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APERTURE OF PLANE CURVES

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ABSTRACT. For any given C^∞ immersion $\mathbf{r} : S^1 \to \mathbb{R}^2$ such that the set

$$\mathcal{NS}_{\mathbf{r}} = \mathbb{R}^2 - \cup_{s\in S^1}\left(\mathbf{r}(s) + d\mathbf{r}_s(T_s(S^1))\right)$$

is not empty, a simple geometric model of crystal growth is constructed. It is shown that our geometric model of crystal growth never formulates a polygon while it is growing. Moreover, it is shown also that our model always dissolves to a point.

1. Introduction

Let $\mathbf{r} : S^1 \to \mathbb{R}^2$ be a C^∞ immersion such that the set

$$\mathbb{R}^2 - \bigcup_{s\in S^1}\left(\mathbf{r}(s) + d\mathbf{r}_s(T_s(S^1))\right) \tag{1.1}$$

is not the empty set, where $T_{\mathbf{r}(s)}\mathbb{R}^2$ is identified with $\mathbb{R}^2$. The perspective projection of the given plane curve $\mathbf{r}(S^1)$ from any point of (1.1) does not give the silhouette of $\mathbf{r}(S^1)$ because it is non-singular. By this reason, the set (1.1) is called the *no-silhouette* of $\mathbf{r}$ and is denoted by $\mathcal{NS}_{\mathbf{r}}$ (see Figure 1). The notion of no-silhouette was first defined and studied from the viewpoint

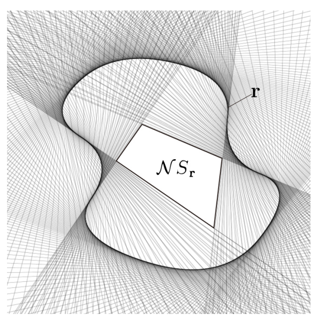

FIGURE 1. The no-silhouette $\mathcal{N}S_{\mathbf{r}}$.

of perspective projection in [10]. In [11] it has been shown that the topological closure of no-silhouette is a Wulff shape, which is the well-known geometric model of crystal at equilibrium introduced by G. Wulff in [14].

In this paper, we show that by rotating all tangent lines about their tangent points simultaneously with the same angle, we always obtain a geometric model of crystal growth (Proposition

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6), our model never formulates a polygon while it is growing (Theorem 1), our model always dissolves to a point (Theorems 2), and our model is growing in a relatively simple way when the given $\mathbf{r}$ has no inflection points (Theorem 3).

For any C^∞ immersion $\mathbf{r} : S^1 \to \mathbb{R}^2$ and any real number θ, define the new set

$$\mathcal{NS}_{\theta,\mathbf{r}} = \mathbb{R}^2 - \bigcup_{s \in S^1} \left(\mathbf{r}(s) + R_\theta\left(d\mathbf{r}_s(T_s(S^1))\right)\right),$$

where $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ is the rotation defined by $R_\theta(x, y) = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$ (see Figure 2). When the given $\mathbf{r}$ has its no-silhouette $\mathcal{NS}_{\mathbf{r}}$, by definition, it follows that

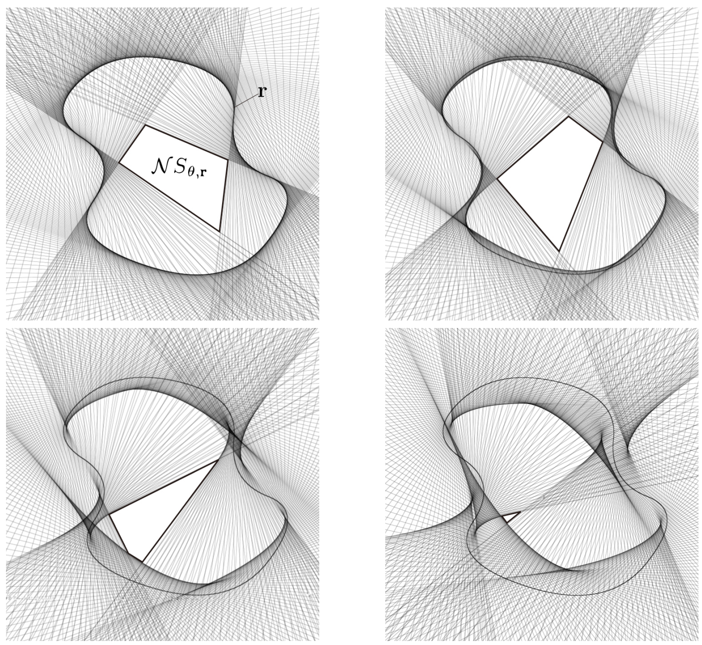

FIGURE 2. $\mathcal{N}S_{\theta,\mathbf{r}}$ for several θs. Left top : $\theta = 0$, right top : $\theta = \pi/12$, left bottom : $\theta = \pi/6$, right bottom : $\theta = \pi/4$.

$\mathcal{NS}_{\mathbf{r}} = \mathcal{NS}_{0,\mathbf{r}}$.

Lemma 1.1. *For any C^∞ immersion $\mathbf{r} : S^1 \to \mathbb{R}^2$, $\mathcal{NS}_{\frac{\pi}{2},\mathbf{r}}$ is the empty set.*

Proof of Lemma 1.1 For any point $P \in \mathbb{R}^2$, let $F_P : S^1 \to \mathbb{R}$ be the function defined by

$$F_P(s) = (P - \mathbf{r}(s)) \cdot (P - \mathbf{r}(s)), \tag{1.2}$$

where the dot in the center stands for the scalar product of two vectors. Since F_P is a C^∞ function and S^1 is compact, there exist the maximum and the minimum of the set of images $\left\{F_P(s) \mid s \in S^1\right\}$. Let s_1 (resp., s_2) be a point of S^1 at which F_P attains its maximum (resp., minimum). Then, both s_1 and s_2 are critical points of F_P. Thus, differentiating (1.2) with respect to s yields that the vector $(P - \mathbf{r}(s_i))$ is perpendicular to the tangent line to $\mathbf{r}$ at $\mathbf{r}(s_i)$. It follows that $P \in \left(\mathbf{r}(s_i) + R_{\frac{\pi}{2}}(d\mathbf{r}_{s_i}(T_{s_i}S^1)\right)$. □

In Section 2, it turns out that with respect to the Pompeiu-Hausdorff metric the topological closure of $\mathcal{NS}_{\theta,\mathbf{r}}$ varies continuously depending on θ while $\mathcal{NS}_{\theta,\mathbf{r}}$ is not empty (Proposition 7). Therefore, by Lemma 1.1, the following notion of aperture angle $\theta_{\mathbf{r}}$ ($0 < \theta_{\mathbf{r}} \leq \frac{\pi}{2}$) is well-defined.

Definition 1. Let $\mathbf{r} : S^1 \to \mathbb{R}^2$ be a C^∞ immersion with its no-silhouette $\mathcal{NS}_{\mathbf{r}}$. Then, $\theta_{\mathbf{r}}$ $(0 < \theta_{\mathbf{r}} \le \frac{\pi}{2})$ is defined as the largest angle which satisfies $\mathcal{NS}_{\theta,\mathbf{r}} \neq \emptyset$ for any θ $(0 \le \theta < \theta_{\mathbf{r}})$. The angle $\theta_{\mathbf{r}}$ is called the *aperture angle* of the given $\mathbf{r}$.

In Section 2, it turns out also that $\overline{\mathcal{NS}_{\theta,\mathbf{r}}}$ is a Wulff shape for any θ $(0 \le \theta < \theta_{\mathbf{r}})$, where $\overline{\mathcal{NS}_{\theta,\mathbf{r}}}$ stands for the topological closure of $\mathcal{NS}_{\theta,\mathbf{r}}$ (Proposition 6). We are interested in how the Wulff shape $\overline{\mathcal{NS}_{\theta,\mathbf{r}}}$ dissolves as θ goes to $\theta_{\mathbf{r}}$ from 0.

Theorem 1. *Let $\mathbf{r} : S^1 \to \mathbb{R}^2$ be a C^∞ immersion with its no-silhouette $\mathcal{NS}_{\mathbf{r}}$. Then, for any θ $(0 < \theta < \theta_{\mathbf{r}})$, $\overline{\mathcal{NS}_{\theta},\mathbf{r}}$ is never a polygon even if the given $\overline{\mathcal{NS}_{\mathbf{r}}}$ is a polygon.*

By Theorem 1, none of $\overline{\mathcal{NS}_{\frac{\pi}{12},\mathbf{r}}}$, $\overline{\mathcal{NS}_{\frac{\pi}{6},\mathbf{r}}}$ and $\overline{\mathcal{NS}_{\frac{\pi}{4},\mathbf{r}}}$ in Figure 2 is a polygon although $\overline{\mathcal{NS}_{0,\mathbf{r}}}$ is a polygon constructed by four tangent lines to $\mathbf{r}$ at four inflection points.

Theorem 2. *Let $\mathbf{r} : S^1 \to \mathbb{R}^2$ be a C^∞ immersion with its no-silhouette $\mathcal{NS}_{\mathbf{r}}$. Then, there exists the unique point $P_{\mathbf{r}} \in \mathbb{R}^2$ such that, for any sequence $\{\theta_i\}_{i=1,2,\ldots} \subset [0, \theta_{\mathbf{r}})$ satisfying $\lim_{i\to\infty} \theta_i = \theta_{\mathbf{r}}$, the following holds:*

$$\lim_{i\to\infty} d_H(\overline{\mathcal{NS}_{\theta_i},\mathbf{r}}, P_{\mathbf{r}}) = 0.$$

.

Here, $d_H : \mathcal{H}(\mathbb{R}^2) \times \mathcal{H}(\mathbb{R}^2) \to \mathbb{R}$ is the Pompeiu-Hausdorff metric (for the Pompeiu-Hausdorff metric, see Section 2). Theorem 2 justifies the following definition.

Definition 2. Let $\mathbf{r} : S^1 \to \mathbb{R}^2$ be a C^∞ immersion with its no-silhouette $\mathcal{NS}_{\mathbf{r}}$. Then, the set $\cup_{\theta\in[0,\theta_{\mathbf{r}})}\overline{\mathcal{NS}_{\theta,\mathbf{r}}}$ is called the *aperture* of $\mathbf{r}$ and the unique point $P_{\mathbf{r}} = \lim_{\theta\to\theta_{\mathbf{r}}} \overline{\mathcal{NS}_{\theta,\mathbf{r}}}$ is called the *aperture point* of $\mathbf{r}$. Here, $\theta_{\mathbf{r}}$ $(0 < \theta_{\mathbf{r}} \le \frac{\pi}{2})$ is the aperture angle of $\mathbf{r}$.

The simplest example is a circle. The aperture of a circle is the topological closure of its inside region and the aperture point of it is its center. In this case, the aperture angle is $\pi/2$. In general, in the case of curves with no inflection points, the crystal growth is relatively simpler than in the case of curves with inflections as follows.

Theorem 3. *Let $\mathbf{r} : S^1 \to \mathbb{R}^2$ be a C^∞ immersion with its no-silhouette $\mathcal{NS}_{\mathbf{r}}$. Suppose that $\mathbf{r}$ has no inflection points. Then, for any two θ_1, θ_2 satisfying $0 \le \theta_1 < \theta_2 < \theta_{\mathbf{r}}$, the following inclusion holds:*

$$\mathcal{NS}_{\theta_1,\mathbf{r}} \supset \mathcal{NS}_{\theta_2,\mathbf{r}}.$$

Figure 2 shows that in general it is impossible to expect the same property for a curve with inflection points.

In Section 2, preliminaries are given. Theorems 1, 2 and 3 are proved in Sections 3, 4 and 5 respectively.

2. Preliminaries

2.1. **Spherical curves.** Let $\widetilde{\mathbf{r}} : S^1 \to S^2$ be a C^∞ immersion. Let $\widetilde{\mathbf{t}} : S^1 \to S^2$ be the mapping defined by

$$\widetilde{\mathbf{t}}(s) = \frac{\widetilde{\mathbf{r}}'(s)}{||\widetilde{\mathbf{r}}'(s)||},$$

where $\widetilde{\mathbf{r}}'(s)$ stands for differentiating $\widetilde{\mathbf{r}}(s)$ with respect to $s \in S^1$. Let $\widetilde{\mathbf{n}} : S^1 \to S^2$ be the mapping defined by

$$\det\left(\widetilde{\mathbf{r}}(s), \widetilde{\mathbf{t}}(s), \widetilde{\mathbf{n}}(s)\right) = 1.$$

The mapping $\widetilde{\mathbf{n}} : S^1 \to S^2$ is called the *spherical dual of* $\widetilde{\mathbf{r}}$. The singularities of $\widetilde{\mathbf{n}}$ belong to the class of Legendrian singularities which are relatively well-investigated (for instance, see [1, 2, 3]). Let U be an open arc of S^1. Suppose that $||\widetilde{\mathbf{r}}'(s)|| = 1$ for any $s \in U$. Then, for the orthogonal moving frame $\{\mathbf{r}(s), \mathbf{t}(s), \mathbf{n}(s)\}$, $(s \in U)$, the following Serre-Frenet type formula has been known.

Lemma 2.1 ([7, 8]).

$$\begin{cases} \widetilde{\mathbf{r}}'(s) &= \widetilde{\mathbf{t}}(s) \\ \widetilde{\mathbf{t}}'(s) &= -\widetilde{\mathbf{r}}(s) + \kappa_g(\theta)\widetilde{\mathbf{n}}(s) \\ \widetilde{\mathbf{n}}'(s) &= -\kappa_g(\theta)\widetilde{\mathbf{t}}(s). \end{cases}$$

Here, $\kappa_g(\theta)$ is defined by

$$\kappa_g(\theta) = \det\left(\widetilde{\mathbf{r}}(s), \widetilde{\mathbf{t}}(s), \widetilde{\mathbf{t}}'(s)\right).$$

Let N be the north pole $(0,0,1)$ of the unit sphere $S^2 \subset \mathbb{R}^3$ and let $S^2_{N,+}$ be the northern hemisphere $\{P \in S^2 \mid N \cdot P > 0\}$, where $N \cdot P$ stands for the scalar product of two vectors $N, P \in \mathbb{R}^3$. Then, define the mapping $\alpha_N : S^2_{N,+} \to \mathbb{R}^2 \times \{1\}$, which is called the *central projection*, as follows:

$$\alpha_N(P_1, P_2, P_3) = \left(\frac{P_1}{P_3}, \frac{P_2}{P_3}, 1\right),$$

where $P = (P_1, P_2, P_3) \in S^2_{N,+}$. Let $\mathbf{r} : S^1 \to \mathbb{R}^2$ be a C^∞ immersion. Then, from $\mathbf{r}$ we can naturally obtain a spherical curve $\widetilde{\mathbf{r}} : S^1 \to S^2$ as follows:

$$\widetilde{\mathbf{r}} = \alpha_N^{-1} \circ Id \circ \mathbf{r},$$

where $Id : \mathbb{R}^2 \to \mathbb{R}^2 \times \{1\}$ is the mapping defined by $Id(P) = (P, 1)$. For any $s \in S^1$, let $GC_{\widetilde{\mathbf{r}}(s)}$ be the intersection $(\mathbb{R}\widetilde{\mathbf{r}}(s) + \mathbb{R}\widetilde{\mathbf{t}}(s)) \cap S^2$. The following clearly holds:

Lemma 2.2. *By the central projection $\alpha_N : S^2_{N,+} \to \mathbb{R}^2 \times \{1\}$, $GC_{\widetilde{\mathbf{r}}(s)} \cap S^2_{N,+}$ is mapped to the line $\mathbf{r}(s) + d\mathbf{r}_s(T_s(S^1))$.*

One of the merit of considering inside the sphere S^2 is the following:

Lemma 2.3 ([10]). *Let $\widetilde{\mathbf{r}} : S^1 \to S^2$ be a Legendrian mapping. Then, the following two are equivalent conditions.*

(1) *The set*

$$S^2 - \bigcup_{s \in S^2} GC_{\widetilde{r}(s)}$$

is not empty and N is inside this open set.

(2) *The connected subset $\{\widetilde{\mathbf{n}}(s) \mid s \in S^1\}$ is inside $S^2_{N,+}$, where $\widetilde{\mathbf{n}}$ is the dual of $\widetilde{\mathbf{r}}$.*

Let $\Psi_N : S^2 - \{\pm N\} \to S^2$ be the mapping defined by

$$\Psi_N(P) = \frac{1}{\sqrt{1 - (N \cdot P)^2}}(N - (N \cdot P)P).$$

The mapping Ψ_N is very useful for studying spherical pedals, pedal unfoldings of spherical pedals, hedgehogs, and Wulff shapes (see [7, 8, 9, 10, 11]). There is also a hyperbolic version of Ψ_N ([6]). The fundamental properties of Ψ_N is as follows:

(1) For any $P \in S^2 - \{\pm N\}$, the equality $P \cdot \Psi_N(P) = 0$ holds,
(2) for any $P \in S^2 - \{\pm N\}$, the property $\Psi_N(P) \in \mathbb{R}N + \mathbb{R}P$ holds,
(3) for any $P \in S^2 - \{\pm N\}$, the property $N \cdot \Psi_N(P) > 0$ holds,

(4) the restriction $\Psi_N|_{S^2_{N,+}-\{N\}} : S^2_{N,+} - \{N\} \to S^2_{N,+} - \{N\}$ is a C^∞ diffeomorphism.

By these properties, we have the following:

Lemma 2.4. *The mapping* $\alpha_N \circ \Psi_N \circ \alpha_N^{-1}\colon \mathbb{R}^2 \times \{1\} - \{N\} \to \mathbb{R}^2 \times \{1\} - \{N\}$ *is the inversion of* $\mathbb{R}^2 \times \{1\} - \{N\}$ *with respect to* N*.*

2.2. **Spherical polar sets and the spherical polar transform.** For any point P of S^2, we let $H(P)$ be the following set:

$$H(P) = \{Q \in S^2 \mid P \cdot Q \geq 0\}.$$

Here, the dot in the center stands for the scalar product of $P, Q \in \mathbb{R}^3$.

Definition 3 ([11])**.** Let W be a subset of S^2. Then, the set

$$\bigcap_{P \in W} H(P)$$

is called the *spherical polar set* of W and is denoted by W°.

Figure 3 illustrates Definition 3. It is clear that $W^\circ = \cap_{P\in W} H(P)$ is closed for any $W \subset S^2$.

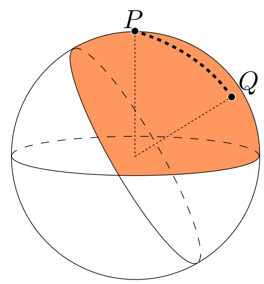

FIGURE 3. Spherical polar set $\{P, Q\}^\circ = (PQ)^\circ$.

Definition 4 ([11])**.** A subset $W \subset S^2$ is said to be *hemispherical* if there exists a point $P \in S^2$ such that $H(P) \cap W = \emptyset$.

Figure 4 illustrates Definition 4.

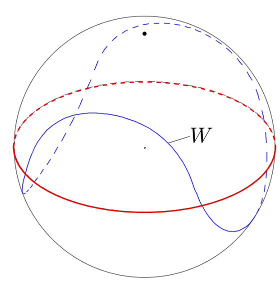

FIGURE 4. Not hemispherical $W \subset S^2$.

Definition 5 ([11])**.** A hemispherical subset $W \subset S^2$ is said to be *spherical convex* if $PQ \subset W$ for any $P, Q \in W$.

Here, PQ stands for the following arc:

$$PQ = \left\{ \frac{(1-t)P + tQ}{||(1-t)P + tQ||} \in S^2 \;\middle|\; 0 \leq t \leq 1 \right\}.$$

Note that $||(1-t)P + tQ|| \neq 0$ for any $P, Q \in W$ and any $t \in [0,1]$ if $W \subset S^2$ is hemispherical. Note further that W° is spherical convex if W is hemispherical and it has an interior point.

Definition 6 ([11])**.** Let W be a hemispherical subset of S^2. Then, the *spherical convex hull of* W (denoted by s-conv(W)) is the following set.

$$\text{s-conv}(W) = \left\{ \frac{\sum_{i=1}^{k} t_i P_i}{||\sum_{i=1}^{k} t_i P_i||} \;\middle|\; P_i \in W,\ \sum_{i=1}^{k} t_i = 1,\ t_i \geq 0, k \in \mathbb{N} \right\}.$$

Lemma 2.5 (Lemma 2.5 of [11])**.** *For any hemispherical finite subset $W = \{P_1, \ldots, P_k\} \subset S^{n+1}$, the following holds:*

$$\left\{ \frac{\sum_{i=1}^{k} t_i P_i}{||\sum_{i=1}^{k} t_i P_i||} \;\middle|\; P_i \in W,\ \sum_{i=1}^{k} t_i = 1,\ t_i \geq 0 \right\}^\circ = H(P_1) \cap \cdots \cap H(P_k).$$

Lemma 2.5 is called *Maehara's lemma* (see [11]).

Definition 7 ([4])**.** Let (X, d) be a complete metric space.

(1) Let x be a point of X and let B a non-empty compact subset of X. Define
$$d(x, B) = \min\{d(x, y) \mid y \in B\}.$$
Then, $d(x, B)$ is called the *distance from the point x to the set B*.

(2) Let A, B be two non-empty compact subsets of X. Define
$$d(A, B) = \max\{d(x, B) \mid x \in A\}.$$
Then, $d(A, B)$ is called the *distance from the set A to the set B*.

(3) Let A, B be two non-empty compact subsets of X. Define
$$d_H(A, B) = \max\{d(A, B), d(B, A)\}.$$
Then, $d_H(A, B)$ is called the *Pompeiu-Hausdorff distance between A and B*.

Let (X, d) be a complete metric space. The set consisting of non-empty compact subsets of X is denoted by $\mathcal{H}(X)$, which is the metric space with respect to the Pompeiu-Hausdorff metric $d_H : \mathcal{H}(X) \times \mathcal{H}(X) \to \mathbb{R}_+ \cup \{0\}$, where d_H is the metric naturally induced by the Pompeiu-Hausdorff distance. It is well-known also that the metric space $(\mathcal{H}(X), d_H)$ is complete. For more details on the complete metric space $(\mathcal{H}(X), d_H)$, see for instance [4, 5].

Definition 8. Let $\bigcirc : \mathcal{H}(S^2) \to \mathcal{H}(S^2)$ be the mapping defined by

$$\bigcirc(A) = A^\circ.$$

The mapping $\bigcirc : \mathcal{H}(S^2) \to \mathcal{H}(S^2)$ is called the *spherical polar transform.*

Proposition 1. *The spherical polar transform is continuous with respect to the Pompeiu-Hausdorff metric.*

Proof of Proposition 1 Let $\{A_i\}_{i=1,2,\ldots} \subset \mathcal{H}(S^2)$ be a convergent sequence, and set $A = \lim_{i\to\infty} A_i$. In order to prove Proposition 1, it is sufficient to show that $A^\circ = \lim_{i\to\infty} A_i^\circ$.

Suppose that there exists a real number $\varepsilon > 0$ such that for any $n \in \mathbb{N}$ there exists an i_n ($i_n > n$) such that $d_H(A_{i_n}^\circ, A^\circ) > \varepsilon$. Then, by Definition 7, it follows that for any $n \in \mathbb{N}$, at least one of $d(A_{i_n}^\circ, A^\circ) > \varepsilon$ and $d(A^\circ, A_{i_n}^\circ) > \varepsilon$ holds. By taking a subsequence if necessary, from the first we may assume that one of the following holds:

(1) $d(A_{i_n}^\circ, A^\circ) > \varepsilon$ for any $n \in \mathbb{N}$.
(2) $d(A^\circ, A_{i_n}^\circ) > \varepsilon$ for any $n \in \mathbb{N}$,

We first show that (1) implies a contradiction. By Definition 7, it follows that for any $n \in \mathbb{N}$ there exists a point $x_n \in A_{i_n}^\circ$ such that $d(x_n, A^\circ) > \varepsilon$. Again by Definition 7, it follows that for any $n \in \mathbb{N}$ there exists a point $x_n \in A_{i_n}^\circ$ such that the inequality $d(x_n, y) > \varepsilon$ holds for any $y \in A^\circ$. It is known that A can be characterized as follows ([4]).

$$A = \left\{ P \in S^2 \mid \exists P_n \in A_{i_n} (n \in \mathbb{N}) \text{ such that } \lim_{n\to\infty} P_n = P \right\}. \tag{2.1}$$

Let P be a point of A. By (2.1), for any $n \in \mathbb{N}$ we may choose a point $P_n \in A_{i_n}$ such that $\lim_{n\to\infty} P_n = P$. Then, since $x_n \in A_{i_n}^\circ$, it follows that $x_n \cdot P_n \geq 0$. Since S^2 is compact, there exists a convergent subsequence $\{x_{j_n}\}_{n=1,2,\ldots}$ of the sequence $\{x_n\}_{n=1,2,\ldots}$. Set $x = \lim_{n\to\infty} x_{j_n}$. Then, the inequality $d(x_n, y) > \varepsilon$ implies the inequality $d(x, y) \geq \varepsilon$ for any $y \in A^\circ$. On the other hand, the inequality $x_n \cdot P_n \geq 0$ implies the inequality $x \cdot P \geq 0$ for any $P \in A$. Therefore, the point x is an element of A° such that the inequality $d(x, y) \geq \varepsilon$ holds for any $y \in A^\circ$. This is a contradiction.

We next show that (2) implies a contradiction. By the same argument as in (1), we have that for any $n \in \mathbb{N}$ there exists a point $x_n \in A^\circ$ such that the inequality $d(x_n, y_n) > \varepsilon$ for any $y_n \in A_{i_n}^\circ$. This implies that there exists an $M \in \mathbb{N}$ such that for any $n \in \mathbb{N}$ there exists $P_n \in A_{i_n}$ such that $x_n \cdot P_n < -\frac{\varepsilon}{M}$. Since S^2 is compact, there exists a subsequence $\{j_n\}_{n=1,2,\ldots}$ of $\mathbb{N}$ such that both $\{x_{j_n}\}_{n=1,2,\ldots}$, $\{P_{j_n}\}_{n=1,2,\ldots}$ are convergent sequences. Set $x = \lim_{n\to\infty} x_{j_n}$ and $P = \lim_{n\to\infty} P_{j_n}$. Then, the inequality $x_n \cdot P_n < -\frac{\varepsilon}{M}$ implies the inequality $x \cdot P \leq -\frac{\varepsilon}{M}$. On the other hand, since A° is compact, x belongs to A°. Moreover, by (2.1), P belongs to A. Hence, by Definition 3, the scalar product $x \cdot P$ must be non-negative. Therefore, we have a contradiction. □

2.3. **Wulff shapes.** Let $\mathbb{R}_+$ be the set $\{\lambda \in \mathbb{R} \mid \lambda > 0\}$ and let $h : S^1 \to \mathbb{R}_+$ be a continuous function. For any $s \in S^1 \subset \mathbb{R}^2$, set

$$\Gamma_{h,s} = \{P \in \mathbb{R}^2 \mid P \cdot s \leq h(s)\},$$

where the dot in the center stands for the scalar product of two vectors $P, s \in \mathbb{R}^2$. The following set is called the *Wulff shape associated with the support function* h (see Figure 5):

$$\mathcal{W}_h = \bigcap_{s\in S^1} \Gamma_{h,s}.$$

For any crystal at equilibrium the shape of it can be constructed as the Wulff shape $\mathcal{W}_h$ by an appropriate support function h ([14]). It is clear that any Wulff shape $\mathcal{W}_h$ is a convex body (namely, it is compact, convex and the origin of $\mathbb{R}^2$ is contained in $\mathcal{W}_h$ as an interior point). It has been known that its converse, too, holds as follows.

Proposition 2 (p. 573 of [13])**.** *Let W be a subset of $\mathbb{R}^2$. Then, there exists a parallel translation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T(W)$ is the Wulff shape associated with an appropriate support function if and only if W is a convex body.*

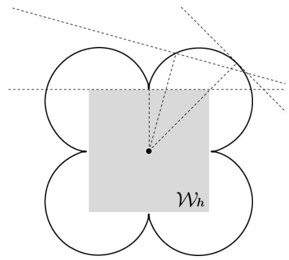

FIGURE 5. The Wulff shape associated with the support function h.

Proposition 3 (Theorem 1.1 of [11]). *Let $\{\mathcal{W}_{h_i}\}_{i=1,2,\ldots}$ be a Cauchy sequence of Wulff shapes in $\mathcal{H}_{\mathrm{conv}}(\mathbb{R}^2)$ with respect to the Pompeiu-Hausdorff metric d_H. Suppose that $\lim_{i\to\infty}\mathcal{W}_{h_i}$ does not have an interior point. Then, it must be a point or a segment.*

Proposition 4 (Theorem 1.2 of [11]). *Let $h: S^1 \to \mathbb{R}_+$ be a continuous function. Then, for the Wulff shape $\mathcal{W}_h$, the set $Id^{-1}\circ\alpha_N\left(\left(\alpha_N^{-1}\circ Id(\mathcal{W}_h)\right)^{\circ}\right)$ is the Wulff shape associated with an appropriate support function.*

The Wulff shape $Id^{-1}\circ\alpha_N\left(\left(\alpha_N^{-1}\circ Id(\mathcal{W}_h)\right)^{\circ}\right)$ is called the *dual Wulff shape of* $\mathcal{W}_h$.

Proposition 5 (Theorem 1.3 of [11]). *Let $h: S^1 \to \mathbb{R}_+$ be a function of class C^1. Then, the Wulff shape $\mathcal{W}_h$ is never a polygon.*

Proposition 6. *Let $\mathbf{r}: S^1 \to \mathbb{R}^2$ be a C^∞ immersion with its no-silhouette $\mathcal{NS}_{\mathbf{r}}$. Then, for any $\theta\in[0,\theta_{\mathbf{r}})$, there exists a parallel translation $T_\theta:\mathbb{R}^2\to\mathbb{R}^2$ such that $T_\theta(\overline{\mathcal{NS}_{\theta,\mathbf{r}}})$ is a Wulff shape $\mathcal{W}_{h_\theta}$ by an appropriate support function $h_\theta: S^1\to\mathbb{R}_+$.*

Proof of Proposition 6 We first show that $\mathcal{NS}_{\theta,\mathbf{r}}$ is an open set for any $\theta\in[0,\theta_{\mathbf{r}})$. Let P be a point of $\mathcal{NS}_{\theta,\mathbf{r}}$. Suppose that for any positive integer n, there exists a point

$$P_n \in D(P,\frac{1}{n}) \cap \left(\cup_{s\in S^1}\left(\mathbf{r}(s)+R_\theta\left(d\mathbf{r}_s(T_s(S^1))\right)\right)\right),$$

where $D(P,\frac{1}{n})$ is the disc $D(P,\frac{1}{n}) = \{Q\in\mathbb{R}^2 \mid ||P-Q||\le\frac{1}{n}\}$. Then, since S^1 is compact, by taking a subsequence if necessary, we may assume that there exists a convergent sequence $s_n\in S^1$ $(n\in\mathbb{N})$ such that P_n belongs to $D(P,\frac{1}{n})\cap\left(\mathbf{r}(s_n)+R_\theta\left(d\mathbf{r}_{s_n}(T_{s_n}(S^1))\right)\right)$. Then, we have that $P\in\mathbf{r}(s)+R_\theta\left(d\mathbf{r}_s(T_s(S^1))\right)$ where $s=\lim_{i\to\infty}s_i$, which implies $P\notin\mathcal{NS}_{\theta,\mathbf{r}}$. Hence, $\mathcal{NS}_{\theta,\mathbf{r}}$ is an open set.

Since $\theta<\theta_{\mathbf{r}}$, it follows that $\mathcal{NS}_{\theta,\mathbf{r}}\neq\emptyset$. Let P be a point of $\mathcal{NS}_{\theta,\mathbf{r}}$. Let

$$P_s\in\mathbf{r}(s)+R_\theta d\mathbf{r}_s(T_s(S^1))$$

be the point such that the vector PP_s is perpendicular to the line $\mathbf{r}(s)+R_\theta d\mathbf{r}_s(T_s(S^1))$. Then, by obtaining the concrete expression of P_s, it follows that the mapping $f: S^1\to\mathbb{R}^2$ defined by $f(s)=P_s$ is of class C^∞. By Subsection 2.1 and [7], the mapping $f:S^1\to\mathbb{R}^2$ is exactly the pedal curve of the family of lines $\left\{\mathbf{r}(s)+R_\theta d\mathbf{r}_s(T_s(S^1))\right\}_{s\in S^1}$ relative to the pedal point $P\in\mathcal{NS}_{\theta,\mathbf{r}}$. Let $I:\mathbb{R}^2-\{P\}\to\mathbb{R}^2-\{P\}$ be the plane inversion defined by $I(Q)=P-\frac{1}{||Q-P||^2}(Q-P)$. Since $P\in\mathcal{NS}_{\theta,\mathbf{r}}$, the composed mapping $\mathbf{n}=I\circ f$ is well-defined and of class C^∞. The mapping $\mathbf{n}$ is exactly the dual curve of the family of lines $\left\{\mathbf{r}(s)+R_\theta d\mathbf{r}_s(T_s(S^1))\right\}_{s\in S^1}$ relative to the point $P\in\mathcal{NS}_{\theta,\mathbf{r}}$. Let the boundary of convex hull of $\mathbf{n}(S^1)$ be denoted by $\partial\mathrm{conv}\left(\mathbf{n}(S^1)\right)$.

Then, by the construction, $\partial\mathrm{conv}\left(\mathbf{n}(S^1)\right)$ intersect the half line $\{P+\lambda s \mid \lambda \in \mathbb{R}_+\}$ exactly at one point for any $s \in S^1$. Thus, the composed image $I\left(\partial\mathrm{conv}\left(\mathbf{n}(S^1)\right)\right)$ intersect the half line $\{P+\lambda s \mid \lambda \in \mathbb{R}_+\}$ exactly at one point for any $s \in S^1$. Moreover, the intersecting points depend on s continuously. Hence, by corresponding $s \in S^1$ to the distance between P and the unique intersecting point $I\left(\partial\mathrm{conv}\left(\mathbf{n}(S^1)\right)\right) \cap \{P+\lambda s \mid \lambda \in \mathbb{R}_+\}$, we obtain the well-defined continuous function $h_\theta : S^1 \to \mathbb{R}_+$. Since $\mathbf{n}$ is of class C^∞, it is easily seen that the obtained function h_θ satisfies the assumption of Theorem 6.3 in [11]. Let $T_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be the parallel translation given by $T_\theta(x,y) = (x,y) - P$. Then, by Theorem 6.3 of [11], it follows that

$$T_\theta(\overline{\mathcal{NS}_{\theta,\mathbf{r}}}) = \mathcal{W}_{h_\theta}.$$

□

Proposition 7. *Let $\mathbf{r} : S^1 \to \mathbb{R}^2$ be a C^∞ immersion with its no-silhouette $\mathcal{NS}_{\mathbf{r}}$. Then, the map $\omega : [0,\theta_{\mathbf{r}}) \to \mathcal{H}_{\mathrm{conv}}(\mathbb{R}^2)$ defined by $\omega(\theta) = \overline{\mathcal{NS}_{\theta,\mathbf{r}}}$ is continuous,*

Proof of Proposition 7 Let $C^0(S^1,\mathbb{R}_+)$ be the set consisting of continuous functions from S^1 to $\mathbb{R}_+$. The set $C^0(S^1,\mathbb{R}_+)$ is a (non-complete) metric space with respect to the metric

$$d_{\mathrm{norm}}(h_1,h_2) = \max_{s\in S^1} |h_1(s) - h_2(s)|.$$

Let $\Gamma : [0,\theta_{\mathbf{r}}) \to C^0(S^1,\mathbb{R}_+)$ (resp. $\Omega : C^0(S^1,\mathbb{R}_+) \to \mathcal{H}_{\mathrm{conv}}(\mathbb{R}^2)$) be the mapping defined by $\Gamma(\theta) = h_\theta$ (resp. $\Omega(h) = \mathcal{W}_h$), where h_θ is the continuous function defined in the proof of Proposition 6. Then, in order to show that ω is continuous, it is sufficient to show that both Γ, Ω are continuous.

We first show that Γ is continuous. Let $\widetilde{h} : S^1 \to \mathbb{R}_+$ be the function defined by

$$\widetilde{h}(\cos\lambda, \sin\lambda) = ||P - I\left(\partial\mathrm{conv}\left(\mathbf{n}(S^1)\right)\right) \cap \{P + t(\cos\lambda,\sin\lambda) \mid t \in \mathbb{R}_+\}||,$$

where the set $I\left(\partial\mathrm{conv}\left(\mathbf{n}(S^1)\right)\right) \cap \{P + t(\cos\lambda,\sin\lambda) \mid t \in \mathbb{R}_+\}$, which appeared in the proof of Proposition 6, is a one point set and it is regarded as a point. By obtaining the concrete expression of $\mathbf{n}$ given in the proof of Proposition 6, it is easily seen that $\mathbf{n}$ is smoothly depending on $\theta \in [0,\theta_{\mathbf{r}})$. Thus, $\widetilde{h}$ is continuously depending on $\theta \in [0,\theta_{\mathbf{r}})$. Since I is a C^∞ diffeomorphism of $\mathbb{R}^2 - \{P\}$, it follows that h_θ is continuously depending on $\theta \in [0,\theta_{\mathbf{r}})$. Hence, Γ is a continuous mapping.

We next show that Ω is continuous. Let $\{h_i\}_{i=1,2,\ldots} \subset C^0(S^1,\mathbb{R}_+)$ be a convergent sequence to an element of $C^0(S^1,\mathbb{R}_+)$. Set $h = \lim_{i\to\infty} h_i$. We also set

$$W = \left\{ P \in \mathbb{R}^2 \;\middle|\; \exists P_i \in \mathcal{W}_{h_i}\ (i \in \mathbb{N});\ \lim_{i\to\infty} P_i = P \right\}.$$

Then, it is easily seen that $\mathbb{R}^2 - W$ is an open set. Thus, W is a closed set.

We show $\mathcal{W}_h = W$. Let P be an interior point of $\mathcal{W}_h$. Then, since $h = \lim_{i\to\infty} h_i$, P must be an interior point of $\mathcal{W}_{h_i}$ for any sufficiently large i. Thus, P is contained in W. Since both $\mathcal{W}_h$ and W are closed, it follows that $\mathcal{W}_h \subset W$. Next, Let Q be a point of W. Suppose that Q is not contained in $\mathcal{W}_h$. Then, there exists $s_0 \in S^1$ such that $(Q \cdot s_0) > h(s_0)$, where $(Q \cdot s_0)$ stands for the scalar product of two vectors $Q, s_0 \in \mathbb{R}^2$. Set $\varepsilon = (Q \cdot s_0) - h(s_0) > 0$. Since $h = \lim_{i\to\infty} h_i$, it follows that $(Q \cdot s_0) - h_i(s_0) > \frac{\varepsilon}{2}$ for any sufficiently large i. This contradicts to the assumption that $Q \in W$. Hence, we have that $W \subset \mathcal{W}_h$, and it follows that $\mathcal{W}_h = W$.

The remaining part of the proof that Ω is continuous is to show the following:

$$\lim_{i\to\infty} d_H(W, \mathcal{W}_{h_i}) = 0. \tag{2.2}$$

In order to show (2.2), by the construction of W, it is sufficient to show that $\{\mathcal{W}_{h_i}\}_{i=1,2,\ldots}$ is a Cauchy sequence of $\mathcal{H}(\mathbb{R}^2)$. Since $\{h_i\}_{i=1,2,\ldots}$ is a Cauchy sequence of $C^0(S^1,\mathbb{R}_+)$, it is clear that $\{\mathcal{W}_{h_i}\}_{i=1,2,\ldots}$ is a Cauchy sequence. Therefore, we have that $\lim_{i\to\infty} d_H(W, \mathcal{W}_{h_i}) = 0$ and it follows that Ω is continuous. $\square$

3. Proof of Theorem 1

By Proposition 6, there exists a parallel translation $T_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T_\theta\left(\overline{\mathcal{NS}_{\theta,\mathbf{r}}}\right)$ is a Wulff shape. In particular, $T_\theta\left(\overline{\mathcal{NS}_{\theta,\mathbf{r}}}\right)$ contains the origin as an interior point. Set $\widetilde{\mathbf{r}} = \alpha_N^{-1} \circ Id \circ T_\theta \circ \mathbf{r}$ and $\widetilde{\mathbf{n}}_\theta(s) = \cos\theta\widetilde{\mathbf{n}}(s) - \sin\theta\widetilde{\mathbf{t}}(s)$ for $s \in S^1$. We investigate the singularities of $\widetilde{\mathbf{n}}_\theta$. Let U be an open arc of S^1. By using the arc-length parameter of $\widetilde{\mathbf{r}}|_U$, without loss of generality, from the first we may assume that $||\widetilde{\mathbf{r}}'(s)|| = 1$ for $s \in U$. Then, by Lemma 2.1, we have the following:

$$\widetilde{\mathbf{n}}_\theta'(s) = -\kappa_g(s)\cos\theta\,\widetilde{\mathbf{t}}(s) + \sin\theta\,\widetilde{\mathbf{r}}(s) - \kappa_g(s)\sin\theta\,\widetilde{\mathbf{n}}(s).$$

Since the angle θ satisfies $0 < \theta < \theta_{\mathbf{r}} \le \frac{\pi}{2}$ in Theorem 1, it follows that $\sin\theta \neq 0$. Therefore, $\widetilde{\mathbf{n}}_\theta$ is non-singular even at the point $s \in S^1$ such that $\kappa_g(s) = 0$.

Next, we show that $\widetilde{\mathbf{n}}_\theta(s)\cdot N > 0$ for any $s \in S^1$. Let the dual of $\widetilde{\mathbf{n}}_\theta$ be denoted by $\widetilde{\mathbf{r}}_\theta$. Then, it follows that $\widetilde{\mathbf{r}}_\theta$ is a Legendrian mapping and the following equality holds.

$$S^2_{N,+} \bigcap \left(S^2 - \bigcup_{s\in S^1} GH_{\widetilde{\mathbf{r}}_\theta} \right) = \alpha_N^{-1} \circ Id \circ \mathcal{NS}_{\theta,\mathbf{r}}.$$

Since $\theta < \theta_{\mathbf{r}}$, by Lemma 2.3, we have that $\widetilde{\mathbf{n}}_\theta(s)\cdot N > 0$ for any $s \in S^1$. Thus, the spherical convex hull of $\{\widetilde{\mathbf{n}}_\theta(s)) \mid s \in S^1\}$ is well-defined. Since $\widetilde{\mathbf{n}}_\theta$ is non-singular, the boundary of s-conv($\{\widetilde{\mathbf{n}}_\theta(s)) \mid s \in S^1\}$) is a submanifold of class C^1 (for instance see [12, 15]). By the property (4) of Ψ_N, the boundary of Ψ_N(s-conv($\{\widetilde{\mathbf{n}}_\theta(s)) \mid s \in S^1$)) is a submanifold of class C^1. It follows that the boundary of $Id^{-1} \circ \alpha_N \circ \Psi_N$(s-conv($\{\widetilde{\mathbf{n}}_\theta(s)) \mid s \in S^1$)) is a submanifold of class C^1.

On the other hand, by constructions, it follows that $T_\theta(\overline{\mathcal{NS}_{\theta,\mathbf{r}}})$ is a Wulff shape $\mathcal{W}_h$ with the support function h whose graph with respect to the polar coordinate expression is the boundary of $Id^{-1} \circ \alpha_N \circ \Psi_N$(s-conv($\{\widetilde{\mathbf{n}}_\theta(s)) \mid s \in S^1$)).

Therefore, the support function h for the Wulff shape $T_\theta(\overline{\mathcal{NS}_{\theta,\mathbf{r}}})$ is of class C^1 and it follows that $\mathcal{W}_h$ is never a polygon by Proposition 5. $\square$

4. Proof of Theorem 2

By Proposition 6, for any $i \in \mathbb{N}$ there exists a parallel translation $T_{\theta_i} : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T_{\theta_i}\left(\overline{\mathcal{NS}_{\theta_i,\mathbf{r}}}\right)$ is a Wulff shape $\mathcal{W}_{h_i}$ by an appropriate support function h_i. By Proposition 4, for any $i \in \mathbb{N}$ the set $Id^{-1} \circ \alpha_N\left(\left(\alpha_N^{-1} \circ Id(\mathcal{W}_{h_i})\right)^\circ\right)$ is a Wulff shape too. Thus, by Proposition 2, it follows that both $\alpha_N^{-1} \circ Id(\mathcal{W}_{h_i})$ and $\left(\alpha_N^{-1} \circ Id(\mathcal{W}_{h_i})\right)^\circ$ belong to $\mathcal{H}(S^2)$ for any $i \in \mathbb{N}$. Moreover, by Proposition 7, we may assume that $\{T_{\theta_i}\}_{i=1,2,\ldots}$ is a Cauchy sequence. Thus, we may assume that both $\left\{\alpha_N^{-1} \circ Id(\mathcal{W}_{h_i})\right\}_{i=1,2,\ldots}$ and $\left\{\left(\alpha_N^{-1} \circ Id(\mathcal{W}_{h_i})\right)^\circ\right\}_{i=1,2,\ldots}$ are Cauchy sequences.

By Proposition 3, $\lim_{i\to\infty} \overline{\mathcal{NS}_{\theta_i,\mathbf{r}}}$ is a point or segment. Suppose that it is a segment. Let $P_1, P_2 \in S^2$ be two boundary points of this segment. Then, by Proposition 1 and Lemma 2.5, we have the following:

$$\lim_{i\to\infty} \left(\alpha_N^{-1} \circ Id(\mathcal{W}_{h_i})\right)^\circ = H(P_1) \cap H(P_2).$$

Let $\widetilde{\mathbf{n}}_{\theta_{\mathbf{r}}} : S^1 \to S^2$ be the C^∞ mapping defined by $\widetilde{\mathbf{n}}_{\theta_{\mathbf{r}}}(s) = \cos\theta_{\mathbf{r}}\widetilde{\mathbf{n}}(s) - \sin\theta_{\mathbf{r}}\widetilde{\mathbf{t}}(s)$ for any $s \in S^1$, where $\widetilde{\mathbf{n}}$ and $\widetilde{\mathbf{t}}$ are the same C^∞ mapping as in the proof of Theorem 1. Then, notice that $\widetilde{\mathbf{n}}_{\theta_{\mathbf{r}}}(S^1) \subset H(P_1) \cap H(P_2)$. For any j ($j = 1, 2$), we let the set $\{Q \in S^2 \mid P_j \cdot Q = 0\}$ be denoted by $\partial H(P_j)$. Then, the intersection $\partial H(P_1) \cap \partial H(P_2)$ consists of two antipodal points Q_1, Q_2. By Lemma 2.3 and Proposition 2, there exists $s_1, s_2 \in S^1$ $(s_1 \neq s_2)$ such that $\widetilde{\mathbf{n}}_{\theta_{\mathbf{r}}}(s_1) = Q_1$, $\widetilde{\mathbf{n}}_{\theta_{\mathbf{r}}}(s_2) = Q_2$.

On the other hand, since $0 \leq \theta_{\mathbf{r}} \leq \frac{\pi}{2}$, similarly as in the proof of Theorem 1, it follows that $\widetilde{\mathbf{n}}_{\theta_{\mathbf{r}}}$ is non-singular. Thus, we have a contradiction. $\square$

5. Proof of Theorem 3

For any θ $(0 \leq \theta < \theta_{\mathbf{r}})$ and any $s \in S^1$, set

$$\ell_{\theta,s} = \mathbf{r}(s) + R_\theta\left(d\mathbf{r}_s(T_sS^1)\right).$$

Let $f_{\theta,s}(x,y)$ be the affine function which define $\ell_{\theta,s}$. Set

$$H^+_{\theta,s} = \{(x,y) \in \mathbb{R}^2 \mid f_{\theta,s}(x,y) > 0\}, \quad H^-_{\theta,s} = \{(x,y) \in \mathbb{R}^2 \mid f_{\theta,s}(x,y) < 0\}.$$

Then, since $\overline{\mathcal{NS}_{\theta,\mathbf{r}}}$ is a convex body for any θ $(0 \leq \theta < \theta_{\mathbf{r}})$, it follows that one of

$$\mathcal{NS}_{\theta,\mathbf{r}} = \cap_{s\in S^1} H^+_{\theta,s} \ \text{ or } \ \mathcal{NS}_{\theta,\mathbf{r}} = \cap_{s\in S^1} H^-_{\theta,s}$$

holds. By Proposition 6, we may assume that the following holds for any θ $(0 \leq \theta < \theta_{\mathbf{r}})$.

$$\mathcal{NS}_{\theta,\mathbf{r}} = \bigcap_{s\in S^1} H^+_{\theta,s}.$$

Since $\mathbf{r}$ does not have inflection points, it follows that $\mathcal{NS}_{0,\mathbf{r}}$ contains $\mathcal{NS}_{\theta,\mathbf{r}}$ for any θ such that $0 \leq \theta < \theta_{\mathbf{r}}$. Thus, for any θ $(0 \leq \theta < \theta_{\mathbf{r}})$, we have the following:

$$\begin{aligned} \mathcal{NS}_{\theta,\mathbf{r}} &= \mathcal{NS}_{\theta,\mathbf{r}} \cap \mathcal{NS}_{0,\mathbf{r}} \\ &= \left(\bigcap_{s\in S^1} H^+_{\theta,s}\right) \bigcap \mathcal{NS}_{0,\mathbf{r}} \\ &= \bigcap_{s\in S^1} \left(H^+_{\theta,s} \bigcap \mathcal{NS}_{0,\mathbf{r}}\right). \end{aligned}$$

Since $\mathbf{r}$ does not have inflection points, we have that $H^+_{\theta_1,s} \cap \mathcal{NS}_{0,\mathbf{r}}$ contains $H^+_{\theta_2,s} \cap \mathcal{NS}_{0,\mathbf{r}}$ for any two $\theta_1, \theta_2 \in [0, \theta_{\mathbf{r}})$ satisfying $0 \leq \theta_1 < \theta_2 < \theta_{\mathbf{r}}$. It follows that $\mathcal{NS}_{\theta_1,\mathbf{r}} \supset \mathcal{NS}_{\theta_2,\mathbf{r}}$ if $0 \leq \theta_1 < \theta_2 < \theta_{\mathbf{r}}$. $\square$

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