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SIMPLE DYNAMICS AND INTEGRABILITY FOR SINGULARITIES OF HOLOMORPHIC FOLIATIONS IN DIMENSION TWO

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Abstract. In this paper we study the dynamics of a holomorphic vector field near a singular point in dimension two. We consider those for which the set of separatrices is finite and the orbits are closed off this analytic set. We assume that none of the singularities arising in the reduction of the foliation has a zero eigenvalue. Under these hypotheses we prove that one of the following cases occurs: (i) there is a holomorphic first integral, (ii) the induced foliation is a pull-back of a hyperbolic linear singularity, (iii) there is a formal Liouvillian first integral. For a germ with closed leaves off the set of separatrices we prove that the existence of a holomorphic first integral is equivalent to the existence of some closed leaf arbitrarily close to the singularity. For this we do not need to assume any non-degeneracy hypothesis on the reduction of singularities. We also study some examples illustrating our results and we prove a characterization of pull-backs of hyperbolic singularities in terms of the dynamics of the leaves off the set of separatrices.

1. Introduction and main results

In this paper we resume the subject of *dynamics versus integrability* for a singularity of holomorphic vector field in dimension two (see [9, 27]). Some references in this subject are results of H. Poincaré, G. Darboux ([13]) (for polynomial vector fields in the complex plane) and more recently [16].

A modern starting point is the following theorem of Mattei-Moussu ([16]): *A germ of a holomorphic vector field at the origin of $\mathbb{C}^2$ admits a holomorphic first integral if, and only if, it has only finitely many leaves accumulating at the singularity and all other leaves are closed.* Also notable is the point of view adopted in [1] where the authors suppose the existence of an uniform bound for the volume of the orbits of the vector field. A holomorphic vector field X defined in a neighborhood $U \subset \mathbb{C}^2$ of the origin $0 \in \mathbb{C}^2$, with an isolated singularity at the origin, defines a germ of holomorphic foliation with a singularity at the origin, and conversely. In this paper we shall adopt the foliation terminology. We shall refer to a germ of a holomorphic foliation $\mathcal{F}$ as induced such a pair (X, U) where X is a holomorphic vector field defined a neighborhood U of the origin $0 \in \mathbb{C}^2$, singular at the origin $X(0) = 0$. Recall that a *separatrix* is an invariant irreducible analytic curve containing the singularity. Throughout this paper we will only consider germs of foliations with a finite number of separatrices, called *non-dicritical* singularities. In this case, we shall say that a leaf of $\mathcal{F}$ (i.e., an orbit of (X, U) for U small enough) *is closed off the set of separatrices* if either it is a separatrix, or it is not a separatrix but accumulates only at the union of separatrices. In few words, it accumulates at no leaf which is not contained in a separatrix. We then characterize those germs of foliations, under the additional hypothesis that they belong to the class of *generalized curves*, meaning that the reduction of singularities does not exhibit final singularities with a null eigenvalue. Before stating our first result we shall state a few notions. Recall that a germ of a singular holomorphic foliation $\mathcal{F}$ at the origin $0 \in \mathbb{C}^2$ is defined by a germ of a holomorphic one-form ω at the origin. We shall assume that $\text{sing}(\omega) = \{0\}$. A holomorphic

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first integral for $\mathcal{F}$ is a germ of a (non-constant) holomorphic function $\mathcal{O}_2 \ni f\colon (\mathbb{C}^2,0) \to (\mathbb{C},0)$ such that $\omega \wedge df = 0$. In terms of Saito-De Rham division lemma, this is equivalent to say that $\omega = g df$ for some germ $g \in \mathcal{O}_2$, provided that we take $f \in \mathcal{O}_2$ as a reduced germ. The function g is necessarily a unit. Thus, if we write $\eta = \frac{dg}{g}$ then we have a germ of a *closed* holomorphic one-form such that $d\omega = \eta \wedge \omega$. In general, the germ $\mathcal{F}$ admits a *Liouvillian first integral* if there is a closed *meromorphic* one-form germ η such that $d\omega = \eta \wedge \omega$. Such a form η is called a *generalized integrating factor* for ω. In this case we say that the first integral is the Liouvillian function F defined by the differential algebraic equation $dF = \frac{\omega}{\exp \int \eta}$. This is all discussed in [31, 28]. We shall now introduce a slightly more general notion:

Definition 1.1. We shall say that $\mathcal{F}$ admits a *formal Liouvillian first integral* $\hat{F}$ if there is a *formal generalized integrating factor* $\hat{\eta}$ which is a formal closed meromorphic one-form such that $(*)\, d\omega = \hat{\eta} \wedge \omega$.

We may rewrite (*) as $d(\frac{\omega}{\exp \int \hat{\eta}}) = 0$, so that the formal Liouvillian first integral is defined by $d\hat{F} = \frac{\omega}{\exp \int \hat{\eta}}$. By a *formal meromorphic* one-form we mean a formal expression $\hat{\eta} = \hat{A} dx + \hat{B} dy$ where $\hat{A}, \hat{B}$ are quotient of formal functions $\hat{A} = \hat{a_1}/\hat{a_2}, \hat{B} = \hat{b_1}/\hat{b_2}$, $\hat{a_j}, \hat{b_j} \in \hat{\mathcal{O}_2}$ ([12]). With these notions we can state:

Theorem 1.2. *Let $\mathcal{F}$ be a germ of a non-dicritical generalized curve at $0 \in \mathbb{C}^2$. Assume that the leaves of $\mathcal{F}$ are closed off the set of separatrices. Then we have three possibilities:*

(1) *$\mathcal{F}$ admits a holomorphic first integral.*
(2) *$\mathcal{F}$ is a holomorphic pull-back of a hyperbolic (linearizable) singularity.*
(3) *$\mathcal{F}$ admits a formal Liouvillian first integral.*

Possibility (3) really occurs, indeed, there is a number of examples which correspond to this last situation. We shall refer to these foliations as *of formal Liouvillian type*. Some information about these foliations is given in § 5. Indeed, the formal one-form $\hat{\eta}$ is actually convergent except in the so called *exceptional case*, which we will detail later on.

The foliation is already in case (2) if some singularity in the reduction of the singularities of the foliation is a non-resonant singularity. More generally, we are in case (2) if there is some non-resonant map in the virtual holonomy group of any separatrix of $\mathcal{F}$. Indeed, from the proof we give for Theorem 1.2 we obtain:

Theorem 1.3. *For a germ of a generalized curve holomorphic foliation $\mathcal{F}$ at the origin $0 \in \mathbb{C}^2$ assume that the following conditions are true:*

(1) *There is only a finite number of separatrices and all leaves are closed off the set of separatrices.*
(2) *Some separatrix has a holonomy map which is not a resonant map.*

Then $\mathcal{F}$ is the pull-back of a hyperbolic singularity.

We stress that the second hypothesis means that there is some separatrix of $\mathcal{F}$ whose local holonomy is of the form $f(z) = e^{2\pi\sqrt{-1}\lambda} z + a_{k+1} z^{k+1} + \ldots$, where $\lambda \in \mathbb{C} \setminus \mathbb{Q}$. We may assume, instead of (2), the weaker condition that the *virtual* holonomy of some separatrix contains some non-resonant map.

The hypotheses in Theorems 1.2 and 1.3 depend on the concept of reduction of singularities, detailed in Section 2. In short, $\mathcal{F}$ is a *generalized curve* if its reduction of singularities only produces singularities with non-zero eigenvalues. It is *non-dicritical* if there are only finitely many separatrices. The necessity of the generalized curve hypothesis in Theorems 1.2 and 1.3 is discussed in Examples 5.3 and 5.4.

We may conclude that we are in case (1) if arbitrarily close to the union of the separatrices we can find some closed leaf. Indeed, for the next result we do not need to assume that the singularity is a generalized curve:

Theorem 1.4. *Let $\mathcal{F}$ be a germ of a non-dicritical foliation at $0 \in \mathbb{C}^2$. Assume that the leaves of $\mathcal{F}$ are closed off the set of separatrices and there is a closed leaf arbitrarily close to the origin. Then $\mathcal{F}$ admits a holomorphic first integral.*

Outline of the proofs:

The proofs are based on a product of two points:

- A description of subgroups of germs of one-dimensional complex diffeomorphisms with closed orbits off the fixed point: these groups are finite, abelian linearizable generated by a hyperbolic map and a periodic (rational) rotation, or solvable discrete (cf. Proposition 4.2).
- A description of the singularities in the reduction of singularities of $\mathcal{F}$ by the blowing-up process.

We apply the above to the holonomy groups arising in the reduction of singularities of $\mathcal{F}$ and to some enriched groups called *virtual holonomy groups*. The possible combinations of these larger groups are also studied in order to prove that they are all solvable of a same type. For this we consider the connection between two such groups associated to adjacent components of the exceptional divisor of the reduction of singularities. This connection is given by the so called *Dulac correspondence* in suitable cases. When there is a closed leaf arbitrarily close to the singular point it is proven that all these groups have a closed orbit and then are finite. This is the case that correspond to the holomorphic first integral (cf. [16], [9]). It is also proven that if some of these virtual holonomy groups contains a map whose linear part is not periodic, then it must be hyperbolic and all these groups are abelian generated by a hyperbolic map and a rational rotation. This case corresponds to (2) in Theorem 1.2 via techniques from [5]. Finally, in the remaining case all the singularities in the reduction process are resonant as well as all the holonomies are solvable. In this case, by techniques from [26] or [21] we are able to construct a formal Liouvillian first integral. This construction is detailed in the Appendix § 9.

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2. Reduction of singularities in dimension two ([30])

Fix now a germ of holomorphic foliation with a singularity at the origin $0 \in \mathbb{C}^2$. Choose a representative $\mathcal{F}(U)$ for the germ $\mathcal{F}$, defined in an open neighborhood U of the origin, such that 0 is the only singularity of $\mathcal{F}(U)$ in U. The *Theorem of reduction of singularities* of Seidenberg ([30]) asserts the existence of a proper holomorphic map $\sigma\colon \tilde{U} \to U$ which is a finite composition of quadratic blowing-up's, starting with a blowing-up at the origin, such that the pull-back foliation $\tilde{\mathcal{F}} := \sigma^*\mathcal{F}$ of $\mathcal{F}$ by σ satisfies:

(1) The *exceptional divisor* $E(\mathcal{F}) = \sigma^{-1}(0) \subset \widetilde{U}$ can be written as $E(\mathcal{F}) = \bigcup_{j=1}^m D_j$, where each irreducible component D_j is diffeomorphic to an embedded projective line $\mathbb{C}P(1)$ introduced as a divisor of the successive blowing-up's ([7]).

(2) $\operatorname{sing}\tilde{\mathcal{F}} \subset \mathrm{E}$ is a finite set, and any singularity $\tilde{p} \in \operatorname{sing}\tilde{\mathcal{F}}$ is *irreducible* i.e., belongs to one of the following categories:
   (a) $xdy - \lambda ydx + \text{h.o.t.} = 0$ and λ is not a positive rational number, i.e. $\lambda \notin \mathbb{Q}_+$ (*non-degenerate singularity*),
   (b) $y^{p+1}dx - [x(1+\lambda y^p) + \text{h.o.t.}]\, dy = 0$, $p \geq 1$. This case is called a *saddle-node* ([18]).

A singularity is a *generalized curve* if its reduction of singularities produces only non-degenerate (i.e., no saddle-node) singularities ([6]). We call the lifted foliation $\tilde{\mathcal{F}}$ the *desingularization* or *reduction of singularities* of $\mathcal{F}$. The foliation is non-dicritical iff $E(\mathcal{F})$ is invariant by $\tilde{\mathcal{F}}$. Any two components D_i and D_j, $i \neq j$, of the exceptional divisor, intersect (transversely) at at most one point, which is called a *corner*. There are no triple intersection points.

3. Holonomy and virtual holonomy groups

Let now $\mathcal{F}$ be a holomorphic foliation with (isolated) singularities on a complex surface M (we have in mind here, the result of a reduction of singularities process). Denote by $\operatorname{sing}(\mathcal{F})$ the singular set of $\mathcal{F}$. Given a leaf L_0 of $\mathcal{F}$ we choose any base point $p \in L_0 \subset M \setminus \operatorname{sing}(\mathcal{F})$ and a transverse disc $\Sigma_p \Subset M$ to $\mathcal{F}$ centered at p. The holonomy group of the leaf L_0 with respect to the disc Σ_p and to the base point p is image of the representation $\mathrm{Hol}\colon \pi_1(\mathrm{L_0}, \mathrm{p}) \to \mathrm{Diff}(\Sigma_\mathrm{p}, \mathrm{p})$ obtained by lifting closed paths in L_0 with base point p, to paths in the leaves of $\mathcal{F}$, starting at points $z \in \Sigma_p$, by means of a transverse fibration to $\mathcal{F}$ containing the disc Σ_p ([4]). Given a point $z \in \Sigma_p$ we denote the leaf through z by L_z. Given a closed path $\gamma \in \pi_1(L_0, p)$ we denote by $\tilde{\gamma}_z$ its lift to the leaf L_z and starting (the lifted path) at the point z. Then the image of the corresponding holonomy map is $h_{[\gamma]}(z) = \tilde{\gamma}_z(1)$, i.e., the final point of the lifted path $\tilde{\gamma}_z$. This defines a diffeomorphism germ map $h_{[\gamma]}\colon (\Sigma_p, p) \to (\Sigma_p, p)$ and also a group homomorphism $\mathrm{Hol}\colon \pi_1(\mathrm{L_0}, \mathrm{p}) \to \mathrm{Diff}(\Sigma_\mathrm{p}, \mathrm{p})$. The image $\mathrm{Hol}(\mathcal{F}, \mathrm{L_0}, \Sigma_\mathrm{p}, \mathrm{p}) \subset \mathrm{Diff}(\Sigma_\mathrm{p}, \mathrm{p})$ of such homomorphism is called the *holonomy group* of the leaf L_0 with respect to Σ_p and p. By considering any parametrization $z\colon (\Sigma_p, p) \to (\mathbb{D}, 0)$ we may identify (in a non-canonical way) the holonomy group with a subgroup of $\mathrm{Diff}(\mathbb{C}, 0)$. It is clear from the construction that the maps in the holonomy group preserve the leaves of the foliation. Nevertheless, this property can be shared by a larger group that may therefore contain more information about the foliation in a neighborhood of the leaf. The *virtual holonomy group* of the leaf with respect to the transverse section Σ_p and base point p is defined as ([5], [8])

$$\mathrm{Hol}^{\mathrm{virt}}(\mathcal{F}, \Sigma_\mathrm{p}, \mathrm{p}) = \{\mathrm{f} \in \mathrm{Diff}(\Sigma_\mathrm{p}, \mathrm{p}) \big| \mathrm{L_z} = \mathrm{L_{f(z)}}, \forall \mathrm{z} \in (\Sigma_\mathrm{p}, \mathrm{p})\}.$$

The virtual holonomy group contains the holonomy group and consists of the map germs that preserve the leaves of the foliation.

Fix now a germ of holomorphic foliation with a singularity at the origin $0 \in \mathbb{C}^2$, with a representative $\mathcal{F}(U)$ as above. Let Γ be a separatrix of $\mathcal{F}$. By Newton-Puiseaux parametrization theorem, $\Gamma \setminus \{0\}$ is biholomorphic to a punctured disc $\mathbb{D}^* = \mathbb{D} \setminus \{0\}$. In particular, we may choose a loop $\gamma \in \Gamma \setminus \{0\}$ generating the (local) fundamental group $\pi_1(\Gamma \setminus \{0\})$. The corresponding holonomy map h_γ is defined in terms of a germ of complex diffeomorphism at the origin of a local disc Σ transverse to $\mathcal{F}$ and centered at a non-singular point $q \in \Gamma \setminus \{0\}$. This map is well-defined up to conjugacy by germs of holomorphic diffeomorphisms, and is generically referred to as *local holonomy* of the separatrix Γ. The connection between the dynamics of the leaves and the local holonomy is stated as follows:

Lemma 3.1. *Let $\mathcal{F}$ be a germ of a holomorphic foliation at the origin $0 \in \mathbb{C}^2$. Assume that $\mathcal{F}$ has only a finite number of separatrices and that there is a neighborhood V of the origin such that on V each leaf of the foliation is closed off the set of separatrices. Let $\Gamma \subset V$ be a separatrix of $\mathcal{F}\big|_V$, $p \in \Gamma \setminus \{0\}$ and Σ_p a small disc transverse to the foliation and centered at p.*

Then:

(1) *The orbits of the local holonomy of* Γ *and of the virtual holonomy group of* Γ *are closed off the origin.*
(2) *A leaf that accumulates at* Γ *properly and is a closed leaf in* V*, induces for the virtual holonomy group* $\mathrm{Hol}^{\mathrm{virt}}(\mathcal{F}, \Sigma_{\mathrm{p}}, \mathrm{p})$ *a pseudo-orbit which is closed.*

In what follows we consider the following situation: $\mathcal{F}$ is a foliation as in Theorem 1.2. We perform the reduction of singularities for $\mathcal{F}$ obtaining:

(1) A proper map $\sigma\colon \tilde{U} \to U$ which is a finite composition of quadratic blow-ups.
(2) A foliation $\tilde{\mathcal{F}} = \sigma^*(\mathcal{F})$ with only irreducible singularities of non-degenerate type.
(3) An invariant exceptional divisor $E(\mathcal{F}) = \sigma^{-1}(0) = \bigcup\limits_{j=1}^{r} D_j$.

Lemma 3.2. *Let* $q = D_i \cap D_j$ *be a (non-degenerate) corner singularity. Given small transverse discs* Σ_j *and* Σ_i *with* $\Sigma_j \cap D_j = \{q_j\}$ *and* $\Sigma_i \cap D_i = \{q_i\}$*, nonsingular points close enough to* q*, then we have: any local leaf of* $\tilde{\mathcal{F}}$ *that accumulates properly at the origin of* Σ_i *also accumulates properly at the origin of* Σ_j*.*

A combination of Lemmas 3.1 and 3.2 actually shows that:

Proposition 3.3. *Let* $\mathcal{F}$ *be as in Theorem 1.2. Then, all virtual holonomy groups* $\mathrm{Hol}^{\mathrm{virt}}(\tilde{\mathcal{F}}, \mathrm{D_j})$ *of the components of* $D_j \subset E(\mathcal{F})$ *are groups with closed orbits off the origin. If moreover* $\mathcal{F}$ *has a closed leaf arbitrarily close to the origin, then each virtual holonomy group* $\mathrm{Hol}^{\mathrm{virt}}(\tilde{\mathcal{F}}, \mathrm{D_j})$ *exhibits a closed pseudo-orbit arbitrarily close to the origin.*

4. Groups of complex diffeomorphisms

Let $\mathrm{Diff}(\mathbb{C}, 0)$ denote the group of germs at the origin $0 \in \mathbb{C}$ of holomorphic diffeomorphisms. It is a well-known result that a a finite group of germs of complex diffeomorphisms is analytically conjugate to a cyclic group generated by a rational rotation. We shall now study the connection between our dynamical hypothesis and the classification of the possible holonomy groups arising in the reduction of singularities. We start with the case of a sole irreducible singularity. This is done in what follows (cf. Lemma 6.1).

4.1. **Non-resonant maps and Pérez-Marco results.** A germ of a complex diffeomorphism f at the origin $0 \in \mathbb{C}$ writes $f(z) = e^{2\pi\sqrt{-1}\lambda} z + a_{k+1} z^{k+1} + \dots$. The linear part $f'(0) = e^{2\pi\sqrt{-1}\lambda}$ does not depend on the coordinate system. We shall say that the germ $f \in \mathrm{Diff}(\mathbb{C}, 0)$ is *resonant* if $\lambda \in \mathbb{Q}^*$. If $\lambda \notin \mathbb{R}$ then $|f'(0)| \neq 1$ and the germ is *hyperbolic*. In the hyperbolic case the diffeomorphism is *analytically linearizable*, i.e., conjugated to its linear part by a germ of a map ([2]). In particular, its dynamics is one of an attractor or of a repeller. If $|f'(0)| = 1$, then we have $f'(0) = e^{2\pi\sqrt{-1}\theta}$ for some $\theta \in \mathbb{R}$. If $f'(0)$ is a root of the unity (i.e., if $\lambda \in \mathbb{Q}$) then f is called *resonant* and the dynamics of f is well-known ([2, 3]). In particular, if f is not linearizable, the orbits are closed off the origin, but no orbit is closed. If $f'(0)$ is not a root of the unity then we have $\lambda \in \mathbb{R} \setminus \mathbb{Q}$. In this case we shall say that the diffeomorphism if *non-resonant*. Assume that the map is not analytically linearizable. Given a representative defined in an open connected subset $0 \in U \subset \mathbb{C}$ the *stable set* of f in U is defined by $K(U, f) = \bigcap\limits_{j=0}^{\infty} f^{-j}(U)$ According to Pérez-Marco [22, 23])). It is compact, connected and not reduced to $\{0\}$. Any point of $K(U, f) \setminus \{0\}$ is recurrent (that is, a limit point of its orbit). Moreover, *there is an orbit in* $K(U, f)$ *which*

*It is common to refer to a map as a non-resonant map in case $\lambda \in \mathbb{R} \setminus \mathbb{Q}$. This may cause some confusion in our current framework. That is why we only define the resonant maps. All other maps are non-resonant for us.

accumulates at the origin and no non-trivial orbit of f converges to the origin. Such a map f will also be referred to in this paper as a *Pérez-Marco* map germ.

4.2. **Groups with closed orbits off the origin.** We shall now study the case of groups modeling the holonomy and virtual holonomy groups appearing in the reduction of singularities. The following definition will be useful.

Definition 4.1. A group $G \subset \mathrm{Diff}(\mathbb{C}, 0)$ of germs of holomorphic diffeomorphisms will be called *resonant* if each map $g \in G$ is a resonant germ. This is equivalent to the fact that G has a set of generators consisting only of resonant maps.

Denote by $\xi \subset \mathbb{C}$ the subset of roots of the unity. Our main result is:

Proposition 4.2. *Let $G \subset \mathrm{Diff}(\mathbb{C}, 0)$ be a finitely generated subgroup such that pseudo-orbits are closed off the origin in any small neighborhood of the origin $0 \in \mathbb{C}$.*

Then we have the following possibilities:

1. *G is a finite cyclic group, generated by a rational rotation.*
2. *G is abelian analytically linearizable generated by a periodic rotation and a hyperbolic map.*
3. *G is resonant, either abelian or solvable non-abelian. In the non-abelian case G is formally conjugate to a subgroup of $\{(z \mapsto \frac{az}{(1+bz^k)^{\frac{1}{k}}})\,;\ a \in \xi, b \in \mathbb{C}\}$, for some $k \in \mathbb{N}$. In this case the subgroup $G_1 \subset G$ of maps tangent to the identity is discrete of the form $(z \mapsto \frac{z}{(1+\beta z^k)^{\frac{1}{k}}})\,;\ \beta \in \mathbb{C}$, where all the β belong to a set of type $\{n_1\beta_1 + n_2\beta_2; n_1, n_2 \in \mathbb{Z}\}$ for some $\beta_1, \beta_2 \in \mathbb{C}$.*

In particular, if G contains some non-resonant map, then it is as in (2).

Proof. By Nakai density theorem, the group G must be solvable. In particular, G is abelian or it is formally conjugate to a subgroup of the group $\mathbb{H}_k = \{(z \mapsto \frac{az}{(1+bz^k)^{\frac{1}{k}}})\,;\ a \neq 0, b \in \mathbb{C}\}$, for some $k \in \mathbb{N}$ ([10], [15]). Notice that $\mathbb{H}_k$ is a finite ramified covering of the group of homographies $\mathbb{H}_1$ by a map $z \mapsto z^k$ ([10]). If G is finite then G is as in (1) as it is well-known. Assume that G contains some hyperbolic diffeomorphism, say a map $f \in G$ whose multiplier is of the form $f'(0) = e^{2\pi\sqrt{-1}\alpha}$ where $\alpha \in \mathbb{C} \setminus \mathbb{R}$. In this case we claim that G is abelian. Indeed, assume that G is not abelian. Then G contains some nontrivial commutator and therefore some nontrivial flat element $g \in G$, $g = z + cz^\ell + \text{h.o.t.}$ for some $c \neq 0$. By what we have observed above there is a homography fixing the origin $T(z) = \frac{\lambda z}{1+\mu z}$ such that $(f(z))^k = T(z^k)$. From this we get $f'(0) = \lambda^{\frac{1}{k}}$. Since f is hyperbolic we have that $1 \neq \lambda = T'(0)$. Therefore T is conjugated to a linear map by another homography. Consequently, we may assume that $f(z) = f'(0).z$ and $g(z) = \frac{z}{(1+\beta z^k)^{\frac{1}{k}}}$. By a ramified covering map (ramified change of coordinates) $Z = \frac{1}{z^k}$ we consider the subgroup corresponding to $\langle f, g\rangle$ and which is generated by a homothety $(Z \mapsto \mu Z)$, with $|\mu| \neq 1$, and a translation $(Z \mapsto Z + \beta)$. It is well known that such a group has no orbit closed off the origin. The same then holds for the group G that contains the subgroup generated by f, g above, contradiction.

The above shows that in case G contains a hyperbolic map, it must be abelian, without flat elements. Since it contains a hyperbolic (analytically linearizable) map, the group G is analytically linearizable, so that it embeds as a subgroup of the multiplicative group $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$. Again, because G has orbits closed off the origin, G must then be generated by a hyperbolic map and a rational rotation (see the proof of Lemma 8 in [5] for a similar situation). The group G is then as in (2).

Now for the final part of the proposition, we may therefore proceed assuming that G contains no hyperbolic map. We claim:

Claim 4.3. *The group G contains no non-resonant map $f \in G$, i.e., there is no map $f \in G$ with multiplier $f'(0) = e^{2\pi\sqrt{-1}\theta}$ where $\theta \in \mathbb{R} \setminus \mathbb{Q}$.*

proof of the Claim. Assume by contradiction that there is $f \in G$ a nonresonant map. If f is analytically linearizable then no orbit is closed off the origin, indeed such orbits are dense on circles centered at the origin in some linearizing coordinates. Thus this case is excluded. Assume therefore that $f \in G$ is a Pérez-Marco map. In this case by Pérez-Marco result in Section 4.1 there is a pseudo-orbit which is not closed off the origin, contradiction. This case is also excluded then. □

Assume now that G is not abelian. Let us now conclude that the group is as in (3). Every map in the group G is resonant. We embed $G \hookrightarrow \mathbb{H}_k = \{(z \mapsto \frac{\alpha z}{(1+\beta z^k)^{\frac{1}{k}}})\,;\ \alpha \neq 0, \beta \in \mathbb{C}\}$. This embedding is analytic unless the group is *exceptional*, in which case it already has the desired form (cf. [10] page 460 Theorem 1, see also Example 5.7). Assume then that the embedding is analytic. Given any map $g \in G$ we write $g(z) = \frac{az}{(1+bz^k)^{\frac{1}{k}}} \in G$. Since g is resonant we have $a \in \xi$. Since G is solvable, the subgroup $G_1 \subset G$ of flat elements, is abelian and analytically conjugated to a group of the form $(z \mapsto \frac{z}{(1+\beta z^k)^{1/k}})\,;\ \beta \in \mathbb{C}$. In particular, G_1 acts like a group of translations in the line $\mathbb{C}$. Since the orbits of G are closed orbits off the origin, we conclude that G_1 must be discrete so that all the β belong to a set of type $\{n_1\beta_1 + n_2\beta_2; n_1, n_2 \in \mathbb{Z}\}$ for some $\beta_1, \beta_2 \in \mathbb{C}$. This shows that G is as in (3). □

From the proof of Proposition 4.2 we actually get:

Corollary 4.4. *Let $G \subset \mathrm{Diff}(\mathbb{C}, 0)$ be a (not necessarily finitely generated) subgroup such that pseudo-orbits are closed off the origin in any small neighborhood of the origin $0 \in \mathbb{C}$. Then:*

1. *Any finitely generated subgroup $H \subset G$ with a non-trivial closed pseudo-orbit is finite.*
2. *If the group G contains a map which is not a resonant map then G is abelian linearizable generated by a hyperbolic attractor and a periodic rotation.*

Proof. We apply Proposition 4.2. If a subgroup $H \subset G$ contains a non-trivial closed pseudo-orbit then it cannot contain any flat element (i.e., any element tangent to the identity). In particular, H is abelian and its resonant maps are periodic. Moreover, there are no non-resonant maps: a non-resonant map $f \in H$ is of the form $f(z) = e^{2\pi i\lambda}z + a_{k+1}z^{k+1} + \dots$ with $\lambda \notin \mathbb{Q}$. If $\lambda \in \mathbb{C} \setminus \mathbb{R}$ then f is hyperbolic and linearizable. This map cannot have a finite orbit off the origin. If $\lambda \in \mathbb{R} \setminus \mathbb{Q}$ then by the proof of Proposition 4.2 we know that f cannot have all its orbits closed off the origin. We conclude that H is abelian consisting only of periodic maps. If H is finitely generated then it is finite. This proves the first part of the lemma. Let us now assume that G contains some map $f \in G$ which is non-resonant. This map is necessarily hyperbolic as we have seen above. But then G is abelian by Proposition 4.2 because in all other cases the group G is resonant. Applying the result of this same proposition we conclude that G is generated by g and some rational rotation. □

5. Examples

In this section we perform a construction and give some examples related to our main results. We also discuss some possible extensions and a related question.

Example 5.1. We shall now construct an example of a fully-resonant foliation $\mathcal{F}$ with closed leaves off the origin, non-dicritical and a generalized curve, but without a holomorphic first integral[†]. Fixed $a \in \mathbb{C} \setminus \{0\}$ we consider the subgroup $G \subset \mathbb{H}_2$ of maps of the form $z \mapsto \frac{\xi z}{\sqrt{1+naz^2}}$ where $n \in \mathbb{Z}$ and $\xi^4 = 1$. The group G has discrete pseudo-orbits off the origin, indeed, it is generated by the periodic maps $f(z) = \frac{iz}{\sqrt{1+az^2}}$ and $g(z) = iz$, where $i^2 = -1$. The group G is finitely generated by periodic maps (f has order 4 and g has order 2), but it has infinite order because $g \circ f(z) = \frac{-z}{\sqrt{1+az^2}}$.

The map f is conjugate to the local holonomy map of the separatrix $(y = 0)$ of a linearizable saddle-type singularity q_f with a holomorphic first integral, say of the form $xdy + 4ydx = 0$. Similarly g is conjugate to the holonomy of a separatrix $(y = 0)$ of a linearizable saddle singularity q_g with holomorphic first integral, of the form $xdy + 2ydx = 0$. Finally, the map $h = (g \circ f)^{-1}$ is conjugate to the holonomy of a separatrix $(y = 0)$ of a non-linearizable resonant saddle-type singularity q_h of the form $\omega_{k,\ell} = kxdy + \ell y(1 + \frac{\sqrt{-1}}{2\pi} x^\ell y^k)dx = 0$ where $\ell = 1$ and $k = 2$.

According to [14] we can construct a germ of a holomorphic foliation $\mathcal{F}$ at the origin $0 \in \mathbb{C}^2$, having three separatrices contained in lines, and which can be reduced with a single blowing-up at the origin. The blow-up foliation $\tilde{\mathcal{F}}(1)$ then has exactly three singularities in the invariant projective line $E(\mathcal{F})(1)$, and the holonomy group of the leaf $L_0 = E(1) \setminus \operatorname{sing}(\tilde{\mathcal{F}}(1))$ is conjugated to the group generated by f, g and $h = (g \circ f)^{-1}$, which is the group G. The singularities of $\tilde{\mathcal{F}}(1)$ are locally conjugated to q_f, q_g and q_h with the above mentioned separatrices contained in the exceptional divisor. All the dynamics of the foliation $\mathcal{F}$ is then described by its projective holonomy, i.e., by the holonomy of the leaf L_0 of the blow-up foliation $\tilde{\mathcal{F}}(1)$. In particular, $\mathcal{F}$ has closed leaves off the set of separatrices. Nevertheless, because group G is not abelian, $\mathcal{F}$ is not given by a closed meromorphic one-form. The foliation admits a Liouvillian first integral. Indeed, the group G embeds into $\mathbb{H}_2$, $\mathcal{F}$ is non-dicritical reduced with a single blow-up and it is a generalized curve ([29] Chapter I, §5, pages 185-188 or [21]). This is also proved as follows: There is a system of coordinate charts $\{U_j, (x_j, y_j)\}_{j \in J}$ covering a neighborhood of L_0 in the blow-up $\tilde{\mathbb{C}}^2_0$, such that:

- $E(\mathcal{F})(1) \cap U_j = L_0 \cap U_j \subset \{y_j = 0\}$.
- On each open subset U_j the blow-up foliation $\tilde{\mathcal{F}}(1)$ is given by $dy_j = 0$.
- If $U_i \cap U_j \neq \emptyset$ then $U_i \cap U_j$ is connected and in this intersection we have $y_j = \phi_{ij}(y_i)$ for some map $\phi_{ij} \in \mathbb{H}_2$.

Then we can write on each U_j the lifted one-form $\tilde{\omega} = \pi^*(\omega)$ as $\tilde{\omega}\big|_{U_j} = g_j dy_j$ for some meromorphic function g_j on U_j. Then we define $\tilde{\eta}$ on each U_j by $\tilde{\eta}\big|_{U_j} = 2\frac{dy_j}{y_j} + \frac{dg_j}{g_j}$. The extension of $\tilde{\eta}$ to the singularities q_f, q_g and q_h is then proved as in [8] or else [26]. This shows the existence of a closed meromorphic one-form $\tilde{\eta}$ in a neighborhood of the projective line $E(\mathcal{F})(1)$ in the space $\tilde{\mathbb{C}}^2_0$. This form satisfies $d\tilde{\omega} = \tilde{\eta} \wedge \tilde{\omega}$. Projecting this one-form into a one-form η in a neighborhood of the origin $0 \in \mathbb{C}^2$ we get a generalized integrating factor for ω. Thus $\mathcal{F}$ admits a Liouvillian first integral. Another (much more general) way of constructing the form η is given in [21] and it is based on the notion of symmetry for the group G.

Notice that in Example 5.1 above, one of the singularities has a non-periodic holonomy. This seems to be an unavoidable situation if one looks for groups which are not finite, but with closed pseudo-orbits off the origin as projective holonomy groups. This fact together with Theorem 1.3 in [27] and Theorem 1.1 in [9], suggests the following question:

Question 5.2. *Given a germ of a foliation $\mathcal{F}$ at $0 \in \mathbb{C}^2$ such that:*

1. *$\mathcal{F}$ is a non-dicritical generalized curve.*
2. *The leaves of $\mathcal{F}$ are closed off the separatrices.*

[†]I am grateful to the anonymous referee for showing me Example 5.1.

(3) *Each separatrix has a periodic local holonomy map.*

Does $\mathcal{F}$ admit a holomorphic first integral?

Example 5.3. This example suggests the possibility of extending the conclusion of Theorem 1.2 for singularities which are not generalized curves. We consider a germ of a saddle-node singularity $\mathcal{F}$, given by $xdy-y^{k+1}dx+\ldots=0$ at $0\in\mathbb{C}^2$. According to [18] there is a formal diffeomorphism $\hat{\phi}\in\hat{\text{Diff}}(\mathbb{C}^2,0)$ such that $\phi^*(\mathcal{F})$ is given by $\mathcal{S}_{k,a}: x(1+ay^k)dy-y^{k+1}dx=0$, for some $a\in\mathbb{C}$. The *formal model* $\mathcal{S}_{k,a}$ admits the Liouvillian first integral given by the generalized integrating factor $\eta=d\log(xy^{k+1})$. In particular, the saddle-node $\mathcal{F}$ admits a formal Liouvillian first integral. An example with closed leaves off the set of separatrices is given by $\omega=xdy-y^2dx=0$ at the origin $0\in\mathbb{C}^2$. Integration of $\Omega=\frac{1}{xy^2}\omega$ gives the first integral $f=xe^{\frac{-1}{y}}$. The leaves are closed off the strong separatrix $(y=0)$.

Example 5.4. This example is related to Question 5.2 above formulated. We construct a germ of a foliation $\mathcal{F}$ at $0\in\mathbb{C}^2$ such that:

(1) $\mathcal{F}$ is non-dicritical.
(2) The leaves of $\mathcal{F}$ are closed off the separatrices.
(3) Each separatrix has a periodic local holonomy map.
(4) $\mathcal{F}$ does not admit a holomorphic first integral.

Nevertheless:

(4) $\mathcal{F}$ is not a generalized curve.

We consider the subgroup $G\subset\text{Diff}(\mathbb{C},0)$ generated by $f(z)=\frac{z}{1-z}$ and $g(z)=-z$. This group is solvable, finite discrete pseudo-orbits off the origin. Indeed, it leaves invariant the function $\varphi(z)=\cos(\frac{2\pi}{z})$. We show that this group corresponds to the holonomy group of the projective line of the blowing-up of a non-dicritical singularity germ $\mathcal{F}$ at the origin $0\in\mathbb{C}^2$. Indeed, we first consider the map $h=f\circ g$, i.e., $h(z)=\frac{-z}{1+z}$. This is a periodic map since $h\circ h=\text{Id}$. Thus, we have $f\circ g\circ h=\text{Id}$. Moreover, each diffeomorphism above corresponds to the holonomy of a germ of irreducible singularity as follows:

- f is conjugate to the map $z\mapsto\frac{z}{1+2\pi z}$, which is the holonomy map of the strong separatrix $(y=0)$ of the saddle-node $q_f: xdy-y^2dx=0$, evaluated at the transverse disc $\Sigma:(x=1)$.
- $g(z)=-z$ is the holonomy of the separatrix $(y=0)$ of the singularity with holomorphic first integral $q_g: xy^2$.
- h is also the holonomy of a separatrix of a singularity with first integral $q_h: xy^2$.

Then, according to [14] we can construct a germ of a holomorphic foliation at the origin $0\in\mathbb{C}^2$, having three separatrices, and which can be reduced with a single blowing-up at the origin. The foliation $\tilde{\mathcal{F}}(1)$ then has exactly three singularities in the invariant projective line $E(\mathcal{F})(1)$, and the holonomy group of the leaf $L_0=E(\mathcal{F})(1)\setminus\text{sing}(\tilde{\mathcal{F}}(1))$ is conjugated to the group generated by f,g and h, which is the group G. The singularities of $\tilde{\mathcal{F}}(1)$ are locally conjugated to q_f,q_g and q_h. In particular, the saddle-node has its strong manifold contained in the projective line $E(\mathcal{F})(1)$ and the separatrix associated to this singularity at 0 is the central manifold, which has trivial holonomy map. The foliation $\mathcal{F}$ then has closed leaves off the set of separatrices, and periodic holonomy for each of its separatrices. Nevertheless, it does not admit a holomorphic first integral (it is not a generalized curve).

Example 5.5 (resonant singularities cf. [17])**.** According to Martinet-Ramis ([17]) a resonant non-linearizable singularity is formally isomorphic to an unique equation

$$\omega_{p/q,k,\lambda}:=p(1+(\lambda-1)u^k)ydx+q(1+\lambda u^k)xdy,$$

where $p, q, k \in \mathbb{N}, \lambda \in \mathbb{C}$ and $u := x^p y^q$. Moreover $p/q, k, \lambda$ are the formal invariants of the equation. By introducing integral numbers $n, m \in \mathbb{Z}$ such that $mp - nq = 1$ we can rewrite $\omega_{p/q,k,\lambda} = (1+(\lambda - mp)u^k)(pydx + qxdy) + pqu^k(nydx + mxdy)$. This last expression admits the integrating factor $h_{p/q,k,\lambda} = pqxyu^k$, this means that the one-form $\frac{1}{h_{p/q,k,\lambda}}\omega_{p/q,k,\lambda} := \Omega_{p/q,k,\lambda}$ is closed and meromorphic, with poles of order $kp+1$ in $(x=0)$ and $kq+1$ in $(y=0)$. In particular we can state:

Claim 5.6. *There is a single formal meromorphic closed one-form η with simple poles in $(y=0)$ such that $d\omega_{p/q,k,\lambda} = \eta \wedge \omega_{p/q,k,\lambda}$. This form is $\eta = dh_{p/q,k,\lambda}/h_{p/q,k,\lambda}$.*

Proof. Indeed, since $\Omega_{p/q,k,\lambda}$ is closed we conclude that $\eta_0 := dh_{p/q,k,\lambda}/h_{p/q,k,\lambda}$ satisfies the equation $d\omega_{p/q,k,\lambda} = \eta \wedge \omega_{p/q,k,\lambda}$. Now assume that ω is a closed meromorphic formal one-form as in the statement. We have $\eta - \eta_0 = g.\Omega_{p/q,k,\lambda}$ for some meromorphic function g such that $dg \wedge \Omega_{p/q,k,\lambda} = 0$. If g is not constant then $\Omega_{p/q,k,\lambda}$ admits a formal meromorphic first integral. This is not possible, because it does not admit a holomorphic first integral (see for instance [16]). Therefore g must be constant. Because both η and η_0 have simple poles, this implies that $g\Omega_{p/q,k,\lambda}$ has simple poles, therefore $g = 0$. □

Example 5.7 (exceptional case)**.** According to [10] a subgroup $G \subset \text{Diff}(\mathbb{C}, 0)$ is called *exceptional* if it is formally conjugated to a group $G_{\xi,k}, 0 < k \in \mathbb{N}, \xi \in \mathbb{C}$, generated by the maps $f_\xi : z \mapsto \xi z$ and $g_k : z \mapsto \frac{z}{(1-kz^k)^{\frac{1}{k}}}$, with $\xi^k = -1$ and $(1)^{\frac{1}{k}} = 1$. In particular an exceptional group is a solvable non-abelian group, formally conjugated to a discrete subgroup of $\mathbb{H}_k = \{(z \mapsto \frac{az}{(1+bz^k)^{\frac{1}{k}}});\ a \neq 0, b \in \mathbb{C}\}$. A non-exceptional group is *formally rigid* (cf. [10] Theorem 1 page 460)[‡]. Moreover we have:

Any non-abelian solvable subgroup $G \subset \text{Diff}(\mathbb{C}, 0)$ is formally conjugated to a subgroup of some $\mathbb{H}_k$, and this conjugation is analytic if G is not exceptional ([10],[15]).

Thus, in our Proposition 4.2 the only possibility for the group G to be not analytically conjugated to a subgroup of some $\mathbb{H}_k$ is that either G is abelian, or G is exceptional, i.e., formally equivalent to some $G_{\xi,k}$. In the exceptional case the group leaves invariant the formal function $\hat{\phi}(z) = \cos(\frac{2\pi}{z^k})$. We now extend the notion of exceptionality to germs of foliations:

Definition 5.8. A germ of a non-dicritical generalized curve $\mathcal{F}$ at $0 \in \mathbb{C}^2$ will be called *solvable exceptional* if every virtual holonomy group in the reduction of singularities of $\mathcal{F}$ is solvable (possibly abelian), and at least one virtual holonomy is solvable exceptional.

Concrete examples of non-formally rigid exceptional groups are found in [10] and [19], associated to certain cusp singularities. By a result due to Pérez-Marco and Yoccoz [24] any germ of a complex diffeomorphism $f \in \text{Diff}(\mathbb{C}, 0)$ is conjugate to the local holonomy of a separatrix associated to a germ of a non-degenerate holomorphic foliation $\mathcal{F}(f) : xdy - \lambda ydx + \ldots = 0$, having two transverse separatrices. This completes previous results from Martinet-Ramis [17], by solving the "non-resonant" case. Adding to this the (local) synthesis result in [14] we conclude that:

Given an exceptional subgroup $G_{exc} \cong G_{\xi,k}$ there is a germ of a foliation $\mathcal{F}(G_{exc})$ at $0 \in \mathbb{C}^2$ such that:

- *$\mathcal{F}$ is a non-dicritical generalized curve, admitting a reduction with a single blow-up, and the exceptional divisor is an invariant projective line $E(\mathcal{F}) \cong \mathbb{P}^1$.*

[‡]The group G is *formally rigid* if given any formal conjugation with another group G' there is an analytic conjugation.

- *$\mathcal{F}$ exhibits three separatrices, all in general position.*
- *The (reduced) foliation $\tilde{\mathcal{F}} = \tilde{\mathcal{F}}(1)$ exhibits three singularities, all non-degenerate, say $\operatorname{sing}(\tilde{\mathcal{F}}) = \{\tilde{\mathrm{p}}_1, \tilde{\mathrm{p}}_2, \tilde{\mathrm{p}}_3\}$.*
- *The holonomy of the leaf $L_0 = E(\mathcal{F}) \setminus \{\tilde{p}_1, \tilde{p}_2, \tilde{p}_3\}$ is conjugate to the group G_{exc}.*

To the list of properties above we can add:

- Assume that G_{exc} is not formally rigid, more precisely, assume that the formal embedding $G_{exc} \subset \mathbb{H}_k$ cannot be analytic. Then the virtual holonomy $H^{\text{virt}} := \operatorname{Hol}^{\text{virt}}(\tilde{\mathcal{F}}, \mathrm{L}_0)$ of the leaf L_0 of $\tilde{\mathcal{F}}$ is conjugate to G_{exc}.

Indeed, H^{virt} contains G_{exc} and it is also solvable with closed orbits off the origin. If H^{virt} contains properly G_{exc} then H^{virt} is not exceptional, therefore it admits an analytic embedding into some $\mathbb{H}_k$. This embedding gives an analytic embedding of G_{exc} on $\mathbb{H}_k$.

Thus, under the above non-formal rigidity condition we can state:

- The virtual holonomy of the leaf $L_0 = E(\mathcal{F}) \setminus \{\tilde{p}_1, \tilde{p}_2, \tilde{p}_3\}$ is conjugate to the group G_{exc}.

Using the above and material in the Appendix § 9 we can state:

Proposition 5.9. *Let $\mathcal{F}$ be a germ of a solvable exceptional foliation at $0 \in \mathbb{C}^2$. Then $\mathcal{F}$ admits a formal first integral of Liouvillian type $\hat{\Phi}$. This first integral admits a transversely formal development along the separatrices of $\mathcal{F}$. Given a separatrix Γ and a transverse disc Σ to $\mathcal{F}$ and Γ, the restriction $\hat{\Phi}\big|_{\Gamma}$ can be written as $\cos(\frac{2\pi}{x^k})$ in suitable formal coordinates x, for some $k \in \mathbb{N}$.*

6. The irreducible case

Let us consider a germ of a holomorphic foliation $\mathcal{F}$ at the origin $0 \in \mathbb{C}^2$, a germ of an irreducible non-degenerate singularity. In suitable local coordinates we can write $\mathcal{F}$ as given by

$$x(1 + A(x,y))dy - \lambda y(1 + B(x,y))dx = 0,$$

for some holomorphic $A(x,y)$, $B(x,y)$ with $0 \neq \lambda \in \mathbb{C} \setminus \mathbb{Q}_+$, $A(0,0) = B(0,0) = 0$. In the normal form above, the separatrices are the coordinate axes. Let us denote by f the holonomy map (its class up to holomorphic conjugacy) of the separatrix $(y = 0)$. From the correspondence between the leaves of $\mathcal{F}$ and the orbits of f ([16, 17, 24]) and according to the well-known properties of f discussed in § 4.1 (see also [2, 3]) we conclude that the foliation $\mathcal{F}$ exhibits the following characteristics:

Lemma 6.1. *Let $\mathcal{F}$ be a germ of an irreducible non-degenerate singularity at the origin $0 \in \mathbb{C}^2$ as above. We have:*

1. *In the hyperbolic case and in the resonant non-linearizable case, $\lambda \in \mathbb{Q}_-$, all leaves of $\mathcal{F}$ are closed off the set of separatrices, no leaf is closed.*
2. *In the non-resonant (Siegel or Poincaré) case, $\lambda \in \mathbb{R} \setminus \mathbb{Q}$, $\mathcal{F}$ has always some leaves which are recurrent. Moreover, no leaf converges only to the set of separatrices, therefore if a leaf is closed off the set of separatrices then it is already a closed leaf.*

Proof. If the singularity is in the Poincaré domain then, since it is not a resonance (because $\lambda, 1/\lambda \notin \mathbb{N}$) it is analytically linearizable. We may therefore choose local coordinates (x,y) on $(\mathbb{C}^2, 0)$ such that the germ writes as $xdy - \lambda ydx = 0$. The holonomy of one of the coordinate axes with respect to a small disc $\Sigma : \{x = a\}$ is given by $h(y) = \exp(2\pi\sqrt{-1}\lambda)y$. Suppose that λ is irrational then the map h is an irrational rotation, and the leaves (not contained in the set of separatrices) are recurrent, therefore not closed off the set of separatrices.

□

7. Fully resonant singularities

The following notion is useful in our framework.

Definition 7.1 (fully resonant)**.** A germ of a generalized curve $\mathcal{F}$ at the origin $0 \in \mathbb{C}^2$ will be called *fully resonant* if every singularity arising in the reduction of singularities is a resonant singularity.

Lemma 7.2. *Let $\mathcal{F}$ be a germ of a non-dicritical generalized curve in a neighborhood of the origin $0 \in \mathbb{C}^2$. Suppose that for some representative $\mathcal{F}_U$ of $\mathcal{F}$ defined in a neighborhood U of the origin, all leaves are closed off the set of separatrices. Then we have two possibilities:*
(i) *$\mathcal{F}$ is a fully-resonant generalized curve.*
(ii) *The reduction of singularities of $\mathcal{F}$ exhibits some hyperbolic singularity, all the final singularities are linearizable. Moreover, given any separatrix Γ through the origin, and a transverse disc Σ meeting Γ at a point $q \neq 0$, the virtual holonomy group $\mathrm{Hol}^{\mathrm{virt}}(\mathcal{F}, \Sigma, \mathrm{q})$ contains a hyperbolic map. In particular, it is an abelian linearizable group generated by a hyperbolic map and a periodic map.*

Proof. We proceed by induction on the number $r \in \{0, 1, 2, ...\}$ of blowing-ups in the reduction of singularities for the germ $\mathcal{F}$.
Case 1. ($r = 0$). In this case the singularity is already irreducible. The result follows from Lemma 6.1.
Case 2 (Induction step). Assume that the result is proved for foliation germs that admit a reduction of singularities with a number of blowing-ups less greater than or equal to r. Suppose that the fixed germ $\mathcal{F}$ admits a reduction of singularities consisting of $r+1$ blowing-ups. Then we perform a first blow-up $\sigma_1 \colon \tilde{U}(1) \to U$ at the origin and obtain a lifted foliation $\tilde{\mathcal{F}}(1) = \sigma_1^*(\mathcal{F})$ with (first) exceptional divisor $E(\mathcal{F})(1) = \sigma_1^{-1}(0)$ consisting of a single embedded invariant projective line in $\tilde{U}(1)$ (by hypothesis the exceptional divisor is invariant by $\tilde{\mathcal{F}}(1)$). Given a leaf L of $\mathcal{F}$ in U we denote by $\tilde{L}(1)$ the lifting $\tilde{L}(1) = \sigma_1^{-1}(L)$ of L to $\tilde{U}(1)$ by the map $\sigma_1 \colon \tilde{U}(1) \to U$. Now, if a leaf L of $\mathcal{F}$ in U is closed in $U \setminus \mathrm{sep}(\mathcal{F}, \mathrm{U})$, then its lift $\tilde{L}(1)$ is closed in $\tilde{U}(1) \setminus \mathrm{sep}(\tilde{\mathcal{F}}(1), \tilde{\mathrm{U}}(1))$ (notice that for each singularity $\tilde{p} \in \mathrm{sing}(\tilde{\mathcal{F}}(1)) \subset \mathrm{E}(1)$ the set of local separatrices of $\tilde{\mathcal{F}}(1)$ through $\tilde{p}$ is formed by $E(\mathcal{F})(1)$ union the local branches through $\tilde{p}$, of the strict transform by $\sigma(1)$ of $\mathrm{sep}(\mathcal{F}, \mathrm{U})$). Given a singularity $\tilde{p} \in \mathrm{sing}(\tilde{\mathcal{F}}(1)) \subset \mathrm{E}(1)$ of $\tilde{\mathcal{F}}(1)$, since the blow-up map is proper, we can conclude that for any small enough neighborhood $\tilde{W}_{\tilde{p}}$ of $\tilde{p}$ in $\tilde{U}(1)$, a leaf $\tilde{L}_0$ of the restriction $\tilde{\mathcal{F}}(1)\big|_{\tilde{W}_{\tilde{p}}}$ is closed in $\tilde{W}_{\tilde{p}} \setminus \mathrm{sep}(\tilde{\mathcal{F}}(1), \tilde{\mathrm{p}})$ provided that it projects into a piece of leaf $\sigma_1(\tilde{L}_0)$ which is contained in a leaf L of $\mathcal{F}$ that is closed in $U \setminus \mathrm{sep}(\mathcal{F}, \mathrm{U})$. Furthermore, since the blow-up map defines a biholomorphism between $\mathbb{C}^2 \setminus \{0\}$ and the complement of the exceptional divisor $\tilde{\mathbb{C}}_0^2 \setminus E(1)$, we conclude that:

The leaves of $\tilde{\mathcal{F}}(1)\big|_{\tilde{W}_{\tilde{p}}}$ are closed off the set of local separatrices of $\tilde{\mathcal{F}}(1)$ through $\tilde{p}$. Thus, by the induction hypothesis, each singularity $\tilde{p} \in \mathrm{sing}(\tilde{\mathcal{F}}(1))$ in the first blow-up is fully-resonant or its reduction of singularities exhibits some hyperbolic singularity, all the final singularities are linearizable. Moreover, given any separatrix $\tilde{\Gamma}_{\tilde{p}}$ through this singularity, and a transverse disc $\tilde{\Sigma}_{\tilde{p}}$ meeting $\tilde{\Gamma}_{\tilde{p}}$ at a point $\tilde{p} \neq \tilde{q} = \tilde{\Sigma}_{\tilde{p}} \cap \Gamma_{\tilde{p}}$, the virtual holonomy group $\mathrm{Hol}^{\mathrm{virt}}(\tilde{\mathcal{F}}(1), \Sigma_{\tilde{\mathrm{p}}}, \tilde{\mathrm{q}})$ is an abelian linearizable group generated by a hyperbolic map and a periodic map.

We have then two possibilities:
(a) All singularities in the first blow-up are fully-resonant. In this case, the original singularity is fully-resonant.

(b) Some singularity $\tilde{p} \in \mathrm{sing}(\tilde{\mathcal{F}}(1))$ in the first blow-up is not fully-resonant.
We shall consider this second possibility:

Claim 7.3. *Given a singularity $\tilde{p}_1 \in \operatorname{sing}(\tilde{\mathcal{F}}(1)) \subset \mathrm{E}(1)$ its reduction of singularities only produces linearizable singularities. Moreover, given any separatrix $\Gamma_{\tilde{p}_1}$ through $\tilde{p}_1$, and a transverse disc Σ meeting $\Gamma_{\tilde{p}_1}$ at a point $\tilde{p}_1 \neq \tilde{q}_1 = \Sigma \cap \Gamma_{\tilde{p}_1}$, the virtual holonomy group $\mathrm{Hol}^{\mathrm{virt}}(\tilde{\mathcal{F}}(1), \Sigma, \tilde{\mathrm{q}}_1)$ is an abelian linearizable group generated by a hyperbolic map and a periodic map.*

Proof. At first sight it may seem that this is a straightforward consequence of the Induction hypothesis. Nevertheless, it is not clear that we are dealing with a singularity which is not fully-resonant. Let us see how to study the case $\tilde{p}_1$ is fully resonant. Since $E(\mathcal{F})(1)$ is invariant, the hyperbolic element in the virtual holonomy of the separatrix through $\tilde{p}$ contained in $E(\mathcal{F})(1)$ induces a hyperbolic element on the virtual holonomy of the separatrix through $\tilde{p}_1$ contained in the exceptional divisor $E(\mathcal{F})(1)$. This is done as follows. Given two points $\tilde{q}$ and $\tilde{q}_1$, close to $\tilde{p}$ and $\tilde{p}_1$ respectively, and transverse discs Σ and Σ_1 meeting $E(\mathcal{F})(1)$ at these points respectively, we can choose a simple path $\alpha\colon [0,1] \to E(1) \setminus \operatorname{sing}(\tilde{\mathcal{F}}(1))$ from $\tilde{q}$ to $\tilde{q}_1$. The holonomy map $h_\alpha\colon (\Sigma, \tilde{q}) \to (\Sigma_1, \tilde{q}_1)$ associated to the path α (recall that $E(\mathcal{F})(1) \setminus \operatorname{sing}(\tilde{\mathcal{F}}(1))$ is a leaf of $\tilde{\mathcal{F}}(1)$), induces a natural morphism for the virtual holonomy groups

$$\alpha^*\colon \mathrm{Hol}^{\mathrm{virt}}(\tilde{\mathcal{F}}(1), \Sigma_1, \tilde{\mathrm{q}}_1) \to \mathrm{Hol}^{\mathrm{virt}}(\tilde{\mathcal{F}}(1), \Sigma, \tilde{\mathrm{q}}),$$

by $\alpha^* : h \mapsto h_\alpha^{-1} \circ h \circ h_\alpha$. Since $h_{\alpha^{-1}} = (h_\alpha)^{-1}$ in terms of holonomy maps, we conclude that the above morphism is actually an isomorphism between the virtual holonomy groups. Thus the virtual holonomy group $\mathrm{Hol}^{\mathrm{virt}}(\tilde{\mathcal{F}}(1)_{\tilde{\mathrm{p}}_1}, \mathrm{E}(\mathcal{F})(1), \Sigma_{\tilde{\mathrm{p}}_1}, \tilde{\mathrm{p}}_1)$ contains a hyperbolic map. Now we can use the Dulac correspondence in order to "pass" this hyperbolic map from the above virtual holonomy (of the separatrix contained in $E(\mathcal{F})(1)$) to the virtual holonomy of any separatrix of $\tilde{\mathcal{F}}_{\tilde{p}_1}$ (see the Appendix § 9). Indeed, because $\mathrm{Hol}^{\mathrm{virt}}(\tilde{\mathcal{F}}(1)_{\tilde{\mathrm{p}}_1}, \mathrm{E}(\mathcal{F})(1), \Sigma_{\tilde{\mathrm{q}}_1}, \tilde{\mathrm{q}}_1)$ contains a hyperbolic element, according to Proposition 4.2 it must be linearizable, generated by this hyperbolic map and a periodic map. This already implies that all local holonomies arising in the reduction of singularities of $\tilde{q}$ are linearizable, therefore the corresponding singularities are linearizable. Because the singularities are linearizable, the Dulac map allows to pass the hyperbolic attractor from $E(\mathcal{F})(1)$ to any separatrix through $\tilde{q}$, proving in this way that any separatrix through $\tilde{q}$ contains a hyperbolic attractor in its virtual holonomy group[§]. Thus, also the virtual holonomy group associated to the separatrix $\tilde{\Gamma}$ of $\tilde{\mathcal{F}}(1)$ through $\tilde{p}_1$ contains some hyperbolic map. $\square$

Now consider any separatrix Γ of $\mathcal{F}$ through the origin. Since the projective line $E(\mathcal{F})(1)$ in the first blow-up is invariant, the lift $\tilde{\Gamma}$ is the separatrix of some singularity $\tilde{p}$ of $\tilde{\mathcal{F}}(1)$. If $\mathcal{F}$ is not fully-resonant, then by the above, we conclude that the virtual holonomy group associated to this separatrix $\tilde{\Gamma}$ contains a hyperbolic map. Recall that the blow-up is a diffeomorphism off the origin and off the exceptional divisor, so that the maps in the virtual holonomy of $\tilde{\Gamma}$ induce maps in the disc Σ transverse to Γ in $\mathbb{C}^2$, but which are defined only in the punctured disc, i.e., off the origin. Nevertheless, since these projected maps are one-to-one, the classical Riemann extension theorem for bounded holomorphic maps shows that indeed such maps induce germs of diffeomorphisms defined in the disc Σ. These diffeomorphisms are the virtual holonomy maps of the separatrix Γ of $\tilde{\mathcal{F}}(1)$ evaluated at the transverse section Σ. Hence, by projecting the maps in $\mathrm{Hol}^{\mathrm{virt}}(\tilde{\mathcal{F}}(1), \Sigma, \tilde{\mathrm{q}})$ we obtain hyperbolic maps in this virtual holonomy group as stated. Now the Induction Principle applies to finish the proof of the lemma. $\square$

[§]The details of the construction of the Dulac map and the "passage" of (virtual) holonomy maps to virtual holonomy maps on adjacent components are are found in the Appendix § 9 and extensively explained in [8] and in [25] §2.3, pages 371 to 374.

8. Germs of foliations with closed leaves off the set of separatrices

In this section we prove Theorems 1.2, 1.3 and 1.4. We rely on Lemma 7.2 and on Lemmas 8.1 and 8.9 below.

Lemma 8.1. *Let $\mathcal{F}$ be a foliation germ as in Theorem 1.2. Then the following are equivalent:*

(1) *$\mathcal{F}$ admits a holomorphic first integral in some neighborhood of the origin.*
(2) *$\mathcal{F}$ is fully-resonant, has a closed leaf arbitrarily close to the origin and all singularities in the reduction of singularities are linearizable.*

Proof. Since (1) implies (2) is well-known (cf.[16],[27]), we prove the converse. Assume then that $\mathcal{F}$ (is as in Theorem 1.2 and moreover) has a closed leaf arbitrarily close to the origin and that all final singularities in the reduction process are resonant and linearizable. We must prove that $\mathcal{F}$ admits a holomorphic first integral.

We proceed by induction on the number $r \in \{0, 1, 2, ...\}$ of blow-ups in the reduction of singularities for the germ $\mathcal{F}$.

Case 1. ($r = 0$). In this case the singularity is already irreducible and resonant linearizable. Since it is resonant, it admits a holomorphic first integral.

Case 2. ($r - 1 \implies r$). Assume that the result is proved for foliation germs that admit a reduction of singularities with a number of blow-ups smaller than r. Suppose that the fixed germ $\mathcal{F}$ admits a reduction of singularities consisting of r blow-ups. Let U be a small connected neighborhood of the origin where the leaves of $\mathcal{F}$ are closed off the set of separatrices. We also assume that for U arbitrarily small the foliation $\mathcal{F}$ exhibits a closed leaf in U. Then we proceed as in the proof of Lemma 7.2 from where we import the notation. Thus we perform a first blow-up $\sigma_1 : \tilde{U}(1) \to U$ at the origin and obtain a lifted foliation $\tilde{\mathcal{F}}(1) = \sigma_1^*(\mathcal{F})$ with (first) exceptional divisor $E(\mathcal{F})(1) = \sigma_1^{-1}(0)$ consisting of a single embedded invariant projective line in $\tilde{U}(1)$ (by hypothesis the exceptional divisor is invariant by $\tilde{\mathcal{F}}(1)$). Given a leaf L of $\mathcal{F}$ in U we denote by $\tilde{L}(1)$ the lifting $\tilde{L}(1) = \sigma_1^{-1}(L)$ of L to $\tilde{U}(1)$ by the map $\sigma_1 : \tilde{U}(1) \to U$. Now, if a leaf L of $\mathcal{F}$ in U is closed in $U \setminus \mathrm{sep}(\mathcal{F}, \mathrm{U})$, then its lift $\tilde{L}(1)$ is closed in $\tilde{U}(1) \setminus \mathrm{sep}(\tilde{\mathcal{F}}(1), \tilde{\mathrm{U}}(1))$ (notice that for each singularity $\tilde{p} \in \mathrm{sing}(\tilde{\mathcal{F}}(1)) \subset \mathrm{E}(1)$ the set of local separatrices of $\tilde{\mathcal{F}}(1)$ through $\tilde{p}$ is formed by $E(\mathcal{F})(1)$ union the local branches through $\tilde{p}$, of the strict transform by $\sigma(1)$ of $\mathrm{sep}(\mathcal{F}, \mathrm{U})$).

Given a singularity $\tilde{p} \in \mathrm{sing}(\tilde{\mathcal{F}}) \subset \mathrm{E}$ of $\tilde{\mathcal{F}}$, since the blow-up map is proper, we can conclude that for any small enough neighborhood $\tilde{W}_{\tilde{p}}$ of $\tilde{p}$ in $\tilde{U}$, a leaf $\tilde{L}_0$ of the restriction $\tilde{\mathcal{F}}\big|_{\tilde{W}_{\tilde{p}}}$ is closed in $\tilde{W}_{\tilde{p}}$ provided that it projects into a piece of leaf $\pi(\tilde{L}_0)$ which is contained in a leaf L of $\mathcal{F}$ that is closed in U. Similarly, a leaf $\tilde{L}_0$ is closed in $\tilde{W}_{\tilde{p}} \setminus E$ provided that it projects into a piece of leaf $\pi(\tilde{L}_0)$ which is contained in a leaf L of $\mathcal{F}$ that is closed in $U \setminus \{0\}$. By the Induction hypothesis, each singularity $\tilde{p} \in \mathrm{sing}(\tilde{\mathcal{F}})$ admits a holomorphic first integral say $\tilde{f}_{\tilde{p}}$ defined in $\tilde{W}_{\tilde{p}}$ if this last is small enough. Now we analyze the holonomy of the leaf $L_0 := E(\mathcal{F}) \setminus \mathrm{sing}(\tilde{\mathcal{F}})$. Choose a regular point $\tilde{q} \in E_0$ and a small transverse disc Σ to L_0 centered at $\tilde{q}$. The corresponding holonomy group representation will be denoted by $H := \mathrm{Hol}(\tilde{\mathcal{F}}, \Sigma, \tilde{\mathrm{q}}) \subset \mathrm{Diff}(\Sigma, \tilde{\mathrm{q}})$. We know that this group is finitely generated and by the invariance of $E(\mathcal{F})$ and the above argumentation and Lemma 3.1, we know that actually, *the orbits of the holonomy group H of the exceptional divisor are closed off the origin, one of which is closed*. Applying Corollary 4.4 we conclude that the holonomy group is finite. Since the virtual holonomy group preserves the leaves of the foliation, the arguments above already show that *the orbits of the virtual holonomy group H^{virt} are closed off the origin, one of which is closed*. The problem is we still do not know that the virtual holonomy group is finitely generated. Nevertheless, from Corollary 4.4 we obtain:

Claim 8.2. *Any finitely generated subgroup H of the virtual holonomy group H^{virt} is a finite group.*

Let us then proceed as follows: given the singularities $\{\tilde{p}_1, ..., \tilde{p}_m\} = \mathrm{sing}(\tilde{\mathcal{F}}) \subset \mathrm{E}$, by induction hypothesis each singularity admits a local holomorphic first integral. Thus, there are small discs $D_j \subset E$, centered at the $\tilde{p}_j$ and such that in a neighborhood V_j of $\tilde{p}_j$ in the blow-up space $\tilde{\mathbb{C}_0^2}$, of product type $V_j = D_j \times \mathbb{D}_\epsilon$, we have a holomorphic first integral $g_j \colon V_j \to \mathbb{C}$, with $g_j(\tilde{p}_j) = 0$. Fix now a point $\tilde{p}_0 \in E \setminus \mathrm{sing}(\tilde{\mathcal{F}})$. Since $E(\mathcal{F})$ has the topology of the 2-sphere, we may choose a simply-connected domain $A_j \subset E$ such that $A_j \cap \{\tilde{p}_0, \tilde{p}_1, ..., \tilde{p}_m\} = \{\tilde{p}_0, \tilde{p}_j\}$, for every $j = 1, ..., m$. Since A_j is simply-connected, we may extend the local holomorphic first integral g_j to a holomorphic first integral $\tilde{g}_j$ for $\tilde{\mathcal{F}}$ in a neighborhood U_j of $D_j \cup A_j$, we may assume that U_j contains V_j. We observe that $\tilde{g}_j$ can be chosen to be *primitive*, i.e., it has connected fibers, therefore it cannot be written as $\tilde{g}_j = h^n$, for some holomorphic function h with $n \geq 2$. Now, given a local transverse section Σ_0 centered at $\tilde{p}_0$ and contained in U_j, we may introduce the *invariance group* of the restriction $g_j^0 := \tilde{g}_j\big|_{\Sigma_0}$ as the group

$$\mathrm{Inv(g_j^0) := \{f \in Diff(\Sigma_0, \tilde{p}_0),\, g_j^0 \circ f = g_j^0\}.}$$

In other words, the invariance group of g_j^0 is the group of germs of maps that preserve the fibers of g_j^0. Clearly $\mathrm{Inv(g_j^0)}$ is a finite (resonant) group ([16] Proposition 1.1. page 475). Let us now denote by $\mathrm{Inv}(\tilde{\mathcal{F}}, \Sigma_0) \subset \mathrm{Diff}(\Sigma_0, \tilde{\mathrm{p}}_0)$ the subgroup generated by the invariance groups $\mathrm{Inv(g_j^0), j = 1, ..., m}$. We call $\mathrm{Inv}(\tilde{\mathcal{F}}, \Sigma_0)$ the *global invariance group* of $\tilde{\mathcal{F}}$ with respect to $(\Sigma_0, \tilde{p}_0)$. Then, from the above we immediately obtain:

Claim 8.3. $\mathrm{Inv}(\tilde{\mathcal{F}}, \Sigma_0)$ *is a finite group.*

Proof. Indeed, first notice that $\mathrm{Inv}(\tilde{\mathcal{F}}, \Sigma_0)$ is finitely generated (by periodic maps). Since $\mathrm{Inv}(\tilde{\mathcal{F}}, \Sigma_0)$ preserves the leaves of $\tilde{\mathcal{F}}$ (recall that $\tilde{g}_j$ was chosen to be primitive) we have that $\mathrm{Inv}(\tilde{\mathcal{F}}, \Sigma_0) \subset \mathrm{Hol^{virt}}(\tilde{\mathcal{F}}, \Sigma_0, \tilde{\mathrm{p}}_0)$ and therefore by Corollary 4.4 $\mathrm{Inv}(\tilde{\mathcal{F}}, \Sigma_0)$ is a finite group. □

Notice that this global invariance group contains in a natural way the local invariance groups of the local first integrals g_j. Therefore, as observed in [16], once we have proved that the global invariance group $\mathrm{Inv}(\tilde{\mathcal{F}}, \Sigma_0)$ is finite, together with the fact that the singularities in $E(\mathcal{F})$ exhibit local holomorphic first integrals, we conclude as in [16] that the foliation $\tilde{\mathcal{F}}$ and therefore the foliation $\mathcal{F}$ has a holomorphic first integral.

□

As a consequence of the proof of the above lemma we have:

Lemma 8.4. *Let $\mathcal{F}$ be a foliation germ as in Theorem 1.2. Assume that $\mathcal{F}$ has a closed leaf arbitrarily close to the origin. Then $\mathcal{F}$ admits a holomorphic first integral in some neighborhood of the origin.*

Proof. The proof is based on Lemma 8.1 above and in the following claims:

Claim 8.5. *The foliation $\mathcal{F}$ is fully-resonant.*

Proof. Indeed, this is a consequence of the fact that any singularity in the reduction of singularities is such that the local induced foliation has closed leaves off the set of local separatrices and of Lemma 6.1. □

Claim 8.6. *Each virtual holonomy group in the reduction of singularities of $\mathcal{F}$ exhibits a closed pseudo-orbit arbitrarily close to the origin.*

Proof. This is a consequence of (what we have observed in the proof of) Proposition 3.3. □

Then, we conclude, as in the proof of Lemma 8.1, that each local holonomy map of a separatrix of a singularity in the reduction of singularities of $\mathcal{F}$, is a finite periodic map. This implies that all the singularities of the reduction of $\mathcal{F}$ are linearizable (and resonant). Applying now Lemma 8.1 we conclude. □

Lemma 8.7. *Let $\mathcal{F}$ be a foliation germ as in Theorem 1.2. Assume that some separatrix Γ of $\mathcal{F}$ contains some hyperbolic map in its virtual holonomy group. Then $\mathcal{F}$ is given by a closed meromorphic one-form with simple poles.*

Proof. We proceed by induction on the number $r \in \{0, 1, 2, ...\}$ of blowing-ups in the reduction of singularities for the germ $\mathcal{F}$.
Case 1. ($r = 0$). In this case the singularity is already irreducible. Since it is not a saddle-node it can be written as $xdy - \lambda ydx + \ldots = 0$ for some $\lambda \in \mathbb{C} \setminus \mathbb{Q}_+$. We claim:

Claim 8.8. *The singularity is not a resonant singularity, i.e., $\lambda \notin \mathbb{Q}$.*

Proof of the Claim. Assume that we have $\lambda = -n/m \in \mathbb{Q}_-$ for some $n, m \in \mathbb{N}$ with $< n, m >= 1$. In this case we have two possibilities.

(1) The singularity is analytically linearizable. In this case we can write $nxdy + mydx = 0$. Then we have a holomorphic first integral $f = x^m y^n$ and any virtual holonomy map must preserve the fibers of f. This implies that any virtual holonomy map is actually a finite periodic map. This case is therefore excluded.

(2) The singularity is not analytically linearizable. As we have seen in Example 5.5, by [17] the foliation is formally isomorphic to an unique equation

$$\omega_{p/q,k,\lambda} := p(1 + (\lambda - 1)u^k)ydx + q(1 + \lambda u^k)xdy,$$

where $p, q, k \in \mathbb{N}, \lambda \in \mathbb{C}$ and $u := x^p y^q$ and $mp - nq = 1$. We can rewrite

$$\omega_{p/q,k,\lambda} = (1 + (\lambda - mp)u^k)(pydx + qxdy) + pqu^k(nydx + mxdy).$$

This last expression admits the integrating factor $h_{p/q,k,\lambda} = pqxyu^k$, this means that the one-form $\frac{1}{h_{p/q,k,\lambda}}\omega_{p/q,k,\lambda} := \Omega_{p/q,k,\lambda}$ is closed and meromorphic, with poles of order $kp+1$ in $(x = 0)$ and $kq + 1$ in $(y = 0)$. Now, if there is a hyperbolic map in the virtual holonomy of one of the separatrices (given by the axes) then this map clearly forces the closed meromorphic one-form to have simple poles along that separatrix, which is not the case, contradiction.

□

Because the singularity is non-resonant we have $\lambda \in \mathbb{C} \setminus \mathbb{Q}$. We claim that this singularity is a hyperbolic singularity, i.e., $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Indeed, if $\lambda \in \mathbb{R} \setminus \mathbb{Q}$ then we have two possibilities.

(i) The singularity is analytically linearizable. In this case we may assume that it is of the form $xdy - \lambda ydx = 0$. Then, the leaves are not closed off the origin, because a typical leaf has as closure the three-dimensional manifold given by $|y||x|^{-\lambda} = c$ for some $c > 0$.

(ii) The singularity is not analytically linearizable. In this case we must have $\lambda \in \mathbb{R}_-$ and the foliation is in the so called *Siegel domain*. In particular, there exactly are two separatrices and we may assume that it is of the form $xdy - \lambda y(1 + A(x, y))dx =$ for some $A(x, y)$ holomorphic with $A(0, 0) = 0$. Such a singularity has a local holonomy map for the separatrix $(y = 0)$ of the form $f(y) = \exp(2\pi\lambda)y + \ldots$. In particular, such a holonomy map is not a resonant map. By Lemma 6.1 or also by the considerations in the proof of Proposition 4.2 we know that the only possibility compatible with the fact that the leaves of $\mathcal{F}$ are closed off the origin, is that f is a hyperbolic map, i.e., $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

We conclude that the singularity is hyperbolic and that any separatrix has a hyperbolic holonomy map. The singularity is linearizable as $xdy - \lambda ydx = 0$ in suitable local coordinates. In these coordinates the foliation is given by the closed one-form $\Omega_\lambda = \frac{dy}{y} - \lambda\frac{dx}{x}$.

Case 2 (Induction step). Assume that the result is proved for foliation germs that admit a reduction of singularities with a number of blowing-ups less greater than or equal to r. Suppose that the fixed germ $\mathcal{F}$ admits a reduction of singularities consisting of $r+1$ blowing-ups. Then we perform a first blow-up $\sigma_1\colon \tilde{U}(1) \to U$ at the origin and obtain a lifted foliation $\tilde{\mathcal{F}}(1) = \sigma_1^*(\mathcal{F})$ with (first) exceptional divisor $E(\mathcal{F})(1) = \sigma_1^{-1}(0)$ consisting of a single embedded invariant projective line in $\tilde{U}(1)$ (by hypothesis the exceptional divisor is invariant by $\tilde{\mathcal{F}}(1)$). Given a leaf L of $\mathcal{F}$ in U we denote by $\tilde{L}(1)$ the lifting $\tilde{L}(1) = \sigma_1^{-1}(L)$ of L to $\tilde{U}(1)$ by the map $\sigma_1\colon \tilde{U}(1) \to U$. By hypothesis $\mathcal{F}$ has some separatrix Γ containing a hyperbolic map in its virtual holonomy. Let then $\tilde{p} \in \operatorname{sing}(\tilde{\mathcal{F}}(1))$ be a singularity exhibiting some separatrix $\tilde{\Gamma}_{\tilde{p}} = \overline{\sigma_1^{-1}(\Gamma \setminus \{0\})}$ not contained in the projective line $E(\mathcal{F})(1)$ and having a hyperbolic map in its virtual holonomy. By the Induction hypothesis the germ $\tilde{\mathcal{F}}(1)_{\tilde{p}}$ induced by $\tilde{\mathcal{F}}(1)$ at $\tilde{p}$, is given by a simple poles closed meromorphic one-form say $\tilde{\Omega}_{\tilde{p}}$. Since $\tilde{E}(1)$ is invariant, it contains a separatrix of the germ $\tilde{\mathcal{F}}(1)_{\tilde{p}}$. Because of the form $\tilde{\Omega}_{\tilde{p}}$, all the separatrices of $\tilde{\mathcal{F}}(1)_{\tilde{p}}$ contain hyperbolic maps in their virtual holonomy groups. Therefore, the separatrix of $\tilde{\mathcal{F}}(1)_{\tilde{p}}$ contained in $E(\mathcal{F})(1)$, contains a hyperbolic map for its virtual holonomy group. Thanks to the invariance of $E(\mathcal{F})(1)$ for $\tilde{\mathcal{F}}(1)$ this implies that each singularity $\tilde{q}$ of $\tilde{\mathcal{F}}(1)$ in $E(\mathcal{F})(1)$ contains a hyperbolic map in the virtual holonomy of the corresponding separatrix contained in $E(\mathcal{F})(1)$. Then, again by Induction hypothesis, each singularity $\tilde{q} \in E(1) \cap \operatorname{sing}(\tilde{\mathcal{F}}(1))$ is given by a closed meromorphic one-form $\tilde{\Omega}_{\tilde{q}}$ having simple poles. Now we focus on the leaf $L_0 = E(1) \setminus \operatorname{sing}(\tilde{\mathcal{F}}(1))$ and on its virtual holonomy group, which we shall denote simply by $\operatorname{Hol}^{\mathrm{virt}}(\tilde{\mathcal{F}}(1), \mathrm{L}_0)$. This leaf contains therefore hyperbolic maps in its virtual holonomy group. In view of Proposition 4.2 the group $\operatorname{Hol}^{\mathrm{virt}}(\tilde{\mathcal{F}}(1), \mathrm{L}_0)$ is abelian linearizable. Using this and the well-known techniques from [5] we can construct a simple poles closed meromorphic one-form $\tilde{\Omega}$ in a neighborhood of $E(\mathcal{F})(1)$, which defines $\tilde{\mathcal{F}}(1)$. Projecting this one-form onto a neighborhood of the origin $0 \in \mathbb{C}^2$, we obtain a closed meromorphic one-form Ω with simple poles, defining $\mathcal{F}$. The lemma is proved by Induction. $\square$

Lemma 8.9. *Let $\mathcal{F}$ be a foliation germ as in Theorem 1.2. Assume that $\mathcal{F}$ is not fully resonant. Then:*

(1) *Each separatrix contains some hyperbolic map in its virtual holonomy group.*
(2) *$\mathcal{F}$ is given by a closed meromorphic one form with simple poles.*

Proof. This is essentially a direct consequence of the lemma above. The idea is the following. Since $\mathcal{F}$ is not fully-resonant, it contains some singularity which is not resonant. As in the proof of Lemma 8.7, this singularity must be hyperbolic. The local holonomies of the separatrices of this singularity then are hyperbolic maps, which induce hyperbolic maps on the virtual holonomy of each separatrix of the foliation. By Lemma 8.7 we conclude. $\square$

Lemma 8.10. *Let $\mathcal{F}$ be a foliation germ as in Theorem 1.2. Assume that $\mathcal{F}$ is fully resonant. Then $\mathcal{F}$ admits a formal Liouvillian first integral.*

Proof. First we recall that all the virtual holonomy groups in the reduction of $\mathcal{F}$ are groups with closed orbits off the origin. Then, according to Proposition 4.2 these groups are solvable. Moreover, by hypothesis, there are no saddle-nodes in the reduction of singularities and all the projective lines are invariant. Then, as already mentioned in Example 5.1, using the techniques from [26], [8] or the more general techniques from [21] we can construct a formal generalized integrating factor for $\mathcal{F}$. We give the detailed proof in the Appendix § 9. □

Proof of Theorem 1.2. Let $\mathcal{F}$ be a germ of a non-dicritical generalized curve at $0 \in \mathbb{C}^2$. Assume that the leaves of $\mathcal{F}$ are closed off the set of separatrices. By hypothesis, there is a neighborhood U of the origin where the leaves are all closed off the set of separatrices. According to Lemma 8.9 we have only two possibilities:

(1) $\mathcal{F}$ is fully resonant.
(2) $\mathcal{F}$ is contains some hyperbolic singularity in its reduction of singularities and:
  (a) Each separatrix contains some hyperbolic map in its virtual holonomy group.
  (b) $\mathcal{F}$ is given by a closed meromorphic one form with simple poles.

We study the different possibilities:

Possibility 1. The singularity is fully-resonant. In this case, by Lemma 8.10 $\mathcal{F}$ admits a formal Liouvillian first integral.

Possibility 2. The singularity is a generalized curve which is not fully-resonant. Moreover, we have:

(1) $\mathcal{F}$ is given by a closed formal meromorphic one form with simple poles,
(2) Given any separatrix Γ through the origin, and a transverse disc Σ meeting Γ at a point $q \neq 0$, the virtual holonomy group $\mathrm{Hol}^{\mathrm{virt}}(\mathcal{F}, \Sigma, \mathrm{q})$ is an abelian linearizable group generated by a hyperbolic map and a periodic map.

As in [5] (page 440, paragraph after the proof Lemma 8) we can conclude that $\mathcal{F}$ is indeed a holomorphic pull-back of a linear hyperbolic singularity $xdy - \lambda ydx = 0$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$. This ends the proof of Theorem 1.2. □

Proof of Theorem 1.3. According to Lemma 8.7 $\mathcal{F}$ is given by a closed meromorphic one-form with simple poles. The rest of the proof goes as in final part of the above proof of Theorem 1.2. □

Proof of Theorem 1.4. If we already know that $\mathcal{F}$ is a generalized curve then this is just the result of Lemma 8.4. Let us then prove that this is the case. Recall that, by hypothesis $\mathcal{F}$ is non-dicritical, its leaves are closed off the set of separatrices and $\mathcal{F}$ has a leaf which is closed on each small neighborhood of the origin. Assume that there is a saddle-node in the reduction of singularities of $\mathcal{F}$. Then the strong manifold of this saddle-node exhibits a non-trivial holonomy tangent to the identity, say of the form $z \mapsto z + a_{k+1}z^{k+1} + \dots$ for some $a_{k+1} \neq 0, k \in \mathbb{N}$. This map has no closed orbit. Because the exceptional divisor is invariant and connected, and thanks to Lemma 3.2, any given closed leaf must approach a saddle-node singularity by at least one of its separatrices. If it approaches by the strong separatrix then we have a contradiction with the above holonomy map dynamics. Therefore, the closed leaf must approach the saddle-node through the central separatrix. Nevertheless, thanks to the local description of the saddle-node, it is well-known that any leaf not contained in a separatrix and that accumulates properly at the central separatrix also accumulates properly at the strong separatrix. Therefore, again, we have a contradiction. This shows that the existence of a saddle-node is not possible under the additional hypothesis of existence of a closed leaf sufficiently close to the original singularity. Thus $\mathcal{F}$ is indeed a generalized curve. □

9. Appendix: Construction of generalized integrating factors

We shall now detail the construction of the formal generalized integrating factor indicated in the proof of Lemma 8.10. We shall adopt the notation of that section. We shall also denote by H_j (respectively, by $H_j^{\rm virt}$) the holonomy group (respectively, the virtual holonomy group) of the component D_j of the divisor $E(\mathcal{F})$, $j=1,...,r$, which is by hypothesis invariant. We also denote by $D_j^* = D_j \setminus \operatorname{sing}(\mathcal{F})$. The virtual holonomy group $H_j^{\rm virt}$ has closed pseudo-orbits off the origin. This group is therefore solvable in the terms of Proposition 4.2. Fixed a regular point $q_j \in D_j - \operatorname{sing}(\tilde{\mathcal{F}}) \cap \rm D_j$, a small transverse disk $\Sigma_j \cong \mathbb{D}$, $\Sigma_j \cap D_j = \{q_j\}$ we have holonomy and virtual holonomy identifications $\operatorname{Hol}(\tilde{\mathcal{F}}, \rm D_j, \Sigma_j) \cong H_j \subset \operatorname{Diff}(\mathbb{C},0)$ and $\operatorname{Hol}^{\rm virt}(\tilde{\mathcal{F}}, \rm D_j, \Sigma_j) \cong H_j^{\rm virt} \subset \operatorname{Diff}(\mathbb{C},0)$. We recall the following result from groups of germs of complex diffeomorphisms in dimension one ([10], [21]):

Lemma 9.1. *Let $H \subset \operatorname{Diff}(\mathbb{C},0)$ be a subgroup. Then:*

(1) *H is abelian $\Leftrightarrow$ there exists a formal vector field ξ in one complex variable which is H-invariant, i.e., $g * \hat{\xi} = \hat{\xi}$, $\forall\, g \in H$.*
(2) *H is solvable $\Leftrightarrow$ there is a formal vector field $\hat{\xi}$ in one complex variable which is H-projectively invariant, i.e., for each $g \in H$ we have $g * \hat{\xi} = c_g \cdot \hat{\xi}$ for some $c_g \in \mathbb{C}^*$.*

As a consequence we have the following possibilities for $H_j^{\rm virt}$:

(a) H_j is abelian $\Rightarrow$ there exists a formal vector field $\hat{\xi}_j$ in one complex variable $y_j \in \Sigma_j$, $y_j(\Sigma_j) = \mathbb{D}$, $y_j(q_j) = 0$, $\hat{\xi}_j$ writes in some formal coordinates $\hat{\xi}_j(\hat{z}) = \dfrac{\hat{z}^{k+1}}{1+a\hat{z}^k}\dfrac{d}{d\hat{z}}$ such that: (a*) $g * \hat{\xi}_j = \hat{\xi}_j$, $\forall\, g \in H_j$,
(b) H_j is solvable non abelian $\Rightarrow$ there exists a formal vector field $\hat{\xi}_j$ such that: (b*) $g * \hat{\xi}_j = c_j \cdot \hat{\xi}_j$, $c_g \in \mathbb{C}^*, \forall\, g \in H_j$ and $c_g \neq 1$ for some $g \in H_j$. The vector field $\hat{\xi}_j$ writes in some formal coordinate $\hat{z}$ as $\hat{\xi}_j(\hat{z}) = \hat{z}^{k+1}\dfrac{d}{d\hat{z}}$.

Definition 9.2 (normalizing coordinates)**.** Let $H \subset \operatorname{Diff}(\mathbb{C},0)$ be solvable and $\hat{\xi}$ a projectively invariant as in Lemma 9.1 above. The vector field $\hat{\xi}_j$ writes in some formal coordinate $\hat{z}$ as $\hat{\xi}_j(\hat{z}) = \dfrac{\hat{z}^{k+1}}{1+a\hat{z}^k}\dfrac{d}{d\hat{z}}$. Such coordinates are called *normalizing coordinates* for the group G.

Let ω be a holomorphic one-form defining $\mathcal{F}$ in a neighborhood $U \subset \mathbb{C}^2$ of the origin. Denote by $\tilde{\omega}$ the lift of ω by the reduction of singularities for $\mathcal{F}$, i.e., $\tilde{\omega} = \sigma^*(\omega)$ where $\sigma\colon \tilde{U} \to U$ is the morphism described in Section 2.

Lemma 9.3. *There exists a transversely formal 1-form $\hat{\eta}_j$ defined over D_j^* such that $d\tilde{\omega} = \hat{\eta}_j \wedge \tilde{\omega}$, $d\hat{\eta}_j = 0$, $\hat{\eta}_j$ has simple poles along D_j^* and along $(\tilde{\omega})_\infty \cup (\tilde{\omega})_0$.*

Moreover, if $C \subset (\tilde{\omega})_\infty \cup (\tilde{\omega})_0$ is an irreducible component with $C \cap D_j \neq \emptyset$, then either $\rm Res_C \hat{\eta}_j = -ord((\tilde{\omega})_\infty, C)$, or $\rm Res_C \hat{\eta}_j = ord(\tilde{\omega})_0$.

Proof. First we assume that H_j is abelian. We consider $\hat{\xi}_j$ as in (a) above. Condition (a*) allows as to extend $\hat{\xi}_j$ as a transversely formal global section $\hat{\tau}_j$ of the sheaf $\widehat{\operatorname{Sim}}(\mathcal{F}, D_j^*)$ of transversely formal symmetries associated to $\tilde{\mathcal{F}}$, over the open curve D_j^*. Indeed, this is just the usual *holonomy extension* of $\hat{\xi}$ as a constant vector field along the plaques of $\mathcal{F}$ near D_j^*. Then $\hat{h}_j = \tilde{\omega}(\hat{\tau}_j)$ is a transversely formal function defined over D_j^* and which satisfies $d\tilde{\omega} = \dfrac{d\hat{h}_j}{\hat{h}_j} \wedge \tilde{\omega}$ [21], so that we take $\hat{\eta}_j = \dfrac{d\hat{h}_j}{\hat{h}_j}$. This 1-form clearly satisfies the required properties.

Now we assume that H_j is solvable non abelian. We consider $\hat{\xi}_j$ as in (b). Condition (b*) allows the construction of a section $\hat{\tau}_j$ of the quotient sheaf $\widehat{\mathrm{Sim}}(\mathcal{F}, D_j^*)/\mathbb{C}^*$. Thus $\dfrac{d(\tilde{\omega}(\hat{\tau}_j))}{\tilde{\omega}(\hat{\tau}_j)} = \hat{\eta}_j$ is well-defined over D_j^* and has the required properties [21]. $\square$

Now we prove that $\hat{\eta}_j$ constructed in Lemma 9.3, extends to the singularities in $D_j \cap \mathrm{sing}(\tilde{\mathcal{F}})$. Let then $q_o \in \mathrm{sing}\tilde{\mathcal{F}} \cap \mathrm{D_j}$ be a singularity. If it is a corner, say $q_o = D_i \cap D_j$ is a corner then it has two separatrices, contained in D_i and D_j. Since q_o is not a saddle-node we have three distinct cases to consider:

(1) q_o admits a formal first integral. In this case by [16] q_o admits a holomorphic first integral so that $\tilde{\omega}$ admits a holomorphic integrating factor around q_o and q_o is analytically linearizable.

(2) q_o is non-resonant of the form $xdy - \lambda ydx + \text{h.o.t.} = 0$, $\lambda \notin \mathbb{Q}$: In this case the local holonomy around q_o is a non-periodic linear part so that H_j is analytically normalizable and we may assume that ($\widehat{\xi}_j$ and therefore) $\hat{\eta}_j$ is convergent.

(3) q_o is resonant not formally linearizable: In this case q_o admits the so called Martinet-Ramis formal normal forms [17]. In particular the 1-form $\tilde{\omega}$ admits a formal integrating factor $\hat{h}$ defined at q_o; that is, (*) $d\left(\dfrac{\tilde{\omega}}{\hat{h}}\right) = 0$ and $\hat{h}$ is a formal series at q_o. This equation (*) exhibits resommation properties for $\hat{h}$ so that by a Briot-Bouquet type argument [17],[18] $\hat{h}$ can be written $\hat{h}(x,y) = \sum\limits_{j=0}^{+\infty} a_j(x)y^j$, where $(x,y) \in U$ is a local coordinate centered at q_o, such that $D_j \cap U = \{y = 0\}$, $D_i \cap U = \{x = 0\}$, $a_j(x)$ is a holomorphic function converging in a small disk $\mathbb{D}_{q_o} \subset D_j$ centered at q_o, not depending on $j \in \mathbb{N}$.

Thus, in any of the three cases above, we conclude that there exists a transversely formal 1-form $\hat{\eta}_{q_o}$ defined over a small disk $q_o \in \mathbb{D}_{q_o} \subset D_j$ and with simple poles along the separatrices (so along D_i and D_j), such that $d\hat{\eta}_{q_o} = 0$ and $d\tilde{\omega} = \hat{\eta}_{q_o} \wedge \tilde{\omega}$. The difference $\hat{\eta}_j - \hat{\eta}_{q_o}$ writes therefore as $\hat{\eta}_j - \hat{\eta}_{q_o} = \hat{h} \cdot \tilde{\omega}$ for some transversely formal integrating factor $\hat{h}$ for $\tilde{\omega}$ (i.e., $d(\hat{h} \cdot \tilde{\omega}) = 0$) defined over the punctured disc $\mathbb{D}^*_{q_o} = \mathbb{D}_{q_o} \setminus \{q_o\}$.

Now we consider these three cases separately.

Case (1): There exists a local chart $(x,y) \in U$, $x(q_o) = y(q_o) = 0$ such that

$$\tilde{\omega}(x,y) = g(x,y)(nxdy + mydx)$$

for some $n, m \in \mathbb{N}^*$ and some holomorphic $g \in \mathcal{O}_2$. We consider the 1-form

$$\hat{\eta}_{q_o} = \frac{dg}{g} + \frac{d(xy)}{xy} = \frac{dg}{g} + \frac{dx}{x} + \frac{dy}{y},$$

which is meromorphic in U. Let also $\omega_o = n\dfrac{dy}{y} + \dfrac{dx}{x}$. Then we have $\hat{\eta}_j - \hat{\eta}_{q_o} = \hat{h} \cdot \tilde{\omega} = (\hat{h}xyg)\omega_o$ and since $d(\hat{h} \cdot \tilde{\omega}) = 0 = d\omega_o$ it follows that $d(\hat{h}xyg) \wedge \omega_o = 0$, that is, $\hat{f} = \hat{h}xyg$ is a transversely formal first integral for $\tilde{\mathcal{F}}$ over $\mathbb{D}^*_{q_o}$. Since $f_o = x^m y^n$ is already a *primitive* first integral for $\tilde{\mathcal{F}}$ around q_o (if we choose $\langle n, m \rangle = 1$) it follows that $\hat{f} = \hat{l}(f_o)$ for some one variable formal expression, that is, $\hat{f} = \hat{l}(x^m y^n)$ and since $\hat{f}$ is defined as a transversely formal expression over $\mathbb{D}^*_{q_o}$ which contains points of the form $(x, 0)$, $x \neq 0$ and since $x^m y^n = 0$ at these points, it follows that $\hat{l}$ is a formal series on the disk $\mathbb{D} \subset \mathbb{C}$ and therefore $\hat{f}$ extends as a transversely formal first integral along $\mathbb{D}_{q_o}$. It follows that ($\hat{h}$ and therefore) $\hat{\eta}_j$ extends as a transversely formal object to $\mathbb{D}_{q_o}$.

Case (2): There exists a formal linearization for $\tilde{\mathcal{F}}$ at q_o,

$$\tilde{\omega}(x,y) = g(x,y)(xdy - \lambda y(1+b(x,y))dx),$$

$b \in \mathcal{O}_2 \notin g$, $\lambda \in \mathbb{C}\backslash\mathbb{Q}$, $b(0,0) = 0$, (x,y) is a holomorphic chart and we can find a formal chart $(x,\widehat{Y})$ at q_o, with $\widehat{Y}(x,y) = \sum\limits_{j=1}^{+\infty} a_j(x)y^j$, $a_j(x)$ holomorphic in $\mathbb{D}_{q_o}$, $\forall\, j$, such that, $\tilde{\omega}(x,\widehat{Y}) = \widehat{G}(x,\widehat{Y})\cdot(xd\widehat{Y} - \lambda\widehat{Y}\,dx)$ is linearized. We define $\hat{\eta}_{q_o} = \frac{d\widehat{G}}{\widehat{G}} + \frac{d(x\widehat{Y})}{x\widehat{Y}} = \frac{d\widehat{G}}{\widehat{G}} + \frac{dx}{x} + \frac{d\widehat{Y}}{\widehat{Y}}$ and $\widehat{\omega}_o = \dfrac{d\widehat{Y}}{\widehat{Y}} - \lambda\frac{dx}{x}$.

Therefore we may write $\hat{\eta}_j - \hat{\eta}_{q_o} = (\hat{h}\cdot x\cdot Y\cdot\widehat{G})\cdot\hat{\omega}_0$ and $d(\hat{h}\cdot x\cdot\widehat{Y}\cdot\widehat{G})\wedge\hat{\omega}_0 = 0$. We put $\hat{f} := \hat{h}x\cdot\widehat{Y}\cdot\widehat{G}$ and write $\hat{f} = \sum\limits_{j=0}^{+\infty} f_j(x)y^j$ where $f_j(x)$ is holomorphic in $\mathbb{D}^*_{q_o}$, $\forall\, j$. Then $d\hat{f}\wedge\hat{\omega}_0 = 0$ gives $(*)$ $x\hat{f}_x + \lambda\widehat{Y}\cdot\hat{f}_{\widehat{Y}} = 0$ over $\mathbb{D}^*_{q_o}$; where by definition (notice that $\frac{\partial\widehat{Y}}{\partial x}$ and $\frac{\partial\widehat{Y}}{\partial y}$ are invertible elements of the ring of formal power series):

$$\hat{f}_x := \frac{\partial\hat{f}}{\partial x} = \sum_{j=0}^{+\infty} f_j'(x)y^j,\quad \hat{f}_y := \frac{\partial\hat{f}}{\partial y} = \sum_{j=1}^{+\infty} jf_j(x)y^{j-1}\ \ \hat{f}_{\widehat{Y}} := \hat{f}_x\bigg(\frac{\partial\widehat{Y}}{\partial x}\bigg)^{-1} + \hat{f}_y\bigg(\frac{\partial\widehat{Y}}{\partial y}\bigg)^{-1}$$

and

$$\frac{\partial\widehat{Y}}{\partial x} := \sum_{j=1}^{+\infty} a_j'(x)y^j,\quad \frac{\partial\widehat{Y}}{\partial y} := \sum_{j=1}^{+\infty} ja_j(x)y^{j-1}.$$

Thus by (*) we conclude that $\hat{f}_x = 0$, $\hat{f}_{\widehat{Y}} = 0 \Rightarrow \hat{f}_x = 0$, $\hat{f}_y = 0 \Rightarrow \hat{f} = f_o$ is a constant and therefore $\hat{f}$ extends naturally to $\mathbb{D}_{q_o}$. This shows that ($\hat{h}$ and therefore) $\hat{\eta}_j$ extends to $\mathbb{D}_{q_o}$.

Case (3): In this case we have local holomorphic coordinates $(x,y)\in U$ centered at q_o, such that $\tilde{\omega}(x,y) = g(x,y)[nxdy + my(1+b(x,y))dx]$ where $n,m\in\mathbb{N}^*$, $\langle n,m\rangle = 1$, $g,b\in\mathcal{O}_2$, $b(0,0) = 0$ [17]. According to [17] and also from what we have observed above we may choose a formal coordinate system $(x,\widehat{Y})$ at q_0, $\widehat{Y} = \sum\limits_{j=1}^{+\infty} a_j(x)y^j$, $a_j\in\mathcal{O}(\mathbb{D}_{q_o})\ \forall\, j$, such that if $\lambda = n/m$ then

$$\tilde{\omega}(x,\widehat{Y}) = \widehat{G}(x,\widehat{Y})[n(1+(\lambda-1)(x^m\widehat{Y}^n)^k)xd\widehat{Y} + m(1+\lambda(x^m\widehat{Y}^m)^k)\widehat{Y}\,dx].$$

In this case we define

$$\hat{\eta}_{q_o} = d\log[\widehat{G}(x,\widehat{Y})\cdot x^{m+1}\widehat{Y}^{m+1}] = \frac{d\widehat{G}}{\widehat{G}} + (m+1)\frac{dx}{x} + (n+1)\frac{d\widehat{Y}}{\widehat{Y}}$$

and

$$\widehat{\omega}_o = \frac{\tilde{\omega}}{\widehat{G}x^{n+1}\widehat{Y}^{n+1}} = -d\bigg(\frac{\widehat{Y}}{x^m\widehat{Y}^n}\bigg) + (x^m\widehat{Y}^n)^{k-1}\bigg[n(\lambda-1)\frac{d\widehat{Y}}{\widehat{Y}} + \lambda.m\frac{dx}{x}\bigg].$$

Since $\lambda = n/m$ it is a straightforward calculation to show that $d\hat{\omega}_o = 0$ and therefore if

$$\hat{f} = h\cdot\widehat{G}x^{m+1}\widehat{Y}^{n+1},$$

then $d\hat{v}\wedge\hat{\omega}_o = 0$. As in the case above, the fact that $\hat{\omega}_o$ admits no first integral outside one separatrix implies that $\hat{f}$ is constant and therefore $\hat{\eta}$ extends to $\mathbb{D}_{q_o}$. But we remark that $\hat{\eta}-\hat{\eta}_{q_o}$ has simple poles along $\mathbb{D}_{q_o}\subset D_j$ and $\hat{\omega}_o$ has poles of order $n+1\geq 2$ along $\mathbb{D}_{q_o}$, so that $\hat{\eta}-\hat{\eta}_{q_o} =$ const.. $\hat{\omega}_o \Rightarrow$ const.. $= 0$ and therefore we have in fact concluded that if q_o is of type **(3)** then $\hat{\eta}_j$ extends as $\hat{\eta}_j = \hat{\eta}_{q_o}$ to q_o.

Summarizing the above discussion we obtain:

Proposition 9.4. *Given any component $D_j \subset E(\mathcal{F})$ there exists a transversely formal generalized integrating factor $\hat{\eta}_j$ for $\tilde{\omega}$ defined over D_j which also satisfies: the formal polar set $(\hat{\eta}_j)_\infty$ has order one.*

Now it remains to show how to construct the forms $\hat{\eta}_j$ in a compatible way, i.e., such that if $D_i \cap D_j = \{q\}$ then both forms bind up into a transversely form defined in $D_i \cup D_j$. For this we need the solvability of the virtual holonomy group H_j^{virt}, not only of the holonomy group H_j. The idea is basically the following: Take a component $D_i \subset E(\mathcal{F})$ that meets D_j at a corner singularity $q = D_i \cap D_j$. We may assume that q is resonant so that we are in Case 1 or 3 of the above argumentation. The difference $\hat{\alpha}_{ij} := \hat{\eta}_i - \hat{\eta}_j$ is a formal closed meromorphic one-form at q such that $\hat{\alpha}_{ij} \wedge \tilde{\omega} = 0$. Moreover, $\hat{\alpha}_{ij}$ is zero or it has only simple poles. Thus, we may assume that we are just in Case 1 of the above argumentation, i.e., that $\tilde{\mathcal{F}}$ has a holomorphic first integral at q. In this case the so called Dulac correspondence is defined as follows:

Choose a small neighborhood $\tilde{U}$ of q, where we take small transverse sections Σ_j to D_j and Σ_i to D_i. Denote by $\mathcal{F}(\Sigma_j)$ the collection of subsets $E \subset \Sigma_j$ such that E is contained in some leaf of $\tilde{\mathcal{F}}\big|_{\tilde{U}}$. Define $\mathcal{F}(\Sigma_i)$ in a similar way. Roughly speaking, the Dulac correspondence is a multivalued correspondence $\mathcal{D}_q : \Sigma_j \to \Sigma_i$, which is obtained by tracing the local leaves of $\tilde{\mathcal{F}}\big|_{\tilde{U}}$. Given any $x \in \Sigma_j$ the set of intersections of the local leaf of $\tilde{\mathcal{F}}\big|_{\tilde{U}}$ that contains x, with the transverse section Σ_j, is denoted by $L_x \cap \Sigma_j \in \mathcal{F}(\Sigma_j)$. The correspondence $\mathcal{D}_q$ associates to any point $z \in L_x \cap \Sigma_j$, the subset $\mathcal{D}_q(z) \subset L_x \cap \Sigma_i \in \mathcal{F}(\Sigma_i)$, usually defined by the some local normal form of $\tilde{\mathcal{F}}$ in $\tilde{U}$.

Given an element $h \in \mathrm{Hol}^{\mathrm{virt}}(\tilde{\mathcal{F}}, \mathrm{D_j}, \Sigma_\mathrm{j})$, we associate h with a collection of elements

$$\{h^{\mathcal{D}}\} \subset \mathrm{Diff}(\Sigma_\mathrm{i}, \mathrm{q_i}) \subset \mathrm{Hol}^{\mathrm{virt}}(\tilde{\mathcal{F}}, \mathrm{D_i}, \Sigma_\mathrm{i}),$$

each of which satisfies the following relation

$$h^{\mathcal{D}} \circ \mathcal{D}_q = \mathcal{D}_q \circ h\,,$$

called the *adjunction equation*. We remark that the adjunction equation is not exactly an equation, but rather an equality of sets or correspondences. More precisely, given any element $h \in \mathrm{Hol}^{\mathrm{virt}}(\tilde{\mathcal{F}}, \mathrm{D_j}, \Sigma_\mathrm{j})$, each diffeomorphism $h^{\mathcal{D}} \in \mathrm{Hol}^{\mathrm{virt}}(\tilde{\mathcal{F}}, \mathrm{D_i}, \Sigma_\mathrm{i})$ must satisfy, for every $x \in \Sigma_i$, the equality of sets $h^{\mathcal{D}}(\mathcal{D}_q(x)) = \mathcal{D}_q(h(x))$, where $\mathcal{D}_q(x) \subset L_x \cap \Sigma_i$ and $\mathcal{D}_q(h(x)) \subset L_x \cap \Sigma_j$ are subsets as above. This adjunction is adequately defined for the special case of singularities $\{q\} = D_i \cap D_j$ we are considering as we shall see in what follows. There are local holomorphic coordinates $(x, y) \in \tilde{U}$ such that $D_i \cap \tilde{U} = \{x = 0\}$, $\;D_j \cap \tilde{U} = \{y = 0\}$, and such that $\tilde{\mathcal{F}}\big|_{\tilde{U}}$ is given in the *normal form* as $nxdy + mydx = 0$ and $q : x = y = 0$, where $n/m \in \mathbb{Q}_+$ and $\langle n, m\rangle = 1$. We fix the local transverse sections as $\Sigma_j = \{x = 1\}$ and $\Sigma_i = \{y = 1\}$, such that $\Sigma_i \cap D_i = q_i \neq q$ and $\Sigma_j \cap D_j = q_j \neq q$. The local leaves of the foliation are given by $x^m y^n = \text{const}$. The Dulac correspondence is the correspondence obtained by following these leaves

$$\mathcal{D}_q \colon \Sigma_j \to \Sigma_i,\ \mathcal{D}_q(x) = \{x^{m/n}\}.$$

from a local transverse section Σ_j to D_j to another transverse section Σ_i to D_i. Let be given a map f in the virtual holonomy H_i^{virt} of D_i. We search for a well-defined map $f^{\mathcal{D}_q} \in H_j^{\mathrm{virt}}$ in the virtual holonomy of D_j, such that it satisfies the " adjunction equation" $f^{\mathcal{D}_q} \circ \mathcal{D}_q = \mathcal{D}_q \circ f$.

The fact that we can construct both $\hat{\eta}_i$ and $\hat{\eta}_j$ in a compatible way, i.e., such that $\hat{\eta}_i$ and $\hat{\eta}_j$ agree as formal objects at q is a consequence of the following: (1) $\hat{\eta}_i$ and $\hat{\eta}_j$ are constructed in a compatible way (agreeing) with the virtual holonomy groups H_i^{virt} and H_j^{virt} respectively. (2) these virtual holonomy groups are related by the Dulac correspondence. Indeed, we can embed the virtual holonomy group of D_j into the virtual holonomy group of D_i. Thus, the solvability of the group $H_i^{\rm virt}$ means that, in a certain sense, both virtual holonomy groups are solvable and simultaneously written in formal normalizing coordinates. In particular, we can already choose the form $\hat{\eta}_i$ in such a way that it agrees with $\hat{\eta}_j$ as formal objects at q. The details of this construction and compatibility conditions are thoroughly discussed in [21]. There the author mentions the so called *zone holomorphe, zone logarithmique,...*. To such a zone, denoted by $\mathcal{Z}$, the author associates a holonomy pseudo-group $\mathrm{Hol}(\mathcal{Z}, \mathrm{f}_{\mathcal{Z}})$ which measures the obstruction to the integration of the foliation in a neighborhood of the zone $\mathcal{Z}$. The main point is that under our hypothesis, both components D_i and D_j are accumulated by analytic leaves and therefore both exhibit solvable virtual holonomy groups. On the other hand, any generalized holonomy $\mathrm{Hol}(\mathcal{Z}, \mathrm{f}_{\mathcal{Z}})$ constructed in [21] is contained in the virtual holonomy. This implies that the conditions of [21] are automatically satisfied by Proposition 3.3. Now we can finish the argumentation just by observing that from the above discussion we already conclude from Proposition 3.3 that the forms $\hat{\eta}_j$ can be constructed in a compatible way, resulting into a global transversely formal one-form $\tilde{\hat{\eta}}$ along the divisor $D = \bigcup\limits_j D_j$. Blowing down this one-form we obtain a transversely formal generalized integrating factor $\hat{\eta}$ for ω in a neighborhood of the origin $0 \in \mathbb{C}^2$.

Sketch of the proof of Proposition 5.9. We perform the reduction of singularities of the foliation $\mathcal{F}$. The first step is:

Claim 9.5. *All the virtual holonomy groups are exceptional, isomorphic.*

The next step is:

Claim 9.6. *There is a transversely formal function $\hat{\Phi}_j$ defined along $D_j^* = D_j \setminus \mathrm{sing}(\tilde{\mathcal{F}})$, with the property below: Given any point $q \in D_j^*$ and a transverse disc Σ_q with $\Sigma_{\tilde{D}} \cap D_j = \{q\}$, we choose a formal normalizing coordinate $\hat{x}_q \in \Sigma_q$, centered at q, for the virtual holonomy group* $\mathrm{Hol}^{\rm virt}(\tilde{\mathcal{F}}, \mathrm{D_j}, \Sigma_\mathrm{q}), \mathrm{q})$. *Then we have $\hat{\Phi}_j\big|_{\Sigma_q}(\hat{x}_q) = \cos(\frac{2\pi}{\hat{x}_q^{k_j}})$.*

In the case $\mathrm{Hol}^{\rm virt}(\tilde{\mathcal{F}}, \mathrm{D_j}, \Sigma_\mathrm{q}, \mathrm{q})$ is exceptional we define $\hat{\Phi}_j\big|_{\Sigma_q}$ as $\hat{\Phi}_j(\hat{x}_q) = \cos(\frac{2\pi}{\hat{x}_q^{k_j}})$. Then:

Claim 9.7. *The function $\hat{\Phi}_j$ extends to each singularity $p \in D_j \cap \mathrm{sing}(\tilde{\mathcal{F}})$, the result is a transversely formal Liouvillian function along D_j which is a first integral for $\tilde{\mathcal{F}}$.*

Proceeding as in the proof of Proposition 9.4 we obtain:

Claim 9.8. *Given any corner $p = D_i \cap D_j$ there is a constant $c_{ij} \in \mathbb{C}$ such that at p we have $\hat{\Phi}_i = c_{ij}\hat{\Phi}_j$ as formal objects.*

Since the exceptional divisor $E(\mathcal{F})$ contains no cycles we may choose a globally defined transversely formal function $\hat{\Phi}$ along $E(\mathcal{F})$ by suitable choices of constants $c_j \in \mathbb{C}$ and setting $\hat{\Phi} = c_j\hat{\Phi}_j$ whenever it makes sense. Blowing down $\hat{\Phi}$ we obtain the desired formal Liouvillian first integral. This completes the proof of Lemma 8.10. □

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