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ON STRATIFIED MORSE THEORY: FROM TOPOLOGY TO CONSTRUCTIBLE SHEAVES

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Abstract. Stratified Morse theory is the generalization of usual Morse theory to functions on stratified spaces. There are versions for the topological type, homotopy type or (co)homology. A standard reference is the book of Goresky-MacPherson which primarily treats the topological type. Corresponding results about the homotopy type or cohomology may be expected to be consequences but in fact usually one needs some extra information, in particular in the case of cohomology of constructible sheaves, as we will see in this paper.

Introduction

This paper is based on a talk given at the conference "Geometry and topology of singular spaces" (10/29 - 11/02, 2012) in Luminy/Marseille, France, on the occasion of David Trotman's 60th birthday.

We will study the relation between stratified Morse theory concerning the topological type and cohomology, including the cohomology of constructible sheaves. It is quite instructive to look at classical Morse theory first, because already here one has to pay attention - in this case the geometry is so clear that it may seem pedantic to emphasize this point but one sees where one should be careful in more general situations.

Stratified Morse theory is the generalization of usual Morse theory to functions on stratified spaces. A standard reference is the book of Goresky - MacPherson [GM2]. The transition from topology to constructible sheaves in full generality is indicated there in an appendix ([GM2] II 6.A, p. 222-224). Cf. [Ms], too.

The main purpose of the present paper is to make this step more explicit, showing that the setup in [GM2] is indeed strong enough to enable the transition, with some extra care.

In fact there are more direct ways to get the statements about cohomology of constructible sheaves: directly, see Kashiwara-Schapira [KS] or Schürmann [S], or using some weaker version of stratified Morse theory which is sufficient for this purpose [H2].

In special cases one can argue more simply, as we will see. This holds especially for singular cohomology, or for homotopy groups which are discussed in [GM2]. Even in this case, however, one has to be careful, too, and we take the opportunity for some corresponding comments.

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We also take the opportunity to adjust the technique of Moving the Wall which has been developed and used by Goresky - MacPherson, see [GM2] I 4.3, p. 71f.

1. Classical Morse theory

We start with usual Morse theory which is well-known, see e.g. [Ma]. We treat this case because we want to stress some point which we will encounter in the general case, too, it can be more easily discussed in this simple context.
In particular, we will see that it is not completely true that the usual statements about the topological type imply the ones about homotopy or cohomology groups.

Let M be a C^∞ manifold of dimension n and $f : M \to \mathbb{R}$ a C^∞ function. Let us assume that f has isolated critical points which are non-degenerate. Put $M_a := \{p \in M \mid f(p) \le a\}$. Let $a < b$ be regular values, $f^{-1}([a, b])$ compact. We want to compare M_a with M_b.

First suppose that $f^{-1}([a, b])$ contains no critical point. Then we have that M_b is homeomorphic (and even diffeomorphic) to M_a.

As a consequence we have that M_a and M_b have the same homotopy type.
Furthermore, we obtain that $H^k(M_b; \mathbb{Z}) \simeq H^k(M_a; \mathbb{Z})$ for all k.
More precisely: if $h : M_a \to M_b$ is a homeomorphism we obtain that

$$h^* : H^k(M_b; \mathbb{Z}) \to H^k(M_a, \mathbb{Z})$$

is bijective for all k.
In fact we want that it is the inclusion $i : M_a \to M_b$ which induces bijective mappings for all k. This is needed e.g. if one wants to reformulate the cohomological result by saying that $H^k(M_b, M_a; \mathbb{Z}) = 0$ for all k; similarly for homotopy groups.

But this is not obvious, the best is to go back to the proof and show that $i \sim h$ (homotopic). This implies that i is a homotopy equivalence, i.e. M_a is a weak deformation retract of M_b (see [Sp] 1.4, p. 30), which is in turn sufficient to show that $H^k(M_b, M_a; \mathbb{Z}) = 0$ for all k.
Furthermore, we want to have that $H^k(M_b, \mathcal{S}) \simeq H^k(M_a, \mathcal{S})$ if $\mathcal{S}$ is a locally constant sheaf (of abelian groups) on M_b. Here the situation is even worse: h induces isomorphisms

$$H^k(M_b, \mathcal{S}) \to H^k(M_a, h^*\mathcal{S}),$$

and we cannot simply replace $h^*\mathcal{S}$ by $\mathcal{S}|M_a$. But if i is a homotopy equivalence we have that $i^* : H^k(M_b, \mathcal{S}) \to H^k(M_a, \mathcal{S})$ is an isomorphism for all k, see [H1] Theorem 2.6.

So let us recall how one can obtain the homeomorphism h: Choose a vector field v on M with compact support such that $df_x(v(x)) = b - a$ for $x \in f^{-1}([a, b])$. Let σ be the corresponding flow, $h_t(p) := \sigma(p, t)$, $0 \le t \le 1$. Then (h_t) defines a one-parameter family of homeomorphisms $M_a \to M_{a+t(b-a)}$ with $h_0 = id$. In particular, $h := h_1$ is a homeomorphism of M_a onto M_b. Since $M_{a+t(b-a)} \subset M_b, 0 \le t \le 1$, we have that i is homotopic to h.

So M_a is, in particular, a weak deformation retract of M_b. In fact, M_a is even a strong deformation retract of M_b (see [Sp] loc. cit.). This is not completely obvious: Note that h^{-1} cannot be a retraction (except for the trivial case $M_a = M_b$) because otherwise $h^{-1} \circ i = id$ which would imply that i is bijective.

But (M_b, M_a) is a polyhedral pair, cf. [Mu] Theorem 10.6, p. 103, so M_a is a strong deformation retract of M_b if and only if M_a is a weak deformation retract of M_b : This equivalence follows from a homotopy extension property, cf. [Sp] Cor. 1.4.10, Theorem 1.4.11, p. 31, which holds, in particular, in the case of polyhedral pairs, cf. [Sp] Cor. 3.2.5, p. 118.

That we have a strong deformation retract can in our case also be shown directly using the flow σ above, of course.

Now we pass to the case where $f^{-1}([a,b])$ contains exactly one non-degenerate critical point p and λ is defined to be the corresponding index. Then we have that M_b is homeomorphic to a space obtained from M_a by attaching a handle of index λ, i.e. $D^\lambda \times D^{n-\lambda}$ along $S^{\lambda-1} \times D^{n-\lambda}$. Here we have the same problem when passing to cohomology: We want that

$$H^k(M_b, M_a; \mathbb{Z}) \simeq H^k(D^\lambda \times D^{n-\lambda}, S^{\lambda-1} \times D^{n-\lambda}; \mathbb{Z}) \simeq H^k(D^\lambda, S^{\lambda-1}; \mathbb{Z}) \simeq \mathbb{Z}$$

if $k = \lambda$ and $= 0$ if $k \neq \lambda$.

So we look at the proof more closely. It is sufficient to show that there is a space X with $M_a \subset X \subset M_b$ such that there is a homeomorphism $h : X \to M_b$ which is homotopic to the inclusion i and such that X is obtained from M_a by attaching a handle of index λ:
Then $H^k(M_b, X; \mathbb{Z}) = 0$ for all k, hence $H^k(M_b, M_a; \mathbb{Z}) \simeq H^k(X, M_a; \mathbb{Z})$. By excision,

$$H^k(X, M_a; \mathbb{Z}) \simeq H^k(D^\lambda \times D^{n-\lambda}, S^{\lambda-1} \times D^{n-\lambda}; \mathbb{Z}) \simeq \mathbb{Z}$$

if $k = \lambda$ and $= 0$ if $k \neq \lambda$.

Such a space X can be found as follows: Choose a suitable closed neighbourhood U of p, a and b sufficiently close to the critical value. Put $X := M_a \cup (U \cap M_b)$. Then

$$(U \cap \{a \leq f \leq b\}, U \cap \{f = a\})$$

is homeomorphic to $(D^\lambda \times D^{n-\lambda}, S^{\lambda-1} \times D^{n-\lambda})$.

If we look at a locally constant sheaf $\mathcal{S}$ instead of $\mathbb{Z}$ we do not meet new difficulties: As before we can deduce $H^k(M_b, X; \mathcal{S}) = 0$ for all k. Then, by excision:

$$H^k(X, M_a; \mathcal{S}) \simeq H^k(U \cap \{a \leq f \leq b\}, U \cap \{f = a\}; \mathcal{S}),$$

and U is contractible, which implies that $\mathcal{S}|U$ is isomorphic to the constant sheaf $\mathcal{S}_p$ (as usual, $\mathcal{S}_p$ denotes the stalk of $\mathcal{S}$ at p). So $H^k(M_b, M_a; \mathcal{S}) \simeq \mathcal{S}_p$ for $k = \lambda$ and $= 0$ if $k \neq \lambda$.

2. Decomposed homotopy equivalence

Now let us prepare the case of stratified Morse theory.

Let I be a partially ordered set (denoted by $\mathcal{S}$ in [GM2] I 1.1, p. 36). Let X be an I-decomposed space, i.e. a topological space with a locally finite decomposition (= partition) into locally closed subsets $S_{(i)}, i \in I$, such that $S_{(i)} \cap \bar{S}_{(j)} \neq \emptyset \Leftrightarrow i \leq j$. Similarly let Y be an I-decomposed space with subsets $R_{(i)}$. An I-decomposed map $f : X \to Y$ is a continuous map such that $f(S_{(i)}) \subset R_{(i)}$ for all i. See [GM2] I 1.1, p. 36. A homotopy F between I-decomposed maps $f_0, f_1 : X \to Y$ is a homotopy such that $F(S_{(i)} \times [0,1]) \subset R_{(i)}$ for all i.

We will fix I and speak of decomposed instead of I-decomposed.

It is now straightforward to define a decomposed homotopy equivalence and a decomposed weak/strong deformation retract.

An important ingredient in [GM2] is the technique of Moving the Wall which is based on Thom's first isotopy lemma. In fact there are two versions of Moving the Wall in [GM2], here we will concentrate on the first one. The moving is parametrized by a parameter t. In the corresponding theorem ([GM2] I 4.3, p. 72) the parameter space is $\mathbb{R}$. However, in later applications obviously $[0, 1]$ is taken as a parameter space. Therefore it is appropriate to modify Theorem I 4.3 of [GM2] as follows. Note that we weaken the properness hypothesis, too. In order to facilitate the comparison we use the notations of [GM2]:

Let M, N be smooth manifolds, $f : M \to N$ smooth, $Z \subset M$ a Whitney stratified closed subset, see [GM2] I 1.2, p. 37. Then Z is a space which is decomposed by the strata; so I is the corresponding index set. Subsets of Z are naturally decomposed, too. Let $-\infty \leq \alpha < 0$, $1 < \beta \leq \infty$, $Y \subset N \times]\alpha, \beta[$ a closed Whitney stratified subset such the projection on the second factor yields a stratified submersion $\pi : Y \to]\alpha, \beta[$, cf. [GM2] I 1.5, p. 41. Assume that for each $(p, t) \in Y$ with $p \in f(Z)$, $t \in [0, 1]$, and each non-zero characteristic covector $\lambda \in T_p^* N$ of $f|Z : Z \to N$, we have $\lambda | T_p S_t \neq 0$, where S is the stratum of Y which contains (p, t) and $S_t = \pi^{-1}(t) \cap S$. Recall that a covector $\lambda \in T_p^* N$, $p \in N$, is characteristic if and only if for all $z \in Z \cap f^{-1}(p)$ we have that $f^* \lambda | T_z S = 0$, where S is the stratum of Z which contains z, see [GM2] I 1.9, p. 46, together with [GM2] I 1.8, p. 44.

Furthermore assume that the mapping $(Z \times]\alpha, \beta[) \cap (f \times id_{\mathbb{R}})^{-1}(Y) \to]\alpha, \beta[$ given by the projection onto the second factor is proper.
Put $Y_t := \{q \in N \mid (q, t) \in Y\}$.

Now we have the following modified version of Moving the Wall ([GM2] I Theorem 4.3), cf. [S] Lemma 4.3.5, p. 267, too:

Theorem 2.1: Under these hypotheses there is a decomposed homeomorphism

$$h : Z \cap f^{-1}(Y_0) \to Z \cap f^{-1}(Y_1)$$

which preserves the Whitney stratification of both sides and is smooth on each stratum.

Note that these spaces must be compact!

Proof. We may assume that α, β are arbitrarily near to 0 resp. 1. Then we may assume that the assumption about characteristic covectors holds for all $t \in]\alpha, \beta[$ instead of $t \in [0, 1]$, by continuity. This means that we have the hypotheses of [GM2] loc. cit. with $]\alpha, \beta[$ instead of $\mathbb{R}$, except for a weaker properness assumption.

Since $]\alpha, \beta[$ is diffeomorphic to $\mathbb{R}$ we may reduce to $]\alpha, \beta[= \mathbb{R}$ by base change.

Now proceed similarly as in the proof loc. cit.:

Our hypothesis guarantees that $f \times id_{\mathbb{R}}|_{Z \times \mathbb{R}}$ is transverse to Y in the stratified sense (cf. [GM2] I 1.3.1, p. 38), hence $(Z \times \mathbb{R}) \cap (f \times id_{\mathbb{R}})^{-1}(Y)$ inherits an induced stratification, and that $\pi \circ (f \times id_{\mathbb{R}}) : (Z \times \mathbb{R}) \cap (f \times id_{\mathbb{R}})^{-1}(Y) \to \mathbb{R}$ (projection onto the second factor) is a proper stratified submersion. Then apply Thom's first isotopy lemma, see [GM2] I 1.5, p. 41, with $\mathbb{R}$ instead of $\mathbb{R}^n$, $f =$ canonical projection. $\square$

In order to handle certain situations where we get difficulties with the compactness assumption involved above it is useful to have

Remark 2.2: Suppose moreover that there is a closed subset Y_+ of N such that $Y_+ \times]\alpha, \beta[$ is a union of strata of Y of the form $S \times]\alpha, \beta[$. Then we may achieve that $h|Z \cap f^{-1}(Y_+)$ is the identity.

In order to prove this we need the following complement to Thom's first isotopy lemma ([M], [GM2] I 1.5, p. 41):

Theorem 2.3 (see [M] if $X_+ = \emptyset$): Suppose that M is a smooth manifold and that $X \subset M \times \mathbb{R}$ is a Whitney stratified subset. Let $f : X \to \mathbb{R}$ be the restriction of the projection onto the second factor. Let X_+ be a closed subset of M such that $X_+ \times \mathbb{R}$ is a union of strata of X of the form $S \times \mathbb{R}$. Assume that f is a proper stratified submersion. Then there is a stratum preserving homeomorphism $H : f^{-1}(\{0\}) \times \mathbb{R} \to X$ such that:

(1) $f(H(p, t)) = t$ for $p \in f^{-1}(\{0\}), t \in \mathbb{R}$,
(2) $H((q, 0), t) = (q, t)$ for $q \in X_+, t \in \mathbb{R}$.

Proof. The isotopy lemma is proved in [M] using a vector field which is constructed inductively with respect to the strata. On $X_+ \times \mathbb{R}$ choose the obvious one, using control data for $X_+ \times \mathbb{R}$ which come from control data for X_+. □

Because of the difficulty when passing from topological type to homotopy or cohomology groups mentioned in the first section a statement about the homotopy type is appropriate, too:

Theorem 2.4: Beyond the hypotheses of Theorem 2.1 suppose that $Y_t \subset Y_1$ for $0 \le t \le 1$. Then $Z \cap f^{-1}(Y_0)$ is a decomposed weak deformation retract of $Z \cap f^{-1}(Y_1)$. Cf. [S] loc. cit., too.

Proof. The proof of Theorem 2.1 shows that we may assume that $]\alpha, \beta[= \mathbb{R}$ and that the assumption about covectors holds with $t \in \mathbb{R}$ instead of $t \in [0, 1]$. So we can apply Thom's isotopy lemma to $pr : (Z \times \mathbb{R}) \cap (f \times id_{\mathbb{R}})^{-1}(Y) \to \mathbb{R}$, where pr is the projection onto the second factor, and get a homeomorphism

$$H : pr^{-1}(\{0\}) \times \mathbb{R} \to (Z \times \mathbb{R}) \cap (f \times id_{\mathbb{R}})^{-1}(Y)$$

such that $f(H(p, t)) = t$ for $p \in pr^{-1}(\{0\}), t \in \mathbb{R}$.
We may achieve that $H(p, 0) = p$ for all such p because H is obtained by integration of a vector field. Note that $pr^{-1}(\{0\}) = (Z \cap f^{-1}(Y_0)) \times \{0\}$, and H can be written as

$$H((q, 0), t) = (H'(q, t), t)$$

with a continuous mapping $H' : (Z \cap f^{-1}(Y_0)) \times \mathbb{R} \to Z$. Then

$$H'(q, t) \in Z \cap f^{-1}(Y_t) \subset Z \cap f^{-1}(Y_1)$$

for $t \in [0, 1]$. Put $h_t : Z \cap f^{-1}(Y_0) \to Z \cap f^{-1}(Y_1) : h_t(q) := H'(q, t)$. Then H' yields the desired homotopy between the the inclusion h_0 and a homeomorphism h_1. □

We have a remark similar to Remark 2.2:

Remark 2.5: Suppose moreover that Y_+ is a closed subset of N such that $Y_+ \times]\alpha, \beta[$ is a union of strata of Y. Then there is a decomposed homotopy H' between the inclusion and a homeomorphism $h : Z \cap f^{-1}(Y_0) \to Z \cap f^{-1}(Y_1)$ such that $H'(p, t) = p$ for all $p \in Z \cap f^{-1}(Y_+)$.

The proof is as before but apply Theorem 2.3 instead of the usual Thom's first isotopy lemma.

It is not clear whether one can get a decomposed strong deformation retract by this method.

We need some preparation for dealing with constructible sheaves.

A constructible sheaf on the decomposed space $X = \bigcup_i S_{(i)}$ is a sheaf which is locally constant on each $S_{(i)}$. A constructible sheaf complex is a nonnegative complex of sheaves whose cohomology sheaves are constructible on X. We do not impose any finiteness condition.

We have the following general fact: If $\mathcal{S}$ is a sheaf complex on a topological space Y and $f : X \to Y$ is continuous we get induced homomorphisms $\mathbb{H}^k(Y, \mathcal{S}) \to \mathbb{H}^k(X, f^*\mathcal{S})$.
In particular, if f is a homeomorphism it induces isomorphisms.
Here $\mathbb{H}^k$ denotes the k-th hypercohomology group.

Theorem 2.6: Let $\mathcal{S}$ be a constructible sheaf complex on the decomposed space Y.

a) Let $f_0, f_1 : X \to Y$ be decomposed maps which are decomposed homotopic. Then $f_0^*\mathcal{S}$ and $f_1^*\mathcal{S}$ are quasiisomorphic, and the mappings $f_i^* : \mathbb{H}^k(Y, \mathcal{S}) \to \mathbb{H}^k(X, f_i^*\mathcal{S})$, $i = 0, 1$, coincide (if we identify $\mathbb{H}^k(X, f_i^*\mathcal{S}), i = 0, 1$).
b) If $f : X \to Y$ is a decomposed homotopy equivalence we have that the mappings
$$f^* : \mathbb{H}^k(Y, \mathcal{S}) \to \mathbb{H}^k(X, f^*(\mathcal{S}))$$
are isomorphisms.
c) In particular, if $X \subset Y$ is a decomposed weak deformation retract we have that the mappings $\mathbb{H}^k(Y, \mathcal{S}) \to \mathbb{H}^k(X, \mathcal{S})$ are isomorphisms for all k.
d) If (X, X_1) and (Y, Y_1) are pairs of spaces and X resp. X_1 is a decomposed weak deformation retract of Y resp. Y_1 we have that $\mathbb{H}^k(Y, Y_1; \mathcal{S}) \simeq \mathbb{H}^k(X, X_1; \mathcal{S})$ for all k.

Proof. a) The case where $\mathcal{S}$ consists of a single sheaf can be attacked in an elementary way, cf. [H1] Theorem 2.2, 2.7. In general we argue as follows: Let $p : X \times [0, 1] \to X$ be the projection, and let $i_t : X \to X \times [0, 1]$ be defined by $i_t(x) := (x, t)$. Let $\mathcal{T}$ be a constructible sheaf complex on the I-decomposed space $X \times [0, 1]$. By [KS] Prop. 2.7.8, p. 122, we have $\mathcal{T} \sim p^*\mathcal{Q}$ with $\mathcal{Q} = Rp_*\mathcal{T}$, where $\sim$ denotes "quasiisomorphic". So $\mathbb{H}^k(X \times [0, 1], \mathcal{T}) \to \mathbb{H}^k(X, i_t^*\mathcal{T})$ can be rewritten as $\mathbb{H}^k(X, (Rp_*)p^*\mathcal{Q}) \to \mathbb{H}^k(X, i_t^*p^*\mathcal{Q})$. This mapping is induced by $(Rp_*)p^*\mathcal{Q} \to i_t^*p^*\mathcal{Q}$ which is independent of $t \in \{0, 1\}$ because $i_t^*p^*\mathcal{Q} \sim \mathcal{Q}$. So $i_0^*\mathcal{T} \sim i_1^*\mathcal{T}$, and
$$\mathbb{H}^k(X \times [0, 1], \mathcal{T}) \to \mathbb{H}^k(X, i_t^*\mathcal{T})$$
is independent of $t \in \{0, 1\}$ under the corresponding identification of cohomology.

Now let $F : X \times [0, 1] \to Y$ be a decomposed homotopy between f_0 and f_1. Then $f_t = F \circ i_t$, $t = 0, 1$, so $f_t^*\mathcal{S} = i_t^*F^*\mathcal{S}$, $t = 0, 1$, are quasiisomorphic: put $\mathcal{T} := F^*\mathcal{S}$ above. Furthermore look at the composition $\mathbb{H}^k(Y, \mathcal{S}) \to \mathbb{H}^k(X \times [0, 1], F^*\mathcal{S}) \to \mathbb{H}^k(X, i_t^*F^*\mathcal{S})$. Here the right arrow is independent of t, see above.
The rest (b - d) is easy. □

3. Stratified Morse theory

a) The Main Theorem of Goresky-MacPherson

Now pass to stratified Morse theory in the sense of Goresky-MacPherson [GM2] which constitutes a deep generalization of usual Morse theory. Let Z be a Whitney stratified subset of a manifold M, see [GM2] I 1.2, p. 37, $\hat{f} : M \to \mathbb{R}$ smooth, $f := \hat{f}|Z$. Let $Z_c := \{f \leq c\}$.

In [GM2] it is supposed that f is proper (see [GM2] I 3.1, p. 61). Note that this does not imply that Z_c is compact, for this we need an extra assumption:

$$f \text{ is bounded from below.} \tag{*}$$

However we will not assume that (*) is fulfilled and weaken the properness assumption: Let $a < b$ be fixed. Then we assume that there are a_1, b_1 such that $a_1 < a < b < b_1$ and that $f^{-1}([a_1, b_1])$ is compact.

Let us begin with the easiest case:

Theorem 3.1: Suppose that $[a, b]$ contains no critical value.
a) Z_a is homeomorphic to Z_b, the homeomorphism being decomposed, compatible with the stratifications.
b) Z_a is a decomposed strong deformation retract of Z_b.
Note that Z_a, Z_b are stratified in an obvious way.

As in the case of classical Morse theory we need b) if we want to show the vanishing of relative homotopy or cohomology groups.

Proof. We assume without loss of generality that $[a, b] = [0, 1]$.

a) Similar to [GM2] I 7.2, p. 90, we may use the technique of Moving the Wall as modified in Theorem 2.1.

We can choose $\alpha < 0, \beta > 1$ sufficiently near to 0 resp. 1 so that t is not a critical value, $t \in [\alpha, \beta]$.

First suppose that (*) is fulfilled.

Then $Y := \{(y, t) \,|\, y \in \mathbb{R}, \alpha < t < \beta, y \leq t\}$. The hypothesis of Theorem 2.1 (Moving the Wall) is fulfilled, and we get the assertion. Note that the properness assumption is guaranteed because of (*), whereas the projection $Y \to]\alpha, \beta[$, $(y, t) \mapsto t$, is not proper.
Note that we cannot take $\mathbb{R}$ here instead of $[0, 1]$ and $]\alpha, \beta[$ because then the condition on covectors may not be satisfied because of critical points of f. Also, if we modify Y_t by taking $Y_t := Y_0$ for $t \leq 0$, $Y_t := Y_1$ for $t \geq 1$ we have to introduce the strata $\{(0, 0)\}$ resp. $\{(1, 1)\}$ in Y which are not mapped submersively to $\mathbb{R}$. So we need our modified version of Moving the Wall (Theorem 2.1).

If assumption (*) does not hold we take a different Y: $Y := \{(y, t) \,|\, \alpha < t < \beta, \alpha \leq y \leq t\}$. Now the hypothesis of Remark 2.2 is fulfilled, and we obtain a decomposed homeomorphism $h : f^{-1}([\alpha, 0]) \to f^{-1}([\alpha, 1])$ such that $h|f^{-1}(\{\alpha\}) = id$. We glue with $f^{-1}(]\infty, \alpha])$ in order to obtain the desired decomposed homeomorphism $Z_0 \to Z_1$.
Alternative: Use Thom's first isotopy lemma ([GM2] I 1.5, p. 41) more directly. Choose $\alpha < 0$ close to 0. There is a decomposed homeomorphism $H : f^{-1}(\{\alpha\}) \times [\alpha, 1] \to f^{-1}([\alpha, 1])$ such that $f(H(p, t)) = t$ for all (p, t), $H(p, \alpha) = p$. Now the homeomorphism $h : Z_0 \to Z_1$ is defined as follows: $h(p) := p$ if $f(p) \leq \alpha$, $h(p) := H(q, (1 - \frac{1}{\alpha})t + 1)$ if $f(p) > \alpha$, $p = H(q, t)$.

b) Use moreover Theorem 2.4 in order to obtain a weak decomposed deformation retract. In the case where (*) is not fulfilled use Remark 2.5, too.

In order to obtain a strong decomposed deformation retract we use again Thom's isotopy lemma directly. Let H' be, similarly as in the alternative above, a decomposed homeomorphism $f^{-1}(\{0\}) \times [0, 1] \to f^{-1}([0, 1])$ such that $f(H'(p, t)) = t$ for all (p, t), $H'(p, 0) = p$. It is sufficient to show that $f^{-1}(\{0\})$ is a strong decomposed deformation retract of $f^{-1}([0, 1])$. Using H'

this amounts to proving that $f^{-1}(\{0\}) \times \{0\}$ is a strong decomposed deformation retract of $f^{-1}(\{0\}) \times [0,1]$ which is obvious. $\square$

Now suppose that $f^{-1}([a,b])$ contains exactly one critical point p. Let S be the stratum which contains p. Assume that p is a nondepraved critical point of f, see [GM2] I 2.3, p. 55. This involves a condition on $f|S$ which holds automatically if the critical point of $f|S$ is non-degenerate or if S and $f|S$ are real analytic, see [GM2] I 2.3, 2.4. Moreover it is demanded that the critical point p of f is normally nondegenerate (called nondegenerate in [GM2]), i.e. $d\hat{f}_p|T \neq 0$ for every generalized tangent space to Z at p, $T \neq T_pS$. Furthermore we call p a nondegenerate point of index λ if p is a nondegenerate point of $f|S$ of index λ and p is normally nondegenerate, too.

Put $v := f(p)$. We may take a, b as close to v as we wish, namely $a = v - \epsilon, b = v + \epsilon$, where $\epsilon > 0$ can be taken arbitrarily small.

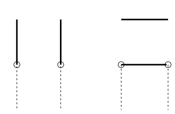
In order to express the main theorem use the following notations, see [GM2] I 3.3-3.6, pp. 62-65:

If (A, B) is a pair of decomposed topological spaces such that Z_b is decomposed homeomorphic to a space obtained from Z_a by attaching A along B we say that (A, B) is a Morse data for f at p.

Example: $(A, B) := (f^{-1}([a, b]), f^{-1}(a))$: "coarse" Morse data.

Morse data (A, B) are not well-defined (this even holds for the homotopy type of A/B):

Examples: a) $Z = \mathbb{Z}$, $f(x) = x$, $v = 0$. Then $(\emptyset, \emptyset)$ as well as $(\{0\}, \emptyset)$ are Morse data for f at 0.
b) $Z = \{0,1\} \times [-1,1], f(x,y) = y, v = 0$. Then not only $(\{0,1\} \times [0,1], \{0,1\} \times \{0\})$ but also $([0,1] \times \{0,1\}, \{0,1\} \times \{0\})$ is Morse data for f at $(0,0)$ (it is harmless to regard the regular point $(0,0)$ as a critical one, too).
In the following drawings A consists of the fat lines and B of the encircled points. On the left side the whole space is Z, on the right side the whole space is homeomorphic to Z.



Choose a Riemannian metric which is the canonical one with respect to some local coordinates near p, and let r be the square of the distance from p.

Let U be a suitable closed neighbourhood of p in Z: $U := Z \cap \{r \leq \delta\}$, $\delta > 0$ small. Choose ϵ above small compared with δ. Then the coarse Morse data of $f|U$ at p is called the local Morse data of f at p. The local Morse data of $f|S$ at p are called the tangential, the local Morse data of $f|N$ at p the normal Morse data at p, where N is a normal slice at p, see [GM2] I 1.4, p. 41. It is of the form $N = N^* \cap \{r \leq \delta\}$, N^* being the intersection of Z and some submanifold of M.

In a first step it is shown that local Morse data is Morse data. More precisely:

Theorem 3.2: a) $(Z_a \cup U) \cap Z_b$ is homeomorphic to Z_b, the homeomorphism being decomposed ([GM2] I 7.6, p. 95),

b) $(Z_a \cup U) \cap Z_b$ is a decomposed strong deformation retract of Z_b.

Again b) is needed, too, in order to pass to the vanishing of relative homotopy or cohomology groups.

Proof. a) Use Moving the Wall, see [GM2] I 7.6, i.e. use Theorem 2.1.

We encounter the same difficulties as in the proof of Theorem 3.1a), so we assume first (*).

Note that $Y_t, t \in [0,1]$, is depicted on [GM2] p. 96, it is obvious how to define Y_t for $t < 0$ close to 0 and $t > 1$ close to 1.

In general replace Y_t by its intersection with $\{(x,y) \,|\, y \geq c\}$ for a suitable c and proceed as in the proof of Theorem 3.1a).

Or: Apply the methods of [H2]. By [H2] Lemma 3.6 we have that (f,r) is submersive along $\{r = \epsilon, a \leq f \leq b\}$. By the Preparatory theorem (Theorem 1.2) of [H2] we get our statement.

b) If we apply Moving the Wall in the proof of a) we can use Theorem 2.4 in order to show that we have a decomposed weak deformation retract. If (*) is not fulfilled use Remark 2.5, too. Or apply the Preparatory Theorem of [H2] loc. cit. □

Now the Main Theorem says:

Theorem 3.3 ([GM2] I 3.7, p. 65): Local Morse data is homeomorphic to Tangential Morse data $\times$ Normal Morse data.

In particular, the product Tangential Morse data $\times$ Normal Morse data is a Morse data - a consequence which can be proved directly much more easily, as proved in [H2] (Theorem 1.9) (see also King [K] Theorem 5).

As we will see in the next section, the Main Theorem has corresponding consequences for singular cohomology groups and simple cases of constructible sheaves. For treating constructible sheaves in general one needs to look at the proof again, see section 5. Applications will be given in section 6.

Remark 3.4: In [GM2], stratified Morse theory is mainly applied to homotopy groups or homotopy type instead of cohomology. In particular, Lefschetz type theorems are proved. Here one needs the following argument: If the local Morse data is k-connected the same holds for the pair $(Z_{\leq b}, Z_{\leq a})$, too. But here one needs Theorem 3.2b), as in the case of singular cohomology which will be treated in section 4a.

b) Variants

There are variants of the Main Theorem of [GM2] developed in the same book.

Relative case: Suppose that $g : X \to Z$ is a proper stratified mapping, i.e. X is Whitney stratified, too, and each stratum of X is mapped submersively to a stratum of Z. We consider X as a decomposed space, the decomposition being given by the stratification. Let f be as before. Put $X_a := X \cap \{f \circ g \leq a\}$.
Relative local Morse data: inverse image of local Morse data of f under g. Relative normal Morse data: local relative Morse data of $f|N$, N being a normal slice, under g.

Theorem 3.5: Local relative Morse data is Morse data, more precisely, there is a decomposed homeomorphism $h : X_a \cup (X \cap \{f \circ g \leq b, r \circ g \leq \epsilon\}) \to X_b$ ([GM2] I 9.4, p. 115).
Moreover we can achieve that $h \sim i$, i inclusion, via a decomposed homotopy, so we have a decomposed weak deformation retract.

The proof is based on Moving the Wall again.

Theorem 3.6 (Main Theorem in relative case) ([GM2] I 9.5, p. 116): Local relative Morse data is homeomorphic to Tangential Morse data of f × Relative normal Morse data.

Nonproper case: Suppose that X is an open subset of Z which is a union of strata. We can define local nonproper Morse data similarly as before, using the inclusion of X in Z instead of g. Similarly: nonproper normal Morse data. See [GM2] I 10.3, p. 120.

Again we have that nonproper local Morse data are Morse data, see [GM2] I 10.4, p. 120. Moreover, $X_a \cup (X \cap \{f \leq b, r \leq \epsilon\})$ is a decomposed weak deformation retract of X_b.

Main Theorem in the nonproper case: the formulation is straightforward ([GM2] I Theorem 10.5, p. 121).

c) Additional remarks

Instead of Z_a we can also study $Z_{<a} := \{p \in Z \,|\, f(p) < a\}$. This will be useful when treating intersection cohomology.

Theorem 3.7: Suppose that $[a, b]$ contains no critical value.
a) $Z_{<a}$ is homeomorphic to $Z_{<b}$, the homeomorphism being decomposed and compatible with the stratifications.
b) $Z_{<a}$ is a weak decomposed deformation retract of $Z_{<b}$.
Of course, $Z_{<a}, Z_{<b}$ are stratified in an obvious way.

It is not true that $Z_{<a}$ is a retract of $Z_{<b}$ if f is surjective: if r is a retraction, we must have $r(z) = z$ for $z \in Z_{<a}$, hence for $z \in Z_a$ by continuity, which contradicts $r(Z_{<b}) \subset Z_{<a}$.

Proof. a) This follows from Theorem 3.1 a) because the homeomorphism there preserves strata. So the homeomorphism is obtained by the technique of Moving the Wall.
b) This follows by application of Theorem 2.4 resp. Remark 2.5. □

In fact we can compare the spaces $Z_{<a}$ and Z_a:

Theorem 3.8: Suppose that $[a, b]$ contains no critical value. Then Z_a is a strong decomposed deformation retract of $Z_{<b}$.

Proof. This is obvious by Thom's first isotopy lemma because $f^{-1}(\{a\}) \times \{a\}$ is a strong decomposed deformation retract of $f^{-1}(\{a\}) \times [a, b[$. But it does not follow from Theorem 2.4. □

4. Transition to cohomology

The assumptions are those of section 3.

a) Cohomology with integral coefficients

If $f^{-1}([a,b])$ contains no critical points, $H^k(Z_b;\mathbb{Z}) \simeq H^k(Z_a;\mathbb{Z})$ for all k. As in classical Morse theory, the isomorphism is induced by the inclusion but one needs Theorem 3.1b) rather than Theorem 3.1a) to see this: Z_a is a deformation retract of Z_b.

If $f^{-1}([a,b])$ contains exactly one critical point p which is non-degenerate of index λ,

$$H^k(Z_b, Z_a;\mathbb{Z}) \simeq H^{k-\lambda}(N \cap \{a \le f \le b\}, N \cap \{f = a\};\mathbb{Z})$$

Here one needs more information than that the product Tangential $\times$ Normal Morse data is Morse data. We need Theorem 3.2b), too:
$H^k(Z_b, Z_a \cup (U \cap \{a \le f \le b\});\mathbb{Z}) = 0$ for all k,
so the exact cohomology sequence of a triple gives

$$\begin{aligned} H^k(Z_b, Z_a;\mathbb{Z}) &\simeq H^k(Z_a \cup (U \cap \{a \le f \le b\}), Z_a;\mathbb{Z}) \simeq H^k(U \cap \{a \le f \le b\}, U \cap \{f = a\};\mathbb{Z}) \\ &\simeq H^k((D^\lambda \times D^{m-\lambda}, S^{\lambda-1} \times D^{m-\lambda}) \times (N \cap \{a \le f \le b\}, N \cap \{f = a\});\mathbb{Z}) \\ &\simeq H^{k-\lambda}(N \cap \{a \le f \le b\}, N \cap \{f = a\};\mathbb{Z}) \end{aligned}$$

where m denotes the dimension of the stratum which contains p.
Here we have used the Main Theorem (Theorem 3.3).

b) Relative case

Suppose first that $[a,b]$ contains no critical value. Then X_a is a weak deformation retract of X_b, so $H^k(X_b, X_a;\mathbb{Z}) = 0$. Here argue as in a) with $f \circ g$ instead of f.

If $f^{-1}([a,b])$ contains exactly one non-degenerate critical point of index λ,

$$X_a \cup \{f \circ g \le b, r \circ g \le \delta\}$$

is a decomposed weak deformation retract of X_b, hence $H^k(X_b, X_a \cup \{f \circ g \le b, r \circ g \le \delta\};\mathbb{Z}) = 0$. Now use the Main Theorem in the relative case and apply Künneth. So

$$H^k(X_b, X_a;\mathbb{Z}) \simeq H^{k-\lambda}(g^{-1}(N \cap \{a \le f \le b\}), g^{-1}(N \cap \{f = a\});\mathbb{Z})$$

c) Nonproper case

Similarly as before we get:

If $[a,b]$ contains no critical value of f we have that $H^k(X_b, X_a;\mathbb{Z}) = 0$.
If $f^{-1}([a,b])$ contains exactly one non-degenerate critical point of index λ,

$$H^k(X_b, X_a;\mathbb{Z}) \simeq H^{k-\lambda}(N \cap X \cap \{a \le f \le b\}, N \cap X \cap \{f = a\});\mathbb{Z}).$$

d) Intersection cohomology

Let p be any perversity. Then the corresponding intersection cohomology can be defined on a purely n-dimensional pseudomanifold Z using the Deligne intersection complex

$$IC_p(Z) = IC_p(Z;\mathbb{Z})$$

which is constructible. Then look at $IH_p^k(Z;\mathbb{Z}) := \mathbb{H}^{k-n}(Z, IC_p(Z))$. See [GM1] p. 98.

Now let Z be as before, Z being purely n-dimensional. In order to have a pseudomanifold we need that there are no strata of codimension 1.

For the same reason we cannot take Z_a directly. So look at $Z_{<a} := Z \cap \{f < a\}$ instead.

By Theorem 2.6 and 3.7 we obtain $IH^k_p(Z_{<b};\mathbb{Z}) \simeq IH^k_p(Z_{<a};\mathbb{Z})$ for all k if $[a,b]$ contains no critical value.

Note that $IH^k_p(Z_{<a};\mathbb{Z}) \simeq \mathbb{H}^{k-n}(Z_a, IC_p(Z))$ if a is a regular value, by Theorem 5.2 below.

Now assume that $f^{-1}([a,b])$ contains exactly one non-degenerate critical point p of index λ. Let d be the dimension of the stratum S which contains p.

Let us look at

$$IH^k_p(Z_{<b}, Z_{<a};\mathbb{Z}) := \mathbb{H}^{k-n}(Z_{<b}, Z_{<a}; IC_p(Z)) \simeq \mathbb{H}^{k-n}(Z_b, Z_a; IC_p(Z)).$$

The Main Theorem of Goresky-MacPherson implies, using Theorem 3.2b) and 2.6c), that

$$\mathbb{H}^{k-n}(Z_b, Z_a; IC_p(Z)) \simeq$$
$$\mathbb{H}^{k-n}((D^\lambda \times D^{d-\lambda}, S^{\lambda-1} \times D^{d-\lambda}) \times (N \cap \{a \le f \le b\}, N \cap \{f = a\}), IC_p(S \times N^*)).$$

Here N^* is chosen as in the definition of a normal slice, it contains N.

Note that first we should take a pull-back of $IC_p(Z)$ on the right hand side but the Deligne intersection complex can be characterized axiomatically, see [GM1] §4, p. 107.

Let $i : N^* \to S \times N^*$ be defined by $q \mapsto (p,q)$. Then we have

$$\begin{aligned}&\mathbb{H}^{k-n}((D^\lambda \times D^{d-\lambda}, S^{\lambda-1} \times D^{d-\lambda}) \times (N \cap \{a \le f \le b\}, N \cap \{f = a\}); IC_p(S \times N^*)\\ &\simeq \mathbb{H}^{k-n-\lambda}(N \cap \{a \le f \le b\}, N \cap \{f = a\}); i^* IC_p(S \times N^*)).\end{aligned}$$

Here we argue as in part (iv) of the proof of Theorem 5.4 below, replacing the commutative square there by

$$\begin{array}{ccc} (\overset{o}{D^\lambda} \times D^{d-\lambda}) \times (N \cap \{a < f \le b\}) & \overset{p_1}{\to} & N \cap \{a < f \le b\} \\ \downarrow \pi_1 & & \downarrow \pi_0 \\ \overset{o}{D^\lambda} \times D^{d-\lambda} & \overset{p_0}{\to} & \{p\} \end{array}$$

where p_1 and π_1 are canonical projections.

Then, $i^* IC_p(S \times N') \sim IC_p(N')[d]$, by [GM1] 5.4.1, p. 115.

Finally,

$$\begin{aligned}&\mathbb{H}^{k-n+d-\lambda}(N \cap \{a \le f \le b\}, N \cap \{f = a\}); IC_p(N^*))\\ &\simeq \mathbb{H}^{k-n+d-\lambda}(N \cap \{r < \delta, a < f < b\}, N \cap \{r < \delta, a < f < a'\}); IC_p(N^*))\\ &\simeq IH^{k-\lambda}_p(N \cap \{r < \delta, a < f < b\}, N \cap \{r < \delta, a < f < a'\}; \mathbb{Z})\end{aligned}$$

where $a' > a$ is sufficiently close to a.

In total,

$$IH^k_p(Z_{<b}, Z_{<a};\mathbb{Z}) \simeq IH^{k-\lambda}_p(N \cap \{r < \delta, a < f < b\}, N \cap \{r < \delta, a < f < a'\}; \mathbb{Z})$$

e) Locally constant coefficients

Let $\mathcal{L}$ be a locally constant sheaf on Z. Then $H^k(Z_b, Z_a; \mathcal{L}) = 0$ if $[a,b]$ contains no critical value: use Theorem 2.6 and Theorem 3.1b).

If there is just one critical point in $f^{-1}([a,b])$ which is non-degenerate of index λ we have $H^k(Z_b, Z_a; \mathcal{L}) \simeq H^k(U \cap Z \cap \{a \le f \le b\}, U \cap Z \cap \{f = a\}; \mathcal{L})$. Now $U \cap Z$ is contractible, so $\mathcal{L}|U \cap Z$ is constant, therefore

$$H^k(U \cap Z \cap \{a \le f \le b\}, U \cap Z \cap \{f = a\}; \mathcal{L}) \simeq H^k(U \cap Z \cap \{a \le f \le b\}, U \cap Z \cap \{f = a\}; \mathcal{L}_p).$$

Now we can continue as in the case of constant coefficients:

$$H^k(Z_b, Z_a; \mathcal{L}) \simeq H^{k-\lambda}(N \cap \{a \leq f \leq b\}, N \cap \{f = a\}; \mathcal{L}_p) \simeq$$
$$H^{k-\lambda}(N \cap \{a \leq f \leq b\}, N \cap \{f = a\}; \mathcal{L}).$$

5. Stratified Morse theory for constructible sheaves

Let $\mathcal{S}$ be a constructible sheaf complex on the decomposed space Z. So the cohomology groups of $\mathcal{S}$ are locally constant along the strata.

We take up the assumptions of the beginning of section 3.

By Theorem 3.1 b) and 2.6 we obtain immediately:

Theorem 5.1: $\mathbb{H}^k(Z_b, Z_a; \mathcal{S}) = 0$ for all k if $[a, b]$ contains no critical values.

We can also compare the cohomology of Z_a and $Z_{<a}$:

Theorem 5.2: If a is a regular value, the inclusion induces isomorphisms

$$\mathbb{H}^k(Z_a, \mathcal{S}) \simeq \mathbb{H}^k(Z_{<a}, \mathcal{S})$$

for all k.

Proof. It is an exercise to prove this using Theorem 5.1 and Theorem 3.7: Let $a' < a$ and $b > a$ sufficiently close to a so that $[a', b]$ contains no critical value. Then $\mathbb{H}^k(Z_{<b}, \mathcal{S}) \simeq \mathbb{H}^k(Z_{<a}, \mathcal{S})$, $\mathbb{H}^k(Z_a, \mathcal{S}) \simeq \mathbb{H}^k(Z_{a'}, \mathcal{S})$, which implies our statement.
Or: $Z_{a'}$ is a strong decomposed deformation retract of $Z_{<a}$, see Theorem 3.8. By Theorem 2.6 we have $\mathbb{H}^k(Z_{<a}, Z_{a'}; \mathcal{S}) = 0$ for all k. Finally use Theorem 5.1, too. □

Now suppose that there is just one critical point p in $f^{-1}([a, b])$ with $a < f(p) < b$ which is non-degenerate of index λ.
Then we can also pass to (co)homology, see e.g. [GM2] II Remark (2) after Theorem 6.4, p. 211: conclusion for $H_i(Z_b, Z_a; \mathbb{Z})$, but again one has to be more careful!

Let r be chosen as in section 3, $U := Z \cap \{r \leq \delta\}$, where $\delta > 0$ is sufficiently small, $v := f(p)$, $\epsilon > 0$ small compared with δ, $a := v - \epsilon, b := v + \epsilon$.

Using Theorem 3.2 and 2.6 we obtain first:

Theorem 5.3: $\mathbb{H}^k(Z_b, Z_a \cup (U \cap Z_b); \mathcal{S}) = 0$ for all k.

By excision, $\mathbb{H}(Z_b, Z_a; \mathcal{S}) \simeq \mathbb{H}^k(Z_a \cup (U \cap Z_b), Z_a; \mathcal{S}) \simeq \mathbb{H}^k(U \cap \{a \leq f \leq b\}, U \cap \{f = a\}; \mathcal{S})$.

The final aim is to show that

$$\mathbb{H}^k(Z_b, Z_a; \mathcal{S}) \simeq \mathbb{H}^{k-\lambda}(N \cap \{a \leq f \leq b\}, N \cap \{f = a\}; \mathcal{S})$$

(i) By Theorem 3.3 (Main Theorem of Goresky-MacPherson) we have a homeomorphism

$$h : (U \cap S \cap \{a \leq f \leq b\}, U \cap S \cap \{f = a\}) \times (N \cap \{a \leq f \leq b\}, N \cap \{f = a\})$$
$$\to (U \cap \{a \leq f \leq b\}, U \cap \{f = a\})$$

This implies:

$$\begin{gathered}\mathbb{H}^k(U \cap \{a \leq f \leq b\}, U \cap \{f = a\}; \mathcal{S}) \\ \simeq \mathbb{H}^k((U \cap S \cap \{a \leq f \leq b\}, U \cap S \cap \{f = a\}) \times (N \cap \{a \leq f \leq b\}, N \cap \{f = a\}), h^*\mathcal{S}) \\ \simeq \mathbb{H}^{k-\lambda}(N \cap \{a \leq f \leq b\}, N \cap \{f = a\}), i^*h^*\mathcal{S})\end{gathered}$$

where

$$i : N \cap \{a \leq f \leq b\} \to (U \cap S \cap \{a \leq f \leq b\}) \times (N \cap \{a \leq f \leq b\})$$

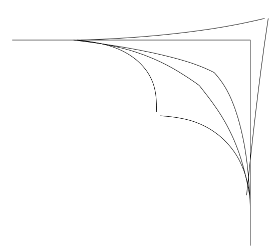
is defined by $x \mapsto (p, x)$.

There are cases where we can replace $i^*h^*\mathcal{S}$ by $\mathcal{S}$ without difficulty: if $\mathcal{S}$ is constant or merely locally constant (because we are dealing with a small neighbourhood). Similarly for the intersection cohomology complex which extends a constant sheaf on the union of the maximal strata of Z. See Section 4. But in other situations - e.g. if we look at an open subspace X of Z and a locally constant sheaf on this space or at the intersection cohomology complex extending a locally constant sheaf, see Section 6 - we must be more careful and look at the proof of Goresky-MacPherson's Main Theorem:

(ii) One considers a pair (A, B) of subspaces of Z which is more easily seen to be homeomorphic to the product of normal and tangential Morse data. The main difficulty is to construct a homeomorphism of the local Morse data onto (A, B). This is obtained as a composition of homeomorphisms each of which is obtained by the technique of "moving the wall".

For technical reasons, 2δ will be taken instead of δ, and let us assume $v = 0$.

More precisely: one considers a sequence (A_i, B_i) of subspaces and shows that two subsequent pairs are homeomorphic via a decomposed homeomorphism. In fact one applies the technique of Moving the Wall. This is indicated in [GM2] I 8.4, 8.5, pp. 103-113. In particular one has to describe walls depending on a parameter t which varies not only in $[0, 1]$ but in a slightly larger interval. But it is straightforward in most cases how to do this, except maybe for the stage of "rounding the corner" (I 8.5.1, p. 107) where the family of walls can be extended like follows:



Note that each A_i is defined as the "realization" of a diagram which is a pair of stratified regions in $\mathbb{R}^2$, together with functions to $\mathbb{R}$. In [GM2] pp. 103-106 these diagrams are depicted, with the two regions on the left and right respectively, the functions are written along the coordinate axes. Each time a subspace is indicated which is a union of strata, the realization of which yields B_i. With Moving the Wall one obtains a homeomorphism $A_i \to A_{i+1}$. Since it is stratum preserving it maps B_i homeomorphically onto B_{i+1}.

Note that we can ignore the transition $D_0 \to D_1$ and $D_5 \to D_6$ because nothing happens there. Let us be more specific about Moving the Wall in the other cases. On [GM2] p. 71 it is said that the wall space is taken to be 4-dimensional. We prefer $\mathbb{R}^2$ instead, because in each case only one of the two "pictures" P_a, P_b (left/right) is varied.

Example: $D_6 \to D_7$ (cf. [GM2] pp. 105, 111). Then we have a variation of P_a and get a corresponding subset $Y = \bigcup_{t \in]\alpha,\beta[} (P_a(t) \times \{t\}) \subset \mathbb{R}^2 \times]\alpha, \beta[$. Here $]\alpha, \beta[$ is a small neighbourhood of $[0,1]$. Furthermore, replace Z in "Moving the Wall" by the inverse image of P_b under the mapping on the right hand side, i.e. by $Z \cap \{r \leq 2\delta\}$. The mapping f is replaced by the mapping $(f \circ \pi, f)$ on the left. In the case of other pictures proceed similarly but intersect also by $\{r < 2\delta'\}$, $\delta' > \delta$ near δ, in order to stay in a neighbourhood of p.

There is a technical problem because π is not defined everywhere but extend $f \circ \pi, r \circ \pi, \rho$ outside $\{r \leq 2\delta\}$ arbitrarily: this is harmless because the relevant considerations concern subsets of $\{r \leq 2\delta\}$.

Furthermore, in most cases we can apply Theorem 2.4 to the transition from A_i to A_{i+1} as well as from B_i to B_{i+1} or vice versa. However we cannot proceed in this way for B_i in all cases: it may happen that neither $B_i \subset B_{i+1}$ nor $B_{i+1} \subset B_i$. Therefore we modify the diagrams D_2, D_3, D_4 in order to pass from D_2 to D_3, D_3 to D_4: On the left we have to consider a "region" P_a together with a subregion Q_a. Replace the region P_a by $P'_a := \{(x,y) \,|\, y \geq -\epsilon'\}$ instead, where $\epsilon' > \epsilon$ is sufficiently near to ϵ. Also Q_a is replaced by $Q'_a :=$ closure of the complement of Q_a in P'_a, i.e. by

$$\{(x,y) \,|\, -\epsilon' \leq y \leq -\epsilon\} \text{ in the case of } D_2$$

$$\{(x,y) \,|\, y \geq -\epsilon', x \leq -\frac{3\epsilon}{4} \,\text{or}\, y \leq -\epsilon\} \text{ in the case of } D_3$$

$$\{(x,y) \,|\, y \geq -\epsilon', x \leq -\frac{3\epsilon}{4} \,\text{or}\, y - x \leq -\frac{\epsilon}{4}\} \text{ in the case of } D_4$$

We take obvious stratifications so that the subspaces are unions of strata. Instead of pairs (A_i, B_i), we thus obtain (A'_i, B'_i). Now we have $A'_i = A'_{i+1}$, $B'_i \subset B'_{i+1}$, $i = 2, 3$. By Theorem 2.1, we obtain a homeomorphism $A'_i \to A'_{i+1}$ such that the restriction gives homeomorphisms $B'_i \to B'_{i+1}$, $A_i \to A_{i+1}$, $B_i \to B_{i+1}$. Furthermore, by Theorem 2.4 and 2.6 the inclusion defines isomorphisms $\mathbb{H}^k(A'_{i+1}, B'_{i+1}; \mathcal{S}) \to \mathbb{H}^k(A'_i, B'_i; \mathcal{S})$. By excision, we obtain

$$\mathbb{H}^k(A_{i+1}, B_{i+1}, \mathcal{S}) \simeq \mathbb{H}^k(A_i, B_i; \mathcal{S})$$

for all k. This shows that we obtain isomorphisms for the cohomology groups with the same constructible sheaf $\mathcal{S}$.

So we have $\mathbb{H}^k(U \cap f^{-1}([a,b]), U \cap \{f = a\}, \mathcal{S}) \simeq \mathbb{H}^k(A, B; \mathcal{S})$.

The precise description of (A, B) will be recalled in (iv) below.

In total we now have: $\mathbb{H}^k(Z_b, Z_a, \mathcal{S}) \simeq \mathbb{H}^k(A, B; \mathcal{S})$.

(iii) This result can be obtained more easily using different techniques, as in [H2]. Then we get:
The space Z_a is a decomposed strong deformation retract of $Z'_a := \{f \leq -\epsilon\} \cup E$, with

$$E := \{f \circ \pi \leq -\frac{3\epsilon}{4}, f - f \circ \pi \leq \frac{\epsilon}{4}, \rho \leq \delta, r \circ \pi \leq \delta\} \cup \{f - f \circ \pi \leq -\frac{\epsilon}{4}, f \circ \pi \leq \frac{3\epsilon}{4}, \rho \leq \delta, r \circ \pi \leq \delta\}$$

and

$$Z'_b := \{f \leq -\epsilon\} \cup \{f \circ \pi \leq \frac{3\epsilon}{4}, f - f \circ \pi \leq \frac{\epsilon}{4}, \rho \leq \delta, r \circ \pi \leq \delta\}$$

is a decomposed strong deformation retract of Z_b. See [H2] Prop. 4.4.
Therefore, $\mathbb{H}^k(Z'_a, \mathcal{S}) \simeq \mathbb{H}^k(Z_a, \mathcal{S})$, and $\mathbb{H}^k(Z_b, \mathcal{S}) \simeq \mathbb{H}^k(Z'_b, \mathcal{S})$.

Finally, $Z'_b = Z'_a \cup A$, and $B = Z'_a \cap A$, so $H^k(Z'_b, Z'_a, \mathcal{S}) \simeq H^k(A, B, \mathcal{S})$, which shows again that $H^k(Z_b, Z_a, \mathcal{S}) \simeq H^k(A, B, \mathcal{S})$.

(iv) Now

$$A = Z \cap \{|f - f \circ \pi| \leq \frac{\epsilon}{4}, |f \circ \pi| \leq \frac{3\epsilon}{4}, r \circ \pi \leq \delta, \rho \leq \delta\}$$

$$B = Z \cap \{r \circ \pi \leq \delta, \rho \leq \delta\} \cap (\{|f - f \circ \pi| \leq \frac{\epsilon}{4}, f \circ \pi = -\frac{3\epsilon}{4}\} \cup \{f - f \circ \pi = -\frac{\epsilon}{4}, |f \circ \pi| \leq \frac{3\epsilon}{4}\})$$

cf. [GM2] I Prop. 8.2. p. 101.
Furthermore,

$$\pi : (Z \cap \{r \circ \pi \leq \delta, \rho \leq \delta, |f - f \circ \pi| \leq \frac{\epsilon}{4}\}, Z \cap \{r \circ \pi \leq \delta, \rho \leq \delta, f - f \circ \pi = -\frac{\epsilon}{4}\})$$
$$\to S \cap \{r \leq \delta\}$$

is a fibre bundle pair with contractible base, hence trivial. The fibre pair over p is

$$(N \cap \{|f| \leq \frac{\epsilon}{4}, r \leq \delta\}, N \cap \{f = -\frac{\epsilon}{4}, r \leq \delta\})$$

A trivialization yields a mapping pair

$$pr : (Z \cap \{r \circ \pi \leq \delta, \rho \leq \delta, |f - f \circ \pi| \leq \frac{\epsilon}{4}\}, Z \cap \{r \circ \pi \leq \delta, \rho \leq \delta, f - f \circ \pi = -\frac{\epsilon}{4}\})$$
$$\to (N \cap \{|f| \leq \frac{\epsilon}{4}, r \leq \delta\}, N \cap \{f = -\frac{\epsilon}{4}, r \leq \delta\})$$

The fibres are contractible, and $\mathcal{S}$ is cohomologically locally constant along the fibres. By [KS] Prop. 2.7.8, p. 122, we can conclude that $\mathcal{S}$ is quasiisomorphic to $pr^*\mathcal{T}$ with

$$\mathcal{T} := \mathcal{S}|N \cap \{|f| \leq \frac{\epsilon}{4}, r \leq \delta\} = i_0^*\mathcal{S},$$

where $i_0 : N \cap \{|f| \leq \frac{\epsilon}{4}, r \leq \delta\} \to (Z \cap \{r \circ \pi \leq \delta, \rho \leq \delta, |f - f \circ \pi| \leq \frac{\epsilon}{4}\}$ is the inclusion:

Indeed, $\mathcal{S} \sim pr^*(Rpr_*\mathcal{S})$, by [KS] loc. cit., so $i_0^*\mathcal{S} \sim i_0^* pr^*(Rpr_*\mathcal{S}) \sim Rpr_*\mathcal{S}$ because $pr \circ i_0 = id$.

From now on it is easier to work with cohomology with compact support instead of relative cohomology. Note that

$$A \setminus B = Z \cap \{-\frac{\epsilon}{4} < f - f \circ \pi \leq \frac{\epsilon}{4}, -\frac{3\epsilon}{4} < f \circ \pi \leq \frac{3\epsilon}{4}, r \circ \pi \leq \delta, \rho \leq \delta\}$$

Put $C := S \cap \{-\frac{3\epsilon}{4} < f \leq \frac{3\epsilon}{4}, r \leq \delta\}$, $D := N \cap \{-\frac{\epsilon}{4} < f \leq \frac{\epsilon}{4}, r \leq \delta\}$. Let $\pi_1 : A \setminus B \to C$ and $\pi_0 : D \to \{p\}$ be the restrictions of π, and let $p_1 : A \setminus B \to D$ and $p_0 : C \to \{p\}$ be the restrictions of pr, so that we have a commutative diagram:

$$\begin{array}{ccc} A \setminus B & \xrightarrow{p_1} & D \\ \downarrow \pi_1 & & \downarrow \pi_0 \\ C & \xrightarrow{p_0} & \{p\} \end{array}$$

Then we have:

$$\mathbb{H}^k(A, B, \mathcal{S}) \simeq \mathbb{H}^k(A, B, pr^*\mathcal{T}) \simeq \mathbb{H}^k_c(A \setminus B, p_1^*\mathcal{T}') \simeq H^k(R(p_0)_! R(\pi_1)_! p_1^*\mathcal{T}')$$

where $\mathcal{T}' := \mathcal{T}|D$.

Now

$$R(p_0)_! R(\pi_1)_! p_1^*\mathcal{T}' \sim R(p_0)_!(\mathbb{Z}_C \otimes^L R(\pi_1)_! p_1^*\mathcal{T}')$$
$$\sim R(p_0)_!(\mathbb{Z}_C \otimes^L p_0^* R(\pi_0)_!\mathcal{T}') \sim (R(p_0)_!\mathbb{Z}_C) \otimes^L R(\pi_0)_!\mathcal{T}'$$

The second quasiisomorphism follows by base change, the third one by some kind of projection formula, see [KS] Prop. 2.6.6, p. 113.

Hence

$$\mathbb{H}^k(A,B,\mathcal{S}) \simeq H^k((R(p_0)_!\mathbb{Z}_C) \otimes^L R(\pi_0)_!\mathcal{T}') \simeq \mathbb{H}_c^{k-\lambda}(D,\mathcal{T}')$$
$$\simeq \mathbb{H}^{k-\lambda}(N \cap \{|f| \leq \frac{\epsilon}{4}, r \leq \delta\}, N \cap \{f = -\frac{\epsilon}{4}, r \leq \delta\}, \mathcal{S})$$

Altogether we obtain as final result, with $N = N^* \cap \{r \leq \delta\}$:

Theorem 5.4: $\mathbb{H}^k(Z_b, Z_a; \mathcal{S}) \simeq \mathbb{H}^{k-\lambda}(N \cap \{a \leq f \leq b\}, N \cap \{f = a\}; \mathcal{S})$.

The final result has been shown by J. Schürmann [S] directly, too, with milder conditions on the critical point. See [S] Theorem 5.3.3.

6. Applications of stratified Morse theory for constructible sheaves

a) Cohomology with locally constant coefficients in the relative case

Let us look at the relative case as in section 4. Let $\mathcal{L}$ be a locally constant sheaf on X. Then we obtain:

If $[a,b]$ contains no critical value, $H^k(X_b, X_a; \mathcal{L}) = 0$ for all k.

If $f^{-1}([a,b])$ contains exactly one non-degenerate critical point of index λ,

$$H^k(X_b, X_a; \mathcal{L}) = H^{k-\lambda}(g^{-1}(N \cap \{a \leq f \leq b\}), g^{-1}(N \cap \{f = a\}), \mathcal{L})$$

In order to prove this, proceed as in the last section (Theorem 5.4) with $f \circ g$ instead of f, $r \circ g$ instead of r etc.

Or apply our theorem above to $Rg_*\mathcal{L}$, similarly as in [S] p. 275.

Note that Rg_* commutes with restriction to closed subsets because $Rg_* = Rg_!$, g being proper.

b) Cohomology with locally constant coefficients in the nonproper case

Suppose that X is an open subset of Z which is a union of strata and $\mathcal{L}$ a locally constant sheaf on X. Put $X_a := X \cap Z_a$. Then:

If $[a,b]$ contains no critical value, $H^k(X_b, X_a; \mathcal{L}) = 0$ for all k.

If $f^{-1}[a,b]$ contains exactly one non-degenerate critical point of index λ,

$$H^k(X_b, X_a; \mathcal{L}) = H^{k-\lambda}(N \cap X \cap \{a \leq f \leq b\}, N \cap X \cap \{f = a\}, \mathcal{L})$$

In order to prove this, proceed as in the last section finding decomposed weak deformation retracts.

Or apply Theorem 5.4 to $Rj_*\mathcal{L}$, where $j : X \to Z$ is the inclusion, similarly as in [S] p. 275. But the conclusion is not evident. Note that Rj_* commutes in general with $i^!$, hence with i^* if i is the inclusion of an open but not of a closed subset.

One needs a base change property which is proved in [S] Prop. 4.3.1, p. 261.

We argue in the same way for the normal slice and obtain

$$\mathbb{H}^k(N \cap \{a \leq f \leq b\}, Rj_*\mathcal{L}) \simeq H^k(N \cap X \cap \{a \leq f \leq b\}; \mathcal{L})$$

Similarly with $f = a$ instead of $a \leq f \leq b$.

c) Intersection cohomology with coefficients in a locally constant sheaf

Note that the locally constant sheaf has to be given outside codimension 2. We assume that Z is pure-dimensional. Again the reduction to the local case does not allow to assume that the locally constant sheaf is constant when applying the Main Theorem.

Similarly as in section 4 we obtain, using the complex $IC_p(Z;\mathcal{L})$:

$IH_p^k(Z_{<b}, Z_{<a};\mathcal{L}) = 0$ if $[a,b]$ does not contain critical values.

If there is exactly one non-degenerate critical point of index λ in $f^{-1}([a,b])$:

$$IH_p^k(Z_{<b}, Z_{<a};\mathcal{L}) = IH_p^{k-\lambda}(N \cap \{a < f < b, r < \delta\}, N \cap \{a < f < a', r < \delta\};\mathcal{L})$$

where $a' > a$ is sufficiently close to a.

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