k+1); R_\nu)$ coincides with the group of classes which are anti-invariant under the deck transformation of the covering map $\pi : BSO(k+1) \to BO(k+1)$ (i.e., those which the deck transformation sends to their negatives) and hence $H^*(BO(k+1); R_\nu)$ is zero when k is even.*

Proof. Note that the local system R_ν is the same as R_π, because $w_1(\nu) = w_1(\pi)$. From the Leray spectral sequence applied to the covering π it follows that

$$H^*(BSO(k+1); R) = H^*(BO(k+1); \pi_*(R)).$$

Here $\pi_*(R)$ is the push-forward of the untwisted local system R on $BSO(k+1)$. Hence this is locally $R \oplus R$ at each point, and the non-trivial loop-class acts on it by interchanging the summands. Therefore it can be decomposed as the sum of the invariant and anti-invariant part: $\pi_*(R) = R \oplus R_\pi$. Thus

$$\begin{aligned} H^*(BSO(k+1); R) &= H^*(BO(k+1); \pi_*(R)) = \\ &= H^*(BO(k+1); R) \oplus H^*(BO(k+1); R_\nu). \end{aligned}$$

Since the groups $H^*(BSO(k+1); R)$ and $H^*(BO(k+1); R)$ are isomorphic (both groups are generated by the Pontrjagin classes) we obtain that $H^*(BO(k+1); R_\nu) = 0$. □

Completely analogously one obtains the twisted cohomologies of $BO(k)$ for k even.

Lemma 2. *If $k = 2t$, then $H^*(BO(k); R_\nu) = \chi \cup R[p_1, \dots, p_t]$, where χ is the twisted Euler class.* □

Now knowing where the Kazarian spectral sequence for prim maps and for k even converges to let us have a look at its starting term.

The E_1 term of the Kazarian spectral sequence can be described in this (prim maps) case as follows. The r-th column contains the twisted cohomology groups of the pair (D_r, S_r), where D_r and S_r are the total spaces of the disc bundle and the sphere bundle associated to the vector bundle $\bar{\xi}_r$, which is the universal source bundle over the Σ^{1_r}-points for prim maps. This universal bundle is $\bar{\xi}_r = r(\gamma^k \oplus \varepsilon^1)$ (over $BO(k)$ as its base space), see [9], [7]. The virtual normal bundle ν over the base space of $\bar{\xi}_r$ is stably the same as the canonical bundle γ^k over $BO(k)$. Hence $w_1(\bar{\xi}_r) = rw_1(\gamma^k) = rw_1(\nu)$.

Using the twisted Thom isomorphism again, we have

$$H^*(D_r, S_r; R_\nu) \cong H^{*-r(k+1)}(BO(k); R_\nu^{r+1}),$$

where $R_\nu^{r+1} = R_\nu \otimes \cdots \otimes R_\nu$, $r+1$ copies. Recall that we consider the case when k is even, say $k = 2t$. Let us denote by A the ring $R[p_1, \ldots, p_t]$. It is well known that the untwisted cohomology ring of $BO(k)$ is isomorphic to this ring A, with p_j identified with the jth Pontrjagin class. Summarizing the computation we see that each column of the E_1 term of the Kazarian spectral sequence is the ring A but shifted differently: in the columns number $2h$ and $2h+1$ it is shifted by $(2h+1)k$. It follows that this spectral sequence must converge to zero in a very controlled way. Namely the first differential d_1 must be an isomorphism between the p-th and $(p+1)$-th column for $p = 0, 2, 4, \ldots$. Indeed for $p = 0$ this follows from the facts that

- the elements of lowest degree in the 0-th column must be mapped onto those in the next column by d_1 isomorphically because this is the only chance for these elements to disappear, and they do disappear since the E_∞ term vanishes. Hence $d_1(\chi) = U$, where U is the twisted Thom class of the bundle $\bar{\xi}_1$, while χ is the twisted Euler class in $H^*(BO(k); R_\nu)$ (note that in this case R_ν and R_γ are isomorphic).
- the differential d_1 is multiplicative in the sense that $d_1(\chi \cup p_I) = d_1(\chi) \cup p_I$, see [11, Section 13.1].

The argument can then be repeated for each even p.

Second case, general (not necessarily prim) Morin maps, $k = 2t$. The previous spectral sequence (for the prim case) has a $\mathbb{Z}_2$ action corresponding to changing the orientation of the kernel bundle, and we need to know what are the eigenspaces corresponding to the two possible eigenvalues of this action. For this, we need to understand what happens with the coefficient system $R_\nu \otimes R_{\xi_r} = R_{\tilde{\xi}_r}$ for various values of r. From [7] one knows that the target representation is

$$\tilde{\lambda}(\varepsilon, Q) = \varepsilon^{r+1} \oplus Q \oplus \left\lceil \frac{r-1}{2} \right\rceil 1 \oplus \left\lfloor \frac{r-1}{2} \right\rfloor \varepsilon \oplus \left\lfloor \frac{r}{2} \right\rfloor Q \oplus \left\lceil \frac{r}{2} \right\rceil \varepsilon Q$$

for $Q \in O(k)$, $\varepsilon \in O(1)$. When ε changes its sign then $\tilde{\lambda}$ changes orientation exactly when $r + 1 + \lfloor \frac{r-1}{2} \rfloor$ is odd, i.e., when $r \equiv 2, 3 \bmod 4$.

When $r \equiv 0, 1 \bmod 4$, then the action discussed above (induced by $-id : \varepsilon \to \varepsilon$) is identical and the columns as well as the differentials between them remain the same as for prim maps. When $r \equiv 2, 3 \bmod 4$, then the action is changing the signs of all cohomology classes, so these columns in the spectral sequence for general Morin maps vanish.

Since the spectral sequences for prim maps and arbitrary Morin maps can be mapped into each other, and for the columns $r \equiv 0, 1 \bmod 4$ will be mapped onto each other isomorphically we obtain that the differential d_1 will be isomorphism again, and so the final E_∞ term vanishes again. We obtain that the cohomology groups of the Kazarian space for arbitrary Morin maps (with coefficients twisted by $w_1(\nu)$) vanish if k is even (under the assumption that in the coefficient ring 2 is invertible).

If we truncate the previous spectral sequence at the column corresponding to Σ^{1_r}, i.e., we consider the spectral sequence for Σ^{1_r}-maps, then the differential $d_1^{p,*}$

still remains an isomorphism between the p-th and $(p+1)$-th column for p even except when $p = r$ and $r \equiv 0 \bmod 4$.

3.2. **Case of odd k.**

Here, the coefficient system on $BO(k)$ is twisted by $w_1(\gamma^k)$, and in the column corresponding to Σ^{1_r} the coefficients are twisted corresponding to

$$w_1(\tilde{\xi}) = \left(r + 1 + \left\lfloor \frac{r-1}{2} \right\rfloor + \left\lceil \frac{r}{2} \right\rceil \right) w_1(\gamma^1) + (r+1)w_1(\gamma^k) =$$

$$= r w_1(\gamma^1) + (r+1)w_1(\gamma^k).$$

In particular, for no column is the coefficient system twisted trivially. Notice that $H^*(BSO(k); R) \cong R[p_1, \ldots, p_{\lfloor k/2 \rfloor}]$ is isomorphic to $H^*(BO(k) \times \mathbb{R}P^\infty; R)$, with the natural projection $BSO(k) \times \mathbb{S}^\infty \to BO(k) \times \mathbb{R}P^\infty$ inducing an isomorphism. This implies that the groups $H^*(BO(k) \times \mathbb{R}P^\infty; R_\zeta)$ are all 0 except when ζ is a trivial line bundle by the same argument as in Lemma 1. So the Kazarian spectral sequence with coefficients in R_ν starts from the empty state $E_1^{**} = 0$ in this case and hence $H^*(\mathcal{K}_{\Sigma^{1_r}}; R_\nu) = 0$ for all $0 \le r \le \infty$.

3.3. **A geometric argument.**

In addition to the previous computations we prove the following proposition:

Theorem 2. *The image of the forgetting map $\rho : Imm(n,k) \to Fold(n,k)$ from the unoriented cobordism group of immersions into that of fold maps contains only elements of order 2 and the zero element. That is, $2\rho(Imm(n,k)) = 0$.*

Proof. Let $i : M^n \to P^{n+k}$ be a k codimensional immersion. Choose a section v of the normal bundle ν_i transverse to the zero section; we consider v as a vector field along i. Our claim is that the map $j : M \times [-1,1] \to P \times [-1,1]$, $j(x,t) = (i(x) + tv(x), t^2)$ is a fold map if v is small enough. At the points where v is nonzero, the map j is clearly an immersion, and hence the only singular points can be of the form $(p, 0)$ with $p \in M$ and $v(p) = 0$. Since v is transverse to the zero section, the set of points p where $v(p) = 0$ is a k codimensional submanifold Z of M and v establishes an isomorphism between the normal bundle of Z in M and the normal bundle of $i(M)$ in P (restricted to Z). Thus for any point p that satisfies $v(p) = 0$ we can choose coordinate neighbourhoods $U \cong \mathbb{R}^k \times \mathbb{R}^{n-k}$ of p and $V \cong \mathbb{R}^k \times \mathbb{R}^k \times \mathbb{R}^{n-k}$ of $i(p)$ such that on U the immersion i has the form $(x,y) \mapsto (0,x,y)$ with $x \in \mathbb{R}^k, y \in \mathbb{R}^{n-k}$, and the vector field v has the form $(0,x,y) \mapsto (x,0,0)$. In these coordinates

$$j(x,y,t) = (i(x,y) + tv(x,y), t^2) = (tx, x, y, t^2)$$

has the normal form of a $\Sigma^{1,0}$ singularity multiplied by $id_{\mathbb{R}^{n-k}}$. Consequently j is a fold map as claimed. The boundary of j consists of two immersions regularly homotopic to i, proving our initial claim. □

Remark: This means that the embedding $\mathcal{K}_{imm} \to \mathcal{K}_{fold}$ of the Kazarian space of immersions into that of fold maps is 0 modulo 2-torsion in (co)homology with coefficients twisted by ν, and there is only a single way of getting this result in the Kazarian spectral sequence, by having the first differential surjective in cohomology (injective in homology). But the ranks of the corresponding groups are the same, so these surjections are actually isomorphisms. Hence we obtained a geometric proof of the fact that we have seen above, that the differential d_1 in the Kazarian spectral sequence maps the zero column isomorphically onto the next column.

4. Proof of Theorem 1

As a consequence of the computations in Section 3, we see that for k even and $r \not\equiv 0 \bmod 4$, as well as for k odd, the stable space $T\nu$ (where ν is the virtual normal bundle over the Kazarian space $\mathcal{K}_\tau$) has the same (co)homology groups modulo 2-primary groups as a contractible space. By the mod $\mathcal{C}$ Hurewicz theorem (due to Serre [8]), this implies that the stable homotopy groups of $T\nu$ are also the same as those of a point modulo 2-primary groups, that is, they are all 2-primary groups. But the stable homotopy groups of $T\nu$ are the homotopy groups of the classifying space $X_\tau \cong \Omega^\infty S^\infty T\nu$. Applying the mod $\mathcal{C}$ Hurewicz theorem again, we obtain the statement of the Theorem.

When k is even and $r \equiv 0 \bmod 4$, we have

$$H^*(T\nu;\mathbb{Q}) \cong H^{*-k}(\mathcal{K}_{\Sigma^{1_r}};\mathbb{Q}_\nu) \cong H^{*-r(k+1)-k}(BO(k);\mathbb{Q}_\nu) \cong$$
$$\cong H^{*-r(k+1)-2k}(BO(k);\mathbb{Q})$$

as the Kazarian spectral sequence degenerates with only column number r being nonzero. Since stable homotopy groups of any space have the same rank as the rational homology of the same space, we have

$$\dim \pi_{n+k}(X_\tau) \otimes \mathbb{Q} = \dim \pi^S_{n+k}(T\nu) \otimes \mathbb{Q} = \dim H_{n+k}(T\nu;\mathbb{Q}) =$$
$$= \dim H^{n-r(k+1)-k}(BO(k);\mathbb{Q})$$

as claimed.

5. Left-right bordism groups of τ-maps.

Definition 2. *The so called left-right bordism groups of τ-maps were defined in* [11]. *In this case we allow to change the target manifold also by a cobordism, and two τ-maps are equivalent (in this case bordant) if their sources and targets are cobordant and there is a τ map from the cobordism between the sources into that of the targets joining the original maps. The corresponding group is denoted by $Bord_\tau(n,k)$. (Here the singularities in the list τ are those of codimension k maps, the sources are n-dimensional and the target manifolds are $(n+k)$-dimensional.)*

Remark: A version of the Pontrjagin - Thom construction for singular maps implies that

$$Bord_\tau(n,k) \approx \mathfrak{N}_{n+k}(X_\tau).$$

These groups are vector spaces over $\mathbb{Z}_2$. In [12] these groups were computed for the simplest set of multisingularities. Here we shall consider the following versions of these groups:

- The targets and their cobordisms are oriented, but the sources might be non-orientable. These groups are denoted by $Bord_\tau^{target-or}(n,k)$. They are isomorphic to the oriented bordism groups of the unoriented classifying spaces X_τ:

$$Bord_\tau^{target-or}(n,k) \cong \Omega_{n+k}(X_\tau).$$

- Both the target and the source, as well as their bordisms, are oriented. These groups are denoted by $Bord_\tau^{SO}(n,k)$. They are isomorphic to the oriented bordism groups of the oriented version of the classifying space X_τ^{SO}:[1]

$$Bord_\tau^{SO}(n,k) \cong \Omega_{n+k}(X_\tau^{SO}).$$

Theorem 3. *Let τ be the collection of all multisingularities of k codimensional maps, $k \geq 2$, composed from*

- *all Morin monosingularities, or*
- *Σ^{1_s}, $s \leq r$ for some $r \geq 0$, $r \not\equiv 0 \bmod 4$ and k is even, or*
- *Σ^{1_s}, $s \leq r$ for some $r \geq 0$, and k is odd.*

Then the τ-bordism groups with oriented target $Bord_\tau^{target-or}(n,k)$ are isomorphic modulo 2-primary groups to Ω_{n+k}, the oriented cobordism group of $(n+k)$-manifolds. The mapping $Bord_\tau^{target-or}(n,k) \to \Omega_{n+k}$ that associates to a map the cobordism class of its target is a mod 2 *isomorphism.*

Proof. The bordism groups $Bord_\tau^{target-or}(n,k) \cong \Omega_{n+k}(X_\tau)$ can be computed using the Atiyah-Hirzebruch spectral sequence (see [1]). The first page of the spectral sequence is $H_p(X_\tau;\Omega_q)$. Since the homotopy groups of X_τ are 2-primary groups and the space X_τ is $(k-1)$-connected, its reduced integral homology groups are also finite 2-primary groups for $k \geq 2$; hence the first page modulo 2-primary groups vanishes apart from the first column (that is, $p=0$), which corresponds to the cobordism class of the target. The spectral sequence degenerates and we get the statement of the theorem. □

Remark: For $k=1$ the space X_τ might be (and will be) non-simply connected and therefore we cannot use the mod $\mathcal{C}$ Hurewicz theorem of [8] to deduce that its homology groups are 2-primary groups.

The remaining case when k is even and r is divisible by 4 can be handled more conveniently with the following notation. Let A^* be a graded Abelian group of finite type, $A^* = \underset{j}{\oplus} A^j$. Denote by $\mathrm{SP}(A^*)$ the graded skew-commutative ring multiplicatively freely generated by the additive generators of A^*. That is,

$$\mathrm{SP}(A^*) = \left(\wedge(\underset{j \text{ odd}}{\oplus} A^j)\right) \otimes \left(Sym(\underset{j \text{ even}}{\oplus} A^j)\right),$$

where for a vector space V the symmetric algebra $Sym(V)$ is defined as

$$Sym(V) = \bigoplus_{j=0}^{\infty} V^{\otimes j} / (v_1 \otimes \cdots \otimes v_j - v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(j)})_{v_1,\ldots,v_j \in V, \sigma \in S_j},$$

and $\wedge V$ is the free skew algebra generated by V.

Recall that for a topological space X the infinite symmetric power of X is also denoted by $\mathrm{SP}\,X$.

Lemma 3.

$$H^*(\mathrm{SP}\,X;\mathbb{Q}) \cong \mathrm{SP}\,H^*(X;\mathbb{Q}).$$

[1]The space X_τ was not considered in [11], and X_τ^{SO} was denoted there by X_τ.

Proof. By [3, page 472] the space $\mathrm{SP}\, X$ is weakly homotopically equivalent to $\prod_{j\geq 0} K(H_j(X;\mathbb{Z}),j)$. Hence the rational homology groups of $\mathrm{SP}\, X$ are those of $\prod_{j\geq 0} K(\mathbb{Z}^{b_j},j)$, where b_j is the rank of $H_j(X;\mathbb{Z})$. It is a well-known result of Serre (see e.g. [2]) that $H^*(K(\mathbb{Z},j);\mathbb{Q})$ is generated by a j-dimensional free, respectively skew generator depending on whether j is even or odd. Hence the left hand side of the statement of the Lemma is the free skew symmetric algebra on the generators of $H^*(X;\mathbb{Q})$, and this coincides with the right hand side by definition. □

Theorem 4. *Let τ be the collection of all multisingularities of k codimensional maps composed from Σ^{1_s}, $s\leq r$ where k is even and r is divisible by 4. Then the free part of the τ-bordism group with oriented target has dimension*

$$\dim Bord_\tau^{target-or}(n,k)\otimes\mathbb{Q} \;=\; \dim\left(\mathrm{SP}\left(H^*(T\nu;\mathbb{Q})\right)\otimes\Omega_*\right)_{n+k} \;=$$

$$\sum_{\frac{n+k}{4}=j+\sum_l la_l} q_j \prod_{\substack{l\\ a_l>0}} \binom{q_{l-\frac{(k+1)r+2k}{4}}+a_l-1}{a_l},$$

where q_m is the number of partitions of m into positive integers.

Proof. Just as before, we have

$$Bord_\tau^{target-or}(n,k)\otimes\mathbb{Q}\cong\Omega_{n+k}(X_\tau)\otimes\mathbb{Q}\cong\sum_{i=0}^{n+k}H_{n+k-i}(X_\tau;\mathbb{Q})\otimes\Omega_i\cong$$
$$\cong\sum_{j=0}^{\lfloor\frac{n+k}{4}\rfloor}H_{n+k-4j}(\Gamma T\nu;\mathbb{Q})\otimes\mathbb{Q}^{q_j}.$$

For any virtual cell complex L the spaces ΓL and $\mathrm{SP}\, L$ are rationally homotopy equivalent, see e.g. [11, Lemma 81]. In particular, their rational homology groups are isomorphic and we can replace $\Gamma T\nu$ by $\mathrm{SP}\, T\nu$ in the last expression above, obtaining the first claimed equality. Extend the notation q_α by setting $q_\alpha=0$ if α is not an integer. Then

$$\dim H_m(T\nu;\mathbb{Q})=\dim H^{m-(k+1)r-2k}(BO(k);\mathbb{Q})=q_{\frac{m-(k+1)r-2k}{4}};$$

note that in particular $4\mid r$ and $2\mid k$ imply that $\dim H_m(T\nu;\mathbb{Q})=0$ unless $4\mid m$. Therefore, for any n, we have

$$\dim Bord^{target-or}(n,k)\otimes\mathbb{Q}=$$

$$\sum_{j=0}^{\lfloor\frac{n+k}{4}\rfloor}\dim\left(\left(\wedge\left(\bigoplus_{u\text{ odd}}H^u(T\nu;\mathbb{Q})\right)\otimes Sym\left(\bigoplus_{u\text{ even}}H^u(T\nu;\mathbb{Q})\right)\right)_{n+k-4j}\otimes\mathbb{Q}^{q_j}\right)=$$

$$\sum_{j=0}^{\lfloor\frac{n+k}{4}\rfloor}\sum_{\substack{n+k-4j=\\ \sum ub_u}} q_j\prod_{\substack{u\text{even}\\ b_u>0}}\binom{\dim H_u(T\nu;\mathbb{Q})+b_u-1}{b_u}=$$

$$\sum_{\substack{n+k=\\4j+\sum 4la_l}} q_j \prod_{\substack{l\\a_l>0}} \binom{\dim H_{4l}(T\nu;\mathbb{Q})+a_l-1}{a_l} =$$

$$\sum_{\frac{n+k}{4}=j+\sum la_l} q_j \prod_{\substack{l\\a_l>0}} \binom{q_{l-\frac{(k+1)r+2k}{4}}+a_l-1}{a_l},$$

which is our second claimed equality. □

6. Ring structure on the direct sum of oriented cobordism groups of fold maps $\bigoplus_{n,k} Fold^{SO}(n,k)\otimes\mathbb{Q}$

Let us recall first Wells' theorem from [13] on the ring of immersions. By the product of two immersion-cobordism classes $[f : M^m \to \mathbb{R}^{m+k}]$ and $[g : N^n \to \mathbb{R}^{n+l}]$ we mean the cobordism class of the product of the representatives[2]: $[f : M^m \to \mathbb{R}^{m+k}] \times [g : N^n \to \mathbb{R}^{n+l}] = [f \times g : M^m \times N^n \to \mathbb{R}^{m+k} \times \mathbb{R}^{n+l}]$. Let s_i be the characteristic class corresponding to the symmetric polynomial $x_1^{2i} + x_2^{2i} + \dots$, where the total Pontrjagin class is $1 + p_1 + p_2 + \dots = \prod_j (1 + x_j^2)$, see [4].

Theorem 5 (Wells, [13])**.**

$$\bigoplus_{n,k} \mathrm{Imm}(n,k)\otimes\mathbb{Q} = \mathbb{Q}\big[[f_0],[f_1],\dots\big],$$

where $[f_i]$ is the cobordism class of an immersion $f_i : M^{4i+2} \looparrowright \mathbb{R}^{4i+4}$ such that $\langle e \cup s_i(p_1,\dots,p_i),[M]\rangle \neq 0$. Here e denotes the twisted normal Euler class of f_i, and $[M]$ is the twisted integer valued fundamental class of the unoriented manifold M, finally s_i is the characteristic class described above.

Definition 3. *Given two cobordism classes of oriented fold maps $[f : M^m \to \mathbb{R}^{m+k}]$ and $[g : N^n \to \mathbb{R}^{n+l}]$ we define their product as follows. The representatives f and g can be chosen to be immersions. Their product is an immersion, and we define $[f] \times [g]$ to be the oriented fold-cobordism class of this product.*

Theorem 6.

a) The above definition does not depend on the involved choices, that is, it gives a well-defined product on the direct sum $\bigoplus_{n,k} Fold^{SO}(n,k)\otimes\mathbb{Q}$.

b)

$$\bigoplus_{n,k} Fold^{SO}(n,k)\otimes\mathbb{Q} = \mathbb{Q}[g_0,h_1,h_2,\dots],$$

where $h_i : \mathbb{C}P^{2i} \to \mathbb{R}^{6i}$ is any generic map, and g_0 is the inclusion of a point into the line.

Proof. Part a) follows from the facts that:

1) the natural map $H_n(BSO(k);\mathbb{Q}) \to H_n(BO(k);\mathbb{Q})$ is onto;
2) $Fold^{SO}(n,k)\otimes\mathbb{Q} \cong H_n(BO(k);\mathbb{Q})$ (see [11]); and

[2] In [6] a different multiplication was considered: $f * g : M^m \times N^n \to R^{m+k} \times R^{n+l} \times R^1$. For arbitrary Morin maps only that definition seemed to be possible, besides this operation made the singularities multpilicative. Here we consider the most natural product operation, it turns out to be possible to define it for *fold* maps, more precisely for their rational cobordism classes.

3) $Imm^{SO}(n,k)\otimes\mathbb{Q} \cong H_n(BSO(k);\mathbb{Q})$ (see e.g. the proof of Corollary 7 below).

Part b). We have seen in [11] that the Kazarian space $\mathcal{K}_\tau$ for $\tau = \{[\Sigma^0],[\Sigma^{1,0}]\}$ (i.e., $\mathcal{K}_\tau = \mathcal{K}_{fold}$) has the same rational homology groups as $BO(k)$. Recall that

$$H^*(BO(k);\mathbb{Q}) = \mathbb{Q}\big[p_1,\dots,p_{\left[\frac{k}{2}\right]}\big] = \mathbb{Q}\big[s_1,\dots,s_{\left[\frac{k}{2}\right]}\big].$$

It is well known [4] that the cobordism class of a manifold M^{4i} is irreducible in $\Omega_*\otimes\mathbb{Q}$ if and only if $s_i[M^{4i}] \neq 0$. In particular the even complex dimensional projective spaces $\mathbb{C}P^{2i}$ satisfy this property, hence $\Omega_*\otimes\mathbb{Q} = \mathbb{Q}[[\mathbb{C}P^2],[\mathbb{C}P^4],\dots]$.

For the convenience of the reader we give here a short summary of the properties of the Kazarian spaces that we use. We deal with the case of cooriented maps, for the unoriented version of the space replace all occurrences of the group SO with O. The space $\mathcal{K}_\tau$ depends on the set τ, which is the list of allowed singularities. Recall that a τ-map is a map such that all its singularities have a type from the list τ. The space $\mathcal{K}_\tau$ is universal in the following sense. It has a stratification according to the singularity types, each stratum corresponds to an element of τ. Further for each cooriented τ-map $f: M^n \to P^{n+k}$ there arises a map $\kappa_f : M^n \to \mathcal{K}_\tau$ of the source manifold M into the Kazarian space. This map κ_f is transverse to each stratum of $\mathcal{K}_\tau$, hence the preimages of the strata induce a stratification on M which coincides with the one induced by f (that is, the pulled-back strata coincide with the singularity strata of f). Additionally, $\mathcal{K}_\tau$ is the total space of a fibre bundle over BSO,

$$\pi : \mathcal{K}_\tau \to BSO.$$

Pulling back the canonical bundle γ (the limit $\lim\limits_{m\to\infty}\gamma^m$) to $\mathcal{K}_\tau$ one obtains the bundle $\nu = \pi^*\gamma$, which is the universal virtual normal bundle of a τ-map in the sense outlined below. If $f: M^n \to P^{n+k}$ is a τ-map, and ν_f is its virtual normal bundle, then

$$\kappa_f^*\nu \cong \nu_f.$$

Hence there is a homotopically commutative diagram

$$\begin{tikzcd} & \mathcal{K}_\tau \arrow[d, "\pi"] \\ M \arrow[ur, "\kappa_f"] \arrow[r, "\nu_f"] & BSO \end{tikzcd}$$

so κ_f is a lift of the map $\nu_f : M \to BSO$ to $M \to \mathcal{K}_\tau$.

Now recall that in [11] for any τ-map $f: M^n \to \mathbb{R}^{n+k}$ and for any $x \in H^n(\mathcal{K}_\tau;\mathbb{Q})$ the x-characteristic number of f, that is, $x[f] \overset{\text{def}}{=\!=} \langle \kappa_f^*(x),[M^n]\rangle$ has been defined and that these characteristic numbers give an isomorphism:

$$\begin{aligned} H_n(\mathcal{K}_\tau;\mathbb{Q}) = \operatorname{Hom}(H^n(\mathcal{K}_\tau;\mathbb{Q}),\mathbb{Q}) &\longleftarrow Cob_\tau(n,k)\otimes\mathbb{Q} \\ (x \longmapsto x[f]) \quad &\longleftarrow\!\shortmid \quad [f]. \end{aligned}$$

If $h_i : \mathbb{C}P^{2i} \to \mathbb{R}^{6i}$ is any generic map, then it is a fold map. Suppose that $k \geq 2i$. Let us consider the map $h_i^{(k)} = g_0^{k-2i}\cdot h_i$, i.e., the composition $\mathbb{C}P^{2i} \xrightarrow{h_i} \mathbb{R}^{6i} \hookrightarrow \mathbb{R}^{4i+k}$. In [11] we have shown that for $\tau = \{\Sigma^0,\Sigma^{1,0}\}$ (when τ-maps are precisely the fold maps) the space $\mathcal{K}_\tau$ has the same rational homotopy groups as $BO(k)$ does, and $\pi : \mathcal{K}_\tau \to BSO$ is the standard mapping $BO(k) \to BSO$ (induced by $O(k) \overset{(id,\det)}{\to} SO(k+1) \to SO$) up to a rational homotopy. Hence π^* maps all the

Pontrjagin classes $p_i \in H^{4i}(BSO;\mathbb{Q})$ to those in $H^{4i}(BO(k);\mathbb{Q})$ for $i \leq \frac{k}{2}$, while for $i > \frac{k}{2}$ the class p_i goes to zero.

Beyond the characteristic classes s_i mentioned in the previous proof we shall also consider the classes $\bar{s}_i$ that we define as $s_i(\bar{p}_1,\dots)$, where $\bar{p}_j$ are the normal Pontrjagin classes defined by $1+\bar{p}_1+\bar{p}_2+\dots=(1+p_1+p_2+\dots)^{-1}$.

Lemma 4. *For $x=\bar{s}_i$, that is, $x(p_1,\dots)=s_i(\bar{p}_1,\dots)$, the x-characteristic number of $h_i^{(k)}$ is*

$$\bar{s}_i[h_i^{(k)}]=s_i[\mathbb{C}P^{2i}]\neq 0.$$

Proof.

$$\begin{aligned}\langle s_i(p_1,\dots,p_m),[\mathbb{C}P^{2i}]\rangle &= \langle \nu_f^* s_i(\bar{p}_1,\dots,\bar{p}_m),[\mathbb{C}P^{2i}]\rangle = \\ &= \langle \kappa_f^*\pi^* s_i(\bar{p}_1,\dots,\bar{p}_m),[\mathbb{C}P^{2i}]\rangle = \langle \kappa_f^* s_i(\bar{p}_1,\dots,\bar{p}_m),[\mathbb{C}P^{2i}]\rangle = \\ &= \langle \bar{s}_i(\kappa_f^* p_1,\dots,\kappa_f^* p_m),[\mathbb{C}P^{2i}]\rangle = \bar{s}_i[h_i^{(k)}].\end{aligned}$$

□

By the multiplicative property of the classes $\bar{s}_I=\bar{s}_{i_1}\dots\bar{s}_{i_r}$ we have that the cobordism classes of fold maps $g_0^{k-2|I|}h_I$, where $I=(i_1,\dots,i_r)$ is a multiindex $h_I=h_{i_1}\times h_{i_2}\times\dots\times h_{i_r}$ and $|I|=i_1+\dots+i_r$, are linearly independent. Indeed, the matrix $a_{JI}=\left(\bar{s}_J\left[g_0^{k-2|I|}\cdot h_I\right]\right)$ is non-degenerated. Here both multiindices J and I run over all the partitions of all the numbers $0,1,2,\dots,\left[\frac{k}{2}\right]$. Hence

$$\bigoplus_{n,k} Cob_{\Sigma^{1,0}}(n,k)\otimes\mathbb{Q}\cong\mathbb{Q}[g_0,h_1,h_2,\dots].$$

□

Corollary 7. *The ring $\bigoplus_{n,k} Imm^{SO}(n,k)\otimes\mathbb{Q}$ is isomorphic to the direct sum of the rings $\mathbb{Q}[g_0,h_1,h_2,\dots]$ and $\mathbb{Q}[f_0,f_1,f_2,\dots]$. Here f_i is the map defined in Wells' theorem. If $\alpha\in\mathbb{Q}[g_0,h_1,h_2,\dots]$ and $\beta\in\mathbb{Q}[f_0,f_1,f_2,\dots]$, then $\alpha\cdot\beta=0$.*

Proof. By Wells' theorem $Imm^{SO}(n,k)\approx\pi^s_{n+k}(T\gamma_k^{SO})$, where γ_k^{SO} is the universal oriented bundle of rank k. By Serre's theorem

$$\pi^s_{n+k}(T\gamma_k^{SO})\otimes\mathbb{Q}\approx H_{n+k}(T\gamma_k^{SO};\mathbb{Q}).$$

By the Thom isomorphism this is isomorphic to $H_n(BSO(k);\mathbb{Q})$, which is isomorphic to $H^n(BSO(k);\mathbb{Q})$, since $\mathbb{Q}$ is a field. Now let us consider again the double cover $\pi: BSO(k)\to BO(k)$ and the decomposition arising from it (see Lemma 1):

$$H_*(BSO(k);\mathbb{Q})=H_*(BO(k);\mathbb{Q})\oplus H_*(BO(k);\mathbb{Q}_\pi).$$

Wells has shown that the second summand is isomorphic to the cobordism group of unoriented immersion of n-manifolds into $\mathbb{R}^{n+k}$, while in [11] it has been shown that the first summand is isomorphic to the oriented cobordism group of fold maps of oriented n-manifolds into $\mathbb{R}^{n+k}$.

Hence the theorem holds at least additively. But then we have also an isomorphism of rings if we define the multiplication on the direct sum as described in the theorem. This follows from the following obvious lemma.

Lemma 5. *Let A, B, C be rings. Let $\varphi : A \to B$ and $\psi : A \to C$ be ring epimorphisms such that their direct sum $\varphi \oplus \psi : A \to B \oplus C$ is an additive isomorphism. Then $\varphi \oplus \psi$ is a ring isomorphism between A and $B \oplus C$, where the product on the latter is defined so that for $\beta \in B$ and $\gamma \in C$ the equality $\beta \cdot \gamma = 0$ holds.*

Proof. The obtained map is obviously a ring homomorphism and additive isomorphism, hence it is a ring isomorphism. □

□

References

[1] P.E. Conner, E.E. Floyd: Differentiable periodic maps, Springer, Berlin, 1964.
[2] A.T. Fomenko, D.B. Fuchs, V.L. Gutenmacher: Homotopic topology, Akadémiai Kiadó, Budapest, 1986.
[3] A. Hatcher: Algebraic Topology I., http://www.math.cornell.edu/~hatcher/AT/ATpage.html
[4] J.W. Milnor, J.D. Stasheff: Characteristic classes, Princeton University Press, 1974.
[5] B. Morin: Formes canoniques des singularités d'une application différentiable, C.R. Acad. Sci. Paris. 1965. V.260. p. 5662-5665, 6503-6506.
[6] G. Lippner, A. Szűcs: Multiplicative properties of Morin maps, Algebraic and Geometric Topology 10/3 (2010), 1437–1454.
[7] R. Rimányi, A. Szűcs: Pontrjagin–Thom-type construction for maps with singularities, Topology 37 (1998), 1177–1191. DOI: 10.1016/S0040-9383(97)00093-1
[8] J. P. Serre: Groupes d'homotopie et classes de groupes abeliens, Ann. Math. (2) 58 (1953), 258–294. DOI: 10.2307/1969789
[9] A. Szűcs: On the cobordism groups of immersions and embeddings, Math. Proc. Cambr. Phil. Soc. 109 (1991), 343–349. DOI: 10.1017/S0305004100069802
[10] A. Szűcs: On the cobordism group of Morin maps, Acta Math. Hungar. 80/3 (1998), 191–209. DOI: 10.1023/A:1006598609305
[11] A. Szűcs: Cobordism of singular maps, Geometry & Topology 12(4) (2008), 2379–2452. DOI: 10.2140/gt.2008.12.2379
[12] A. Szűcs, T. Terpai: Bordism groups of fold maps, Acta Math. Hungar. 126(4) (2009), 334–351. DOI: 10.1007/s10474-009-9070-5
[13] R. Wells: Cobordism groups of immersions, Topology 5 (1966), 281–294. DOI: 10.1016/0040-9383(66)90011-5

Alfréd Rényi Institute of Mathematics, 13-15 Reáltanoda u. 1053 Budapest, Hungary
E-mail address: `endre@renyi.hu`

Department of Analysis, Eötvös Loránd University, 1/C Pázmány Péter sétány 1117 Budapest, Hungary
E-mail address: `szucs@math.elte.hu`

Alfréd Rényi Institute of Mathematics, 13-15 Reáltanoda u. 1053 Budapest, Hungary
E-mail address: `terpai@renyi.hu`