DEFORMATION OF SINGULARITIES AND ADDITIVE INVARIANTS

GEORGES COMTE

ABSTRACT. In this survey on local additive invariants of real and complex definable singular germs we systematically present classical or more recent invariants of different nature as emerging from a tame degeneracy principle. For this goal, we associate to a given singular germ a specific deformation family whose geometry degenerates in such a way that it eventually gives rise to a list of invariants attached to this germ. Complex analytic invariants, real curvature invariants and motivic type invariants are encompassed under this point of view. We then explain how all these invariants are related to each other as well as we propose a general conjectural principle explaining why such invariants have to be related. This last principle may appear as the incarnation in definable geometry of deep finiteness results of convex geometry, according to which additive invariants in convex geometry are very few.

INTRODUCTION

A beautiful and fruitful principle occurring in several branches of mathematics consists in deforming the object under consideration in order to let appear some invariants attached to this object. In this deformation process, the object X_0 to study becomes the special fibre of a deformation family $(X_\varepsilon)_\varepsilon$ where each fibre X_ε approximates X_0, from a topological, metric or geometric point of view, depending on the nature of the invariant that one aims for X_0 through $(X_\varepsilon)_\varepsilon$.

For instance in Morse theory, where a smooth real valued function $f : M \to \mathbf{R}$ on a smooth manifold M is given, such that $f^{-1}([f(m) - \eta, f(m) + \eta]$ contains no critical point of f but m, the homotopy type of $f^{-1}(] - \infty, f(m) + \eta])$ is given by the homotopy type of $f^{-1}(] - \infty, f(m) - \varepsilon])$, for any ε with $0 < \varepsilon \le \eta$, plus a discrete invariant attached to f at m, namely the index of f at m. In this case the family $(f^{-1}(] - \infty, f(m) - \varepsilon]))_{0 < \varepsilon \le \eta}$ has the same fibres, from the differential point of view, and approximates the special set $f^{-1}(] - \infty, f(m) + \eta])$, from the homotopy type point of view, up to some additional discrete topological invariant.

Another instance of this deformation principle can be found in tropical geometry, where a patchwork polynomial embeds a complex curve X_0 of the complex torus $(\mathbf{C}^*)^2$ into a family (X_ε) of complex curves. This family may be viewed as a curve $\mathscr{X}$ on the non archimedean valued field $\mathbf{C}((\varepsilon^{\mathbf{R}}))$ of Laurent series with exponents in $\mathbf{R}$. Then, by a result of Mikhalkin and Rullgård ([82] and [91]), the amoebas family $\mathscr{A}(X_\varepsilon)$ of (X_ε) has limit (in the Hausdorff metric) the non archimedean amoeba $\mathscr{A}(\mathscr{X})$ of $\mathscr{X}$.

In the theory of sufficiency of jets the aim is to approach a smooth map by its family of Taylor polynomials up to some sufficient degree depending on the kind of equivalence considered for maps (right, left, or V equivalence). Embedding a germ into a convenient deformation is a seminal way of thinking for R. Thom that has been successfully achieved in his cobordism theory or in his works on regular stratifications providing regular trivializations. We could multiply examples in this spirit, old ones as well as recent ones (from recent developments in

general deformation theory itself for instance [1]), but in this introduction we will focus only on two specific examples, that will be developed thereafter: the Milnor fibration and the Lipschitz-Killing curvatures.

The Milnor fibre of a complex singularity. The first of these two examples is provided by the Milnor fibre of a complex singular analytic hypersurface germ $f : (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$. We will assume for simplicity that this singularity is an isolated one, that is to say that we will assume that $0 = f(0)$ is the only critical value of f, at least locally around 0. We denote $B_{(0,\eta)}$ the open ball of radius η, centred at 0 of the ambient space depending upon the context. Now, for $\eta > 0$ small enough and $0 < \varrho \ll \eta$, the family $(f^{-1}(\varepsilon) \cap B_{(0,\eta)})_{0 < |\varepsilon| < \varrho}$ is a smooth bundle, with projection f, over the punctured disc $B_{(0,\varrho)} \setminus \{0\} \subset \mathbf{C}$. The topological type of a fibre

$$X_\varepsilon := f^{-1}(\varepsilon) \cap B_{(0,\eta)}$$

does not depend on the choice of ε, and the homotopy type of this fibre is the homotopy of a finite CW complex of dimension $n-1$, the one of a bouquet of μ spheres S^{n-1}, where μ is called the Milnor number of the fibration (see [86]). On the other hand, the special singular fibre

$$X_0 := f^{-1}(0) \cap B_{(0,\eta)}$$

is contractible, as a germ of a semialgebraic set. It follows that the family $(X_\varepsilon)_{0 < |\varepsilon| < \eta}$ approximates the singular fibre X_0 up to μ cycles that vanish as ε goes to 0. The number μ of these cycles appears as an analytic invariant of the germ of the hypersurface f that is geometrically embodied on the nearby fibres X_ε of the deformation on the singular fibre (see also [23]). In [102], B. Teissier embedded the Milnor number μ in a finite sequence of integers in the following way. For a generic vector space V of $\mathbf{C}^n$ of dimension $n-i$, the Milnor number of the restriction of f to V does not depend on V and is denoted $\mu^{(n-i)}$. In particular $\mu = \mu^{(n)}$ and therefore the sequence $\mu^{(*)} := (\mu_0, \cdots, \mu^{(n-1)}, \mu^{(n)})$ gives a multidimensional version of μ.

We can consider other invariants attached to the Milnor fibre of f, also extending the simple invariant μ: the Lefschetz numbers of the iterates of the monodromy of the Milnor fibration, that we introduce now in order to fix notations in the sequel. The Milnor fibre X_ε may be endowed with an isomorphism M, the monodromy of the Milnor fibre, defined up to homotopy and that induces in an unambiguous way an automorphism, also denoted M, on the cohomology group $H^\ell(X_\varepsilon, \mathbf{C})$

$$M : H^\ell(X_\varepsilon, \mathbf{C}) \to H^\ell(X_\varepsilon, \mathbf{C}), \ell = 0, \cdots, n-1.$$

For the m-th iterate M^m of M, for any $m \geq 0$, one finally defines the Lefschetz number $\Lambda(M^m)$ of M^m by

$$\Lambda(M^m) := \sum_{i=0}^{n-1} (-1)^i tr(M^m, H^i(X_\varepsilon, \mathbf{C})),$$

where tr stands for the trace of endomorphisms. Note that $\Lambda(M^0) = \chi(X_0) = 1 + (-1)^{n-1}\mu$ and that the eigenvalues of M are roots of unity (see for instance [99]).

A more convenient deformation of $f^{-1}(0)$ than the family $(f^{-1}(\varepsilon) \cap B_{(0,\eta)})_\varepsilon$, at least for the practical computation of the topological invariants we just have introduced, is provided by an adapted resolution of the singularity of f at 0. To define such a resolution and fix the notations used in Section 3, let us consider $\sigma : (M, \sigma^{-1}(0)) \to (\mathbf{C}^n, 0)$ a proper birational map which is an isomorphism over the (germ of the) complement of $f^{-1}(0)$ in $(\mathbf{C}^n, 0)$, such that $f \circ \sigma$ and the jacobian determinant jac σ are normal crossings and $\sigma^{-1}(0)$ is a union of components of the divisor $(f \circ \sigma)^{-1}(0)$. We denote by E_j, for $j \in \mathscr{J}$, the irreducible components of $(f \circ \sigma)^{-1}(0)$

---

[1]As recalled by Kontsevich and Soibelman, Gelfand quoted that "any area of mathematics is a kind of deformation theory", see [68].

and assume that E_k are the irreducible components of $\sigma^{-1}(0)$ for $k \in \mathscr{K} \subset \mathscr{J}$. For $j \in \mathscr{J}$ we denote by N_j the multiplicity $mult_{E_j} f \circ \sigma$ of $f \circ \sigma$ along E_j and for $k \in \mathscr{K}$ by ν_k the number $\nu_k = 1 + mult_{E_k} jac\ \sigma$. For any $I \subset \mathscr{J}$, we set $E_I^0 = (\bigcap_{i \in I} E_i) \setminus (\bigcup_{j \in \mathscr{J} \setminus I} E_j)$.

The collection $(E_I^0)_{I \subset \mathscr{J}}$ gives a canonical stratification of the divisor $f \circ \sigma = 0$, compatible with $\sigma = 0$ such that in some affine open subvariety U in M we have $f \circ \sigma(x) = u(x) \prod_{i \in I} x_i^{N_i}$, where u is a unit, that is to say a rational function which does not vanish on U, and $x = (x', (x_i)_{i \in I})$ are local coordinates. Now the nearby fibres X_ε are isomorphic to their lifting

$$\widetilde{X}_\varepsilon := \sigma^{-1}(f^{-1}(\varepsilon) \cap B_{(0,\eta)})$$

in M and the family $(\widetilde{X}_\varepsilon)_{0 < |\varepsilon| < \eta}$ approximates the divisor $\widetilde{X}_0 := \sigma^{-1}(f^{-1}(0) \cap B_{(0,\eta)})$. Of course the geometry of $\widetilde{X}_0$ has apparently nothing to do with the geometry of our starting germ $(f^{-1}(0), 0)$, but as the topological information concerning the singularity of $f : (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$ is carried by the nearby fibres family $(X_\varepsilon)_{0 < |\varepsilon| < \eta}$, all this information is still encoded in the family $(\widetilde{X}_\varepsilon)_{0 < |\varepsilon| < \eta}$, and the discrete data N_j, ν_k although depending on the choice of the resolution, may be combined in order to explicitly compute invariants of the singularity.

Not only μ, the most elementary of our invariants, may be computed in the resolution, but also more elaborated ones such as the Lefschetz numbers of the iterates of the monodromy of the singularity. Indeed, by [1] we have the celebrated A'Campo formulas

$$\Lambda(M^m) = \sum_{i \in \mathscr{K},\ N_i/m} N_i \cdot \chi(E_{\{i\}}^0),\ m \geq 0$$

and in particular

$$1 + (-1)^{n+1}\mu = \chi(X_0) = \Lambda(M^0) = \sum_{i \in \mathscr{K}} N_i \cdot \chi(E_{\{i\}}^0).$$

0.1. *Remark.* Denoting $\bar{X}_0$ the closure of the Milnor fibre of $f : (\mathbf{C}, 0) \to (\mathbf{C}, 0)$, since the boundary $\bar{X}_0 \setminus X_0$ is a compact smooth manifold with odd dimension, we have $\chi(\bar{X}_0 \setminus X_0) = 0$, and in particular $\chi(X_0) = \chi(\bar{X}_0)$. This is why, in the complex case and for topological considerations at the level of the Euler-Poincaré characteristic, the issue of the open or closed nature of balls is not so relevant. In contrast, in the real case, this issue really matters.

Metric invariants coming from convex geometry. The second main example of invariants arising from a deformation that we aim to emphasize and develop here, comes from convex geometry. In this case, starting from a compact convex set of $\mathbf{R}^n$, it is usual to approximate this set by its family of ε-tubular neighbourhoods, $\varepsilon > 0$, since those neighbourhoods remain convex and generally have a more regular shape than the original set. This method is notably used in [101] to generate a finite sequence of metric invariants attached to a compact convex polytope P (the convex hull of a finite number of points) in $\mathbf{R}^n$ (actually in $\mathbf{R}^2$ or $\mathbf{R}^3$ in [101]). It is established in [101] that the volume of the tubular neighbourhood of radius $\varepsilon \geq 0$ of P,

$$T_{P,\varepsilon} := \bigcup_{x \in P} \bar{B}_{(x,\varepsilon)},$$

where $\bar{B}_{(x,\varepsilon)}$ is the closed ball of $\mathbf{R}^n$ centred at x with radius ε, is a polynomial in ε with coefficients $\Lambda_0(P), \cdots, \Lambda_n(P)$ depending only on P and being invariant under isometries of $\mathbf{R}^n$. We have

$$\forall \varepsilon \geq 0,\ Vol_n(T_{P,\varepsilon}) = \sum_{i=0}^n \alpha_i \Lambda_{n-i}(P) \cdot \varepsilon^i, \tag{1}$$

It is convenient to normalize the coefficients $\Lambda_i(P)$ by the introduction, in the equality (1) defining them, of the i-volume α_i of the i-dimensional unit ball.

When $\varepsilon = 0$ in this formula, one gets $\Lambda_n(P) = Vol_n(P)$. On the other hand, denoting $\delta = \max\{|x - y|; x, y \in P\}$ the diameter of P, for any $x \in P$, the inclusions

$$B_{(x,\varepsilon)} \subset T_{P,\varepsilon} \subset B_{(x,\varepsilon+\delta)}$$

show that $Vol_n(T_{P,\varepsilon}) \underset{\varepsilon \to \infty}{\sim} \alpha_n \cdot \varepsilon^n$ and thus $\Lambda_0(P) = 1$. Denoting the Euler-Poincaré characteristic by χ and having in mind further generalizations, the relation $\Lambda_0(P) = 1$ has rather to be considered as $\Lambda_0(P) = \chi(P)$. A direct proof of (1) leads to an expression of the other coefficients $\Lambda_i(P)$ in terms of some geometrical data of P. To give this proof, we set now some notation.

For P a polytope in $\mathbf{R}^n$ of dimension n, generated by $n + 1$ independent points, an affine hyperplane in $\mathbf{R}^n$ generated by n of these points is called a facet of P. The normal vector to a facet F of P is the unit vector orthogonal to F and pointing in the half-space defined by F not containing P. For $i \in \{0, \cdots, n - 1\}$, a i-face of P is the intersection of P with $n - i$ distinct facets of P. We denote $\mathscr{F}_i(P)$ the set of i-faces of P. By convention $\mathscr{F}_n(P) = \{P\}$. For $x \in P$ one consider F_x, the unique face of P of minimal dimension containing x. If $x \in \partial P$ (the boundary of P), the normal exterior cone of P at x, denoted $C(x, P)$, is the $\mathbf{R}_+$-cone of $\mathbf{R}^n$ generated by the normal vectors to the facets of P containing x. By convention $C(x, P) = \{0\}$, for $x \in P \setminus \partial P$.

We note that for F_x of dimension $i \in \{0, \cdots, n - 1\}$, $C(x, P)$ is a $\mathbf{R}_+^\times$-invariant cone of $\mathbf{R}^n$ of dimension $n - i$. Furthermore, for any $y \in F_x$, $C(x, P) = C(y, P)$. One thus defines $C(F, P)$, the exterior normal cone of P along a face F of P, by $C(x, P)$, where x is any point in F. One has $C(P, P) = \{0\}$.

For P a degenerated polytope of $\mathbf{R}^n$, that is to say that the affine subspace $[P]$ of $\mathbf{R}^n$ generated by P is of dimension $< n$, one denotes $C_{[P]}(x, P)$ the exterior normal cone of P at x in $[P]$, since P is of maximal dimension in $[P]$. With this notation, the exterior normal cone of P at x in $\mathbf{R}^n$, denoted $C_{\mathbf{R}^n}(x, P)$, or simply $C(x, P)$ when no confusion is possible, is defined by $C_{[P]}(x, P) \times [P]^\perp$. We finally define $C(F, P)$, for P general, as $C(x, P)$ for any $x \in F$. The exterior normal cone of P depends on the ambient space in which we embed P, but we now define an intrinsic measure attached to the normal exterior cone, the exterior angle.
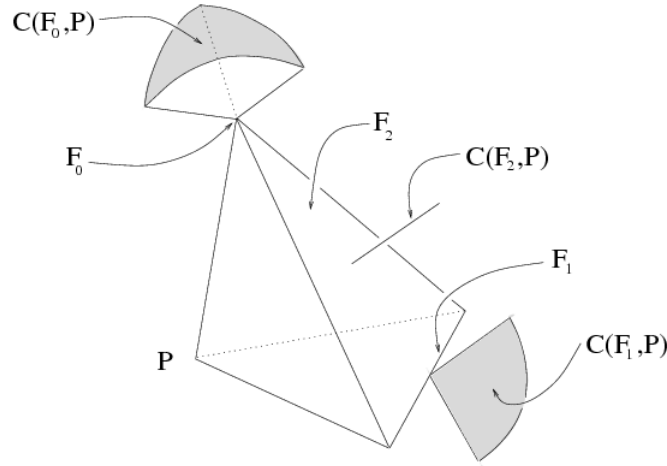

fig.1

0.2. **Definition.** Let P be a polytope of $\mathbf{R}^n$ and $F \in \mathscr{F}_i(P)$. One defines the exterior angle $\gamma(F, P)$ of P along F (see fig.1), by

$$\gamma(F, P) := \frac{1}{\alpha_{n-i}} \cdot Vol_{n-i}(C(F, P) \cap \bar{B}_{(0,1)}) = Vol_{n-i-1}(C(F, P) \cap S_{(0,1)}).$$

By convention $\gamma(P, P) = 1$.

With the definition of the exterior angle, the proof of (1) is trivial.

Proof of equality (1). We observe that

$$Vol_n(T_{P,\varepsilon}) = \sum_{i=0}^{n} \alpha_i \cdot \varepsilon^{n-i} \sum_{F \in \mathscr{F}_i(P)} Vol_i(F) \cdot \gamma(F, P).$$

In particular

$$\Lambda_i(P) = \sum_{F \in \mathscr{F}_i(P)} Vol_i(F) \cdot \gamma(F, P). \tag{2}$$

$\square$

The equality (2) shows how the invariant $\Lambda_i(P)$ captures the concentration $\gamma(F, P)$ of the curvature of the family $(T_{P,\varepsilon})_{\varepsilon>0}$ along the i-dimensional faces of P as ε goes to 0.

In the general convex case and not only in the convex polyhedral case, the equality (1) still holds, defining invariants Λ_i on the set $\mathscr{K}^n$ of convex sets of $\mathbf{R}^n$. A proof of this equality by approximation of a convex set by a sequence of polytopes is given in [97], section 4.2. Another proof is indicated in [40] (3.2.35) and [74], using the Cauchy-Crofton formula, a classical formula in integral geometry, that we recall here.

0.3. **Cauchy-Crofton formula** ([40] 5.11, [41] 2.10.15, 3.2.26, [94] 14.69). *Let $A \subset \mathbf{R}^n$ a $(\mathscr{H}^d, d)$-rectifiable set, where $\mathscr{H}^d$ is the d-dimensional Hausdorff measure. We have*

$$Vol_d(A) = \frac{1}{\beta(d,n)} \int_{\bar{P} \in \overline{G}(n-d,n)} Card(A \cap \bar{P}) \, d\overline{\gamma}_{n-d,n}(\bar{P}), \tag{$\mathscr{C}\mathscr{C}$}$$

with $\overline{G}(n-d,n)$ the Grassmannian of $(n-d)$-dimensional affine planes $\bar{P}$ of $\mathbf{R}^n$, $\overline{\gamma}_{n-d,n}$ its canonical measure and denoting Γ the Euler function, $\beta(d,n)$ the universal constant

$$\Gamma(\frac{n-d+1}{2})\Gamma(\frac{d+1}{2})/\Gamma(\frac{n+1}{2})\Gamma(\frac{1}{2}).$$

One can now prove equality (1) in the general compact convex case.

Proof of equality (1) *in the convex case.* We proceed by induction on the dimension of the ambient space is which our convex compact set K lies. If this dimension is 1, formula (1) is trivial, and if this dimension is $n > 1$, one has

$$Vol_n(T_{K,\varepsilon}) = Vol_n(K) + \int_{r=0}^{\varepsilon} Vol_{n-1}(K^r) \, dr,$$

where K^r is the set of points in $\mathbf{R}^n$ at distance r of K. We compute $Vol_{n-1}(K^r)$ using the Cauchy-Crofton formula.

Noting that $Card(\overline{L} \cap K^r) = 2$ or $Card(\overline{L} \cap K^r) = 0$, up to a $\overline{\gamma}_{1,n}$-null subset of $\overline{G}(1,n)$, we obtain by definition of $\overline{\gamma}_{1,n}$

$$Vol_n(T_{K,\varepsilon}) =$$

$$Vol_n(K) + \int_{r=0}^{\varepsilon} \frac{2}{\beta(1,n)} \int_{H \in G(n-1,n)} Vol_{n-1}(\pi_H(K^r)) \, d\gamma_{n-1,n}(H) \, dr,$$

where $G(n-1, n)$ is the Grassmannian of $(n-1)$-dimensional vector subspace of $\mathbf{R}^n$ equipped with its canonical measure $\gamma_{n-1,n}$ invariant under the action of $O_n(\mathbf{R})$ and π_H is the orthogonal projection onto $H \in G(n-1, n)$.
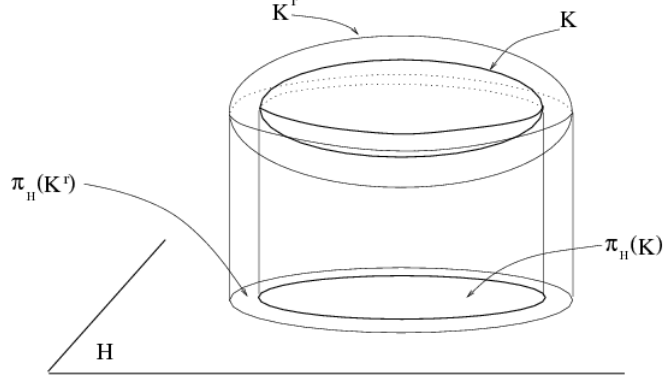

fig.2

By induction hypothesis, the expression of the volume of the tubular neighbourhood of radius r of the convex sets of $\mathbf{R}^{n-1}$ is a polynomial in r. Since $\pi_H(K^r)$ is $T_{\pi_H(K),r}$ in H, we have

$$Vol_n(T_{K,\varepsilon}) = Vol_n(K)$$

$$+ \frac{2}{\beta(1,n)} \int_{r=0}^{\varepsilon} \int_{H \in G(n-1,n)} \sum_{i=0}^{n-1} \alpha_i \Lambda_{n-1-i}(\pi_H(K)) \cdot r^i \; d\gamma_{n-1,n}(H) \; dr$$

$$= Vol_n(K) + \frac{2}{\beta(1,n)} \sum_{i=0}^{n-1} \frac{\alpha_i}{i+1} \cdot \varepsilon^{i+1} \int_{H \in G(n-1,n)} \Lambda_{n-1-i}(\pi_H(K)) \; d\gamma_{n-1,n}(H).$$

$\square$

In [101], the formula (1) is also proved for C^{2+} surfaces, giving a hint for a possible extension of this formula to the smooth case. This extension is due to H. Weyl, who proved in [110] the following statement (see also [74]).

0.4. **Theorem** (Weyl's tubes formula)**.** *Let X be a smooth compact submanifold of $\mathbf{R}^n$ of dimension d. Let $\eta_X > 0$ such that for any ε, $0 < \varepsilon \leq \eta_X$, for any $y \in T_{X,\varepsilon}$, there exists a unique $x \in X$ such that $y \in x + (T_x X)^{\perp}$. Then for any $\varepsilon \leq \eta_X$*

$$Vol_n(T_{X,\varepsilon}) = \sum_{i=0}^{[d/2]} \alpha_{n-d+2i} \Lambda_{d-2i}(X) \cdot \varepsilon^{n-d+2i},$$

where the $\Lambda_k(X)'s$ are invariant under isometric embeddings of X into Riemannian manifolds.

When, on the other hand, X is a non convex union of two polytopes P, Q, $Vol_n(T_{X,\varepsilon})$ is no more necessarily a polynomial in ε. For example for $X_1 = P \cup Q$ where $P = \{(0,0)\} \subset \mathbf{R}^2$ and $Q = \{(0,2)\} \subset \mathbf{R}^2$, and for $1 \leq \varepsilon \leq 2$. In the same way, when X is a singular set, for any $\varepsilon > 0$, $Vol_n(T_{X,\varepsilon})$ is not necessarily a polynomial in ε. For example for

$$X_2 = \{(x,y) \in \mathbf{R}^2; x \geq 0, \;\; (x^2 + y^2 - 1)(x^2 + (y-2)^2 - 1) = 0\},$$

for any sufficiently small $\varepsilon > 0$, $Vol_2(T_{X,\varepsilon}) = (1 + \varepsilon)^2 \arccos(\dfrac{1}{1 + \varepsilon}) - \sqrt{\varepsilon^2 + 2\varepsilon}$ (see fig.3).

0.5. *Remark.* Nevertheless by [22] we know that for X a subanalytic subset of $\mathbf{R}^n$, $Vol(T_{X,\varepsilon})$ is a polynomial in subanalytic functions with variable ε and the logarithms of these functions, and thus that it is defined in some o-minimal structure over the reals.

0.6. *Remark.* The grey areas Σ_1 and Σ_2 in figure 3, counted with multiplicities 1 in $Vol_2(T_{X_i,\varepsilon})$ have non polynomial contributions. But when these areas are counted with multiplicity 2, on one hand, with this modified computation for $Vol_2(T_{X_1,\varepsilon})$, we obtain the sum of the areas of two discs of radius ε centred at P and Q and, on the other hand, with this modified computation for $Vol_2(T_{X_2,\varepsilon})$, we obtain twice the volume of the tubular neighbourhood of radius ε of half a circle minus the volume of a ball of radius ε.
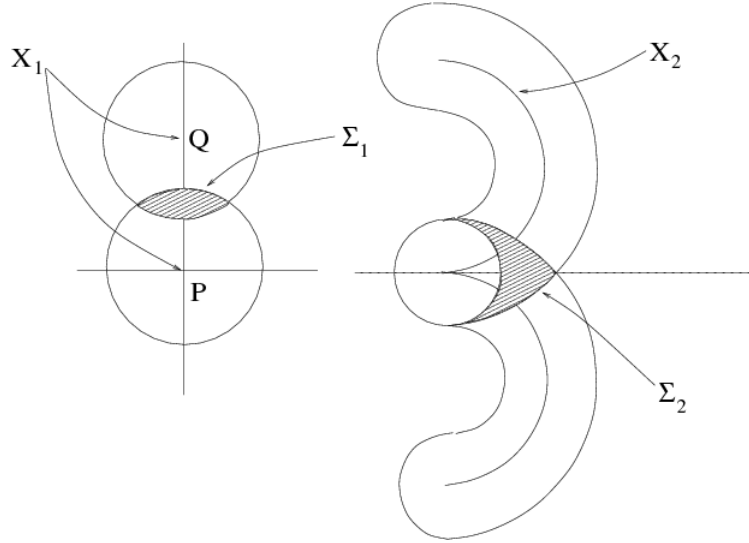

fig.3

In conclusion, a multiple contribution of the volume of the grey areas provides two polynomials in ε. Moreover, we observe that for $j = 1, 2$:

- $\forall x \in \Sigma_j$: $2 = \chi(X_j \cap \bar{B}_{(x,\varepsilon)})$,
- $\forall x \in T_{X_j,\varepsilon} \setminus \Sigma_j$: $1 = \chi(X_j \cap \bar{B}_{(x,\varepsilon)})$,
- $\forall x \in \mathbf{R}^2 \setminus T_{X_j,\varepsilon}$: $\chi(X_j \cap \bar{B}_{(x,\varepsilon)}) = 0$.

It follows that for j=1,2 $\displaystyle\int_{x \in \mathbb{R}^2} \chi(X_j \cap \bar{B}_{(x,\varepsilon)}) \, dx = \int_{x \in T_{X_j,\varepsilon}} \chi(X_j \cap \bar{B}_{(x,\varepsilon)}) \, dx$ is a polynomial in ε. These examples are in the scope of a general fact: the formula (1) can be generalized to compact sets definable in some o-minimal structure over the reals by the formula (1′) given below.

0.7. **Theorem** ([45], [46], [47], [48], [49], [50], [51], [8], [9], [11])**.** *Let X be a compact subset of $\mathbf{R}^n$ definable in some o-minimal structure over the ordered real field. There exist constants*

$\Lambda_0(X), \cdots, \Lambda_n(X)$ *such that for any* $\varepsilon \geq 0$

$$\int_{x \in T_{\varepsilon, X}} \chi(X \cap \bar{B}_{(x,\varepsilon)}) \ dx = \sum_{i=0}^{n} \alpha_i \Lambda_{n-i}(X) \cdot \varepsilon^i. \tag{1'}$$

The real numbers $\Lambda_i(X), i = 0, \cdots, n$ *are called the Lipschitz-Killing curvatures of* X, *they only depend on definable isometric embeddings of* X *into euclidean spaces. Moreover, we have*

$$\Lambda_i(X) = \int_{\bar{P} \in \bar{G}(n-i,n)} \chi(X \cap \bar{P}) \ \frac{d\bar{\gamma}_{n-i,n}(\bar{P})}{\beta(i,n)}. \tag{2'}$$

0.8. *Remarks about* $(1')$ *and* $(2')$. In Theorem 0.7 we assume the set X compact, although for X bounded but not compact the equality $(1')$ together with $(2')$ is still true with χ the Euler-Poincaré characteristic with compact support, usually considered for non-compact definable sets. This characteristic is additive and multiplicative and defined by any finite cell decomposition $\cup_i C_i$ of X by $\chi(X) = \sum_i (-1)^{\dim(C_i)}$ (see [39], p. 69). For simplicity, in what follows we will still consider the compact case.

0.9. *Remark.* The formula $(1')$ is clearly a generalization to the non convex case of the formula (1), since for X compact convex, for any $\varepsilon \geq 0$, for any $x \in T_{X,\varepsilon}$, $\chi(X \cap \bar{B}_{(x,\varepsilon)}) = 1$, thus $\int_{x \in T_{\varepsilon,X}} \chi(X \cap \bar{B}_{(x,\varepsilon)}) \ dx = Vol_n(T_{X,\varepsilon})$. In the same way $(1')$ generalizes Weyl's tube formula to the singular case, since for X smooth, there exists $\eta_X > 0$ such that for any $\varepsilon, 0 < \varepsilon < \eta_X$, for any $x \in T_{X,\varepsilon}$, $\chi(X \cap \bar{B}_{(x,\varepsilon)}) = 1$ and $\int_{x \in T_{\varepsilon,X}} \chi(X \cap \bar{B}_{(x,\varepsilon)}) \ dx = Vol_n(T_{X,\varepsilon})$.

The formula $(1')$ comes from a more general cinematic formula (see [11], [49]). For X and Y two definable sets of $\mathbf{R}^n$

$$\int_{g \in G} \Lambda_k(X \cap g \cdot Y) \ dg = \sum_{i+j=k+n} c_{n,i,j} \cdot \Lambda_i(X) \cdot \Lambda_j(Y)$$

with G the group of isometries of $\mathbf{R}^n$ and $c_{n,i,j}$ universal constants.

The expression of Λ_i given by $(2')$ generalizes to the definable case the representation formula (2) of Λ_i given in the polyhedral case. Furthermore, from $(2')$ we get the following characterization of Λ_0 and Λ_d, $d = \dim(X)$, already obtained from (1) in the compact convex case

$$\Lambda_0(X) = \chi(X)$$

and, using the Cauchy-Crofton formula,

$$\Lambda_d(X) = Vol_d(X).$$

Finally,

$$\Lambda_{d+1}(X) = \cdots = \Lambda_n(X) = 0,$$

for $d < n$.

The last remark made now here is a remark that, having in mind geometric measure theory, we are eager to address: the Euler-Poincaré characteristic being additive for definable sets (see [39]) the equality $(1')$ or $(2')$ shows that the Λ_i's are additive invariants of definable sets, in the following sense

$$\forall i \in \{0, \cdots, n\}, \ \Lambda_i(X \cup Y) = \Lambda_i(X) + \Lambda_i(Y) - \Lambda_i(X \cap Y),$$

for any definable sets X and Y of $\mathbf{R}^n$.

We have now in hand two kinds of deformation of a singular set. When this set is an analytic isolated hypersurface singularity, we may consider its Milnor fibration, providing in particular as invariant the Milnor number of the singularity, and in the more general definable case, we have recalled in details the notion of Lipschitz-Killing curvatures, coming from the deformation family provided by the tubular neighbourhoods. It is worth noting that in these two cases, the deformations considered lead to additive invariants attached to the given germ.

In what follows, we explain how to localize the Lipschitz-Killing curvatures in order to attach to a singular germ a finite sequence of additive invariants (Section 1). We will then explain how all our local invariants are related and how such kind of relation illustrates, in the very general context of definable sets over the reals, the emergence of a well-known principle in convex geometry wherein additive invariants (with some additional properties) may not be so numerous (Section 2). Finally, we will stress the fact that the additive nature of an invariant coming from a deformation allows us to compute this invariant in some adapted scissors ring via some generating zeta function capturing the nature of the deformation. This is the point of view which underlies the work of Denef and Loeser (Section 3).

0.10. *Notation.* As well as in this introduction, in the sequel, $B_{(x,r)}$, $\bar{B}_{(x,r)}$ and $S_{(x,r)}$ are respectively the open ball, the closed ball and the sphere centred at x and with radius r of the real or the complex vector spaces $\mathbf{R}^n$ or $\mathbf{C}^n$. If necessary, to avoid confusion, we emphasize the dimension d of the ambient space to which the ball belongs by denoting $B^d_{(x,r)}$. Definable means definable in some given o-minimal structure expanding the ordered real field $(\mathbf{R}, +, -, \cdot, 0, 1, <)$ (see [14], [39]).

1. Local invariants from the tubular neighbourhoods deformation

The invariants $\Lambda_0, \cdots, \Lambda_n$ defined in the introduction for compact definable sets (or at least bounded definable sets) may be extended to non bounded definable sets as well as they may be localized in order to be attached to any definable germ $(X, 0)$. The extension of the invariants Λ_i to non bounded definable sets has been proposed in [36]. These two possible extensions are similar; they essentially use the fact that near a given point or near infinity the topological types of affine sections of a definable set are finite in number. As we are mainly interested in local singularities, we explain in this section how to localize the sequence $(\Lambda_0, \cdots, \Lambda_n)$ at a given point.

For this goal, let us consider $X \subset \mathbf{R}^n$ a compact definable set. We assume that $0 \in X$ and we denote by X_0 the germ of X at 0, d its dimension. Representing elements $\bar{P}$ of the Grassmann manifold $\bar{G}(n-i, n)$ of $(n-i)$-dimensional affine subspaces of $\mathbf{R}^n$ by pairs $(x, P) \in \mathbf{R}^n \times G(i, n)$, where $x \in P$, and $\bar{P}$ is the affine subspace of $\mathbf{R}^n$ orthogonal to P at x, the measure $\bar{\gamma}_{n-i,n}$ on $\bar{G}(n-i, n)$ is the image through this representation of the product $m \otimes \gamma_{i,n}$, where m is the Lebesgue measure on P and P is identified with $\mathbf{R}^i$. It follows by formula (2') that Λ_i is i-homogeneous, that is to say $\Lambda_i(\lambda \cdot X) = \lambda^i \Lambda_i(X)$, for any $\lambda \in \mathbf{R}^*_+$. In consequence, it is natural to consider the asymptotic behaviour of $\frac{1}{\alpha_i} \Lambda_i(\frac{1}{\varrho} \cdot (X \cap \bar{B}_{(0,\varrho)})) = \frac{1}{\alpha_i \varrho^i} \Lambda_i(X \cap \bar{B}_{(0,\varrho)})$, as $\varrho \to 0$, in order to obtain invariants attached to the germ X_0 of X at 0.

Using standard arguments for the definable family

$$(\frac{1}{\varrho} \cdot (X \cap \bar{B}_{(0,\varrho)}) \cap \bar{P}))_{(\varrho, \bar{P}) \in \mathbf{R}^*_+ \times \bar{G}(n-i,n)}$$

such as Thom-Mather's isotopy lemma or cell decomposition theorem, one knows that, for any fixed $\bar{P} \in \bar{G}(n-i, n)$, the topological type of the family $(\frac{1}{\varrho} \cdot (X \cap \bar{B}_{(0,\varrho)}) \cap \bar{P}))_{\varrho \in \mathbf{R}^*_+}$ is constant for ϱ small enough and therefore the limit of $\chi(\frac{1}{\varrho} \cdot (X \cap \bar{B}_{(0,\varrho)}) \cap \bar{P})$ for $\varrho \to 0$ does exist. Furthermore,

still by finiteness arguments proper to definable sets, the family $(\chi(\frac{1}{\varrho} \cdot (X \cap \bar{B}_{(0,\varrho)}) \cap \bar{P}))_{\bar{P} \in \bar{G}(n-i,n)}$ is bounded with respect to $\bar{P}$.

The next definition follows from these observations (see [21], Theorem 1.3).

1.1. **Definition** (Local Lipschitz-Killing invariants, see [21])**.** Let X be a (compact) definable set of $\mathbf{R}^n$, representing the germ X_0 at $0 \in X$. The limit

$$\Lambda_i^{\ell oc}(X_0) := \lim_{\varrho \to 0} \frac{1}{\alpha_i \cdot \varrho^i} \Lambda_i(X \cap \bar{B}(0, \varrho)) \tag{3}$$

exists and the finite sequence of real numbers $(\Lambda_i^{\ell oc}(X_0))_{i \in \{0,\ldots,n\}}$ is called the sequence of local Lipschitz-Killing invariants of the germ X_0.

1.2. *Remark.* Another kind of localization of the invariants Λ_i have been obtained and studied in [9], by considering the family $(X \cap S_{(0,\varrho)})_{\varrho > 0}$ instead of the family $(X \cap \bar{B}_{(0,\varrho)})_{\varrho > 0}$.

1.3. *Remarks.* For any $i \in \{0, \cdots, n\}$, just as Λ_i, $\Lambda_i^{\ell oc}$ is invariant under isometries of $\mathbf{R}^n$ and defines an additive function on the set of definable germs at the origin of $\mathbf{R}^n$. Moreover $\Lambda_i^{\ell oc}(X_0) = 0$, for $i > d$, since $\Lambda_i = 0$ for definable sets of dimension $< i$ and for any definable compact germ X_0, $\Lambda_0^{\ell oc}(X_0) = 1$, since $\Lambda_0 = \chi$ and a definable germ is contractible. Finally, since by the Cauchy-Crofton formula $(\mathscr{CC})$ and $(2')$ we have

$$\Lambda_d^{\ell oc}(X_0) = \lim_{\varrho \to 0} \frac{Vol_d(X \cap B_{(0,\varrho)})}{Vol_d(B_{(0,\varrho)}^d)}, \tag{4}$$

we observe that $\Lambda_d^{\ell oc}(X_0)$ is by definition the local density $\Theta_d(X_0)$ of X_0, and thus we have obtained, by finiteness arguments leading to Definition 1.1, the following theorem of Kurdyka and Raby.

1.4. **Corollary** ([69], [70], [78])**.** *The local density of definable sets of* $\mathbf{R}^n$ *exists at each point of* $\mathbf{R}^n$.

On figure 4 are represented the data taken into account in the computation of $\Lambda_i^{\ell oc}(X)$.

For $P \in G(i,n)$, we denote by $K_\ell^{P,\varrho}$ the domains of P above which the Euler-Poincaré characteristic of the fibres of $\pi_{P|X \cap \bar{B}(0,\varrho)}$ is constant and equals $\chi_\ell^{P,\varrho} \in \mathbf{Z}$. The quantity $\Lambda_i(X \cap \bar{B}_{(0,\varrho)}^n)$ is then obtained as the mean value over the vector planes P of the sum $\sum_{\ell=1}^{\ell_P} \chi_\ell^{P,\varrho} \cdot Vol_i(K_\ell^{P,\varrho})$. In particular we'd like to stress the fact that are considered in this sum the volumes of the domains $K_\ell^{P,\varrho}$ (in green on figure 4) defined by the critical values of π_P coming from the link $X \cap S_{(0,\varrho)}$. We draw attention to these green domains, far from the origin, in view of other local invariants, the polar invariants, that will be defined in the next section, and for which only the domains $K_\ell^{P,\varrho}$ close to the origin will be considered.
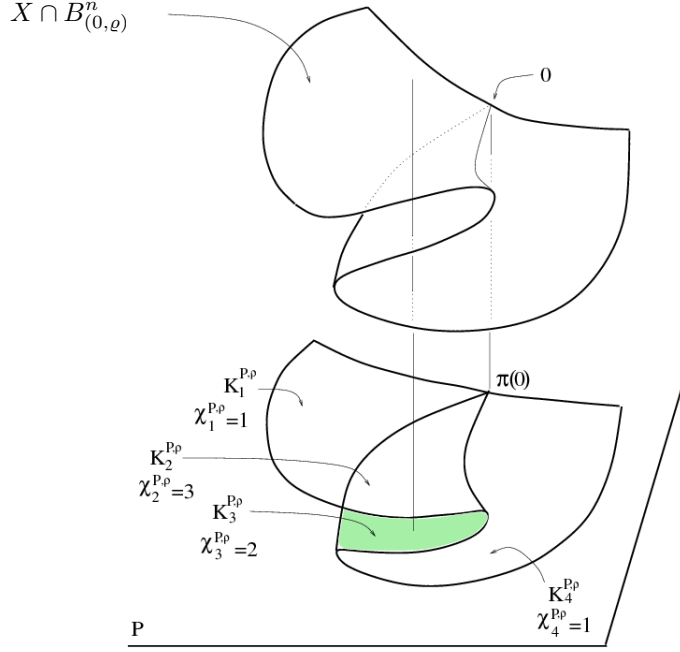
$X \cap B^n_{(0,\varrho)}$

0

$K_1^{P,\rho}$
$\chi_1^{P,\rho}=1$

$K_2^{P,\rho}$
$\chi_2^{P,\rho}=3$

$K_3^{P,\rho}$
$\chi_3^{P,\rho}=2$

$\pi(0)$

$K_4^{P,\rho}$
$\chi_4^{P,\rho}=1$

P

fig.4

2. ADDITIVE INVARIANTS OF SINGULARITIES AND HADWIGER PRINCIPLE

In the previous section we have localized the Lipschitz-Killing invariants. We'd like now to investigate the question of how these invariants are related to classical local invariants of singularities, such as Milnor number, or Milnor numbers of generic plane sections of the singularity (for the complex case). The question of the correspondence of invariants coming from differential geometry and invariants of singularities has been tackled by several authors. In the first section 2.1, we briefly recall some of these works. We then introduce in section 2.2 a sequence of invariants of real singularities that is the real counterpart of classical invariants of complex singularities and we finally relate the localized Lipschitz-Killing invariants to the polar invariants in section 2.3. In section 2.4 we recall results from convex geometry and convex valuation theory giving a strong hint, called here Hadwiger principle, of the reason why such invariants have to be linearly related.

2.1. **Differential geometry of complex and real hypersurface singularities.** The first example we'd like to recall of such a correspondence may be found in [71] (see also [72] and [73]). In [71] R. Langevin relates the concentration of the curvature of the Milnor fibre

$$F_\varepsilon^\eta = f^{-1}(\varepsilon) \cap B_{(0,\eta)}, \quad 0 < \varepsilon \ll \eta \ll 1,$$

as ε and η go to 0 to the Milnor numbers μ and $\mu^{(n-1)}$ of the isolated hypersurface singularity $f : (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$. To present this relation, we need some definitions.

For a given real smooth oriented hypersurface H of $\mathbf{R}^n$ and for $x \in H$, we classically define the Gauss curvature $K(x)$ of H at x by $K(x) := \mathrm{jac}(\nu)$, where $\nu : H \to \mathbf{S}^{n-1}$ is the mapping giving the unit normal vector to H induced by the canonical orientation of $\mathbf{R}^n$ and the given orientation of H. The curvature $K(x)$ can be generalized in the following way to any submanifold

M of $\mathbf{R}^n$ of dimension $d, d \in \{0, \cdots, n-1\}$ (see [42]). Let x be a point of M and denote by $N(x) \simeq \mathbf{S}^{n-d-1}$ the manifold of normal vectors to M at x, and, for $\nu \in N(x)$, by $K(x, \nu)$ the Gauss curvature at x of the projection M_ν of M to $\mathrm{T}_x M \oplus \nu$. Note that the projection M_ν defines at x a smooth hypersurface of $\mathrm{T}_x M \oplus \nu$ oriented by ν. The mean value of $K(x, \nu)$ over $N(x)$ define the desired generalized Gauss curvature.

2.1.1. **Definition** (see [42]). With the above notation, the curvature $K(x)$ of M at x is defined by

$$K(x) := \int_{\nu \in \mathbf{P} N(x)} K(x, \nu) \, d\nu$$

2.1.2. *Remarks.* The curvature $K(x, \nu)$ is ε^{d-n} times the Gauss curvature of the boundary $\partial T_{M,\varepsilon}$ of the ε-neighbourhood of M at $x + \varepsilon\nu$ (see [15]).

In [71], following Milnor, it is observed that for M a smooth complex hypersurface of $\mathbf{C}^n$, $K(x) = (-1)^{n-1}\pi|\mathrm{jac}\gamma_{\mathbf{C}}(x)|^2$, where $\gamma_{\mathbf{C}}$ is the complex Gauss map sending $x \in M$ to the normal complex line $\gamma_{\mathbf{C}}(x) \in \mathbf{P}\mathbf{C}^{n-1}$ to M at x.

In case M is a compact submanifold of $\mathbf{R}^n$, using a so-called exchange formula ([72] Theorem II.1, [74], [35]) relating $\int_M K(x) \, dx$ and the mean value over generic lines L in $\mathbf{R}^n$ of the total index of the projection of M on L, we obtain the Gauss-Bonnet theorem

$$\int_M K(x) \, dx = c(n, d)\chi(M).$$

Applying Definition 2.1.1 to the Milnor fibre F_ε^η of an isolated hypersurface singularity $f : (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$ and using again the exchange formula, we can estimate the concentration of the curvature $K(x)$ of the Milnor fibre F_ε^η as ε and η go to 0. This value is related to the invariants μ and $\mu^{(n-1)}$ of the singularity, thanks to a result of [103], by the following formula

2.1.3. **Theorem** ([71], [72]). *The curvature K of a complex Milnor fibre satisfies*

$$\lim_{\eta \to 0} \lim_{\varepsilon \to 0} \; c(n) \int_{F_\varepsilon^\eta} (-1)^{n-1} K(x) \, dx = \mu + \mu^{(n-1)},$$

where $c(n)$ is a constant depending only on n.

This formula has been generalized to the other terms of the sequence $\mu^{(*)}$ by Loeser in [79] in the following way.

2.1.4. **Theorem** ([79]). *For $k \in \{1, \cdots, n-1\}$, we have*

$$\lim_{\eta \to 0} \lim_{\varepsilon \to 0} \frac{(-1)^{n-k}c(n, k)}{\eta^{2k}} \int_{F_\varepsilon^\eta} c_{n-1-k}(\Omega_{f^{-1}(\varepsilon)}) \wedge \Phi^k = \mu^{(n-k)} + \mu^{(n-k-1)},$$

where $c_{n-1-k}(\Omega_{f^{-1}(\varepsilon)})$ is the $(n-1-k)$-th Chern form of $f^{-1}(\varepsilon)$, Φ the Kähler form of $\mathbf{C}^n$ and as usual $c(n, k)$ a constant depending only on n and k.

A real version of these two last statements has been given by N. Dutertre, in [35] for the real version of Theorem 2.1.3 and in [36] for the real version of Theorem 2.1.4. In [35] (see Theorem 5.6), a real polynomial germ $f : (\mathbf{R}^n, 0) \to (\mathbf{R}, 0)$ having an isolated singularity at 0 is considered and the following equalities are given for the asymptotic behaviour of the Gauss curvature on the real Milnor fibre $F_\varepsilon^\eta = f^{-1}(\varepsilon) \cap B_{(0,\eta)}$.

2.1.5. **Theorem** ([35], Theorem 5.6). *The Gauss curvature K of the real Milnor fibre F_ε^η have the following asymptotic behaviour*

$$\lim_{\eta \to 0} \lim_{\varepsilon \to 0^+} \int_{F_\varepsilon^\eta} K(x) \, dx = \frac{Vol(S^{n-1})}{2} deg_0 \nabla f + \frac{1}{2} \int_{G(n-1,n)} deg_0 \nabla(f_{|P}) \, dP$$

$$\lim_{\eta \to 0} \lim_{\varepsilon \to 0^-} \int_{F_\varepsilon^\eta} K(x) \ dx = -\frac{Vol(S^{n-1})}{2} deg_0 \nabla f + \frac{1}{2} \int_{G(n-1,n)} deg_0 \nabla(f_{|P}) \ dP$$

In [36], the asymptotic behaviour of the symmetric functions $s_0, \cdots, s_{n-1}$ of the curvature of the real Milnor fibre are studied. For a given smooth hypersurface H of $\mathbf{R}^n$, the s_i's are defined by

$$\det(Id + tD\nu(x)) = \sum_{i=0}^{n-1} s_i(x) \cdot t^i = \prod_{i=1}^{n-1}(1 + k_i(x)t),$$

where the k_i's are the principal curvatures of H, that is to say, the eigenvalues of the symmetric morphism $D\nu(x)$. The limits

$$\lim_{\eta \to 0} \lim_{\varepsilon \to 0} \frac{1}{\eta^k} \int_{F_\varepsilon^\eta} s_{n-k}(x) \ dx$$

are then given in terms of the mean value of $deg_0 \nabla(f_{|P})$ for $P \in G(n - k + 1, n)$ and for $P \in G(n - k - 1, n)$ (see [36], Theorem 7.1). In particular, the asymptotic behaviour of the symmetric functions $s_0, \cdots, s_{n-1}$ of the curvature of the real Milnor fibre is related to the Euler-Poincaré characteristic of the real Milnor fibre by the following statement.

2.1.6. **Theorem** ([36], Corollary 7.2)**.** *For n odd,*

$$\chi(F_\varepsilon^\eta) = \sum_{k=0}^{(n-1)/2} c(n,k) \lim_{\eta \to 0} \lim_{\varepsilon \to 0} \frac{1}{\eta^{2k}} \int_{F_\varepsilon^\eta} s_{n-1-2k}(x) \ dx,$$

and for n even

$$\chi(F_\varepsilon^\eta) = \frac{1}{2}\chi(f^{-1}(0) \cap S_{(0,\eta)}^{n-1}) = \sum_{k=0}^{(n-2)/2} c'(n,k) \lim_{\eta \to 0} \lim_{\varepsilon \to 0} \frac{1}{\eta^{2k}} \int_{F_\varepsilon^\eta} s_{n-2-2k}(x) \ dx,$$

2.2. **Local invariants of definable singular germs.** In the present survey we aim to relate the local Lipschitz-Killing invariants $\Lambda_i^{\ell oc}, i = 0, \cdots, n$, coming from the tubular neighbourhoods deformation, to local invariants of definable singular germs of $\mathbf{R}^n$. These germs have not necessarily to be of codimension 1 in $\mathbf{R}^n$, as it is the case in the complex and real statements recalled above in Section 2.1. Therefore we have to define local invariants of singularities attached to definable germs of $\mathbf{R}^n$ of any dimension and try to relate them to the sequence $\Lambda_*^{\ell oc}$. Furthermore those invariants have to extend, to the real setting, classical invariants of complex singularities, such as the sequence $\mu^{(*)}$ in the hypersurface case or the sequence of the local multiplicity of polar varieties in the general case. For this purpose we introduce now a new sequence σ_* of local invariants, called the sequence of polar invariants.

Let, as before, $X \subset \mathbf{R}^n$ be a closed definable set, and assume that X contains the origin of $\mathbf{R}^n$ and that d is the dimension of X at 0. We denote by $\mathscr{C}(X)$ the group of definable constructible functions on X, that is to say the group of definable $\mathbf{Z}$-valued functions on X. These functions f are characterized by the existence of a finite definable partition (X_i) of X (depending on f) such that $f_{|X_i}$ is a constant integer $n_i \in \mathbf{Z}$, for any i. We denote by $\mathscr{C}(X_0)$ the group of germs at the origin of functions of $\mathscr{C}(X)$. For $Y \subset \mathbf{R}^m$ a definable set, $f : X \to Y$ a definable mapping, a definable set $Z \subset X$ and $y \in Y$, we introduce the notation $f_*(\mathbf{1}_Z)(y) := \chi(f^{-1}(y) \cap Z)$ and we then define the following functor from the category of definable sets to the category of groups

$$
\begin{array}{ccc}
X & \rightsquigarrow & \mathscr{C}(X) \\
f \downarrow & & \downarrow f_* \\
Y & \rightsquigarrow & \mathscr{C}(Y)
\end{array}
$$

In [21], Theorem 2.6, it is stated that, for $f = \pi_P$ the (orthogonal) projection onto a generic i-dimensional vector subspace P of $\mathbf{R}^n$, this diagram leads to the following diagram for germs

$$\begin{array}{ccc} X_0 & \rightsquigarrow & \mathscr{C}(X_0) \\ \pi_{P_0} \downarrow & & \downarrow \pi_{P_0 *} \\ P_0 & \rightsquigarrow & \mathscr{C}(P_0) \end{array} \tag{5}$$

where for $Z_0 \subset X_0$ and $y \in P$, $\pi_{P_0 *}(\mathbf{1}_{Z_0})(y)$ is defined by $\chi(\pi_P^{-1}(y) \cap Z \cap \bar{B}_{(0,\varrho)})$, ϱ being sufficiently small and $0 < \|y\| \ll \varrho$. The existence of such a diagram for germs simply amounts to prove that a generic projection of a germ defines a germ [2] and that, for such a projection and for any $c \in \mathbf{Z}$, the germ at 0 of the definable set $\{y \in P; \chi(\pi_P^{-1}(y) \cap Z \cap \bar{B}_{(0,\varrho)}) = c\}$ does not depend on ϱ.

Denoting by $\theta_i(\varphi)$ the integral with respect to the local density Θ_i at $0 \in \mathbf{R}^i$ of a germ $\varphi : P_0 \to \mathbf{Z}$ of constructible function, that is to say

$$\theta_i(\varphi) := \sum_{j=1}^{N} n_j \cdot \Theta_i(K_0^j),$$

when $\varphi = \sum_{j=1}^{N} n_j \cdot \mathbf{1}_{K_0^j}$, for some definable germs $K_0^j \subset P_0$ partitioning P_0, we can define the desired polar invariants $\sigma_i(X_0)$ of X_0.

2.2.1. **Definition** (Polar invariants). With the previous notation, the polar invariants of the definable germ X_0 are

$$\sigma_i(X_0) := \int_{P \in G(i,n)} \theta_i(\pi_{P_0 *}(\mathbf{1}_{X_0})) \, d\gamma_{i,n}(P), \quad i = 0, \cdots, n$$

2.2.2. *Remarks.* Since they are defined as mean values over generic projections, the σ_i's are invariant under the action of isometries of $\mathbf{R}^n$. On the other hand the σ_i's define additive invariants (as well as the $\Lambda_i^{\ell oc}$'s do), since they are defined through the Euler-Poincaré characteristic χ and the local density Θ_i, two additive invariants.

Observe that $\sigma_i(X_0) = 0$, for $i > d$, since a general k-dimensional affine subspace of $\mathbf{R}^n$ does not encounter a definable set of codimension $> k$. We also have $\sigma_0(X_0) = \Lambda_0^{\ell oc}(X_0) = 1$, again by the local conic structure of definable sets and because X_0 is closed. Finally, for $i = d$, one shows that

$$\sigma_d(X_0) = \Theta_d(X_0) = \Lambda_d^{\ell oc}(X_0)$$

(we recall that by the Cauchy-Crofton formula ($\mathscr{C}\mathscr{C}$) and by definition (2') and (3) of Λ_d and of $\Lambda_d^{\ell oc}$, we have $\Theta_d(X_0) = \Lambda_d^{\ell oc}(X_0)$, as already observed for Corollary 1.4). Since the relation

$$\sigma_d(X_0) = \Theta_d(X_0)$$

asserts that the localization Θ_d of the d-volume is σ_d, that is to say, by definition of σ_d, that the localization of the volume may be computed by the mean value over (generic) d-dimensional vector subspaces $P \subset \mathbf{R}^n$ of the number of points in the fibre of the projections of the germ X_0 onto the germ P_0, this relation appears as the local version of the global Cauchy-Crofton formula ($\mathscr{C}\mathscr{C}$). We state it as follows.

---

[2] Let us for instance denote X the blowing-up of $\mathbf{R}^2$ at the origin and $x \in X$ a point of the exceptional divisor of X. We then note that the projection of the germ $(X)_x$ on $\mathbf{R}^2$, along the exceptional divisor of X, does not define a germ of $\mathbf{R}^2$. Indeed, the projection of $X \cap U$, for U a neighbourhood of x in X, defines a germ at the origin of $\mathbf{R}^2$ that depends of U.

2.2.3. **Local Cauchy-Crofton formula** ([18], [19] 1.16, [21] 3.1). *Let X be a definable subset of $\mathbf{R}^n$ of dimension d (containing the origin), let $\mathscr{G}$ be a definable subset of $G(d,n)$ on which transitively acts a subgroup G of $O_n(\mathbf{R})$ and let m be a G-invariant measure on $\mathscr{G}$, such that*
  - *the tangent spaces to the tangent cone of X_0 are in $\mathscr{G}$,*
  - *There exists $P^0 \in \mathscr{G}$ such that $\{g \in G; g \cdot P^0 = P^0\}$, the isotropy group of P^0, transitively acts on the d-dimensional vector subspace P^0 and $m(\mathscr{G}) = m(\mathscr{G} \cap \mathscr{E}_X) = 1$, where $\mathscr{E}_X$ is the generic set of $G(d,n)$ for which the localization (5) is possible.*

   Then, we have

$$\sigma_d^{\mathscr{G}}(X_0) = \Theta_d(X_0), \qquad\qquad (\mathscr{C}\mathscr{C}^{\ell oc})$$

where $\sigma_d^{\mathscr{G}}$ is defined as in Definition 2.2.1, but relatively to $\mathscr{G}$ and m.

   In the case $\mathscr{G} = G(d,n)$ and $G = O_n(\mathbf{R})$, the formula $(\mathscr{C}\mathscr{C}^{\ell oc})$ is just

$$\sigma_d(X_0) = \Theta_d(X_0) = \Lambda_d^{\ell oc}(X_0).$$

   In the case X is a complex analytic subset of $\mathbf{C}^n$, $\mathscr{G} = \widetilde{G}(d/2, n)$ (the $d/2$-dimensional complex vector subspaces of $\mathbf{C}^n$) and $G = U_n(\mathbf{C})$, since by definition the number of points in the fibre of a projection of the germ X_0 onto a generic $d/2$-dimensional complex vector subspace of $\mathbf{C}^n$ is the local multiplicity $e(X, 0)$ of X_0, formula $(\mathscr{C}\mathscr{C}^{\ell oc})$ gives

$$e(X, 0) = \sigma_d^{\tilde{G}(d/2,n)}(X_0) = \Theta_d(X_0).$$

Obtaining the equality $e(X, 0) = \Theta_d(X_0)$ as a by-product of the formula $(\mathscr{C}\mathscr{C}^{\ell oc})$ provides a new proof of Draper's result (see [32]).

2.2.4. *Remark.* When $(X^j)_{j \in \{0, \cdots, k\}}$ is a Whitney stratification of the closed set X (see for instance [105] and [106] for a survey on regularity conditions for stratifications) and $0 \in X^0$, $\sigma_i(X_0) = 1$, for $i \leq \dim(X^0)$ (see [21], Remark 2.9). Therefore, to sum up, when $(X^j)_{j \in \{0, \cdots, k\}}$ is a Whitney stratification of the closed definable set X and d_0 is the dimension of the stratum containing 0, one has

$$\sigma_*(X_0) = (1, \cdots, 1, \sigma_{d_0+1}(X_0), \cdots, \sigma_{d-1}(X_0), \Lambda_d^{\ell oc}(X_0)(X_0) = \Theta_d(X_0), 0, \cdots, 0).$$

   On figure 5 we represent the data taken into account in the computation of the invariant $\sigma_i(X_0)$. Here, contrary to the computation of the $\Lambda_i^{\ell oc}(X_0)$ where all the domains $K_\ell^{P,\varrho}$ matter (see figure 4), only the domains $K_\ell^{P,\varrho}$ having the origin in their adherence (these domains are coloured in red in figure 5) are considered, since only these domains appear as

$$\chi(\pi_P^{-1}(y) \cap Z \cap \bar{B}_{(0,\varrho)}) \cdot \Theta_i((K_\ell^{P,\varrho})_0)$$

in the computation of $\theta(\pi_{P_0*}(\mathbf{1}_{X_0}))$.

   In particular the domains $K_\ell^{P,\varrho}$ (in green on figure 5) defined by the critical values of the projection π_P restricted to the link $X \cap S_{(0,\varrho)}$ are not considered in the definition of $\theta(\pi_{P_0*}(\mathbf{1}_{X_0}))$. One can indeed prove (see [21], Proposition 2.5) that for any generic projection π_P exists $r_P > 0$, such that for all ϱ, $0 < \varrho < r_P$, the discriminant of the restriction of π_P to the link $X \cap S_{(0,\varrho)}$ is at a positive distance from $0 = \pi_P(0)$.
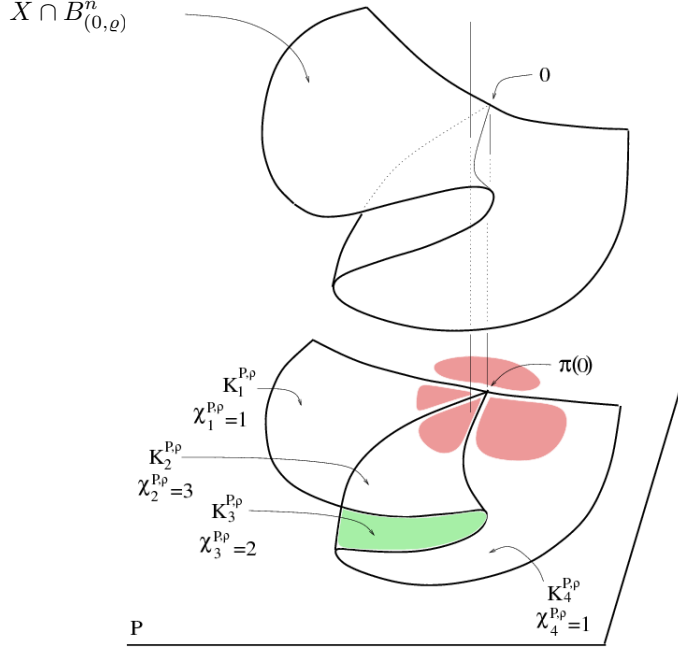
fig.5

Let us now deal with the question of what kind of invariants of complex singularities the sequence σ_* of invariants of real singularities generalizes. For this goal, we consider that X is a complex analytic subset of $\mathbf{C}^n$ of complex dimension d. One may define, like in the real case, the polar invariants of the germ X_0, denoted $\tilde{\sigma}_i$, $i = 0, \cdots, n$. These invariants are defined by generic projections on i-dimensional complex vector subspaces of $\mathbf{C}^n$. In the complex case, assuming $0 \in X$, there exists $r > 0$, such that for y generic in a generic i-dimensional vector space P of $\mathbf{C}^n$ and y sufficiently closed to 0

$$\tilde{\sigma}_i(X_0) = \chi(\pi_P^{-1}(y) \cap X \cap \bar{B}_{(0,r)}).$$

In particular, as already observed, $\tilde{\sigma}_d(X_0)$ is $e(X, 0)$, the local multiplicity of X at 0.

In the case where X is a complex hypersurface $f^{-1}(0)$, given by an analytic function $f : (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$ having at 0 an isolated singularity, one has for a generic y in the germ at 0 of a generic i-dimensional vector space P_0 of $\mathbf{C}^n$

$$\chi(\pi_P^{-1}(y) \cap X \cap \bar{B}_{(0,\eta)}) = \chi(\pi_P^{-1}(0) \cap f^{-1}(\varepsilon) \cap \bar{B}_{(0,\eta)}),$$

where ε is generic in $\mathbf{C}$, sufficiently close to 0 and $0 < |\varepsilon| \ll \eta \ll 1$.

Therefore, for $0 < |\varepsilon| \ll \eta \ll 1$, the integer $\chi(\pi_P^{-1}(0) \cap f^{-1}(\varepsilon) \cap \bar{B}_{(0,\eta)})$ is the Euler-Poincaré characteristic of the Milnor fibre of f restricted to $P^\perp$, that is to say $1 + (-1)^{n-i-1}\mu^{(n-i)}$. In the case where X is a complex analytic hypersurface of $\mathbf{C}^n$ with an isolated singularity at 0, we thus have

$$\tilde{\sigma}_i(X_0) = 1 + (-1)^{n-i-1}\mu^{(n-i)}.$$

For X a complex analytic subset of $\mathbf{C}^n$ of dimension d, d being not necessarily $n-1$, the complex invariants $\tilde{\sigma}_i(X_0)$ have been first considered by Kashiwara in [65] (where the balls are open and not closed as it is the case here). An invariant $E_{X_0}^0$ is then defined in [65] by induction

on the dimension of X_0 using $\tilde{\sigma}_i$. This invariant is studied in [33], [34] and in [12] where a multidimensional version $E_{X_0}^k$ of $E_{X_0}^0$ is given (see also [85]). The definition is the following

$$E_{X_0}^k = \sum_{X^{j_0} \subset \bar{X}^j \setminus X^j, \; \dim(X^j) < \dim(X_0)} E_{\bar{X}^j}^k \cdot \tilde{\sigma}_{k+\dim(X^j)+1}(X_0),$$

where (X^j) is a Whitney stratification of X_0, X^{j_0} the stratum containing 0 and $E_{\{0\}}^k = 1$. The authors then remark that (see also [33], [34])

$$E_{X_0}^k = Eu_{X_0}^k,$$

where $Eu_{X_0}^k = Eu_{(X_0 \cap H)}$, H is a general vector subspace of dimension k of $\mathbf{C}^n$ and where Eu is the local Euler obstruction of X at 0, introduced by MacPherson in [84]. In particular,

$$E_{X_0}^0 = Eu_{X_0}.$$

Let us now denote $\mathscr{P}^i(X_0)$, $i = 0, \cdots, d$, the codimension i polar variety of X_0, that is to say the closure of the critical locus of the projection of the regular part of X_0 to a generic vector space of $\mathbf{C}^n$ of dimension $d - i + 1$. The following relation between invariants $E_{X_0}^k$ and the local multiplicity of the polar varieties $\mathscr{P}^i(X_0)$ is obtained in [12]

$$(-1)^i (E_{X_0}^{\dim(X_0)-i-1} - E_{X_0}^{\dim(X_0)-i}) = e(\mathscr{P}^i(X_0), 0),$$

which in turn gives (see also [85], [75], [76], [77], [34])

$$Eu_{X_0} = \sum_{i=0}^{d-1} (-1)^i e(\mathscr{P}^i(X_0), 0),$$

where $e(\mathscr{P}^i(X_0), 0)$ is, as before, the local multiplicity at 0 of the codimension i polar variety $\mathscr{P}^i(X_0)$ of X_0 at 0.

All the invariants $\tilde{\sigma}_i(X_y)$, $E_{X_y}^i$, $e(\mathscr{P}^i(X_y), y)$, viewed as functions of the base-point y, enjoy the same remarkable property: they can detect subtle variations of the geometry of an analytic family (X_y), in the sense that the family (X_y) may be Whitney stratified with y staying in the same stratum if and only if these invariants are constant with respect to the parameter y. Without proof, it is actually stated in [33] Proposition 1, [34] Theorem II.2.7 page 30, and [12], that the invariants $\tilde{\sigma}_i(X_y)$ are constant as y varies in a stratum of a Whitney stratification of X_0 (see also [21] Corollary 4.5). And in [58], [87], [104] it is proved that the constancy of the multiplicities $e(\mathscr{P}^i(X_y), y)$ as y varies in a stratum of a stratification of X_0, is equivalent to the Whitney regularity of this stratification, giving also a proof, considering the relations between $e(\mathscr{P}^i(X_y), y)$ and $\tilde{\sigma}_i(X_y)$ stated above, of the constancy of $y \mapsto \tilde{\sigma}_i(X_y)$ along Whitney strata.

We sum-up these results in the following theorem, where $e(\Delta^i(X_y), y)$ is the local multiplicity at y of the discriminant $\Delta^i(X_y)$ associated to $\mathscr{P}^i(X_y)$, that is the image of $\mathscr{P}^i(X_y)$ under the generic projection that gives rise to $\mathscr{P}^i(X_y)$.

2.2.5. **Theorem** ([58], [87], [77], [104]). *Let X_0 be a complex analytic germ at 0 of $\mathbf{C}^n$ endowed with a stratification (X^j). The following statements are equivalent*
  (1) *The stratification (X^j) is a Whitney stratification.*
  (2) *The functions $X^j \ni y \mapsto e(\mathscr{P}^i(X_y^\ell), y)$, for $i = 0, \cdots, d-1$ and any pairs (X^j, X^ℓ) such that $X^j \subset \overline{X^\ell}$, are constant.*
  (3) *The functions $X^j \ni y \mapsto e(\Delta^i(X_y^\ell), y)$, for $i = 0, \cdots, d-1$ and for any pairs (X^j, X^ℓ) such that $X^j \subset \overline{X^\ell}$, are constant.*
  (4) *The functions $y \ni X^j \mapsto \tilde{\sigma}_i(X_y)$, for $i = 1, \cdots, d$ are constant.*

2.2.6. *Remark.* In the real case the functions $y \mapsto \sigma_i(X_y)$ are not **Z**-valued functions as in the complex case, but **R**-valued functions and in general one can not stratify a compact definable set in such a way that the restriction of these functions to the strata are constant. However, it is proved in [21] Theorems 4.9 and 4.10, that Verdier regularity for a stratification implies continuity of the restriction of $y \mapsto \sigma_i(X_y)$ to the strata of this stratification. Since, in the complex setting, Verdier regularity is the same as Whitney regularity, this result is the real counterpart of Theorem 2.2.5. Note that in the real case one can not expect that the continuity or even the constancy of the functions $y \mapsto \sigma_i(X_y)$ in restriction to the strata of a given stratification implies a convenient regularity condition for this stratification (see the introduction of [21]).

As a conclusion of this section, the complex version $\tilde{\sigma}_i$ of the real polar invariants σ_i of definable singularities plays a central role in singularity theory since they let us compute classical invariants of singularities and since their constancy, with respect to the parameter of an analytic family, as well as the constancy of other classical invariants related to them, means that the family does not change its geometry. Since our polar invariants σ_i appear now as the real counterpart of classical complex invariants, we'd like to understand in the sequel how they are related to the local Lipschitz-Killing invariants $\Lambda_i^{\ell oc}(X_0)$ coming from the differential geometry of the deformation of the germ X_0 through its tubular neighbourhoods family.

This is the goal of the next section.

2.3. **Multidimensional local Cauchy-Crofton formula.** The local Cauchy-Crofton formula $(\mathscr{CC}^{\ell oc})$ given at 2.2.3 already equals σ_d and $\Lambda_d^{\ell oc}$ over definable germs. This relation suggests a more general relation between the $\Lambda_i^{\ell oc}$'s and the σ_j's. We actually can prove that each invariant of one family is a linear combination of the invariants of the other family. The precise statement is given by the following formula $(\mathscr{CC}_{mult}^{\ell oc})$.

2.3.1. **Multidimensional local Cauchy-Crofton formula** ([21] Theorem 3.1)**.** *There exist real numbers* $(m_i^j)_{1 \leq i,j \leq n, i<j}$ *such that, for any definable germ* X_0, *one has*

$$\begin{pmatrix} \Lambda_1^{\ell oc}(X_0) \\ \vdots \\ \Lambda_n^{\ell oc}(X_0) \end{pmatrix} = \begin{pmatrix} 1 & m_1^2 & \dots & m_1^{n-1} & m_1^n \\ 0 & 1 & \dots & m_2^{n-1} & m_2^n \\ \vdots & & & & \vdots \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \sigma_1(X_0) \\ \vdots \\ \sigma_n(X_0) \end{pmatrix} \qquad (\mathscr{CC}_{mult}^{\ell oc})$$

These constant real numbers are given by

$$m_i^j = \frac{\alpha_j}{\alpha_{j-i} \cdot \alpha_i} \binom{i}{j} - \frac{\alpha_{j-1}}{\alpha_{j-1-i} \cdot \alpha_i} \binom{i}{j-1},$$

for $i + 1 \leq j \leq n$.

2.3.2. *Remark.* Applied to a d-dimensional definable germ X_0, the last *a priori* non-trivial equality provided by formula $(\mathscr{CC}_{mult}^{\ell oc})$, involving the d-th line of the matrices, is

$$\Theta_d(X_0) = \Lambda_d^{\ell oc}(X_0) = \sigma_d(X_0),$$

which is the local Cauchy-Crofton formula $(\mathscr{CC}^{\ell oc})$. The local Cauchy-Crofton formula $(\mathscr{CC}^{\ell oc})$ expresses the d-density of a d-dimensional germ as the mean value of the number of points in the intersection of this germ with a $(n-d)$-dimensional affine space of $\mathbf{R}^n$. This number of points may be viewed as the Euler-Poincaré characteristic of this intersection. Now, since for d-dimensional germs the d-density is the last invariant of the sequence $\Lambda_*^{\ell oc}$ and since formula

($\mathscr{CC}_{mult}^{\ell oc}$) expresses all invariants $\Lambda_i^{\ell oc}$ in terms of the mean values of the Euler-Poincaré characteristics of the multidimensional plane sections of our germ, we see in formula ($\mathscr{CC}_{mult}^{\ell oc}$) a multimensional version of the local Cauchy-Crofton formula.

2.4. **Valuations theory and Hadwiger principle.** In the previous section 2.3, with formula ($\mathscr{CC}_{mult}^{\ell oc}$), we have answered the question: *how are the local Lipschitz-Killing invariants and the polar invariants related?* In this section we would like to risk some speculative and maybe prospective insights about formulas similar to ($\mathscr{CC}_{mult}^{\ell oc}$), that is to say formulas linearly relating two families of local invariants of singularities. We include in this scope the complex formulas presented in Section 2.1. For this goal we first recall some definitions and celebrated statements from convex geometry, since it appears that from the theory of valuations on convex bodies one can draw precious lessons on the question: *why our additive invariants are linearly dependent?*

We have already observed that the additive functions Λ_i, $\Lambda_i^{\ell oc}$ and σ_j are invariant under isometries of $\mathbf{R}^n$. Therefore the first question we would like to address to convex geometry is the following: to what extend those invariants are models of additive and rigid motion invariants?

The systematic study of additive invariants (of compact convex sets of $\mathbf{R}^n$) has been inaugurated by Hadwiger and his school and motivated by Hilbert's third problem (solved by Dehn by introducing the so-called Dehn invariants) consisting in classifying scissors invariants of polytopes (see for instance [13] for a quick introduction to Hilbert's third problem). One of the most striking results in this field is Hadwiger's theorem that characterises the set of additive and rigid motion invariant functions (on the set of compact convex subsets of $\mathbf{R}^n$) as the vector space spanned by the Λ_i's. We give now the needful definitions to state Hadwiger's theorem and the still-open question of its extension to the spherical case (for more details one can refer to [83], [97] or [98]).

We denote by $\mathscr{K}^n$ (resp. $\mathscr{K}\mathbf{S}^{n-1}$) the set of compact convex sets of $\mathbf{R}^n$ (resp. of $\mathbf{S}^{n-1}$, that is to say the intersection of the sphere $\mathbf{S}^{n-1}$ and conic compact convex sets of $\mathscr{K}^n$ with vertex the origin of $\mathbf{R}^n$). A function $v : \mathscr{K}^n \to \mathbf{R}$ (resp. $v : \mathscr{K}\mathbf{S}^{n-1} \to \mathbf{R}$) is called a valuation (resp. a spherical valuation) when $v(\emptyset) = 0$ and for any $K, L \in \mathscr{K}^n$ (resp. $K, L \in \mathscr{K}\mathbf{S}^{n-1}$) such that $K \cup L \in \mathscr{K}^n$ (resp. $K \cup L \in \mathscr{K}\mathbf{S}^{n-1}$), one has the additivity property

$$v(K \cup L) = v(K) + v(L) - v(K \cap L).$$

One says that a valuation v on $\mathscr{K}^n$ (resp. $\mathscr{K}\mathbf{S}^{n-1}$) is continuous when it is continuous with respect to the Hausdorff metric on $\mathscr{K}^n$ (resp. on $\mathscr{K}\mathbf{S}^{n-1}$). A valuation v on $\mathscr{K}^n$ (resp. $\mathscr{K}\mathbf{S}^{n-1}$) is called simple when the restriction of v to convex sets with empty interior is zero. Let G be a subgroup of the orthogonal group $O_n(\mathbf{R})$. A valuation v on $\mathscr{K}^n$ (resp. $\mathscr{K}\mathbf{S}^{n-1}$) is G-*invariante* when it is invariant under the action of translations of $\mathbf{R}^n$ and the action of G on $\mathscr{K}^n$ (resp. the action of G on $\mathscr{K}\mathbf{S}^{n-1}$).

The Hadwiger theorem emphasizes the central role played in convex geometry by the Lipschitz-Killing invariants as additive rigid motion invariants.

2.4.1. **Theorem** ([55], [66]). *A basis of the vector space of $SO_n(\mathbf{R})$-invariant and continuous valuations on $\mathscr{K}^n$ is $(\Lambda_0 = 1, \Lambda_1, \cdots, \Lambda_n = Vol_n)$.*

Equivalently (by an easy induction argument) a basis of the vector space of continuous and $SO_n(\mathbf{R})$-invariant simple valuations on $\mathscr{K}^n$ is $\Lambda_n = Vol_n$.

This statement forces a family of $n + 2$ additive, continuous and $SO_n(\mathbf{R})$-invariant functions on euclidean convex bodies to be linearly dependent. The second formulation of Hadwiger's theorem, concerning the space of simple valuations, enables to address the question of such a rigid structure of the space of valuations in the setting of spherical convex geometry. This question of a spherical version of Hadwiger's result has been address by Gruber and Schneider.

2.4.2. *Question* ([53] Problem 74, [83] Problem 14.3). Is a simple, continuous and $O_n(\mathbf{R})$-invariant valuation on $\mathscr{K}\mathbf{S}^{n-1}$ a multiple of the $(n-1)$-volume on S^{n-1}?

2.4.3. *Remark.* In the case $n \leq 3$ a positive answer to this question given in [83], Theorem 14.4, and in the easy case where the simple valuation has constant sign one also has a positive answer given in [95] Theorem 6.2, and [96]. Note that in this last case the continuity is not required and that the valuation is *a priori* defined only on convex spherical polytopes.

This difficult and still unsolved problem naturally appears as soon as one consider the localizations $\Lambda_i^{\ell oc}$ of Λ_i and their relation with other classical additive invariants such as the σ_j's. Indeed, the question of why and how such invariants are related falls within the framework of Question 2.4.2. Let's clarify this principled position.

The invariants $(\Lambda_i^{\ell oc})_{i \in \{0, \cdots, n\}}$ define spherical $O_n(\mathbf{R})$-invariant and continuous valuations $(\widehat{\Lambda}_i)_{i \in \{0, \cdots, n\}}$ on the convex sets of $\mathbf{S}^{n-1}$ by the formula

$$\widehat{\Lambda}_i(K) := \Lambda_i^{\ell oc}(\widehat{K}_0) = \frac{1}{\alpha_i}\Lambda_i(\widehat{K} \cap \bar{B}_{(0,1)}), \tag{$\widehat{8}$}$$

where K is a convex set of $\mathbf{S}^{n-1}$, that is to say the trace in $\mathbf{S}^{n-1}$ of the cone $\widehat{K} = \mathbf{R}_+ \cdot K$ with vertex the origin of $\mathbf{R}^n$. Another possible finite sequence of continuous and $O_n(\mathbf{R})$-invariant spherical valuations on convex polytopes of $\mathbf{S}^{n-1}$ is

$$\Xi_i(P) := \sum_{F \in \mathscr{F}_i(P)} Vol_i(F) \cdot \gamma(\widehat{F}, \widehat{P}) = Vol_i(S^i(0,1)) \sum_{F \in \mathscr{F}_i(P)} \Theta_i(\widehat{F}_0) \cdot \gamma(\widehat{F}, \widehat{P}), \tag{$\widehat{9}$}$$

where $P \subset S^{n-1}$ is a spherical polytope, that is to say that $\mathbf{R}_+ \cdot P = \widehat{P}$ is the intersection of a finite number of closed half vector spaces of $\mathbf{R}^n$, $\mathscr{F}_i(P)$ the set of all i-dimensional faces of P (the $(i+1)$-dimensional faces of $\widehat{P}$) and $\gamma(\widehat{F}, \widehat{P})$ the external angle of $\widehat{P}$ along $\widehat{F}$. The valuations Ξ_i are the natural spherical substitutes of the euclidean Lipschitz-Killing curvatures Λ_i according to formula (2).

Finally the polar invariants $(\sigma_i)_{i \in \{0, \cdots, n\}}$ also define continuous and $O_n(\mathbf{R})$-invariants spherical valuations $(\widehat{\sigma}_i)_{i \in \{0, \cdots, n\}}$ on the convex sets of $\mathbf{S}^{n-1}$, according to a formula of the same type that formula $(\widehat{8})$

$$\widehat{\sigma}_i(K) := \sigma_i(\widehat{K}_0). \tag{$\widehat{10}$}$$

These three families of continuous and $O_n(\mathbf{R})$-invariant spherical valuations

$$(\widehat{\Lambda}_i)_{i \in \{0, \cdots, n\}}, (\Xi_i)_{i \in \{0, \cdots, n\}}, (\widehat{\sigma}_i)_{i \in \{0, \cdots, n\}}$$

being linearly independent families in the space of spherical valuations, a positive answer to Question 2.4.2 would have for direct consequence that each element of one family is a linear combination of elements of any of the other two families.

Therefore, in restriction to polyhedral cones, each element of the family $(\Lambda_i^{\ell oc})_{i \in \{0, \cdots, n\}}$ could be expressed as a linear combination (with universal coefficients) of elements of the family $(\sigma_i)_{i \in \{0, \cdots, n\}}$ and conversely. Despite the absence of any positive answer to Question 2.4.2 for $n > 3$, this linear dependence is proved in [21] (Theorem A4 and A5) over the set of convex polytopes (and in [21], section 3.1, even over the set of definable cones). It is actually shown that each invariant $\Lambda_i^{\ell oc}$ and each invariant σ_j may be expressed as a linear combination (with universal coefficients) of elements of the family $(\Xi_i)_{i \in \{0, \cdots, n\}}$. The coefficients involved in such linear combinations may be explicitly computed by considering the case of polytopes.

In conclusion, an anticipating positive answer to this question would imply the existence of a finite number of independent models for additive, continuous and $O_n(\mathbf{R})$-invariant functions on convex and conic germs. Consequently, solely following this principle and restricted at least

to convex cones, our local invariants σ_j and $\Lambda_i^{\ell oc}$ would be automatically linearly dependent. In the next step, in order to extend a relation formula involving some valuations from the set of finite union of convex conic polytopes to the set of general conic definable set one has to prove some general statement according to which the normal cycle of a definable conic set may be approximate by the normal cycles of a family of finite union of convex conic polytopes. We do not want to go more into technical details and even define the notion of normal cycle introduced by Fu; we just point out that this issue has been tackled recently in [44]. Finally to extend a relation formula from the set of conic definable set to the set of all definable germs one just has to use the local conic structure of definable germs and the deformation on the tangent cone (see [21], [38]).

To finish to shed light on convex geometry as an area from which some strong relations between singularity invariants may be understood, let us remark that the following generalization of Hadwiger's theorem 2.4.1 has been obtained by Alesker.

2.4.4. **Theorem.** *Let G be a compact subgroup of $O_n(\mathbf{R})$.*

(1) *The vector space $Val^G(\mathscr{K}^n)$ of continuous, translation and G-invariant valuations on $\mathscr{K}^n$ has finite dimension if and only if G acts transitively on $\mathbf{S}^{n-1}$ (see [2] Theorem 8.1, [4] Proposition 2.6).*

(2) *One can endowed the vector space $Val^G(\mathscr{K}^n)$ with a product (see [3], [5]) providing a graded algebra structure (the graduation coming from the homogeneity degree of the valuations) and*

$$\begin{aligned}
\mathbf{R}[x]/(x^{n+1}) &\rightarrow Val^{O_n(\mathbf{R})}(\mathscr{K}^n) = Val^{SO_n(\mathbf{R})}(\mathscr{K}^n) \\
x &\mapsto \Lambda_1
\end{aligned}$$

is an isomorphism of graded algebras (see [3], Theorem 2.6).

3. Generating additive invariants via generating functions

The possibility of generating invariants from a deformation of a singular set into a family of approximating and less complicated sets is perfectly illustrated by the work developed by Denef and Loeser consisting in stating that some generating series attached to a singular germ are rational. Such generating series have their coefficients in some convenient ring reflecting the special properties of the invariants to highlight, such as additivity, multiplicativity, analytic invariance, the relation with some specific group action and so forth, and on the other hand each of these coefficients is attached to a single element of the deformation family. It follows that such a generating series captures the geometrical aspect that one aims to focus on through the deformation family as well as its rationality indicates that asymptotically this geometry specializes on the geometry of the special fibre approximated by the deformation family. Indeed, being rational strongly expresses that a series is encoded by a finite amount of data concentrated in its higher coefficients.

To be more explicit we now roughly describe how Denef and Loeser define the notion of motivic Milnor fibre (for far more complete and precise introductions to motivic integration which is the central tool of the theory, and to motivic invariants in general, the reader may refer to [10], [16], [17], [25], [28], [29], [30], [31], [52], [54], [56], [57], [80], [81], [107]).

3.1. **The complex case.** A possible starting point of the theory of motivic invariants may be attributed to the works of Igusa (see [62], [63], [64]) on zeta functions introduced by Weil (see [109]). In the works of Igusa the rationality of some generating Poincaré series is proved. These series, called zeta functions, have for coefficients the number of points in $\mathscr{O}/\mathfrak{M}^m$ of $f = 0$ mod $\mathfrak{M}^m$, for f a n-ary polynomial with coefficients in the valuation ring $\mathscr{O}$ of some discrete

valuation field of characteristic zero, with maximal ideal $\mathfrak{M}$ and finite residue field of cardinal q. This result amounts to prove the rationality (as a function of q^{-s}, $s \in \mathbf{C}$, $\Re e(s) > 0$) of an integral of type

$$\int_{\mathscr{O}^n} |f|^s |ds| \tag{11}$$

(the Igusa local zeta function) which is achieve using a convenient resolution of singularities of f, as described in the introduction (see also [24] for comparable statements on Serre's series and a strategy based on Macintyre's proof of elimination of quantifiers as an alternative to Hironaka's resolution of singularities). In the opposite direction, the rational function, expressed in terms of the data of a resolution of f, associated to a polynomial germ $f : (\mathbf{C}, 0) \to (\mathbf{C}, 0)$ by the expression provided by the computation of Igusa's integral in the discrete valuation field case let Denef and Loeser define intrinsic invariants attached to the complex germ $(f^{-1}(0), 0)$, called topological zeta functions (see [26]).

Another key milestone in the systematic use of discrete valuation fields (here with finite residue field and more explicitely in the p-adic context) have been reached in Batyrev's paper [7], where it is shown that two birationally equivalent Calabi-Yau manifolds over $\mathbf{C}$ have the same Betti numbers. Indeed, by Weil's conjectures, these Betti numbers are obtained from the rational expression of the local zeta functions having for coefficients the number of points in the reductions modulo p^m of the manifolds into consideration (viewed as defined over $\mathbf{Q}_p$ when they are defined over $\mathbf{Q} \subset \mathbf{C}$) and on the other hand, these local zeta functions may be computed by Igusa's integrals over these manifolds. Being birationally equivalent, these manifolds provide the same integrals.

Kontsevich, in his seminal talk [67], extended this method (consisting in shifting a complex geometric problem in a discrete valuation field setting) in the equicharacteristic setting by developing an integration theory in particular over $\mathbf{C}[[t]]$. Note that the theory may be developed in great generality and not only in equicharacteristic zero (see [81], [16]). The idea of Kontsevich was to define an integration theory over arc spaces, say $\mathbf{C}[[t]]$, by considering a measure with values in the Grothendieck ring $K_0(\mathrm{Var}_\mathbf{C})$ of algebraic varieties over $\mathbf{C}$ (localized by the multiplicative set generated by the class $\mathbb{L}$ of $\mathbf{A}^1$ in $K_0(\mathrm{Var}_\mathbf{C})$). The main tool in this context being a change of variables formula that allows computation of integrals through morphisms, and in particular through a morphism given by a resolution of singularities. Formally the ring $K_0(\mathrm{Var}_\mathbf{C})$ is the free abelian group generated by isomorphism classes $[X]$ of varieties X over $\mathbf{C}$, with the relations

$$[X \setminus Y] = [X] - [Y],$$

for Y closed in X, the product of ring being given by the product of varieties (see for instance [88]). Denoting $\mathbb{L}$ the class of $\mathbf{A}^1$ in $K_0(\mathrm{Var}_\mathbf{C})$, we then denote $\mathscr{M}_\mathbf{C}$ the localization $K_0(\mathrm{Var}_\mathbf{C})[\mathbb{L}^{-1}]$. Any additive and multiplicative invariant on $\mathrm{Var}_\mathbf{C}$ with non zero value at $\mathbf{A}^1$, such as the Euler-Poincaré characteristic or the Hodge characteristic (both with compact support), factorizes through the universal additive and multiplicative map $\mathrm{Var}_\mathbf{C} \ni X \mapsto [X] \in \mathscr{M}_\mathbf{C}$. Now we equip the space $\mathscr{L}(\mathbf{C}^n, 0)$ of formal arcs of $\mathbf{C}^n$ passing through 0 at 0 with the above-mentioned measure that provides a σ-additive measure, with values in a completion $\hat{\mathscr{M}}_\mathbf{C}$ of $\mathscr{M}_\mathbf{C}$, for sets of the boolean algebra of the so-called constructible sets of $\mathscr{L}(\mathbf{C}^n, 0)$. Finally denoting $\mathscr{L}_m(\mathbf{C}^n, 0)$, $m \geq 0$, the set of polynomial arcs of $\mathbf{C}^n$ of degree $\leq m$, passing through 0 at 0, and for $f : (\mathbf{A}^n, 0) \to (\mathbf{A}^1, 0)$ a morphism having a (isolated) singularity at the origin, inducing the morphism $f_m : \mathscr{L}_m(\mathbf{C}^n, 0) \to \mathscr{L}(\mathbf{C}, 0)$, we denote

$$\mathscr{X}_{m,0,1} := \{\varphi \in \mathscr{L}_m(\mathbf{C}^n, 0); (f_m \circ \varphi)(t) = t^m + \text{high order terms}\},$$

and we define (see for instance [31]) the motivic zeta function of f with formal variable T by

$$Z_f(T) := \sum_{m \geq 1} [\mathscr{X}_{m,0,1}] \, \mathbb{L}^{-mn} T^m.$$

This generating series appears as the $\mathbf{C}[[t]]$-substitute of p-adic zeta functions introduced by Weil, and whose rationality, following Igusa, amounts to compute an integral of type (11). Inspired by the analogy of $Z_f(T)$ with Igusa integrals, and also using a resolution of singularities of f as presented in the introduction and using the Kontsevich change of variables formula for this resolution applied to the coefficients of Z_f viewed as measures (in the ring $\hat{\mathscr{M}}_{\mathbf{C}}$) of the constructible sets $\mathscr{X}_{m,0,1}$, Denef and Loeser proved (see [27], [28], [31]) the rationality of $Z_f(T)$. With the notation given in the introduction, we then have

$$Z_f(T) = \sum_{I \cap \mathscr{K} \neq \emptyset} (\mathbb{L} - 1)^{|I|-1} [\widetilde{E}_I^0] \prod_{i \in I} \frac{\mathbb{L}^{-\nu_i} T^{N_i}}{1 - \mathbb{L}^{-\nu_i} T^{N_i}} \tag{12}$$

where $\widetilde{E}_I^0$ is a covering of E_I^0 defined in the following way. Let U be some affine open subset of M such that on U, $f \circ \sigma(x) = u(x) \prod_{i \in I} x_i^{N_i}$, with u a unit. Then $\widetilde{E}_I^0$ is obtained by gluing along $E_I^0 \cap U$ the sets

$$\{(x, z) \in (E_I^0 \cap U) \times \mathbf{A}^1 ; z^{m_I} \cdot u(x) = 1\},$$

where $m_I = \gcd(N_i)_{i \in I}$.

3.1.1. *Question* (Monodromy conjecture). We do not know how the poles (in some sense) of the rational expression of Z_f relates on the eigenvalues of the monodromy function M associated to the singular germ $f : (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$. The Monodromy Conjecture of Igusa, that has been stated in many different forms after Igusa, asserts that when $\mathbb{L}^\nu - T^N$ indeed appears as denominator of the rational expression of Z_f viewed as an element of the ring generated by $\hat{\mathscr{M}}_{\mathbf{C}}$ and $T^N/(\mathbb{L}^\nu - T^N)$, $\nu, N > 0$, then $e^{2i\pi\nu/N}$ is an eigenvalue of M (see for instance [25] and the references given in this article for additional classical references).

3.1.2. *Remark*. The rationality of Z_f illustrates again a deformation principle; the family $(\mathscr{X}_{m,0,1})_{m \geq 1}$ may be considered as a family of tubular neighbourhoods in $\mathscr{L}(\mathbf{C}^n, 0)$ around the singular fibre $\mathscr{X}_0 = \{f = 0\}$ and in a neighbourhood of the origin, with respect to the ultrametric distance given by the order of arcs. Now the rationality of Z_f expresses the regularity of the degeneracy of the geometry of $\mathscr{X}_{m,0,1}$ onto the geometry of $\mathscr{X}_0$. Following this principle, the rational expression (12) of Z_f is supposed to concentrate the part of the geometrical information encoded in $(\mathscr{X}_{m,0,1})_{m \geq 1}$ that accumulates at infinity in the series Z_f.

This is achieved in particular by the following observation (see [27], [30], [31]): the negative of the constant term of the formal expansion as a power series in $1/T$ of the rational expression of Z_f given by formula (12) defines the following element in $\hat{\mathscr{M}}_{\mathbf{C}}$

$$S_f := \sum_{I \cap \mathscr{K} \neq \emptyset} (\mathbb{L} - 1)^{|I|-1} [\widetilde{E}_I^0],$$

called the motivic Milnor fibre of $f = 0$ at the singular point 0 of f. Taking the realization of S_f under the morphism $\chi : \hat{\mathscr{M}}_{\mathbf{C}} \to \mathbf{Z}$ (note that $\chi(\mathbb{L}) = 1$) gives, in particular, by the A'Campo formula recalled in the introduction,

$$\chi(S_f) = \sum_{i \in \mathscr{K}} N_i \cdot \chi(E_{\{i\}}^0) = \chi(X_0) = 1 + (-1)^{n-1} \mu. \tag{13}$$

3.1.3. *Remark.* Generally speaking, taking the constant term in the expansion of a rational function Z as a power series in $1/T$, amounts to consider $\lim_{T\to\infty} Z(T)$ (in a setting where this makes sense). A process that gives an increasing importance to the m-th coefficient of Z as m itself increases. On the other hand, the coefficients of Z_f may be directly interpreted at the level of the Euler-Poincaré characteristic, which could be seen as the first topological degree of realization of $K_0(\mathrm{Var}_{\mathbf{C}})$, and it turns out that the sequence $(\chi(\mathscr{X}_{m,0,1}))_{m\geq 1}$ has a strong regularity since it is in fact periodic. Indeed, one has by [31] Theorem 1.1 (we recall that M is the monodromy map and Λ the Lefschetz number)

$$\chi(\mathscr{X}_{m,0,1}) = \Lambda(M^m), \ \forall m \geq 1 \tag{14}$$

and by quasi-unipotence of M (see [99] I.1.2) there exists $N > 1$ such that the order of the eigenvalues of M divides N. It follows from (14) that $\chi(\mathscr{X}_{m+N,0,1}) = \chi(\mathscr{X}_{m,0,1}), m \geq 1$.

Now formula (13), showing that the motivic Milnor fibre has a realization, via the Euler-Poincaré characteristic, on the Euler-Poincaré characteristic of the set-theoretic Milnor fibre, is a direct consequence of formulas (14) (note in fact that the proofs of (13) and (14), using A'Campo's formulas and a resolution of singularities, are essentially the same and thus gives comparable statements). Indeed, as noticed by Loeser (personal communication), working with χ instead of formal classes of $K_0(\mathrm{Var}_{\mathbf{C}})$, on gets $\mathbb{L} = \chi(\mathbf{A}^1) = 1$ and thus by definition of Z_f, the series $\chi(Z_f)$, realization of the class Z_f under the Euler-Poincaré characteristic is

$$\chi(Z_f) = \sum_{m\geq 1} \chi(\mathscr{X}_{m,0,1})T^m$$

that gives in turn, by formula (14),

$$\chi(Z_f) = \sum_{m\geq 1} \Lambda(M^m)T^m = \sum_{m=1}^{N} \Lambda(M^m)\sum_{k\geq 0} T^{m+kN} = \sum_{m=1}^{N} \Lambda(M^m)\frac{T^m}{1-T^N}.$$

Since $\chi(S_f) = -\lim_{T\to\infty} \chi(Z_f)$, on finally find again that

$$\chi(S_f) = \Lambda(M^N) = \Lambda(Id) = \chi(X_0) = \chi(\bar{X}_0) = 1 + (-1)^{n-1}\mu.$$

3.1.4. *Remark.* One may consider a more specific Grothendieck ring, that is to say a ring with more relations, in order to take into account the monodromy action on the Milnor fibre. In this equivariant and more pertinent ring equalities (12) and (14) are still true (see [30], [31] Section 2.9)

3.1.5. *Remark.* In [61], Hrushovski and Loeser gave a proof of equality (14) without using a resolution of singularity, and therefore without using A'Campo's formulas. Since a computation of Z_f in terms of the data associated to a particular resolution of the singularities of f leads to the simple observation that one computes in this way an expression already provided by A'Campo's formulas, the original proof of (14) may, in some sense, appear as a not direct proof. The proof proposed in [61] uses étale cohomology of non-archimedean spaces and motivic integration in the model theoretic version of [59] and [60].

3.1.6. *Remark.* To finish with the complex case, let us note that in [93] and [100] a mixed Hodge structure on the Milnor fibre $f^{-1}(t)$ at infinity ($|t| \gg 1$) has been defined by a deformation process, letting t goes to infinity (see also [92]). In [89] and [90] a corresponding motivic Milnor fibre S_f^∞ has then be defined.

3.2. **The real case.** A real version of (12), giving rise to a real version of (13) has been obtained in [20] (see also [43]). In the real case, a singular germ

$$f : (\mathbf{R}^n, 0) \to (\mathbf{R}, 0)$$

defines two smooth bundles $(f^{-1}(\varepsilon) \cap B_{(0,\eta)})_{0 < -\varepsilon \ll \eta \ll 1}$ and $(f^{-1}(\varepsilon) \cap B_{(0,\eta)})_{0 < \varepsilon \ll \eta \ll 1}$ and as well as $\mathscr{X}_{m,0,-1}$ and $\mathscr{X}_{m,0,1}$, it is natural to consider the two sets

$$\mathscr{X}_{m,0,>} := \{\varphi \in \mathscr{L}_m(\mathbf{R}^n, 0); f_m \circ \varphi = at^m + \text{high order terms}, a > 0\},$$

and

$$\mathscr{X}_{m,0,<} := \{\varphi \in \mathscr{L}_m(\mathbf{R}^n, 0); f_m \circ \varphi = at^m + \text{high order terms}, a < 0\}.$$

Let us denote X_0^{-1} and X_0^{+1} the fibre $f^{-1}(\varepsilon) \cap B_{(0,\eta)}$ for respectively $\varepsilon < 0$ and $\varepsilon > 0$. While $\mathscr{X}_{m,0,a}$, for $a \in \mathbf{C}^\times$, is a constructible set having a class in $K_0(\mathrm{Var}_{\mathbf{C}})$, the sets $\mathscr{X}_{m,0,<}$ and $\mathscr{X}_{m,0,>}$ are real semialgebraic sets and unfortunately, the Grothendieck ring of real semialgebraic sets is the trivial ring $\mathbf{Z}$, since semialgebraic sets admit semialgebraic cells decomposition.

Therefore, in the real case, since we have to deal with two signed Milnor fibres, we cannot mimic the construction of $K_0(\mathrm{Var}_{\mathbf{C}})$. To overcome this issue, in [43] we proposed to work in the Grothendieck ring of real (basic) semialgebraic formulas, $K_0(\mathrm{BS}_{\mathbf{R}})$. In this ring no semialgebraic isomorphism relations between semialgebraic sets, but algebraic isomorphim relations between sets given by algebraic formulas, are imposed and distinct real semialgebraic formulas having the same set of real points in $\mathbf{R}^n$ may have different classes. In particular, a first order basic formula in the language of ordered rings with parameters from $\mathbf{R}$ may have a nonzero class in $K_0(\mathrm{BS}_{\mathbf{R}})$ whereas no real point satisfies it. The ring $K_0(\mathrm{BS}_{\mathbf{R}})$ may be sent to the more convenient ring $K_0(\mathrm{Var}_{\mathbf{R}}) \otimes \mathbb{Z}[\frac{1}{2}]$, where explicit computations of classes of basic semialgebraic formula are possible as long as computations of classes of real algebraic formulas in the classical Grothendieck ring of real algebraic varieties $K_0(\mathrm{Var}_{\mathbf{R}})$ are possible.

In this setting, since $\mathscr{X}_{m,0,>}$ and $\mathscr{X}_{m,0,<}$ are given by explicit basic semialgebraic formulas, they do have natural classes in $K_0(\mathrm{BS}_{\mathbf{R}})$ and this allows the consideration of the associated zeta series

$$Z_f^? = \sum_{m \geq 1} [\mathscr{X}_{m,0,?}] \, \mathbb{L}^{-mn} T^m \in (K_0(\mathrm{Var}_{\mathbf{R}}) \otimes \mathbb{Z}[\frac{1}{2}])[\mathbb{L}^{-1}][[T]], \ ? \in \{-1, +1, <, >\}.$$

It is then proved, with the same strategy as in the complex case (using a resolution of singularities of f and the Kontsevich change of variables in motivic integration) that the real zeta function $Z_f^?$ is a rational function that can be expressed as

$$Z_f^?(T) = \sum_{I \cap \mathscr{K} \neq \emptyset} (\mathbb{L} - 1)^{|I|-1} [\widetilde{E}_I^{0,?}] \prod_{i \in I} \frac{\mathbb{L}^{-\nu_i} T^{N_i}}{1 - \mathbb{L}^{-\nu_i} T^{N_i}} \tag{12'}$$

for ? being $-1, +1, >$ or $<$, where $\widetilde{E}_I^{0,\epsilon}$ is defined as the gluing along $E_I^0 \cap U$ of the sets

$$\{(x, t) \in (E_I^0 \cap U) \times \mathbb{R}; \ t^m \cdot u(x) \ !_? \ \},$$

where $!_?$ is $= -1, = 1, > 0$ or < 0 in case ? is respectively $-1, +1, >$ or $<$. The real motivic Milnor ?-fibre $S_f^?$ of f may finally be defined as

$$S_f^? := -\lim_{T \to \infty} Z_f^\epsilon(T) := -\sum_{I \cap \mathscr{K} \neq \emptyset} (-1)^{|I|} [\widetilde{E}_I^{0,?}] (\mathbb{L} - 1)^{|I|-1} \in K_0(\mathrm{Var}_{\mathbb{R}}) \otimes \mathbb{Z}\left[\frac{1}{2}\right].$$

3.2.1. *Remark.* The class $S_f^?$, although having an expression in terms of the data coming from a chosen resolution of f, does not depend of such a choice, since the definition of $Z_f^?$ as nothing to do with any choice of a resolution.

3.2.2. *Remark.* There is no a *priori* obvious reason, from the definition of $Z_f^?(T)$, that the constant term $S_f^?$ in the power series in T^{-1} induced by the rational expression of $Z_f^?(T)$ could be accurately related to the topology of the corresponding set-theoretic Milnor fibre $X_0^?$, that is to say that $S_f^?$ could be the motivic version of the signed Milnor fibre $X_0^?$ of f. In the complex case, it has just been observed that $\chi(S_f)$ is the expression of $\chi(X_0)$ provided by the A'Campo formula. In the real case, taking into account that $\chi(\mathbf{R}) = -1$, the expression of $\chi(S_f^?)$ is

$$\chi(S_f^?) = \sum_{I \cap \mathscr{K} \neq \emptyset} (-2)^{|I|-1} \chi(\widetilde{E}_I^{0,?}),$$

showing a greater complexity than in the complex case where only strata $E_{\{i\}}$ of maximal dimension in the exceptional divisor $\sigma^{-1}(0)$ appear. Despite this increased complexity, in the real case the correspondence still holds, since it is proved in [20] that $\chi(S_f^?)$ is still $\chi(\bar{X}_0^?)$, where $? \in \{-1, +1\}$. This justifies the terminology of motivic real semialgebraic Milnor fibre of f at 0 for $S_f^?$, at least at the first topological level represented by the morphism

$$\chi : K_0(\mathrm{Var}_{\mathbb{R}}) \otimes \mathbb{Z}\left[\frac{1}{2}\right] \to \mathbf{Z}.$$

In order to accurately state the correspondence between the motivic real semialgebraic Milnor fibre and the set-theoretic Milnor fibre we set now the following notation.

3.2.3. *Notation.* Let us denote $Lk(f)$ the link $f^{-1}(0) \cap S(0, \eta)$ of f at the origin, $0 < \eta \ll 1$. We recall that the topology of $Lk(f)$ is the same as the topology of the boundary $f^{-1}(\varepsilon) \cap S(0, \eta)$, $0 < \varepsilon \ll \eta$, of the Milnor fibre $f^{-1}(\varepsilon) \cap B_{(0,\eta)}$, when f has an isolated singularity at 0.

- Let us denote, for $? \in \{<, >\}$, the topological type of $f^{-1}(]0, c_?[) \cap B(0, \eta)$ by $X_0^?$, and the topological type of $f^{-1}(]0, c_?[) \cap \bar{B}(0, \eta)$ by $\bar{X}_0^?$, where $c_< \in]-\eta, 0[$ and $c_> \in]0, \eta[$.

- Let us denote, for $? \in \{<, >\}$, the topological type of $\{f \bar{?} 0\} \cap S(0, \eta)$ by $G_0^?$, where $\bar{?}$ is $\leq$ when $?$ is $<$ and $\bar{?}$ is $\geq$ when $?$ is $>$.

3.2.4. *Remark.* When n is odd, $Lk(f)$ is a smooth odd-dimensional submanifold of $\mathbf{R}^n$ and consequently $\chi(Lk(f)) = 0$. For $? \in \{-1, +1, <, >\}$, we thus have in this situation, that $\chi(X_0^?) = \chi(\bar{X}_0^?)$. This is the situation in the complex setting. When n is even, since $\bar{X}_0^?$ is a compact manifold with boundary $Lk(f)$, one knows from general algebraic topology that

$$\chi(\bar{X}_0^?) = -\chi(X_0^?) = \frac{1}{2}\chi(Lk(f)),$$

for $? \in \{-1, +1, <, >\}$. For general $n \in \mathbb{N}$ and for $? \in \{-1, +1, <, >\}$, we thus have

$$\chi(\bar{X}_0^?) = (-1)^{n+1}\chi(X_0^?).$$

On the other hand we recall that for $? \in \{<, >\}$

$$\chi(G_0^?) = \chi(\bar{X}_0^{\delta_?}),$$

where $\delta_>$ is $+$ and $\delta_<$ is $-$ (see [6], [108]).

We can now state the real version of (13). We have, for $? \in \{-1, +1, <, >\}$

$$\chi(S_f^?) = \sum_{I \cap \mathscr{K} \neq \emptyset} (-2)^{|I|-1} \chi(\widetilde{E}_I^{0,?}) = \chi(\bar{X}_0^?) = (-1)^{n+1}\chi(X_0^?), \tag{13$'$}$$

and for $? \in \{<, >\}$

$$\chi(S_f^?) = \sum_{I \cap \mathscr{K} \neq \emptyset} (-2)^{|I|-1} \chi(\widetilde{E}_I^{0,?}) = -\chi(G_0^?). \tag{13$''$}$$

The formula $(13')$ below is the real analogue of the A'Campo-Denef-Loeser formula (13) for complex hypersurface singularities and thus appears as the extension to the reals of this complex formula, or, in other words, the complex formula is the notably first level of complexity of the more general real formula $(13')$.

3.2.5. *Remark.* In [111], following the construction of Hrushovski and Kazhdan (see [59], [60]), Yin develops a theory of motivic integration for polynomial bounded T-convex valued fields and studies, in this setting, topological zeta functions attached to a function germ, showing that they are rational. This a first step towards a real version of Hrushovski and Loeser work [61], where no resolution of singularities is used, in contrast with [20].

3.2.6. *Questions.* The question of finding a real analogue of the complex monodromy with real analogues of the invariants $\Lambda(M^m)$ is open. Similarly the question of defining a convenient zeta function with coefficients in an adapted Grothendieck ring in order to let appear invariants of type $\Lambda_i^{\ell oc}$ or σ_j $(e(\mathscr{P}^i)$ in the complex case) from a rational expression of this zeta function also naturally arises.

REFERENCES

1. N. A'Campo, *La fonction zêta d'une monodromie*, Comment. Math. Helvetici 50 (1975), 223-248 DOI: 10.1007/BF02565748
2. S. Alesker, *On P. McMullen's conjecture on translation invariant valuations*, Adv. Math. 155 (2000), 239-263 DOI: 10.1006/aima.2000.1918
3. S. Alesker, *The multiplicative structure on continuous polynomial valuations*, GAFA, Geom. Funct. Anal. 14 (1) (2004), 1-26 DOI: 10.1007/s00039-004-0450-2
4. S. Alesker, *Theory of valuations on manifolds: a survey*, GAFA, Geom. Funct. Anal. 17 (2007), 1321-1341 DOI: 10.1007/s00039-007-0631-x
5. S. Alesker, J. H. G. Fu, *Theory of valuations on manifolds, III. Multiplicative structure in the general case*, Trans. Amer. Mathematical Soc. 360 (4) (2008), 1951-1981 DOI: 10.1090/S0002-9947-07-04489-3
6. V.I. Arnol'd, *Index of a singular point of a vector fields, the Petrovski-Oleinik inequality, and mixed Hodge structures*, Funct. Anal. and its Appl. 12 (1978), 1-14 DOI: 10.1007/BF01077558
7. V. Batyrev, *Birational Calabi-Yau n-folds have equal Betti numbers*, New trends in algebraic geometry (Warwick, 1996), London Math. Soc. Lecture Note Ser., 264, Cambridge Univ.Press, Cambridge, (1999), 1-11
8. A. Bernig, L. Bröcker, *Lipschitz-Killing invariants*, Math. Nachr. 245 (2002), 5-25 DOI: 10.1002/1522-2616(200211)245:1⟨5::AID-MANA5⟩3.0.CO;2-E
9. A. Bernig, L. Bröcker, *Courbures intrinsèques dans les catégories analytico-géométriques*, Ann. Inst. Fourier 53 (2003), no. 6, 1897-1924
10. M. Bickle, *A short course on geometric motivic integration*, Motivic integration and its interactions with model theory and non-Archimedean geometry, Volume I, London Math. Soc. Lecture Note Ser., 383, Cambridge Univ. Press, Cambridge, Edited by R. Cluckers, J. Nicaise, J. Sebag, (2011), 189-243
11. L. Bröcker, M. Kuppe, *Integral geometry of tame set*, Geom. Dedicata 82 (2000), no. 1-3, 285-323
12. J. L. Brylinski, A. S. Dubson, M. Kashiwara, *Formule de l'indice pour modules holonomes et obstruction d'Euler locale*, C. R. Acad. Sci. Paris Sér. I Math. 293 (1981), no. 12, 573-576
13. P. Cartier, *Décomposition des polyèdres : le point sur le troisième problème de Hilbert*, Séminaire Bourbaki, 1984-1985, exp. no 646, p. 261-288
14. Z. Chatzidakis, *Introduction notes on the model theory of valued fields*, Motivic integration and its interactions with model theory and non-Archimedean geometry, Volume I, London Math. Soc. Lecture Note Ser., 383, Cambridge Univ. Press, Cambridge, Edited by R. Cluckers, J. Nicaise, J. Sebag, (2011), 189-243
15. S. S. Chern, R. Lashof, *On the total curvature of immersed manifolds*, Amer. J. Math., 79 (1957), 306-318 DOI: 10.2307/2372684
16. R. Cluckers, F. Loeser, *Constructible motivic functions and motivic integration* Invent. Math. 173 (2008), no 1, 23-121
17. R. Cluckers, F. Loeser, *Motivic integration in mixed characteristic with bounded ramification: a summary* Motivic integration and its interactions with model theory and non-Archimedean geometry, Volume I, 305-334, London Math. Soc. Lecture Note Ser., 383, Cambridge Univ. Press, Cambridge, (2011)

18. G. Comte, *Formule de Cauchy-Crofton pour la densité des ensembles sous-analytiques* C. R. Acad. Sci. Paris, t. 328 (1999), Série I, 505-508

19. G. Comte, *Équisingularité réelle : nombres de Lelong et images polaires*, Ann. Scient. Éc. Norm. Sup. 33(6) (2000), 757-788

20. G. Comte, G. Fichou, *Grothendieck ring of semialgebraic formulas and motivic real Milnor fibres*, Geom. & Top. 18 (2014), 963-996

21. G. Comte, M. Merle, *Équisingularité réelle II : invariants locaux et conditions de régularité*, Ann. Scient. Éc. Norm. Sup. 41(2) (2008), 757-788

22. G. Comte, J. -M. Lion, J.-Ph. Rollin, *Nature Log-analytique du volume des sous-analytiques*, Illinois J. Math 44, (4) (2000), 884-888

23. P. Deligne, *Le formalisme des cycles évanescents*, Séminaire de Géométrie Algébrique du Bois Marie, SGA7 XIII, 1967-69, Lecture Notes in Mathematics 340 (1973)

24. J. Denef, *The rationality of the Poincaré series associated to the p-adic points on a variety*, Invent. Math. 77 (1984), no. 1, 1-23

25. J. Denef, *Report on Igusa's local zeta function* Séminaire Bourbaki, Vol. 1990/1991, Astérisque no. 201-203 (1991), Exp. no. 741, 359-386 (1992)

26. J. Denef, F. Loeser, *Caractéristiques d'Euler-Poincaré, fonctions zêta locales et modifications analytiques*, J. Amer. Math. Soc. 5 (1992), no. 4, 705-720

27. J. Denef, F. Loeser, *Motivic Igusa zeta functions* J. Algebraic Geom. 7 (1998), no. 3, 505-537

28. J. Denef, F. Loeser, *Germs of arcs on singular algebraic varieties and motivic integration* Invent. Math. 135 (1999), no. 1, 201-232

29. J. Denef, F. Loeser, *Definable sets, motives and p-adic integrals* J. Amer. Math. Soc. 14 (2001), no. 2, 429-469

30. J. Denef, F. Loeser, *Geometry on arc spaces of algebraic varieties*, European Congress of Mathematics, Vol. I (Barcelona, 2000), 327-348, Progr. Math., 201, Birkhäuser, Basel, (2001) DOI: 10.1007/978-3-0348-8268-2_19

31. J. Denef, F. Loeser, *Lefschetz numbers of iterates of the monodromy and truncated arcs*, Topology 41 (2002), no. 5, 1031-1040

32. R. N. Draper, *Intersection theory in analytic geometry* Math. Ann. 180 (1969), 175-204

33. A. S. Dubson, *Classes caractéristiques des variétés singulières* C. R. Acad. Sci. Paris Sér. A-B 287 (1978), no. 4, 237-240

34. A. S. Dubson, *Calcul des invariants numériques des singularités et des applications* Thèse, Bonn University, (1981)

35. N. Dutertre, *Courbures et singularités réelles*, Comment. Math. Helv. 77(4) (2002), 846-863 DOI: 10.1007/PL00012444

36. N. Dutertre, *A Gauss-Bonnet formula for closed semi-algebraic sets*, Advances in Geometry 8, no 1 (2008), 33-51 DOI: 10.1515/ADVGEOM.2008.003

37. N. Dutertre, *Curvature integrals on the real Milnor fiber*, Comment. Math. Helvetici 83 (2008), 241-288 DOI: 10.4171/CMH/124

38. N. Dutertre, *Euler characteristic and Lipschitz-Killing curvatures of closed semi-algebraic sets*, Geom. Dedicata 158 (2012), 167-189 DOI: 10.1007/s10711-011-9627-7

39. L. van den Dries, *Tame topology and o-minimal structures*, London Mathematical Society Lecture Note Series, 248, Cambridge University Press, Cambridge, (1998) DOI: 10.1017/CBO9780511525919

40. H. Federer, *The (Φ, k) rectifiable subsets of n space*, Trans. Amer. Math. Soc. 62 (1947), 114-192

41. H. Federer, *Geometric measure theory*, Grundlehren Math. Wiss., Vol. 153 (1969) Springer-Verlag

42. W. Fenchel, *On total curvature of riemannian manifolds I*, Journal of London Math. Soc. 15 (1940), 15-22 DOI: 10.1112/jlms/s1-15.1.15

43. G. Fichou, *Motivic invariants of Arc-Symmetric sets and Blow-Nash Equivalence*, Compositio Math. 141 (2005), 655-688 DOI: 10.1112/S0010437X05001168

44. Joseph H.G. Fu, Ryan C. Scott, *Piecewise linear approximation of smooth functions of two variables*, (2013) arVix: 1305.2220

45. H. G. J. Fu, *Tubular neighborhoods in Euclidean spaces*, Duke Math. J. 52 (1985), no. 4, 1025-1046

46. H. G. J. Fu, *Curvature measures and generalized Morse theory*, J. Differential Geom. 30 (1989), no. 3, 619-642

47. H. G. J. Fu, *Monge-Ampère functions I*, Indiana Univ. Math. J. 38 (1989), 745-771 DOI: 10.1512/iumj.1989.38.38035

48. H. G. J. Fu, *Monge-Ampère functions II*, Indiana Univ. Math. J. 38 (1989), 773-789 DOI: 10.1512/iumj.1989.38.38036

49. H. G. J. Fu, *Kinematic formulas in integral geometry*, Indiana Univ. Math. J. 39 (1990), no. 4, 1115-1154

50. H. G. J. Fu, *Curvature of singular spaces via the normal cycle*, Differential geometry: geometry in mathematical physics and related topics (Los Angeles, CA, 1990), 211-221, Proc. Sympos. Pure Math., 54 (1993), Part 2, Amer. Math. Soc., Providence, RI

51. H. G. J. Fu, *Curvature measures of subanalytic sets*, Amer. J. Math. 116 (1994), no. 4, 819-880

52. J. Gordon, Y. Yaffe, *An overview of arithmetic motivic integration* Ottawa lectures on admissible representations of reductive p-adic groups, 113-149, Fields Inst. Monogr., 26, Amer. Math. Soc., Providence, RI, (2009)

53. P. M. Gruber, R. Schneider, *Problems in geometric convexity. In: Contributions to Geometry*, ed. par J. Tölke et J. M. Wills, Birkhäuser Verlag, Basel, (1979), 225-278

54. S. M. Gusein-Zade, I, Luengo, A. Melle-Hernández, *Integration over a space of non-parametrized arcs, and motivic analogues of the monodromy zeta function* Tr. Mat. Inst. Steklova 252 (2006), Geom. Topol., Diskret. Geom. i Teor. Mnozh., 71-82, translation in Proc. Steklov Inst. Math. 2006, no 1 (252), 63-73

55. H. Hadwiger, *Vorlesungen über Inhalt, Oberfläche und Isoperimetrie*, Springer-Verlag, Berlin-Göttingen-Heidelberg (1957)

56. T. Hales, *Can p-adic integrals be computed?* Contributions to automorphic forms, geometry, and number theory, 313-329, Johns Hopkins Univ. Press, Baltimore, MD, (2004)

57. T. Hales, *What is motivic measure?* Bull. Amer. Math. Soc. (N.S.) 42 (2005), no. 2, 119-135

58. J. P. Henry, M. Merle, *Limites de normales, conditions de Whitney et éclatement d'Hironaka*, Proc. Symp. in Pure Math. 40 (1983) (vol. 1), Arcata 1981, Amer. Math. Soc., 575-584

59. E. Hrushovski, D. Kazhdan, *Integration in valued fields, in Algebraic geometry and number theory*, Progress in Mathematics 253, Birkhäuser, (2006), 261-405

60. E. Hrushovski, D. Kazhdan, *The value ring of geometric motivic integration, and the Iwahori Hecke algebra of SL2. With an appendix by Nir Avni*, Geom. Funct. Anal. 17 (2008), 1924-1967 DOI: 10.1007/s00039-007-0648-1

61. E. Hrushovski, F. Loeser, *Monodromy and the Lefschetz fixed point formula*, (2011), to appear in Ann. Sci. École Norm. Sup. arVix: 1111.1954

62. J. Igusa, *Forms of higher degree* Tata Institute of Fundamental Research Lectures on Mathematics and Physics, 59, Tata Institute of Fundamental Research, Bombay, Narosa Publishing House, New Delhi, (1978)

63. J. Igusa, *Complex powers and asymptotic expansions* II. Asymptotic expansions. J. Reine Angew. Math. 278/279 (1975), 307-321

64. J. Igusa, *An introduction to the theory of local zeta functions* AMS/IP Studies in Advanced Mathematics, 14. American Mathematical Society, Providence, RI, International Press, Cambridge, MA, (2000)

65. M. Kashiwara, *Index theorem for a maximally overdetermined system of linear differential equations*, Proc. Japan Acad. 49 (1973), 803-804 DOI: 10.3792/pja/1195519148

66. D. A. Klain, *A short proof of Hadwiger's characterization theorem*, Mathematika 42 (1995), 329-339 DOI: 10.1112/S0025579300014625

67. M. Kontsevich, *Lecture at Orsay*, (December, 7 1995)

68. M. Kontsevich, Y. Soibelman *Deformation Theory I*, preliminary draft http://www.math.ksu.edu/~soibel/Book-vol1.ps

69. K. Kurdyka, G. Raby, *Densité des ensembles sous-analytiques*, Ann. Inst. Fourier 39 (1989), no. 3, 753-771

70. K. Kurdyka, J. -P. Poly, G. Raby, *Moyennes des fonctions sousanalytiques, densité, cône tangent et tranches*, (Trento, 1988), 170-177, Lecture Notes in Math., 1420 (1990), Springer, Berlin

71. R. Langevin, *Courbure et singularités complexes*, Comment. Math. Helvetici 54 (1979), 6-16 DOI: 10.1007/BF02566253

72. R. Langevin, *Singularités complexes, points critiques et intégrales de courbure*, Séminaire P. Lelong-H. Skoda, 18ème-19ème année, (1978-1979), 129-143

73. R. Langevin, Lê Dung Tràng, *Courbure au voisinage d'une singularité*, C. R. Acad. Sci. Paris Sér. A-B 290 (1980), no. 2

74. R. Langevin, Th. Shifrin, *Polar varietes and integral geometry*, Amer. J. Math. 104 (1982), no 3, 553-605

75. Lê Dũng Tráng, B. Teissier, *Variétés polaires locales et classes de Chern des variétés singulières*, Annals of Math. 114 (1981), 457-491 DOI: 10.2307/1971299

76. Lê Dũng Tráng, B. Teissier, *Errata à Variétés polaires locales et classes de Chern des variétés singulières*, Annals of Math. 115 (1982), 668-668 DOI: 10.2307/2007018

77. Lê Dũng Tráng, B. Teissier, *Cycles évanescents et conditions de Whitney*, II. Proc. Symp. in Pure Math. 40 (1983) (vol. 2), Arcata 1981, Amer. Math. Soc., 65-103

78. J.-M. Lion, *Densité des ensembles semi-pfaffiens*, Ann. Fac. Sci. Toulouse Math. 6, 7 (1998), no. 1, 87-92

79. F. Loeser, *Formules intégrales pour certains invariants locaux des espaces analytiques complexes*, Comment. Math. Helv. 59 (1984), no. 2, 204-225

80. F. Loeser, *Seattle lectures on motivic integration* Algebraic geometry-Seattle 2005, Part 2, 745-784, Proc. Sympos. Pure Math., 80, Part 2, Amer. Math. Soc., Providence, RI, (2009)

81. E. Looijenga, *Motivic measures* Séminaire Bourbaki, Vol. 1999/2000, Astérisque no. 276 (2002), 267-297

82. G. Mikhalkin, *Decomposition into pairs-of-pants for complex algebraic hypersurfaces*, Topology 43 (2004), no 5, 1035-1065

83. P. McMullen, R. Schneider, *Valuations on convex bodies, Convexity and its applications*, edited by Peter Gruber and Jörg M. Wills, Boston: Birkhäuser Verlag (1983)

84. R. MacPherson, *Chern classes for singular algebraic varieties*, Ann. of Math. (2) 100 (1974), 423-432 DOI: 10.2307/1971080

85. M. Merle, *Variétés poalires, stratifications de Whitney et classes de Chern des espaces analytiques complexes* (d'après Lê-Teissier), Séminaire Bourbaki, Vol. 1982/83, Exp. no 600, Astérisque 105 (1983), 65-78

86. J. Milnor, *Singular points of complex hypersurfaces*, Ann. of Math. Studies 61 (1968)

87. V. Navarro Aznar, *Stratifications régulières et variétés polaires locales* Manuscrit, (1981)

88. J. Nicaise, J. Sebag, *The Grothendieck rings of varieties*, Motivic integration and its interactions with model theory and non-Archimedean geometry, Volume I, London Math. Soc. Lecture Note Ser., 383, Cambridge Univ. Press, Cambridge, Edited by R. Cluckers, J. Nicaise, J. Sebag, (2011), 145-188

89. M. Raibaut *Fibre de Milnor motivique à l'infini*, C. R. Math. Acad. Sci. Paris, 348(7-8) (2010), 419-422 DOI: 10.1016/j.crma.2010.01.008

90. M. Raibaut *Singularités à l'infini et intégration motivique*, Bull. SMF, 140(1) (2012), 51-100

91. H. Rullgård, *Polynomial amoebas and convexity*, preprint, Stockholm University, (2001) Manuscrit, (1981)

92. C. Sabbah, *Monodromy at infinity and Fourier transform*, Publ. Res. Inst. Math. Sci., 33(4) (1997), 643-685 DOI: 10.2977/prims/1195145150

93. M. Saito, *Mixed Hodge modules* Publ. Res. Inst. Math. Sci., 26(2) (1990), 221-333 DOI: 10.2977/prims/1195171082

94. L. A. Santalo, *Integral geometry and geometric probability*, Encyclopedia of Mathematics and its Applications Vol. 1 (1976), Addison-Wesley Publishing Co.,London-Amsterdam

95. R. Schneider, *Curvatures measures of convex bodies*, Ann. Mat. Pura appl. 116 (1978), 101-134

96. R. Schneider, *A uniqueness theorem for finitely additive invariant measures on a compact homogeneous space*, Rendiconti del Circolo Matematico di Palermo, XXX (1981), 341-344 DOI: 10.1007/BF02844647

97. R. Schneider, *Convex bodies: The Brunn-Minkowski Theory*, Encyclopedia of Mathematics and its Applications 44 (1993), Cambridge University Press DOI: 10.1017/CBO9780511526282

98. R. Schneider, *Integral geometry - Measure theoretic approach and stochastic applications Advanced course on integral geometry*, CRM (1999)

99. SGA 7, *Séminaire de Géométrie Algébrique du Bois-Marie 1967-1969, Groupes de monodromie en géométrie algébrique (SGA 7), Vol. 1*, Springer Lecture Notes in Math. 288 (1972)

100. J. Steenbrink and S. Zucker, *Variation of mixed Hodge structure. I*, Invent. Math., 80(3) (1985), 489-542 DOI: 10.1007/BF01388729

101. J. Steiner, *Über parallele Flächen*, Monatsber. Preuβ Akad. Wissen., Berlin, (1840), Ges. Werke, vol 2 (1882), Reimer, Berlin

102. B. Teissier, *Cycles évanescents, sections panes et conditions de Whitney*, Astérisque 7-8, Soc. Math. France (1973), 285-362

103. B. Teissier, *Introduction to equisingularity problems*, Proc. AMS Symp. in Pure Math. 29, Arcata 1974, (1975)

104. B. Teissier, *Variétés polaires II : Multiplicités polaires, sections planes et conditions de Whitney* Actes de la conférence de géométrie algébrique de la Rábida (1981), Springer Lecture Notes in Math. 961, Springer, Berlin, (1982), 314-491

105. D. Trotman, *Lectures on real stratification theory*, Singularity theory, World Sci. Publ., Hackensack, NJ, (2007), 139-155

106. D. Trotman, *Espaces stratifiés réels*, Stratifications, singularities and differential equations Vol. II (Marseille, 1990; Honolulu, HI, 1990), Travaux en Cours 55, Hermann, Paris, (1997), 93-107

107. W. Veys, *Arc spaces, motivic integration and stringy invariants*, Singularity theory and its applications, 529-572, Adv. Stud. Pure Math., 43, Math. Soc. Japan, Tokyo, 2006

108. C.T.C. Wall, *Topological invariance of the Milnor number mod 2*, Topology 22 (1983), 345-350 DOI: 10.1016/0040-9383(83)90019-8

109. A. Weil, *Sur la formule de Siegel dans la théorie des groupes classiques*, Acta Math. 113 (1965), 1-87 DOI: 10.1007/BF02391774

110. H. Weyl, *On the Volume of Tubes*, Amer. J. Math. 61 (1939)

111. Y. Yin, *Additive invariants in o-minimal valued fields*, (2013) arVix: 1307.0224

Laboratoire de Mathématiques de l'Université de Savoie, UMR CNRS 5127, Bâtiment Chablais, Campus scientifique, 73376 Le Bourget-du-Lac cedex, France

E-mail address: georges.comte@univ-savoie.fr

URL: http://gc83.perso.sfr.fr/