# DETAILED CALCULATIONS OF "THE FUNDAMENTAL GROUP OF THE COMPLEMENT OF THE SINGULAR LOCUS OF LAURICELLA'S $F_C$"

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ABSTRACT. In this paper, we give proofs of Lemma 4.1, Lemma 4.2, Lemma 5.3 and Theorem 5.5 in which omitted calculations are supplied. We also provide detailed explanations of Remark 4.3 and the derivation of the 72 relations in Subsection 5.3. Further, as an appendix, we give calculations in the inductive steps of the proof of Theorem 1.2.

## 3. PROOF OF THEOREM 1.2

For the convenience of the readers, in Appendix A we give an example of the induction steps of the proof of Theorem 1.2.

## 4. PRESENTATION OF $\pi_1(X^{(3)})$

4.2. **Computation of $\pi_1(X^{(3)} \cap H)$.** By computing the monodromy relations, we obtain the following relations:

(0) $\alpha_1\alpha_2\dots\alpha_8 = 1$;
(1) $\alpha_4 = \alpha_5$;
(2) $[\alpha_4, \alpha_6] = 1$;
(3) $(\alpha_3\alpha_4)^2 = (\alpha_4\alpha_3)^2$;
(4) $[\alpha_2, \alpha_4^{-1}\alpha_3\alpha_4] = 1$;
(5) $(\alpha_2\alpha_3\alpha_4\alpha_3^{-1})^2 = (\alpha_3\alpha_4\alpha_3^{-1}\alpha_2)^2$;
(6) $\alpha_6 = \alpha_2\alpha_3\alpha_4(\alpha_2\alpha_3)^{-1}$;
(7) $(\alpha_7\alpha_8)^2 = (\alpha_8\alpha_7)^2$;
(8) $(\tilde{\alpha}_2\alpha_5)^2 = (\alpha_5\tilde{\alpha}_2)^2$;
(9) $\alpha_2\alpha_4\alpha_2^{-1} = \alpha_8^{-1}\alpha_7\alpha_8$;
(10) $[\tilde{\alpha}_7^{-1}\tilde{\tilde{\alpha}}_2\tilde{\alpha}_7, \tilde{\alpha}_8] = 1$;
(11) $[(\tilde{\alpha}_7^{-1}\tilde{\tilde{\alpha}}_2\tilde{\alpha}_7\alpha_1)^{-1}\tilde{\alpha}_8(\tilde{\alpha}_7^{-1}\tilde{\tilde{\alpha}}_2\tilde{\alpha}_7\alpha_1), \tilde{\alpha}_3] = 1$,

and

(0') $[\alpha_1, \alpha_2] = 1$, $[\alpha_6, \alpha_7] = 1$;
(1') same as (1);
(2') $[\alpha_4, \alpha_7] = 1$;
(3') same as (3);
(4') $[\alpha_1, \alpha_4^{-1}\alpha_3\alpha_4] = 1$;
(5') $(\alpha_1\alpha_3\alpha_4\alpha_3^{-1})^2 = (\alpha_3\alpha_4\alpha_3^{-1}\alpha_1)^2$;
(6') $\alpha_7 = \alpha_1\alpha_3\alpha_4(\alpha_1\alpha_3)^{-1}$;
(7') $(\alpha_6\alpha_8)^2 = (\alpha_8\alpha_6)^2$;
(8') $(\tilde{\alpha}'_1\alpha_5)^2 = (\alpha_5\tilde{\alpha}'_1)^2$;
(9') $\alpha_1\alpha_4\alpha_1^{-1} = \alpha_8^{-1}\alpha_6\alpha_8$;

(10') $[\tilde{\alpha}_6'^{-1}\tilde{\tilde{\alpha}}'_1\tilde{\alpha}_6', \tilde{\alpha}_8'] = 1$;
(11') $[(\tilde{\alpha}_6'^{-1}\tilde{\tilde{\alpha}}'_1\tilde{\alpha}_6'\alpha_2)^{-1}\tilde{\alpha}_8(\tilde{\alpha}_6'^{-1}\tilde{\tilde{\alpha}}'_1\tilde{\alpha}_6'\alpha_2), \tilde{\alpha}_3'] = 1$;

where

$$\begin{aligned}
&\tilde{\alpha}_2 = (\alpha_3\alpha_4)^{-1}\alpha_2(\alpha_3\alpha_4), \quad \tilde{\tilde{\alpha}}_2 = (\alpha_3\alpha_4\alpha_5\alpha_6)^{-1}\alpha_2(\alpha_3\alpha_4\alpha_5\alpha_6),\\
&\tilde{\alpha}_3 = \alpha_4^{-1}\alpha_3\alpha_4, \quad \tilde{\alpha}_7 = \alpha_8^{-1}\alpha_7\alpha_8, \quad \tilde{\alpha}_8 = \alpha_7\alpha_8\alpha_7^{-1},\\
&\tilde{\alpha}_1' = (\alpha_3\alpha_4)^{-1}\alpha_1(\alpha_3\alpha_4), \quad \tilde{\tilde{\alpha}}'_1 = (\alpha_3\alpha_4\alpha_5\alpha_7)^{-1}\alpha_1(\alpha_3\alpha_4\alpha_5\alpha_7),\\
&\tilde{\alpha}_3' = \alpha_4^{-1}\alpha_3\alpha_4, \quad \tilde{\alpha}_7' = \alpha_8^{-1}\alpha_6\alpha_8, \quad \tilde{\alpha}_8' = \alpha_6\alpha_8\alpha_6^{-1}.
\end{aligned}$$

Note that the relation (11) (and (11')) obtained as the monodromy around $a_{11} = \infty$ is not needed (see also Remark 4.3).

By using the relations (0), (1), (6) and (6'), we have

$$\pi_1(X^{(3)} \cap H) = \left\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \,\middle|\, \begin{array}{l} (2), (3), (4), (5), (0'), (2'), (4'), (5') \\ (7), (8), (9), (10), (7'), (8'), (9'), (10') \end{array} \right\rangle.$$

We put

$$\beta_1 = \alpha_1, \quad \beta_2 = \alpha_2, \quad \beta_3 = \tilde{\alpha}_3 = \alpha_4^{-1}\alpha_3\alpha_4, \quad \beta_4 = \tilde{\alpha}_4 = \alpha_3\alpha_4\alpha_3^{-1}. \tag{4.3}$$

By the relation (3), $\alpha_i$'s are written as

$$\alpha_1 = \beta_1, \quad \alpha_2 = \beta_2, \quad \alpha_3 = \beta_4^{-1}\beta_3\beta_4, \quad \alpha_4 = \beta_3\beta_4\beta_3^{-1}. \tag{4.4}$$

Thus, $\beta_1$, $\beta_2$, $\beta_3$ and $\beta_4$ form a generator of $\pi_1(X^{(3)} \cap H)$:

$$\pi_1(X^{(3)} \cap H) = \left\langle \beta_1, \beta_2, \beta_3, \beta_4 \,\middle|\, \begin{array}{l} (2), (3), (4), (5), (0'), (2'), (4'), (5') \\ (7), (8), (9), (10), (7'), (8'), (9'), (10') \end{array} \right\rangle.$$

**Lemma 4.1.** *The relations (2), (3), (4), (5), (0'), (2'), (4'), (5') are equivalent to*

(A) $[\beta_i, \beta_j] = 1 \quad (1 \le i < j \le 3)$;
(B) $[\beta_i\beta_4\beta_i^{-1}, \beta_j\beta_4\beta_j^{-1}] = 1 \quad (1 \le i < j \le 3)$;
(C) $(\beta_4\beta_k)^2 = (\beta_k\beta_4)^2 \quad (1 \le k \le 3)$.

*Proof.* Since

$$\begin{aligned}
&[\beta_1, \beta_2] = [\alpha_1, \alpha_2],\ [\beta_1, \beta_3] = [\alpha_1, \alpha_4^{-1}\alpha_3\alpha_4],\ [\beta_2, \beta_3] = [\alpha_2, \alpha_4^{-1}\alpha_3\alpha_4],\\
&[\beta_1\beta_4\beta_1^{-1}, \beta_2\beta_4\beta_2^{-1}] = [\alpha_1\alpha_3\alpha_4\alpha_3^{-1}\alpha_1^{-1}, \alpha_2\alpha_3\alpha_4\alpha_3^{-1}\alpha_2^{-1}] = [\alpha_7, \alpha_6]\\
&\beta_1\beta_4 = \alpha_1\alpha_3\alpha_4\alpha_3^{-1},\ \beta_4\beta_1 = \alpha_3\alpha_4\alpha_3^{-1}\alpha_1,\ \beta_2\beta_4 = \alpha_2\alpha_3\alpha_4\alpha_3^{-1},\ \beta_4\beta_2 = \alpha_3\alpha_4\alpha_3^{-1}\alpha_2,
\end{aligned}$$

the relations (A), $[\beta_1\beta_4\beta_1^{-1}, \beta_2\beta_4\beta_2^{-1}] = 1$, $(\beta_1\beta_4)^2 = (\beta_4\beta_1)^2$ and $(\beta_2\beta_4)^2 = (\beta_4\beta_2)^2$ are equivalent to (4), (5), (0'), (4'), (5'), by the definition.

We show

$$(2), (3), (2') \Longleftrightarrow \begin{cases} [\beta_1\beta_4\beta_1^{-1}, \beta_3\beta_4\beta_3^{-1}] = 1,\\ [\beta_2\beta_4\beta_2^{-1}, \beta_3\beta_4\beta_3^{-1}] = 1,\\ (\beta_3\beta_4)^2 = (\beta_4\beta_3)^2. \end{cases} \tag{$*$}$$

First, we assume (2), (3), (2'). Then we obtain the right-hand side of ($*$), as follows:

$$\begin{aligned}
[\beta_1\beta_4\beta_1^{-1}, \beta_3\beta_4\beta_3^{-1}] &= [\alpha_1 \cdot \alpha_3\alpha_4\alpha_3^{-1} \cdot \alpha_1^{-1}, \alpha_4^{-1}\alpha_3\alpha_4 \cdot \alpha_3\alpha_4\alpha_3^{-1} \cdot (\alpha_4^{-1}\alpha_3\alpha_4)^{-1}],\\
&= [\alpha_7, \alpha_4] = 1,\\
[\beta_2\beta_4\beta_2^{-1}, \beta_3\beta_4\beta_3^{-1}] &= [\alpha_2 \cdot \alpha_3\alpha_4\alpha_3^{-1} \cdot \alpha_2^{-1}, \alpha_4^{-1}\alpha_3\alpha_4 \cdot \alpha_3\alpha_4\alpha_3^{-1} \cdot (\alpha_4^{-1}\alpha_3\alpha_4)^{-1}],\\
&= [\alpha_6, \alpha_4] = 1,\\
(\beta_3\beta_4)^2 &= (\alpha_4^{-1}\alpha_3\alpha_4 \cdot \alpha_3\alpha_4\alpha_3^{-1})^2 = (\alpha_3\alpha_4)^2,\\
(\beta_4\beta_3)^2 &= (\alpha_3\alpha_4\alpha_3^{-1} \cdot \alpha_4^{-1}\alpha_3\alpha_4)^2\\
&= \alpha_3\alpha_4\alpha_3^{-1}\alpha_4^{-1}\alpha_3\alpha_4\alpha_3\alpha_4\alpha_3^{-1}\alpha_4^{-1}\alpha_3\alpha_4 = (\alpha_3\alpha_4)^2.
\end{aligned}$$

Next, we assume the right-hand side of ($*$). We should show three relations

$$[\alpha_4, \alpha_1\alpha_3\alpha_4\alpha_3^{-1}\alpha_1^{-1}] = 1,\ [\alpha_4, \alpha_2\alpha_3\alpha_4\alpha_3^{-1}\alpha_2^{-1}] = 1,\ (\alpha_3\alpha_4)^2 = (\alpha_4\alpha_3)^2.$$

By symmetry between (4.3) and (4.4), the third relation follows from $(\beta_3\beta_4)^2 = (\beta_4\beta_3)^2$ as above. We show the first relation;

$$\begin{aligned}
[\alpha_4, \alpha_1\alpha_3\alpha_4\alpha_3^{-1}\alpha_1^{-1}] &= [\beta_3\beta_4\beta_3^{-1}, \beta_1(\beta_4^{-1}\beta_3\beta_4)(\beta_3\beta_4\beta_3^{-1})(\beta_4^{-1}\beta_3\beta_4)^{-1}\beta_1^{-1}]\\
&= [\beta_3\beta_4\beta_3^{-1}, \beta_1\beta_4\beta_1^{-1}] = 1.
\end{aligned}$$

The second one can be shown in a similar way. □

By this lemma we obtain

$$\pi_1(X^{(3)} \cap H) = \left\langle \beta_1, \beta_2, \beta_3, \beta_4 \;\middle|\; \begin{array}{l} \text{(A), (B), (C)} \\ \text{(7), (8), (9), (10), (7'), (8'), (9'), (10')} \end{array} \right\rangle.$$

Note that by $(\beta_3\beta_4)^2 = (\beta_4\beta_3)^2$, we have

$$\alpha_3\alpha_4 = (\beta_4^{-1}\beta_3\beta_4)(\beta_3\beta_4\beta_3^{-1}) = \beta_3\beta_4.$$

By the proof of Lemma 4.1, we have

$$\begin{aligned}
&\alpha_1 = \beta_1, \quad \alpha_2 = \beta_2, \quad \alpha_3 = \beta_4^{-1}\beta_3\beta_4, \quad \alpha_4 = \alpha_5 = \beta_3\beta_4\beta_3^{-1},\\
&\alpha_6 = \beta_2\beta_4\beta_2^{-1}, \quad \alpha_7 = \beta_1\beta_4\beta_1^{-1},
\end{aligned}$$

and

$$\begin{aligned}
\alpha_8^{-1} &= \alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6\alpha_7 = \beta_1\beta_2 \cdot \beta_3\beta_4 \cdot \beta_3\beta_4\beta_3^{-1} \cdot \beta_2\beta_4\beta_2^{-1} \cdot \beta_1\beta_4\beta_1^{-1}\\
&= \beta_1\beta_2\beta_4\beta_3\beta_4\beta_2\beta_4\beta_2^{-1}\beta_1\beta_4\beta_1^{-1}.
\end{aligned}$$

**Lemma 4.2.** *The relations (7)–(10) and (7')–(10') follow from (A)–(C).*

*Proof.* We show the lemma only for (7)–(10), because the others are shown in a similar way. We assume (A)–(C). First, we rewrite the relations (7), (8), (9), by using $\beta_i$'s:

$$\begin{aligned}
(7) \Leftrightarrow & \alpha_7^{-1}\alpha_8^{-1}\alpha_7^{-1}\alpha_8^{-1} = \alpha_8^{-1}\alpha_7^{-1}\alpha_8^{-1}\alpha_7^{-1} \\
\Leftrightarrow & \beta_1\beta_4^{-1}\beta_2\beta_4\beta_3\beta_4\beta_2\beta_4\beta_1\beta_4\beta_3\beta_4\underline{\beta_2\beta_4\beta_2^{-1}}\beta_1\beta_4\beta_1^{-1} \\
& = \beta_1\beta_2\beta_4\beta_3\beta_4\beta_2\beta_4\beta_1\beta_4\beta_3\beta_4\underline{\beta_2\beta_4\beta_2^{-1}} \\
\Leftrightarrow & \beta_4^{-1}\beta_2\beta_4\beta_3\beta_4\beta_2\beta_4\beta_1\beta_4\beta_3\beta_4\beta_1\beta_4\beta_1^{-1} = \beta_2\beta_4\beta_3\beta_4\beta_2\beta_4\beta_1\beta_4\beta_3\beta_4 \\
\Leftrightarrow & \beta_4\beta_3\beta_4\beta_2\beta_4\beta_1\beta_4\beta_3\underline{\beta_4\beta_1\beta_4\beta_1^{-1}} = \underline{\beta_2^{-1}\beta_4\beta_2\beta_4}\beta_3\beta_4\beta_2\beta_4\beta_1\beta_4\beta_3\beta_4 \\
\Leftrightarrow & \beta_4\beta_3\beta_4\beta_2\beta_4\beta_1\beta_4\beta_3\beta_1^{-1}\beta_4\beta_1\beta_4 = \beta_4\beta_2\beta_4\beta_2^{-1}\beta_3\beta_4\beta_2\beta_4\beta_1\beta_4\beta_3\beta_4 \\
\Leftrightarrow & \beta_3\beta_4\beta_2\beta_4\beta_1\beta_4\beta_3\beta_1^{-1}\beta_4\beta_1 = \beta_2\beta_4\beta_2^{-1}\beta_3\beta_4\beta_2\beta_4\beta_1\beta_4\beta_3,
\end{aligned}$$

$$\begin{aligned}
(8) \Leftrightarrow & ((\alpha_3\alpha_4)^{-1}\alpha_2(\alpha_3\alpha_4)\cdot\alpha_5)^2 = (\alpha_5\cdot(\alpha_3\alpha_4)^{-1}\alpha_2(\alpha_3\alpha_4))^2 \\
\Leftrightarrow & (\beta_4^{-1}\beta_3^{-1}\beta_2\beta_3\beta_4\beta_3\beta_4\beta_3^{-1})^2 = (\beta_3\beta_4\beta_3^{-1}\beta_4^{-1}\beta_3^{-1}\beta_2\beta_3\beta_4)^2 \\
\Leftrightarrow & \beta_4^{-1}\beta_3^{-1}\beta_2\beta_4\beta_2\beta_4\beta_3\beta_4 = \beta_4^{-1}\beta_3^{-1}\beta_4\beta_2\beta_4\beta_2\beta_3\beta_4 \\
\Leftrightarrow & \beta_2\beta_4\beta_2\beta_4 = \beta_4\beta_2\beta_4\beta_2 \quad \text{(this is a relation in (C))},
\end{aligned}$$

$$\begin{aligned}
(9) \Leftrightarrow & \beta_2\beta_3\beta_4\beta_3^{-1}\beta_2^{-1} = \beta_1\beta_2\beta_4\beta_3\beta_4\underline{\beta_2\beta_4\beta_2^{-1}}\beta_1\beta_4\beta_1^{-1}(\beta_1\beta_2\beta_4\beta_3\beta_4\underline{\beta_2\beta_4\beta_2^{-1}})^{-1} \\
\Leftrightarrow & \beta_2\beta_3\beta_4\beta_3^{-1}\beta_2^{-1} = \beta_1\beta_2\beta_4\beta_3\beta_4\beta_1\beta_4\beta_1^{-1}\beta_4^{-1}\beta_3^{-1}\beta_4^{-1}\beta_2^{-1}\beta_1^{-1} \\
\Leftrightarrow & \beta_3\beta_4\beta_3^{-1} = \beta_1\beta_4\beta_3\beta_4\beta_1\beta_4\beta_1^{-1}\beta_4^{-1}\beta_3^{-1}\beta_4^{-1}\beta_1^{-1} \\
\Leftrightarrow & \beta_4 = \beta_3^{-1}\beta_1\beta_4\beta_3\beta_4\beta_1\beta_4\beta_1^{-1}\beta_4^{-1}\beta_3^{-1}\beta_4^{-1}\beta_1^{-1}\beta_3.
\end{aligned}$$

Next, we show (7) and (9), since (8) is already proved. Note that $[\beta_i,\beta_j]=1$ and $[\beta_i\beta_4\beta_i^{-1},\beta_j\beta_4\beta_j^{-1}]=1$ imply $[\beta_i^{-1}\beta_4\beta_i,\beta_j^{-1}\beta_4\beta_j]=1$. The left-hand side of (7) is

$$\begin{aligned}
& \beta_3\beta_4\beta_2\beta_4\beta_1(\beta_3\beta_3^{-1})\beta_4\beta_3\beta_1^{-1}\beta_4\beta_1 = \beta_3\beta_4\beta_2\beta_4\beta_1\beta_3\beta_1^{-1}\beta_4\beta_1\beta_3^{-1}\beta_4\beta_3 \\
& = \beta_3\beta_4\beta_2\beta_4\beta_3\beta_4\beta_1\beta_3^{-1}\beta_4\beta_3 = \beta_3\beta_4\beta_2\beta_4\beta_3\beta_4\beta_3^{-1}\beta_1\beta_4\beta_3,
\end{aligned}$$

and the right-hand side is

$$\begin{aligned}
& \beta_2\beta_4\beta_2^{-1}\beta_3\beta_4(\beta_3^{-1}\beta_3)\beta_2\beta_4\beta_1\beta_4\beta_3 = \beta_3\beta_4\beta_3^{-1}\beta_2\beta_4\beta_2^{-1}\beta_3\beta_2\beta_4\beta_1\beta_4\beta_3 \\
& = \beta_3\beta_4\beta_3^{-1}\beta_2\beta_4\beta_3\beta_4\beta_1\beta_4\beta_3 = \beta_3\beta_4\beta_2\beta_3^{-1}\beta_4\beta_3\beta_4\beta_1\beta_4\beta_3.
\end{aligned}$$

Thus, (7) is equivalent to $\beta_4\beta_3\beta_4\beta_3^{-1} = \beta_3^{-1}\beta_4\beta_3\beta_4$ which is nothing but a relation in (C). The right-hand side of (9) is

$$\begin{aligned}
& \beta_1\underline{\beta_3^{-1}\beta_4\beta_3\beta_4}\beta_1\beta_4\beta_1^{-1}\beta_4^{-1}\beta_3^{-1}\beta_4^{-1}\beta_3\beta_1^{-1} \\
& = \beta_1\beta_4\underline{\beta_3\beta_4\beta_3^{-1}\beta_1\beta_4\beta_1^{-1}}\beta_4^{-1}\beta_3^{-1}\beta_4^{-1}\beta_3\beta_1^{-1} \\
& = \underline{\beta_1\beta_4\beta_1\beta_4}\beta_1^{-1}\beta_3\beta_4\underline{\beta_3^{-1}\beta_4^{-1}\beta_3^{-1}\beta_4^{-1}}\beta_3\beta_1^{-1} \\
& = \beta_4\beta_1\beta_4\beta_1\beta_1^{-1}\beta_3\beta_4\beta_4^{-1}\beta_3^{-1}\beta_4^{-1}\beta_3^{-1}\beta_3\beta_1^{-1} = \beta_4,
\end{aligned}$$

and hence (9) is proved. Finally, we show (10). By using (7), we have

$$\tilde{\alpha}_7\tilde{\alpha}_8\tilde{\alpha}_7^{-1} = \alpha_8^{-1}\alpha_7\alpha_8\alpha_7\alpha_8\alpha_7^{-1}\alpha_8^{-1}\alpha_7^{-1}\alpha_8 = \alpha_8.$$

Thus, the relation (10) is rewritten by $\beta_i$'s as follows:

$$\begin{aligned}
(10) \Leftrightarrow & [\tilde{\tilde{\alpha}}_2, \tilde{\alpha}_7\tilde{\alpha}_8\tilde{\alpha}_7^{-1}] = 1 \ \Leftrightarrow [\tilde{\tilde{\alpha}}_2, \alpha_8^{-1}] = 1 \\
\Leftrightarrow & [(\beta_3\beta_4 \cdot \beta_3\beta_4\beta_3^{-1} \cdot \beta_2\beta_4\beta_2^{-1})^{-1}\beta_2(\beta_3\beta_4 \cdot \beta_3\beta_4\beta_3^{-1} \cdot \beta_2\beta_4\beta_2^{-1}), \\
& \quad \beta_1\beta_2\beta_4\beta_3\beta_4\beta_2\beta_4\beta_2^{-1}\beta_1\beta_4\beta_1^{-1}] = 1 \\
\Leftrightarrow & [(\beta_4\beta_3\beta_4\beta_2\beta_4\beta_2^{-1})^{-1}\beta_2\beta_4\beta_3\beta_4\beta_2\beta_4\beta_2^{-1}, \beta_1\beta_2\beta_4\beta_3\beta_4\beta_2\beta_4\beta_2^{-1}\beta_1\beta_4\beta_1^{-1}] = 1 \\
\Leftrightarrow & [\beta_2, \beta_4\beta_3\beta_4\beta_2\beta_4\beta_2^{-1}\beta_1\beta_2\beta_4\beta_3\beta_4\underline{\beta_2\beta_4\beta_2^{-1}}\beta_1\beta_4\beta_1^{-1}(\beta_4\beta_3\beta_4\underline{\beta_2\beta_4\beta_2^{-1}})^{-1}] = 1 \\
\Leftrightarrow & [\beta_2, \beta_4\beta_3\beta_4\beta_2\beta_4\beta_1\beta_4\beta_3\beta_4\beta_1\beta_4\beta_1^{-1}\beta_4^{-1}\beta_3^{-1}\beta_4^{-1}] = 1.
\end{aligned}$$

Since

$$\begin{aligned}
& \beta_4\beta_3\beta_4\beta_2\beta_4\beta_1\beta_4\beta_3\underline{\beta_4\beta_1\beta_4\beta_1^{-1}}\beta_4^{-1}\beta_3^{-1}\beta_4^{-1} \\
&= \beta_4\beta_3\beta_4\beta_2\beta_4\beta_1\beta_4\beta_3\beta_1^{-1}\beta_4\beta_1\beta_4\beta_4^{-1}\beta_3^{-1}\beta_4^{-1} = \beta_4\beta_3\beta_4\beta_2\beta_4\underline{\beta_1\beta_4\beta_1^{-1}\beta_3\beta_4\beta_3^{-1}}\beta_1\beta_4^{-1} \\
&= \beta_4\beta_3\beta_4\beta_2\beta_4\beta_3\beta_4\beta_3^{-1}\beta_1\beta_4\beta_1^{-1}\beta_1\beta_4^{-1} = \beta_4\beta_3\beta_4\beta_2\beta_4\beta_3\beta_4\beta_3^{-1}\beta_1
\end{aligned}$$

and $[\beta_2, \beta_3^{-1}\beta_1] = 1$, (10) is equivalent to

$$\beta_2 \cdot \beta_4\beta_3\beta_4\beta_2\beta_4\beta_3\beta_4 = \beta_4\beta_3\beta_4\beta_2\beta_4\beta_3\beta_4 \cdot \beta_2.$$

This is shown as

$$\begin{aligned}
& \beta_2 \cdot \beta_4\beta_3\beta_4\beta_2\beta_4\beta_3\beta_4 \cdot \beta_2^{-1} = \beta_2\beta_4(\beta_2^{-1}\beta_2)\beta_3\beta_4\beta_2\beta_4\beta_3\beta_4\beta_2^{-1} \\
&= \beta_2\beta_4\beta_2^{-1}\beta_3\underline{\beta_2\beta_4\beta_2\beta_4}\beta_3\beta_4\beta_2^{-1} = \underline{\beta_2\beta_4\beta_2^{-1}\beta_3\beta_4(\beta_3^{-1}}\beta_3)\beta_2\beta_4\beta_2\beta_3\beta_4\beta_2^{-1} \\
&= \beta_3\beta_4\beta_3^{-1}\beta_2\beta_4\beta_2^{-1}\beta_3\beta_2\beta_4\beta_2\beta_3\beta_4\beta_2^{-1} = \beta_3\beta_4\beta_3^{-1}\beta_2\underline{\beta_4\beta_3\beta_4\beta_3}\beta_2\beta_4\beta_2^{-1} \\
&= \beta_3\beta_4\beta_3^{-1}\beta_2\beta_3\beta_4\beta_3\underline{\beta_4\beta_2\beta_4\beta_2^{-1}} = \beta_3\beta_4\beta_2\beta_3\underline{\beta_3^{-1}\beta_4\beta_3\beta_2^{-1}\beta_4\beta_2}\beta_4 \\
&= \beta_3\beta_4\beta_2\beta_3\beta_2^{-1}\beta_4\beta_2\beta_3^{-1}\beta_4\beta_3\beta_4 = \underline{\beta_3\beta_4\beta_3\beta_4}\beta_2\beta_3^{-1}\beta_4\beta_3\beta_4 \\
&= \beta_4\beta_3\beta_4\beta_3\beta_2\beta_3^{-1}\beta_4\beta_3\beta_4 = \beta_4\beta_3\beta_4\beta_2\beta_4\beta_3\beta_4.
\end{aligned}$$

Therefore, the proof is completed. □

*Remark* 4.3. Note that the relations (11) and (11') follow from others. Indeed, by (10), we have

$$\begin{aligned}
(11) \Leftrightarrow & [\alpha_1^{-1}\tilde{\alpha}_8\alpha_1, \tilde{\alpha}_3] = 1 \Leftrightarrow [\alpha_1^{-1}\tilde{\alpha}_8^{-1}\alpha_1, \tilde{\alpha}_3] = 1 \Leftrightarrow [\alpha_1^{-1} \cdot \alpha_7\alpha_8^{-1}\alpha_7^{-1} \cdot \alpha_1, \tilde{\alpha}_3] = 1 \\
\Leftrightarrow & [\beta_1^{-1} \cdot \beta_1\beta_4\beta_2\beta_4\beta_3\beta_4\beta_2\beta_4\beta_2^{-1} \cdot \beta_1, \beta_3] = 1 \Leftrightarrow [\beta_4\beta_2\beta_4\beta_3\beta_4\beta_2\beta_4\beta_2^{-1}\beta_1, \beta_3] = 1 \\
\Leftrightarrow & [\beta_4\beta_2\beta_4\beta_3\beta_4\beta_2\beta_4, \beta_3] = 1,
\end{aligned}$$

and it is shown by (A)–(C) as follows:

$$\begin{aligned}
& \beta_4\beta_2\beta_4\beta_3\beta_4\beta_2\beta_4 \cdot \beta_3 = (\beta_2\underline{\beta_2^{-1})\beta_4\beta_2\beta_4}\beta_3\beta_4\beta_2(\beta_3^{-1}\beta_3)\beta_4\beta_3 \\
&= \beta_2\beta_4\underline{\beta_2\beta_4\beta_2^{-1}\beta_3\beta_4\beta_3^{-1}}\beta_2\beta_3\beta_4\beta_3 = \beta_2\beta_4\beta_3\beta_4\beta_3^{-1}\beta_2\beta_4\beta_2^{-1}\beta_2\beta_3\beta_4\beta_3 \\
&= \beta_2\beta_4\beta_3\beta_4\beta_2\beta_3^{-1}\underline{\beta_4\beta_3\beta_4\beta_3} = \beta_2\beta_4\beta_3\beta_4\beta_2\beta_3^{-1}\beta_3\beta_4\beta_3\beta_4 \\
&= \beta_2\beta_4\beta_3\underline{\beta_4\beta_2\beta_4(\beta_2^{-1}}\beta_2)\beta_3\beta_4 = \beta_2\beta_4\beta_3\beta_2^{-1}\beta_4\beta_2\beta_4\beta_3\beta_2\beta_4 \\
&= \underline{\beta_2\beta_4\beta_2^{-1}\beta_3\beta_4(\beta_3^{-1}}\beta_3)\beta_2\beta_4\beta_3\beta_2\beta_4 = \beta_3\beta_4\beta_3^{-1}\beta_2\beta_4\beta_2^{-1}\beta_3\beta_2\beta_4\beta_3\beta_2\beta_4 \\
&= \beta_3\beta_4\beta_3^{-1}\beta_2\underline{\beta_4\beta_3\beta_4\beta_3}\beta_2\beta_4 = \beta_3\beta_4\beta_3^{-1}\beta_2\beta_3\beta_4\beta_3\beta_4\beta_2\beta_4 \\
&= \beta_3 \cdot \beta_4\beta_2\beta_4\beta_3\beta_4\beta_2\beta_4.
\end{aligned}$$

5. The covering space —the complement of hyperplane arrangement

**5.3. The case of $n=3$.** We give the proofs of Lemma 5.3, Theorem 5.5, and the derivations of the 72 relations by the Reidemeister-Schreier method.

**Lemma 5.3.** *The following 25 elements form a generator of the free group $K_1$:*

$$\begin{aligned}
&\gamma_0,\ \gamma_1^2,\ \gamma_2^2,\ \gamma_3^2,\ \gamma_1\gamma_0\gamma_1^{-1},\ \gamma_2\gamma_0\gamma_2^{-1},\ \gamma_3\gamma_0\gamma_3^{-1},\\
&\gamma_j\gamma_i\gamma_j^{-1}\gamma_i^{-1},\ \gamma_i\gamma_j\gamma_i\gamma_j^{-1},\ \gamma_i\gamma_j^2\gamma_i^{-1},\ \gamma_i\gamma_j\gamma_0(\gamma_i\gamma_j)^{-1},\\
&\gamma_1\gamma_3\gamma_2(\gamma_1\gamma_2\gamma_3)^{-1},\ \gamma_2\gamma_3\gamma_1(\gamma_1\gamma_2\gamma_3)^{-1},\\
&\gamma_1\gamma_2\gamma_3\gamma_1(\gamma_2\gamma_3)^{-1},\ \gamma_1\gamma_2\gamma_3\gamma_2(\gamma_1\gamma_3)^{-1},\ \gamma_1\gamma_2\gamma_3^2(\gamma_1\gamma_2)^{-1},\\
&\gamma_1\gamma_2\gamma_3\gamma_0(\gamma_1\gamma_2\gamma_3)^{-1},
\end{aligned}\tag{5.2}$$

*where $1\le i<j\le 3$.*

*Proof.* We put $B=\{\gamma_0,\gamma_1,\gamma_2,\gamma_3\}$ which is a generator of $K$. A generator of $K_1$ is given by

$$\{(tb)(\overline{tb})^{-1}\mid t\in T, b\in B, (tb)(\overline{tb})^{-1}\neq 1\}.$$

It is sufficient to compute all $(tb)(\overline{tb})^{-1}$.

(i) In the case $t=1$, since $(tb)(\overline{tb})^{-1}$'s are

$$\gamma_0\overline{\gamma_0}^{-1}=\gamma_0\cdot 1=\gamma_0,\ \gamma_1\overline{\gamma_1}^{-1}=1,\ \gamma_2\overline{\gamma_2}^{-1}=1,\ \gamma_3\overline{\gamma_3}^{-1}=1,$$

we obtain a generator $\gamma_0$.

(ii) In the case $t=\gamma_i$ $(1\le i\le 3)$, $(tb)(\overline{tb})^{-1}$'s are

$$\gamma_i^2(\overline{\gamma_i^2})^{-1},\ \gamma_i\gamma_j(\overline{\gamma_i\gamma_j})^{-1},\ \gamma_i\gamma_0(\overline{\gamma_i\gamma_0})^{-1},$$

where $j\neq i$. Since $\gamma_i^2,\gamma_i\gamma_0^{-1}\gamma_i^{-1}\in K_1$, we have $\overline{\gamma_i^2}=1$ and $\overline{\gamma_i\gamma_0}=\overline{\gamma_i\gamma_0^{-1}\gamma_i^{-1}\cdot\gamma_i\gamma_0}=\gamma_i$. Thus we obtain

$$\gamma_i^2(\overline{\gamma_i^2})^{-1}=\gamma_i^2,\ \gamma_i\gamma_0(\overline{\gamma_i\gamma_0})^{-1}=\gamma_i\gamma_0\gamma_i^{-1}.$$

We compute $\gamma_i\gamma_j(\overline{\gamma_i\gamma_j})^{-1}$. If $i<j$, then $\overline{\gamma_i\gamma_j}=\gamma_i\gamma_j$. If $i>j$, then $\overline{\gamma_i\gamma_j}=\overline{\gamma_j\gamma_i\gamma_j^{-1}\gamma_i^{-1}\cdot\gamma_i\gamma_j}=\gamma_j\gamma_i$. We thus have

$$\gamma_i\gamma_j(\overline{\gamma_i\gamma_j})^{-1}=\begin{cases}1 & (i<j)\\ \gamma_i\gamma_j\gamma_i^{-1}\gamma_j^{-1} & (i>j).\end{cases}$$

Therefore, we obtain generators $\gamma_i^2$, $\gamma_i\gamma_0\gamma_i^{-1}$ $(1\le i\le 3)$ and $\gamma_i\gamma_j\gamma_i^{-1}\gamma_j^{-1}$ $(1\le j<i\le 3)$.

(iii) In the case $t=\gamma_1\gamma_2$, $\overline{tb}$'s are

$$\begin{aligned}
&\overline{\gamma_1\gamma_2\gamma_0}=\overline{\gamma_1\gamma_2\gamma_0(\gamma_1\gamma_2)^{-1}\cdot\gamma_1\gamma_2}=\gamma_1\gamma_2,\\
&\overline{\gamma_1\gamma_2\gamma_1}=\overline{\gamma_1\gamma_2\gamma_1\gamma_2^{-1}\cdot\gamma_2}=\gamma_2,\quad \overline{\gamma_1\gamma_2\gamma_2}=\overline{\gamma_1\gamma_2^2\gamma_1^{-1}\cdot\gamma_1}=\gamma_1,\\
&\overline{\gamma_1\gamma_2\gamma_3}=\gamma_1\gamma_2\gamma_3,
\end{aligned}$$

and hence we obtain generators

$$\begin{aligned}
&\gamma_1\gamma_2\gamma_0(\overline{\gamma_1\gamma_2\gamma_0})^{-1}=\gamma_1\gamma_2\gamma_0(\gamma_1\gamma_2)^{-1},\\
&\gamma_1\gamma_2\gamma_1(\overline{\gamma_1\gamma_2\gamma_1})^{-1}=\gamma_1\gamma_2\gamma_1\gamma_2^{-1},\quad \gamma_1\gamma_2\gamma_2(\overline{\gamma_1\gamma_2\gamma_2})^{-1}=\gamma_1\gamma_2^2\gamma_1^{-1}.
\end{aligned}$$

(iv) In the case $t = \gamma_1\gamma_3$, $\overline{tb}$'s are

$$\overline{\gamma_1\gamma_3\gamma_0} = \overline{\gamma_1\gamma_3\gamma_0(\gamma_1\gamma_3)^{-1}\cdot\gamma_1\gamma_3} = \gamma_1\gamma_3,$$
$$\overline{\gamma_1\gamma_3\gamma_1} = \overline{\gamma_1\gamma_3\gamma_1\gamma_3^{-1}\cdot\gamma_3} = \gamma_3, \quad \overline{\gamma_1\gamma_3\gamma_3} = \overline{\gamma_1\gamma_3^2\gamma_1^{-1}\cdot\gamma_1} = \gamma_1,$$
$$\overline{\gamma_1\gamma_3\gamma_2} = \overline{\gamma_1\gamma_3\gamma_2\gamma_3^{-1}\gamma_2^{-1}\gamma_1^{-1}\cdot\gamma_1\gamma_2\gamma_3} = \gamma_1\gamma_2\gamma_3,$$

so we obtain generators

$$\gamma_1\gamma_3\gamma_0(\overline{\gamma_1\gamma_3\gamma_0})^{-1} = \gamma_1\gamma_3\gamma_0(\gamma_1\gamma_3)^{-1}, \quad \gamma_1\gamma_3\gamma_1(\overline{\gamma_1\gamma_3\gamma_1})^{-1} = \gamma_1\gamma_3\gamma_1\gamma_3^{-1},$$
$$\gamma_1\gamma_3\gamma_2(\overline{\gamma_1\gamma_3\gamma_2})^{-1} = \gamma_1\gamma_3\gamma_2(\gamma_1\gamma_2\gamma_3)^{-1}, \quad \gamma_1\gamma_3\gamma_3(\overline{\gamma_1\gamma_3\gamma_3})^{-1} = \gamma_1\gamma_3^2\gamma_1^{-1}.$$

(v) In the case $t = \gamma_2\gamma_3$, $\overline{tb}$'s are

$$\overline{\gamma_2\gamma_3\gamma_0} = \overline{\gamma_2\gamma_3\gamma_0(\gamma_2\gamma_3)^{-1}\cdot\gamma_2\gamma_3} = \gamma_2\gamma_3,$$
$$\overline{\gamma_2\gamma_3\gamma_1} = \overline{\gamma_2\gamma_3\gamma_1\gamma_3^{-1}\gamma_2^{-1}\gamma_1^{-1}\cdot\gamma_1\gamma_2\gamma_3} = \gamma_1\gamma_2\gamma_3,$$
$$\overline{\gamma_2\gamma_3\gamma_2} = \overline{\gamma_2\gamma_3\gamma_2\gamma_3^{-1}\cdot\gamma_3} = \gamma_3, \quad \overline{\gamma_2\gamma_3\gamma_3} = \overline{\gamma_2\gamma_3^2\gamma_2^{-1}\cdot\gamma_2} = \gamma_2,$$

and hence we obtain generators

$$\gamma_2\gamma_3\gamma_0(\overline{\gamma_2\gamma_3\gamma_0})^{-1} = \gamma_2\gamma_3\gamma_0(\gamma_2\gamma_3)^{-1}, \quad \gamma_2\gamma_3\gamma_1(\overline{\gamma_2\gamma_3\gamma_1})^{-1} = \gamma_2\gamma_3\gamma_1(\gamma_1\gamma_2\gamma_3)^{-1},$$
$$\gamma_2\gamma_3\gamma_2(\overline{\gamma_2\gamma_3\gamma_2})^{-1} = \gamma_2\gamma_3\gamma_2\gamma_3^{-1}, \quad \gamma_2\gamma_3\gamma_3(\overline{\gamma_2\gamma_3\gamma_3})^{-1} = \gamma_2\gamma_3^2\gamma_2^{-1}.$$

(vi) In the case $t = \gamma_1\gamma_2\gamma_3$, $\overline{tb}$'s are

$$\overline{\gamma_1\gamma_2\gamma_3\gamma_0} = \gamma_1\gamma_2\gamma_3, \quad \overline{\gamma_1\gamma_2\gamma_3\gamma_1} = \gamma_2\gamma_3,$$
$$\overline{\gamma_1\gamma_2\gamma_3\gamma_2} = \gamma_1\gamma_3, \quad \overline{\gamma_1\gamma_2\gamma_3\gamma_3} = \gamma_1\gamma_2,$$

so we obtain generators

$$\gamma_1\gamma_2\gamma_3\gamma_0(\overline{\gamma_1\gamma_2\gamma_3\gamma_0})^{-1} = \gamma_1\gamma_2\gamma_3\gamma_0(\gamma_1\gamma_2\gamma_3)^{-1},$$
$$\gamma_1\gamma_2\gamma_3\gamma_1(\overline{\gamma_1\gamma_2\gamma_3\gamma_1})^{-1} = \gamma_1\gamma_2\gamma_3\gamma_1(\gamma_2\gamma_3)^{-1},$$
$$\gamma_1\gamma_2\gamma_3\gamma_2(\overline{\gamma_1\gamma_2\gamma_3\gamma_2})^{-1} = \gamma_1\gamma_2\gamma_3\gamma_2(\gamma_1\gamma_3)^{-1},$$
$$\gamma_1\gamma_2\gamma_3\gamma_3(\overline{\gamma_1\gamma_2\gamma_3\gamma_3})^{-1} = \gamma_1\gamma_2\gamma_3^2(\gamma_1\gamma_2)^{-1}.$$

Therefore, we obtain the 25 generators. □

We put

$$R = \left\{ \begin{array}{cl} \gamma_i\gamma_j\gamma_i^{-1}\gamma_j^{-1} & (1 \le i < j \le 3), \\ \gamma_i\gamma_0\gamma_i^{-1}\gamma_j\gamma_0\gamma_j^{-1}\gamma_i\gamma_0^{-1}\gamma_i^{-1}\gamma_j\gamma_0^{-1}\gamma_j^{-1} & (1 \le i < j \le 3), \\ \gamma_i\gamma_0\gamma_i\gamma_0\gamma_i^{-1}\gamma_0^{-1}\gamma_i^{-1}\gamma_0^{-1} & (1 \le i \le 3) \end{array} \right\}$$

which generates the relations of $G$, that is, $G = \langle \gamma_0, \gamma_1, \gamma_2, \gamma_3 \mid R \rangle$. By the Reidemeister-Schreier method, $G_1$ is presented by the 25 generators (5.2) with relations of the form

$$trt^{-1}, \quad t \in T, \ r \in R.$$

Note that to obtain relations among the generators (5.2), we need to rewrite these relations by them.

We write down the 72($= 8 \cdot 9$) relations.

(i) $r = \gamma_i\gamma_j\gamma_i^{-1}\gamma_j^{-1}$ $(1 \le i < j \le 3)$.

(a) $t=1$. We obtain a relation

$$1=\gamma_j\gamma_i\gamma_j^{-1}\gamma_i^{-1}. \tag{5.3}$$

(b) $t=\gamma_k$. If $k=i$, then we obtain a relation

$$1=\gamma_i\cdot\gamma_i\gamma_j\gamma_i^{-1}\gamma_j^{-1}\cdot\gamma_i^{-1}=\gamma_i^2(\gamma_i\gamma_j\gamma_i\gamma_j^{-1})^{-1}. \tag{5.4}$$

If $k=j$, then we obtain a relation

$$1=\gamma_j\cdot\gamma_i\gamma_j\gamma_i^{-1}\gamma_j^{-1}\cdot\gamma_j^{-1}=\gamma_j\gamma_i\gamma_j^{-1}\gamma_i^{-1}\cdot\gamma_i\gamma_j^2\gamma_i^{-1}\cdot\gamma_j^{-2}. \tag{5.5}$$

If $(i,j,k)=(1,2,3)$, $(1,3,2)$ or $(2,3,1)$, then we obtain relations

$$\begin{aligned}1&=\gamma_3\cdot\gamma_1\gamma_2\gamma_1^{-1}\gamma_2^{-1}\cdot\gamma_3^{-1}\\&=\gamma_3\gamma_1\gamma_3^{-1}\gamma_1^{-1}\cdot\gamma_1\gamma_3\gamma_2(\gamma_1\gamma_2\gamma_3)^{-1}\cdot\left(\gamma_3\gamma_2\gamma_3^{-1}\gamma_2^{-1}\cdot\gamma_2\gamma_3\gamma_1(\gamma_1\gamma_2\gamma_3)^{-1}\right)^{-1},\end{aligned} \tag{5.6}$$

$$1=\gamma_2\cdot\gamma_1\gamma_3\gamma_1^{-1}\gamma_3^{-1}\cdot\gamma_2^{-1}=\gamma_2\gamma_1\gamma_2^{-1}\gamma_1^{-1}\cdot\left(\gamma_2\gamma_3\gamma_1(\gamma_1\gamma_2\gamma_3)^{-1}\right)^{-1}, \tag{5.7}$$

$$1=\gamma_1\cdot\gamma_2\gamma_3\gamma_2^{-1}\gamma_3^{-1}\cdot\gamma_1^{-1}=\left(\gamma_1\gamma_3\gamma_2(\gamma_1\gamma_2\gamma_3)^{-1}\right)^{-1}. \tag{5.8}$$

(c) $t=\gamma_k\gamma_l$. If $(i,j)=(k,l)$, then we obtain relations

$$\begin{aligned}1&=\gamma_i\gamma_j\cdot\gamma_i\gamma_j\gamma_i^{-1}\gamma_j^{-1}\cdot(\gamma_i\gamma_j)^{-1}\\&=\gamma_i\gamma_j\gamma_i\gamma_j^{-1}\cdot\gamma_j^2\cdot\gamma_i^{-2}\cdot(\gamma_i\gamma_j^2\gamma_i^{-1})^{-1}.\end{aligned} \tag{5.9}$$

If $(i,j)=(1,2)$ and $(k,l)=(1,3)$ or $(2,3)$, then we obtain relations

$$\begin{aligned}1&=\gamma_1\gamma_3\cdot\gamma_1\gamma_2\gamma_1^{-1}\gamma_2^{-1}\cdot(\gamma_1\gamma_3)^{-1}\\&=\gamma_1\gamma_3\gamma_1\gamma_3^{-1}\cdot\gamma_3\gamma_2\gamma_3^{-1}\gamma_2^{-1}\cdot\left(\gamma_1\gamma_3\gamma_2(\gamma_1\gamma_2\gamma_3)^{-1}\cdot\gamma_1\gamma_2\gamma_3\gamma_1(\gamma_2\gamma_3)^{-1}\right)^{-1},\end{aligned} \tag{5.10}$$

$$\begin{aligned}1&=\gamma_2\gamma_3\cdot\gamma_1\gamma_2\gamma_1^{-1}\gamma_2^{-1}\cdot(\gamma_2\gamma_3)^{-1}\\&=\gamma_2\gamma_3\gamma_1(\gamma_1\gamma_2\gamma_3)^{-1}\cdot\gamma_1\gamma_2\gamma_3\gamma_2(\gamma_1\gamma_3)^{-1}\cdot\gamma_1\gamma_3\gamma_1^{-1}\gamma_3^{-1}\cdot(\gamma_2\gamma_3\gamma_2\gamma_3^{-1})^{-1}.\end{aligned} \tag{5.11}$$

If $(i,j)=(1,3)$ and $(k,l)=(1,2)$ or $(2,3)$, then we obtain relations

$$1=\gamma_1\gamma_2\cdot\gamma_1\gamma_3\gamma_1^{-1}\gamma_3^{-1}\cdot(\gamma_1\gamma_2)^{-1}=\gamma_1\gamma_2\gamma_1\gamma_2^{-1}\cdot\left(\gamma_1\gamma_2\gamma_3\gamma_1(\gamma_2\gamma_3)^{-1}\right)^{-1}, \tag{5.12}$$

$$\begin{aligned}1&=\gamma_2\gamma_3\cdot\gamma_1\gamma_3\gamma_1^{-1}\gamma_3^{-1}\cdot(\gamma_2\gamma_3)^{-1}\\&=\gamma_2\gamma_3\gamma_1(\gamma_1\gamma_2\gamma_3)^{-1}\cdot\gamma_1\gamma_2\gamma_3^2(\gamma_1\gamma_2)^{-1}\cdot\left(\gamma_2\gamma_3^2\gamma_2^{-1}\cdot\gamma_2\gamma_1\gamma_2^{-1}\gamma_1^{-1}\right)^{-1}.\end{aligned} \tag{5.13}$$

If $(i,j)=(2,3)$ and $(k,l)=(1,2)$ or $(1,3)$, then we obtain relations

$$1=\gamma_1\gamma_2\cdot\gamma_2\gamma_3\gamma_2^{-1}\gamma_3^{-1}\cdot(\gamma_1\gamma_2)^{-1}=\gamma_1\gamma_2^2\gamma_1^{-1}\cdot\left(\gamma_1\gamma_2\gamma_3\gamma_2(\gamma_1\gamma_3)^{-1}\right)^{-1}, \tag{5.14}$$

$$\begin{aligned}1&=\gamma_1\gamma_3\cdot\gamma_2\gamma_3\gamma_2^{-1}\gamma_3^{-1}\cdot(\gamma_1\gamma_3)^{-1}\\&=\gamma_1\gamma_3\gamma_2(\gamma_1\gamma_2\gamma_3)^{-1}\cdot\gamma_1\gamma_2\gamma_3^2(\gamma_1\gamma_2)^{-1}\cdot\left(\gamma_1\gamma_3^2\gamma_1^{-1}\right)^{-1}.\end{aligned} \tag{5.15}$$

(d) $t=\gamma_1\gamma_2\gamma_3$. If $(i,j)=(1,2)$, $(1,3)$ or $(2,3)$, then we obtain relations

$$\begin{aligned}1&=\gamma_1\gamma_2\gamma_3\cdot\gamma_1\gamma_2\gamma_1^{-1}\gamma_2^{-1}\cdot(\gamma_1\gamma_2\gamma_3)^{-1}\\&=\gamma_1\gamma_2\gamma_3\gamma_1(\gamma_2\gamma_3)^{-1}\cdot\gamma_2\gamma_3\gamma_2\gamma_3^{-1}\cdot\left(\gamma_1\gamma_2\gamma_3\gamma_2(\gamma_1\gamma_3)^{-1}\cdot\gamma_1\gamma_3\gamma_1\gamma_3^{-1}\right)^{-1},\end{aligned} \tag{5.16}$$

$$1 = \gamma_1\gamma_2\gamma_3 \cdot \gamma_1\gamma_3\gamma_1^{-1}\gamma_3^{-1} \cdot (\gamma_1\gamma_2\gamma_3)^{-1}$$
$$= \gamma_1\gamma_2\gamma_3\gamma_1(\gamma_2\gamma_3)^{-1} \cdot \gamma_2\gamma_3^2\gamma_2^{-1} \cdot \left(\gamma_1\gamma_2\gamma_3^2(\gamma_1\gamma_2)^{-1} \cdot \gamma_1\gamma_2\gamma_1\gamma_2^{-1}\right)^{-1}, \tag{5.17}$$

$$1 = \gamma_1\gamma_2\gamma_3 \cdot \gamma_2\gamma_3\gamma_2^{-1}\gamma_3^{-1} \cdot (\gamma_1\gamma_2\gamma_3)^{-1}$$
$$= \gamma_1\gamma_2\gamma_3\gamma_2(\gamma_1\gamma_3)^{-1} \cdot \gamma_1\gamma_3^2\gamma_1^{-1} \cdot \left(\gamma_1\gamma_2\gamma_3^2(\gamma_1\gamma_2)^{-1} \cdot \gamma_1\gamma_2^2\gamma_1^{-1}\right)^{-1}. \tag{5.18}$$

(ii) $r = \gamma_i\gamma_0\gamma_i^{-1}\gamma_j\gamma_0\gamma_j^{-1}\gamma_i\gamma_0^{-1}\gamma_i^{-1}\gamma_j\gamma_0^{-1}\gamma_j^{-1}$ $(1 \leq i < j \leq 3)$.

(a) $t = 1$. We obtain a relation

$$1 = \gamma_i\gamma_0\gamma_i^{-1} \cdot \gamma_j\gamma_0\gamma_j^{-1} \cdot \gamma_i\gamma_0^{-1}\gamma_i^{-1} \cdot \gamma_j\gamma_0^{-1}\gamma_j^{-1}. \tag{5.19}$$

(b) $t = \gamma_k$. If $k = i$, then we obtain a relation

$$1 = \gamma_i \cdot \gamma_i\gamma_0\gamma_i^{-1}\gamma_j\gamma_0\gamma_j^{-1}\gamma_i\gamma_0^{-1}\gamma_i^{-1}\gamma_j\gamma_0^{-1}\gamma_j^{-1} \cdot \gamma_i^{-1}$$
$$= \gamma_i^2 \cdot \gamma_0 \cdot \gamma_i^{-2} \cdot \gamma_i\gamma_j\gamma_0(\gamma_i\gamma_j)^{-1} \cdot \gamma_i^2 \cdot \gamma_0^{-1} \cdot \gamma_i^{-2} \cdot \left(\gamma_i\gamma_j\gamma_0(\gamma_i\gamma_j)^{-1}\right)^{-1}. \tag{5.20}$$

If $k = j$, then we obtain a relation

$$1 = \gamma_j \cdot \gamma_i\gamma_0\gamma_i^{-1}\gamma_j\gamma_0\gamma_j^{-1}\gamma_i\gamma_0^{-1}\gamma_i^{-1}\gamma_j\gamma_0^{-1}\gamma_j^{-1} \cdot \gamma_j^{-1}$$
$$= \gamma_j\gamma_i\gamma_j^{-1}\gamma_i^{-1} \cdot \gamma_i\gamma_j\gamma_0(\gamma_i\gamma_j)^{-1} \cdot (\gamma_j\gamma_i\gamma_j^{-1}\gamma_i^{-1})^{-1} \cdot \gamma_j^2 \cdot \gamma_0 \cdot \gamma_j^{-2}$$
$$\cdot \gamma_j\gamma_i\gamma_j^{-1}\gamma_i^{-1} \cdot \left(\gamma_i\gamma_j\gamma_0(\gamma_i\gamma_j)^{-1}\right)^{-1} \cdot (\gamma_j\gamma_i\gamma_j^{-1}\gamma_i^{-1})^{-1} \cdot \gamma_j^2 \cdot \gamma_0^{-1} \cdot \gamma_j^{-2}. \tag{5.21}$$

If $(i, j, k) = (1, 2, 3)$, $(1, 3, 2)$ or $(2, 3, 1)$, then we obtain relations

$$1 = \gamma_3 \cdot \gamma_1\gamma_0\gamma_1^{-1}\gamma_2\gamma_0\gamma_2^{-1}\gamma_1\gamma_0^{-1}\gamma_1^{-1}\gamma_2\gamma_0^{-1}\gamma_2^{-1} \cdot \gamma_3^{-1}$$
$$= \gamma_3\gamma_1\gamma_3^{-1}\gamma_1^{-1} \cdot \gamma_1\gamma_3\gamma_0(\gamma_1\gamma_3)^{-1} \cdot (\gamma_3\gamma_1\gamma_3^{-1}\gamma_1^{-1})^{-1} \cdot \gamma_3\gamma_2\gamma_3^{-1}\gamma_2^{-1}$$
$$\cdot \gamma_2\gamma_3\gamma_0(\gamma_2\gamma_3)^{-1} \cdot (\gamma_3\gamma_2\gamma_3^{-1}\gamma_2^{-1})^{-1} \cdot \gamma_3\gamma_1\gamma_3^{-1}\gamma_1^{-1}$$
$$\cdot \left(\gamma_1\gamma_3\gamma_0(\gamma_1\gamma_3)^{-1}\right)^{-1} \cdot (\gamma_3\gamma_1\gamma_3^{-1}\gamma_1^{-1})^{-1} \cdot \gamma_3\gamma_2\gamma_3^{-1}\gamma_2^{-1}$$
$$\cdot \left(\gamma_2\gamma_3\gamma_0(\gamma_2\gamma_3)^{-1}\right)^{-1} \cdot (\gamma_3\gamma_2\gamma_3^{-1}\gamma_2^{-1})^{-1}, \tag{5.22}$$

$$1 = \gamma_2 \cdot \gamma_1\gamma_0\gamma_1^{-1}\gamma_3\gamma_0\gamma_3^{-1}\gamma_1\gamma_0^{-1}\gamma_1^{-1}\gamma_3\gamma_0^{-1}\gamma_3^{-1} \cdot \gamma_2^{-1}$$
$$= \gamma_2\gamma_1\gamma_2^{-1}\gamma_1^{-1} \cdot \gamma_1\gamma_2\gamma_0(\gamma_1\gamma_2)^{-1} \cdot (\gamma_2\gamma_1\gamma_2^{-1}\gamma_1^{-1})^{-1}$$
$$\cdot \gamma_2\gamma_3\gamma_0(\gamma_2\gamma_3)^{-1} \cdot \gamma_2\gamma_1\gamma_2^{-1}\gamma_1^{-1} \cdot \left(\gamma_1\gamma_2\gamma_0(\gamma_1\gamma_2)^{-1}\right)^{-1}$$
$$\cdot (\gamma_2\gamma_1\gamma_2^{-1}\gamma_1^{-1})^{-1} \cdot \left(\gamma_2\gamma_3\gamma_0(\gamma_2\gamma_3)^{-1}\right)^{-1}, \tag{5.23}$$

$$1 = \gamma_1 \cdot \gamma_2\gamma_0\gamma_2^{-1}\gamma_3\gamma_0\gamma_3^{-1}\gamma_2\gamma_0^{-1}\gamma_2^{-1}\gamma_3\gamma_0^{-1}\gamma_3^{-1} \cdot \gamma_1^{-1}$$
$$= \gamma_1\gamma_2\gamma_0(\gamma_1\gamma_2)^{-1} \cdot \gamma_1\gamma_3\gamma_0(\gamma_1\gamma_3)^{-1}$$
$$\cdot \left(\gamma_1\gamma_2\gamma_0(\gamma_1\gamma_2)^{-1}\right)^{-1} \cdot \left(\gamma_1\gamma_3\gamma_0(\gamma_1\gamma_3)^{-1}\right)^{-1}. \tag{5.24}$$

(c) $t=\gamma_k\gamma_l$. If $(i,j)=(k,l)$, then we obtain a relation

$$
\begin{aligned}
1 &= \gamma_i\gamma_j\cdot\gamma_i\gamma_0\gamma_i^{-1}\gamma_j\gamma_0\gamma_j^{-1}\gamma_i\gamma_0^{-1}\gamma_i^{-1}\gamma_j\gamma_0^{-1}\gamma_j^{-1}\cdot(\gamma_i\gamma_j)^{-1}\\
&= \gamma_i\gamma_j\gamma_i\gamma_j^{-1}\cdot\gamma_j\gamma_0\gamma_j^{-1}\cdot(\gamma_i\gamma_j\gamma_i\gamma_j^{-1})^{-1}\cdot\gamma_i\gamma_j^2\gamma_i^{-1}\\
&\quad\cdot\gamma_i\gamma_0\gamma_i^{-1}\cdot(\gamma_i\gamma_j^2\gamma_i^{-1})^{-1}\cdot\gamma_i\gamma_j\gamma_i\gamma_j^{-1}\cdot(\gamma_j\gamma_0\gamma_j^{-1})^{-1}\\
&\quad\cdot(\gamma_i\gamma_j\gamma_i\gamma_j^{-1})^{-1}\cdot\gamma_i\gamma_j^2\gamma_i^{-1}\cdot(\gamma_i\gamma_0\gamma_i^{-1})^{-1}\cdot(\gamma_i\gamma_j^2\gamma_i^{-1})^{-1}.
\end{aligned}
\tag{5.25}
$$

If $(i,j)=(1,2)$ and $(k,l)=(1,3)$ or $(2,3)$, then we obtain relations

$$
\begin{aligned}
1 &= \gamma_1\gamma_3\cdot\gamma_1\gamma_0\gamma_1^{-1}\gamma_2\gamma_0\gamma_2^{-1}\gamma_1\gamma_0^{-1}\gamma_1^{-1}\gamma_2\gamma_0^{-1}\gamma_2^{-1}\cdot(\gamma_1\gamma_3)^{-1}\\
&= \gamma_1\gamma_3\gamma_1\gamma_3^{-1}\cdot\gamma_3\gamma_0\gamma_3^{-1}\cdot(\gamma_1\gamma_3\gamma_1\gamma_3^{-1})^{-1}\cdot\gamma_1\gamma_3\gamma_2(\gamma_1\gamma_2\gamma_3)^{-1}\\
&\quad\cdot\gamma_1\gamma_2\gamma_3\gamma_0(\gamma_1\gamma_2\gamma_3)^{-1}\cdot\left(\gamma_1\gamma_3\gamma_2(\gamma_1\gamma_2\gamma_3)^{-1}\right)^{-1}\cdot\gamma_1\gamma_3\gamma_1\gamma_3^{-1}\\
&\quad\cdot(\gamma_3\gamma_0\gamma_3^{-1})^{-1}\cdot(\gamma_1\gamma_3\gamma_1\gamma_3^{-1})^{-1}\cdot\gamma_1\gamma_3\gamma_2(\gamma_1\gamma_2\gamma_3)^{-1}\\
&\quad\cdot\left(\gamma_1\gamma_2\gamma_3\gamma_0(\gamma_1\gamma_2\gamma_3)^{-1}\right)^{-1}\cdot\left(\gamma_1\gamma_3\gamma_2(\gamma_1\gamma_2\gamma_3)^{-1}\right)^{-1},
\end{aligned}
\tag{5.26}
$$

$$
\begin{aligned}
1 &= \gamma_2\gamma_3\cdot\gamma_1\gamma_0\gamma_1^{-1}\gamma_2\gamma_0\gamma_2^{-1}\gamma_1\gamma_0^{-1}\gamma_1^{-1}\gamma_2\gamma_0^{-1}\gamma_2^{-1}\cdot(\gamma_2\gamma_3)^{-1}\\
&= \gamma_2\gamma_3\gamma_1(\gamma_1\gamma_2\gamma_3)^{-1}\cdot\gamma_1\gamma_2\gamma_3\gamma_0(\gamma_1\gamma_2\gamma_3)^{-1}\cdot\left(\gamma_2\gamma_3\gamma_1(\gamma_1\gamma_2\gamma_3)^{-1}\right)^{-1}\\
&\quad\cdot\gamma_2\gamma_3\gamma_2\gamma_3^{-1}\cdot\gamma_3\gamma_0\gamma_3^{-1}\cdot(\gamma_2\gamma_3\gamma_2\gamma_3^{-1})^{-1}\cdot\gamma_2\gamma_3\gamma_1(\gamma_1\gamma_2\gamma_3)^{-1}\\
&\quad\cdot\left(\gamma_1\gamma_2\gamma_3\gamma_0(\gamma_1\gamma_2\gamma_3)^{-1}\right)^{-1}\cdot\left(\gamma_2\gamma_3\gamma_1(\gamma_1\gamma_2\gamma_3)^{-1}\right)^{-1}\\
&\quad\cdot\gamma_2\gamma_3\gamma_2\gamma_3^{-1}\cdot(\gamma_3\gamma_0\gamma_3^{-1})^{-1}\cdot(\gamma_2\gamma_3\gamma_2\gamma_3^{-1})^{-1}.
\end{aligned}
\tag{5.27}
$$

If $(i,j)=(1,3)$ and $(k,l)=(1,2)$ or $(2,3)$, then we obtain relations

$$
\begin{aligned}
1 &= \gamma_1\gamma_2\cdot\gamma_1\gamma_0\gamma_1^{-1}\gamma_3\gamma_0\gamma_3^{-1}\gamma_1\gamma_0^{-1}\gamma_1^{-1}\gamma_3\gamma_0^{-1}\gamma_3^{-1}\cdot(\gamma_1\gamma_2)^{-1}\\
&= \gamma_1\gamma_2\gamma_1\gamma_2^{-1}\cdot\gamma_2\gamma_0\gamma_2^{-1}\cdot(\gamma_1\gamma_2\gamma_1\gamma_2^{-1})^{-1}\cdot\gamma_1\gamma_2\gamma_3\gamma_0(\gamma_1\gamma_2\gamma_3)^{-1}\\
&\quad\cdot\gamma_1\gamma_2\gamma_1\gamma_2^{-1}\cdot(\gamma_2\gamma_0\gamma_2^{-1})^{-1}\cdot(\gamma_1\gamma_2\gamma_1\gamma_2^{-1})^{-1}\\
&\quad\cdot\left(\gamma_1\gamma_2\gamma_3\gamma_0(\gamma_1\gamma_2\gamma_3)^{-1}\right)^{-1},
\end{aligned}
\tag{5.28}
$$

$$
\begin{aligned}
1 &= \gamma_2\gamma_3\cdot\gamma_1\gamma_0\gamma_1^{-1}\gamma_3\gamma_0\gamma_3^{-1}\gamma_1\gamma_0^{-1}\gamma_1^{-1}\gamma_3\gamma_0^{-1}\gamma_3^{-1}\cdot(\gamma_2\gamma_3)^{-1}\\
&= \gamma_2\gamma_3\gamma_1(\gamma_1\gamma_2\gamma_3)^{-1}\cdot\gamma_1\gamma_2\gamma_3\gamma_0(\gamma_1\gamma_2\gamma_3)^{-1}\cdot\left(\gamma_2\gamma_3\gamma_1(\gamma_1\gamma_2\gamma_3)^{-1}\right)^{-1}\\
&\quad\cdot\gamma_2\gamma_3^2\gamma_2^{-1}\cdot\gamma_2\gamma_0\gamma_2^{-1}\cdot(\gamma_2\gamma_3^2\gamma_2^{-1})^{-1}\cdot\gamma_2\gamma_3\gamma_1(\gamma_1\gamma_2\gamma_3)^{-1}\\
&\quad\cdot\left(\gamma_1\gamma_2\gamma_3\gamma_0(\gamma_1\gamma_2\gamma_3)^{-1}\right)^{-1}\cdot\left(\gamma_2\gamma_3\gamma_1(\gamma_1\gamma_2\gamma_3)^{-1}\right)^{-1}\\
&\quad\cdot\gamma_2\gamma_3^2\gamma_2^{-1}\cdot(\gamma_2\gamma_0\gamma_2^{-1})^{-1}\cdot(\gamma_2\gamma_3^2\gamma_2^{-1})^{-1}.
\end{aligned}
\tag{5.29}
$$

If $(i,j)=(2,3)$ and $(k,l)=(1,2)$ or $(1,3)$, then we obtain relations

$$
\begin{aligned}
1 &= \gamma_1\gamma_2\cdot\gamma_2\gamma_0\gamma_2^{-1}\gamma_3\gamma_0\gamma_3^{-1}\gamma_2\gamma_0^{-1}\gamma_2^{-1}\gamma_3\gamma_0^{-1}\gamma_3^{-1}\cdot(\gamma_1\gamma_2)^{-1}\\
&= \gamma_1\gamma_2^2\gamma_1^{-1}\cdot\gamma_1\gamma_0\gamma_1^{-1}\cdot(\gamma_1\gamma_2^2\gamma_1^{-1})^{-1}\cdot\gamma_1\gamma_2\gamma_3\gamma_0(\gamma_1\gamma_2\gamma_3)^{-1}\\
&\quad\cdot\gamma_1\gamma_2^2\gamma_1^{-1}\cdot(\gamma_1\gamma_0\gamma_1^{-1})^{-1}\cdot(\gamma_1\gamma_2^2\gamma_1^{-1})^{-1}\cdot\left(\gamma_1\gamma_2\gamma_3\gamma_0(\gamma_1\gamma_2\gamma_3)^{-1}\right)^{-1},
\end{aligned}
\tag{5.30}
$$

$$
\begin{aligned}
1 &= \gamma_1\gamma_3 \cdot \gamma_2\gamma_0\gamma_2^{-1}\gamma_3\gamma_0\gamma_3^{-1}\gamma_2\gamma_0^{-1}\gamma_2^{-1}\gamma_3\gamma_0^{-1}\gamma_3^{-1} \cdot (\gamma_1\gamma_3)^{-1} \\
&= \gamma_1\gamma_3\gamma_2(\gamma_1\gamma_2\gamma_3)^{-1} \cdot \gamma_1\gamma_2\gamma_3\gamma_0(\gamma_1\gamma_2\gamma_3)^{-1} \cdot \big(\gamma_1\gamma_3\gamma_2(\gamma_1\gamma_2\gamma_3)^{-1}\big)^{-1} \\
&\quad \cdot \gamma_1\gamma_3^2\gamma_1^{-1} \cdot \gamma_1\gamma_0\gamma_1^{-1} \cdot (\gamma_1\gamma_3^2\gamma_1^{-1})^{-1} \cdot \gamma_1\gamma_3\gamma_2(\gamma_1\gamma_2\gamma_3)^{-1} \\
&\quad \cdot \big(\gamma_1\gamma_2\gamma_3\gamma_0(\gamma_1\gamma_2\gamma_3)^{-1}\big)^{-1} \cdot \big(\gamma_1\gamma_3\gamma_2(\gamma_1\gamma_2\gamma_3)^{-1}\big)^{-1} \\
&\quad \cdot \gamma_1\gamma_3^2\gamma_1^{-1} \cdot (\gamma_1\gamma_0\gamma_1^{-1})^{-1} \cdot (\gamma_1\gamma_3^2\gamma_1^{-1})^{-1}.
\end{aligned} \tag{5.31}
$$

(d) $t = \gamma_1\gamma_2\gamma_3$. If $(i,j) = (1,2)$, $(1,3)$ or $(2,3)$, then we obtain relations

$$
\begin{aligned}
1 &= \gamma_1\gamma_2\gamma_3 \cdot \gamma_1\gamma_0\gamma_1^{-1}\gamma_2\gamma_0\gamma_2^{-1}\gamma_1\gamma_0^{-1}\gamma_1^{-1}\gamma_2\gamma_0^{-1}\gamma_2^{-1} \cdot (\gamma_1\gamma_2\gamma_3)^{-1} \\
&= \gamma_1\gamma_2\gamma_3\gamma_1(\gamma_2\gamma_3)^{-1} \cdot \gamma_2\gamma_3\gamma_0(\gamma_2\gamma_3)^{-1} \cdot \big(\gamma_1\gamma_2\gamma_3\gamma_1(\gamma_2\gamma_3)^{-1}\big)^{-1} \\
&\quad \cdot \gamma_1\gamma_2\gamma_3\gamma_2(\gamma_1\gamma_3)^{-1} \cdot \gamma_1\gamma_3\gamma_0(\gamma_1\gamma_3)^{-1} \cdot \big(\gamma_1\gamma_2\gamma_3\gamma_2(\gamma_1\gamma_3)^{-1}\big)^{-1} \\
&\quad \cdot \gamma_1\gamma_2\gamma_3\gamma_1(\gamma_2\gamma_3)^{-1} \cdot \big(\gamma_2\gamma_3\gamma_0(\gamma_2\gamma_3)^{-1}\big)^{-1} \cdot \big(\gamma_1\gamma_2\gamma_3\gamma_1(\gamma_2\gamma_3)^{-1}\big)^{-1} \\
&\quad \cdot \gamma_1\gamma_2\gamma_3\gamma_2(\gamma_1\gamma_3)^{-1} \cdot \big(\gamma_1\gamma_3\gamma_0(\gamma_1\gamma_3)^{-1}\big)^{-1} \cdot \big(\gamma_1\gamma_2\gamma_3\gamma_2(\gamma_1\gamma_3)^{-1}\big)^{-1},
\end{aligned} \tag{5.32}
$$

$$
\begin{aligned}
1 &= \gamma_1\gamma_2\gamma_3 \cdot \gamma_1\gamma_0\gamma_1^{-1}\gamma_3\gamma_0\gamma_3^{-1}\gamma_1\gamma_0^{-1}\gamma_1^{-1}\gamma_3\gamma_0^{-1}\gamma_3^{-1} \cdot (\gamma_1\gamma_2\gamma_3)^{-1} \\
&= \gamma_1\gamma_2\gamma_3\gamma_1(\gamma_2\gamma_3)^{-1} \cdot \gamma_2\gamma_3\gamma_0(\gamma_2\gamma_3)^{-1} \cdot \big(\gamma_1\gamma_2\gamma_3\gamma_1(\gamma_2\gamma_3)^{-1}\big)^{-1} \\
&\quad \cdot \gamma_1\gamma_2\gamma_3^2(\gamma_1\gamma_2)^{-1} \cdot \gamma_1\gamma_2\gamma_0(\gamma_1\gamma_2)^{-1} \cdot \big(\gamma_1\gamma_2\gamma_3^2(\gamma_1\gamma_2)^{-1}\big)^{-1} \\
&\quad \cdot \gamma_1\gamma_2\gamma_3\gamma_1(\gamma_2\gamma_3)^{-1} \cdot \big(\gamma_2\gamma_3\gamma_0(\gamma_2\gamma_3)^{-1}\big)^{-1} \cdot \big(\gamma_1\gamma_2\gamma_3\gamma_1(\gamma_2\gamma_3)^{-1}\big)^{-1} \\
&\quad \cdot \gamma_1\gamma_2\gamma_3^2(\gamma_1\gamma_2)^{-1} \cdot \big(\gamma_1\gamma_2\gamma_0(\gamma_1\gamma_2)^{-1}\big)^{-1} \cdot \big(\gamma_1\gamma_2\gamma_3^2(\gamma_1\gamma_2)^{-1}\big)^{-1},
\end{aligned} \tag{5.33}
$$

$$
\begin{aligned}
1 &= \gamma_1\gamma_2\gamma_3 \cdot \gamma_2\gamma_0\gamma_2^{-1}\gamma_3\gamma_0\gamma_3^{-1}\gamma_2\gamma_0^{-1}\gamma_2^{-1}\gamma_3\gamma_0^{-1}\gamma_3^{-1} \cdot (\gamma_1\gamma_2\gamma_3)^{-1} \\
&= \gamma_1\gamma_2\gamma_3\gamma_2(\gamma_1\gamma_3)^{-1} \cdot \gamma_1\gamma_3\gamma_0(\gamma_1\gamma_3)^{-1} \cdot \big(\gamma_1\gamma_2\gamma_3\gamma_2(\gamma_1\gamma_3)^{-1}\big)^{-1} \\
&\quad \cdot \gamma_1\gamma_2\gamma_3^2(\gamma_1\gamma_2)^{-1} \cdot \gamma_1\gamma_2\gamma_0(\gamma_1\gamma_2)^{-1} \cdot \big(\gamma_1\gamma_2\gamma_3^2(\gamma_1\gamma_2)^{-1}\big)^{-1} \\
&\quad \cdot \gamma_1\gamma_2\gamma_3\gamma_2(\gamma_1\gamma_3)^{-1} \cdot \big(\gamma_1\gamma_3\gamma_0(\gamma_1\gamma_3)^{-1}\big)^{-1} \cdot \big(\gamma_1\gamma_2\gamma_3\gamma_2(\gamma_1\gamma_3)^{-1}\big)^{-1} \\
&\quad \cdot \gamma_1\gamma_2\gamma_3^2(\gamma_1\gamma_2)^{-1} \cdot \big(\gamma_1\gamma_2\gamma_0(\gamma_1\gamma_2)^{-1}\big)^{-1} \cdot \big(\gamma_1\gamma_2\gamma_3^2(\gamma_1\gamma_2)^{-1}\big)^{-1}.
\end{aligned} \tag{5.34}
$$

(iii) $r = \gamma_i\gamma_0\gamma_i\gamma_0\gamma_i^{-1}\gamma_0^{-1}\gamma_i^{-1}\gamma_0^{-1}$ $(1 \le i \le 3)$.
(a) $t = 1$. We obtain a relation

$$
\begin{aligned}
1 &= \gamma_i\gamma_0\gamma_i\gamma_0\gamma_i^{-1}\gamma_0^{-1}\gamma_i^{-1}\gamma_0^{-1} \\
&= \gamma_i\gamma_0\gamma_i^{-1} \cdot \gamma_i^2 \cdot \gamma_0 \cdot \gamma_i^{-2} \cdot (\gamma_i\gamma_0\gamma_i^{-1})^{-1} \cdot \gamma_0^{-1}.
\end{aligned} \tag{5.35}
$$

(b) $t = \gamma_k$. If $k < i$, then we obtain a relation

$$
\begin{aligned}
1 &= \gamma_k \cdot \gamma_i\gamma_0\gamma_i\gamma_0\gamma_i^{-1}\gamma_0^{-1}\gamma_i^{-1}\gamma_0^{-1} \cdot \gamma_k^{-1} \\
&= \gamma_k\gamma_i\gamma_0(\gamma_k\gamma_i)^{-1} \cdot \gamma_k\gamma_i^2\gamma_k^{-1} \cdot \gamma_k\gamma_0\gamma_k^{-1} \\
&\quad \cdot \big(\gamma_k\gamma_0\gamma_k^{-1} \cdot \gamma_k\gamma_i\gamma_0(\gamma_k\gamma_i)^{-1} \cdot \gamma_k\gamma_i^2\gamma_k^{-1}\big)^{-1} \qquad (k < i).
\end{aligned} \tag{5.36}
$$

If $k = i$, then we obtain a relation

$$\begin{aligned}1 &= \gamma_i \cdot \gamma_i\gamma_0\gamma_i\gamma_0\gamma_i^{-1}\gamma_0^{-1}\gamma_i^{-1}\gamma_0^{-1} \cdot \gamma_i^{-1} \\ &= \gamma_i^2 \cdot \gamma_0 \cdot \gamma_i\gamma_0\gamma_i^{-1} \cdot \gamma_0^{-1} \cdot \gamma_i^{-2} \cdot (\gamma_i\gamma_0\gamma_i^{-1})^{-1}.\end{aligned} \tag{5.37}$$

If $k > i$, then we obtain a relation

$$\begin{aligned}1 &= \gamma_k \cdot \gamma_i\gamma_0\gamma_i\gamma_0\gamma_i^{-1}\gamma_0^{-1}\gamma_i^{-1}\gamma_0^{-1} \cdot \gamma_k^{-1} \\ &= \gamma_k\gamma_i\gamma_k^{-1}\gamma_i^{-1} \cdot \gamma_i\gamma_k\gamma_0(\gamma_i\gamma_k)^{-1} \cdot \gamma_i\gamma_k\gamma_i\gamma_k^{-1} \cdot \gamma_k\gamma_0\gamma_k^{-1} \\ &\quad \cdot \big(\gamma_k\gamma_0\gamma_k^{-1} \cdot \gamma_k\gamma_i\gamma_k^{-1}\gamma_i^{-1} \cdot \gamma_i\gamma_k\gamma_0(\gamma_i\gamma_k)^{-1} \cdot \gamma_i\gamma_k\gamma_i\gamma_k^{-1}\big)^{-1} \quad (k > i).\end{aligned} \tag{5.38}$$

(c) $t = \gamma_k\gamma_l$. If $i = k$, then we obtain a relation

$$\begin{aligned}1 &= \gamma_i\gamma_l \cdot \gamma_i\gamma_0\gamma_i\gamma_0\gamma_i^{-1}\gamma_0^{-1}\gamma_i^{-1}\gamma_0^{-1} \cdot (\gamma_i\gamma_l)^{-1} \\ &= \gamma_i\gamma_l\gamma_i\gamma_l^{-1} \cdot \gamma_l\gamma_0\gamma_l^{-1} \cdot \gamma_l\gamma_i\gamma_l^{-1}\gamma_i^{-1} \cdot \gamma_i\gamma_l\gamma_0(\gamma_i\gamma_l)^{-1} \\ &\quad \cdot \big(\gamma_i\gamma_l\gamma_0(\gamma_i\gamma_l)^{-1} \cdot \gamma_i\gamma_l\gamma_i\gamma_l^{-1} \cdot \gamma_l\gamma_0\gamma_l^{-1} \cdot \gamma_l\gamma_i\gamma_l^{-1}\gamma_i^{-1}\big)^{-1} \quad (i < l).\end{aligned} \tag{5.39}$$

If $i = l$, then we obtain a relation

$$\begin{aligned}1 &= \gamma_k\gamma_i \cdot \gamma_i\gamma_0\gamma_i\gamma_0\gamma_i^{-1}\gamma_0^{-1}\gamma_i^{-1}\gamma_0^{-1} \cdot (\gamma_k\gamma_i)^{-1} \\ &= \gamma_k\gamma_i^2\gamma_k^{-1} \cdot \gamma_k\gamma_0\gamma_k^{-1} \cdot \gamma_k\gamma_i\gamma_0(\gamma_k\gamma_i)^{-1} \\ &\quad \cdot \big(\gamma_k\gamma_i\gamma_0(\gamma_k\gamma_i)^{-1} \cdot \gamma_k\gamma_i^2\gamma_k^{-1} \cdot \gamma_k\gamma_0\gamma_k^{-1}\big)^{-1} \quad (k < i).\end{aligned} \tag{5.40}$$

If $(i, k, l) = (1, 2, 3)$, $(2, 1, 3)$ or $(3, 1, 2)$, then we obtain relations

$$\begin{aligned}1 &= \gamma_2\gamma_3 \cdot \gamma_1\gamma_0\gamma_1\gamma_0\gamma_1^{-1}\gamma_0^{-1}\gamma_1^{-1}\gamma_0^{-1} \cdot (\gamma_2\gamma_3)^{-1} \\ &= \gamma_2\gamma_3\gamma_1(\gamma_1\gamma_2\gamma_3)^{-1} \cdot \gamma_1\gamma_2\gamma_3\gamma_0(\gamma_1\gamma_2\gamma_3)^{-1} \\ &\quad \cdot \gamma_1\gamma_2\gamma_3\gamma_1(\gamma_2\gamma_3)^{-1} \cdot \gamma_2\gamma_3\gamma_0(\gamma_2\gamma_3)^{-1} \\ &\quad \cdot \big(\gamma_2\gamma_3\gamma_0(\gamma_2\gamma_3)^{-1} \cdot \gamma_2\gamma_3\gamma_1(\gamma_1\gamma_2\gamma_3)^{-1} \\ &\qquad \cdot \gamma_1\gamma_2\gamma_3\gamma_0(\gamma_1\gamma_2\gamma_3)^{-1} \cdot \gamma_1\gamma_2\gamma_3\gamma_1(\gamma_2\gamma_3)^{-1}\big)^{-1},\end{aligned} \tag{5.41}$$

$$\begin{aligned}1 &= \gamma_1\gamma_3 \cdot \gamma_2\gamma_0\gamma_2\gamma_0\gamma_2^{-1}\gamma_0^{-1}\gamma_2^{-1}\gamma_0^{-1} \cdot (\gamma_1\gamma_3)^{-1} \\ &= \gamma_1\gamma_3\gamma_2(\gamma_1\gamma_2\gamma_3)^{-1} \cdot \gamma_1\gamma_2\gamma_3\gamma_0(\gamma_1\gamma_2\gamma_3)^{-1} \\ &\quad \cdot \gamma_1\gamma_2\gamma_3\gamma_2(\gamma_1\gamma_3)^{-1} \cdot \gamma_1\gamma_3\gamma_0(\gamma_1\gamma_3)^{-1} \\ &\quad \cdot \big(\gamma_1\gamma_3\gamma_0(\gamma_1\gamma_3)^{-1} \cdot \gamma_1\gamma_3\gamma_2(\gamma_1\gamma_2\gamma_3)^{-1} \\ &\qquad \cdot \gamma_1\gamma_2\gamma_3\gamma_0(\gamma_1\gamma_2\gamma_3)^{-1} \cdot \gamma_1\gamma_2\gamma_3\gamma_2(\gamma_1\gamma_3)^{-1}\big)^{-1},\end{aligned} \tag{5.42}$$

$$\begin{aligned}1 &= \gamma_1\gamma_2 \cdot \gamma_3\gamma_0\gamma_3\gamma_0\gamma_3^{-1}\gamma_0^{-1}\gamma_3^{-1}\gamma_0^{-1} \cdot (\gamma_1\gamma_2)^{-1} \\ &= \gamma_1\gamma_2\gamma_3\gamma_0(\gamma_1\gamma_2\gamma_3)^{-1} \cdot \gamma_1\gamma_2\gamma_3^2(\gamma_1\gamma_2)^{-1} \cdot \gamma_1\gamma_2\gamma_0(\gamma_1\gamma_2)^{-1} \\ &\quad \cdot \big(\gamma_1\gamma_2\gamma_0(\gamma_1\gamma_2)^{-1} \cdot \gamma_1\gamma_2\gamma_3\gamma_0(\gamma_1\gamma_2\gamma_3)^{-1} \cdot \gamma_1\gamma_2\gamma_3^2(\gamma_1\gamma_2)^{-1}\big)^{-1}.\end{aligned} \tag{5.43}$$

(d) $t = \gamma_1\gamma_2\gamma_3$. We take $k, l$ such that $\{i, k, l\} = \{1, 2, 3\}$ and $k < l$. Then we obtain a relation

$$
\begin{aligned}
1 &= \gamma_1\gamma_2\gamma_3 \cdot \gamma_i\gamma_0\gamma_i\gamma_0\gamma_i^{-1}\gamma_0^{-1}\gamma_i^{-1}\gamma_0^{-1} \cdot (\gamma_1\gamma_2\gamma_3)^{-1} \\
&= \gamma_1\gamma_2\gamma_3\gamma_i(\gamma_k\gamma_l)^{-1} \cdot \gamma_k\gamma_l\gamma_0(\gamma_k\gamma_l)^{-1} \\
&\quad \cdot \gamma_k\gamma_l\gamma_i(\gamma_1\gamma_2\gamma_3)^{-1} \cdot \gamma_1\gamma_2\gamma_3\gamma_0(\gamma_1\gamma_2\gamma_3)^{-1} \\
&\quad \cdot \big(\gamma_1\gamma_2\gamma_3\gamma_0(\gamma_1\gamma_2\gamma_3)^{-1} \cdot \gamma_1\gamma_2\gamma_3\gamma_i(\gamma_k\gamma_l)^{-1} \\
&\qquad \cdot \gamma_k\gamma_l\gamma_0(\gamma_k\gamma_l)^{-1} \cdot \gamma_k\gamma_l\gamma_i(\gamma_1\gamma_2\gamma_3)^{-1}\big)^{-1}.
\end{aligned}
\tag{5.44}
$$

By using the generators (5.1), the relations (5.3)–(5.44) are reduced to the following ones:

$$
\begin{aligned}
&[\lambda_i, \lambda_j] = 1 \quad (1 \le i < j \le 3), \\
&[\lambda_0^{(i)}, \lambda_0^{(j)}] = 1 \quad (1 \le i < j \le 3),
\end{aligned}
$$

$$
[\lambda_0^{(ij)}, \lambda_i\lambda_0\lambda_i^{-1}] = 1,\ [\lambda_0^{(ij)}, \lambda_j\lambda_0\lambda_j^{-1}] = 1 \quad (1 \le i < j \le 3), \tag{5.51}
$$

$$
[\lambda_0^{(12)}, \lambda_0^{(13)}] = 1,\ [\lambda_0^{(12)}, \lambda_0^{(23)}] = 1,\ [\lambda_0^{(13)}, \lambda_0^{(23)}] = 1,
$$

$$
[\lambda_i\lambda_0^{(j)}\lambda_i^{-1}, \lambda_j\lambda_0^{(i)}\lambda_j^{-1}] = 1 \quad (1 \le i < j \le 3), \tag{5.52}
$$

$$
[\lambda_0^{(123)}, \lambda_i\lambda_0^{(j)}\lambda_i^{-1}] = 1 \quad (1 \le i \ne j \le 3), \tag{5.53}
$$

$$
\begin{aligned}
&[\lambda_1\lambda_0^{(23)}\lambda_1^{-1}, \lambda_2\lambda_0^{(13)}\lambda_2^{-1}] = 1, \\
&[\lambda_1\lambda_0^{(23)}\lambda_1^{-1}, \lambda_3\lambda_0^{(12)}\lambda_3^{-1}] = 1,\ [\lambda_2\lambda_0^{(13)}\lambda_2^{-1}, \lambda_3\lambda_0^{(12)}\lambda_3^{-1}] = 1,
\end{aligned}
\tag{5.54}
$$

$$
\begin{aligned}
&\lambda_0^{(i)}\lambda_i\lambda_0 = \lambda_0\lambda_0^{(i)}\lambda_i = \lambda_i\lambda_0\lambda_0^{(i)} \quad (1 \le i \le 3), \\
&\lambda_j\lambda_0^{(i)}\lambda_0^{(ij)} = \lambda_0^{(i)}\lambda_0^{(ij)}\lambda_j = \lambda_0^{(ij)}\lambda_j\lambda_0^{(i)} \quad (1 \le i < j \le 3), \\
&\lambda_i\lambda_0^{(j)}\lambda_0^{(ij)} = \lambda_0^{(j)}\lambda_0^{(ij)}\lambda_i = \lambda_0^{(ij)}\lambda_i\lambda_0^{(j)} \quad (1 \le i < j \le 3), \\
&\lambda_i\lambda_0^{(jk)}\lambda_0^{(123)} = \lambda_0^{(jk)}\lambda_0^{(123)}\lambda_i = \lambda_0^{(123)}\lambda_i\lambda_0^{(jk)} \quad (\{i, j, k\} = \{1, 2, 3\},\ j < k).
\end{aligned}
$$

**Theorem 5.5.** *The fundamental group $\pi_1(\tilde{X}^{(3)}) = G_1$ has a presentation by 11 generators*

$$
\lambda_1, \lambda_2, \lambda_3, \lambda_0, \lambda_0^{(1)}, \lambda_0^{(2)}, \lambda_0^{(3)}, \lambda_0^{(12)}, \lambda_0^{(13)}, \lambda_0^{(23)}, \lambda_0^{(123)},
$$

*and 27 defining relations*

$$[\lambda_i, \lambda_j] = 1 \quad (1 \le i < j \le 3), \tag{5.55}$$

$$[\lambda_0^{(i)}, \lambda_0^{(j)}] = 1 \quad (1 \le i < j \le 3), \tag{5.56}$$

$$[\lambda_0^{(12)}, \lambda_0^{(13)}] = 1,\ [\lambda_0^{(12)}, \lambda_0^{(23)}] = 1,\ [\lambda_0^{(13)}, \lambda_0^{(23)}] = 1, \tag{5.57}$$

$$[\lambda_0^{(ij)}, \lambda_i \lambda_0 \lambda_i^{-1}] = 1 \quad (1 \le i < j \le 3), \tag{5.58}$$

$$\begin{aligned} &[\lambda_0^{(123)}, \lambda_1 \lambda_0^{(2)} \lambda_1^{-1}] = 1, \\ &[\lambda_0^{(123)}, \lambda_2 \lambda_0^{(3)} \lambda_2^{-1}] = 1,\ [\lambda_0^{(123)}, \lambda_3 \lambda_0^{(1)} \lambda_3^{-1}] = 1, \end{aligned} \tag{5.59}$$

$$\begin{aligned} &\lambda_0^{(i)} \lambda_i \lambda_0 = \lambda_0 \lambda_0^{(i)} \lambda_i = \lambda_i \lambda_0 \lambda_0^{(i)} \quad (1 \le i \le 3), \\ &\lambda_i \lambda_0^{(j)} \lambda_0^{(ij)} = \lambda_0^{(j)} \lambda_0^{(ij)} \lambda_i = \lambda_0^{(ij)} \lambda_i \lambda_0^{(j)} \quad (1 \le i < j \le 3), \\ &\lambda_j \lambda_0^{(i)} \lambda_0^{(ij)} = \lambda_0^{(i)} \lambda_0^{(ij)} \lambda_j = \lambda_0^{(ij)} \lambda_j \lambda_0^{(i)} \quad (1 \le i < j \le 3), \\ &\lambda_i \lambda_0^{(jk)} \lambda_0^{(123)} = \lambda_0^{(jk)} \lambda_0^{(123)} \lambda_i = \lambda_0^{(123)} \lambda_i \lambda_0^{(jk)} \\ &\qquad (\{i, j, k\} = \{1, 2, 3\},\ j < k). \end{aligned} \tag{5.60}$$

*Proof.* We need to show that the relations

- the second relation of (5.51),
- (5.52),
- (5.53) for $(i, j) = (3, 2), (1, 3), (2, 1)$, and
- (5.54)

follow from (5.55)–(5.60). First, we consider the second relation of (5.51). By (5.60), we have

$$\lambda_0^{(i)} \lambda_i \lambda_0 = \lambda_0 \lambda_0^{(i)} \lambda_i, \quad \lambda_j \lambda_0^{(i)} \lambda_0^{(ij)} = \lambda_0^{(ij)} \lambda_j \lambda_0^{(i)}.$$

Then (5.58) implies

$$\begin{aligned} &[\lambda_0^{(ij)}, \lambda_j \lambda_0 \lambda_j^{-1}] = \lambda_j \cdot [\lambda_j^{-1} \lambda_0^{(ij)} \lambda_j, \lambda_0] \cdot \lambda_j^{-1} \\ &= \lambda_j \cdot [\lambda_0^{(i)} \lambda_0^{(ij)} {\lambda_0^{(i)}}^{-1}, \lambda_0] \cdot \lambda_j^{-1} = \lambda_j \lambda_0^{(i)} \cdot [\lambda_0^{(ij)}, {\lambda_0^{(i)}}^{-1} \lambda_0 \lambda_0^{(i)}] \cdot (\lambda_j \lambda_0^{(i)})^{-1} \\ &= \lambda_j \lambda_0^{(i)} \cdot [\lambda_0^{(ij)}, \lambda_i \lambda_0 \lambda_i^{-1}] \cdot (\lambda_j \lambda_0^{(i)})^{-1} = \lambda_j \lambda_0^{(i)} \cdot 1 \cdot (\lambda_j \lambda_0^{(i)})^{-1} = 1. \end{aligned}$$

Second, we consider (5.52). By (5.55) and (5.60), we have

$$\lambda_i \lambda_j = \lambda_j \lambda_i, \quad \lambda_0^{(i)} \lambda_i \lambda_0 = \lambda_i \lambda_0 \lambda_0^{(i)}.$$

Then (5.56) implies

$$\begin{aligned} &[\lambda_i \lambda_0^{(j)} \lambda_i^{-1}, \lambda_j \lambda_0^{(i)} \lambda_j^{-1}] = \lambda_i \lambda_j \cdot [\lambda_j^{-1} \lambda_0^{(j)} \lambda_j, \lambda_i^{-1} \lambda_0^{(i)} \lambda_i] \cdot (\lambda_i \lambda_j)^{-1} \\ &= \lambda_i \lambda_j \cdot [\lambda_0 \lambda_0^{(j)} \lambda_0^{-1}, \lambda_0 \lambda_0^{(i)} \lambda_0^{-1}] \cdot (\lambda_i \lambda_j)^{-1} \\ &= \lambda_i \lambda_j \lambda_0 \cdot [\lambda_0^{(j)}, \lambda_0^{(i)}] \cdot (\lambda_i \lambda_j \lambda_0)^{-1} = \lambda_i \lambda_j \lambda_0 \cdot 1 \cdot (\lambda_i \lambda_j \lambda_0)^{-1} = 1. \end{aligned}$$

Third, we consider (5.53) for $(i, j) = (3, 2)$. By (5.60), we have

$$\lambda_0^{(2)} \lambda_0^{(12)} \lambda_1 = \lambda_0^{(12)} \lambda_1 \lambda_0^{(2)}, \quad \lambda_3 \lambda_0^{(12)} \lambda_0^{(123)} = \lambda_0^{(123)} \lambda_3 \lambda_0^{(12)}.$$

Then the first relation of (5.59) implies

$$\begin{aligned}
&[\lambda_0^{(123)}, \lambda_3\lambda_0^{(2)}\lambda_3^{-1}] = \lambda_3 \cdot [\lambda_3^{-1}\lambda_0^{(123)}\lambda_3, \lambda_0^{(2)}] \cdot \lambda_3^{-1} \\
&= \lambda_3 \cdot [\lambda_0^{(12)}\lambda_0^{(123)}\lambda_0^{(12)^{-1}}, \lambda_0^{(2)}] \cdot \lambda_3^{-1} \\
&= \lambda_3\lambda_0^{(12)} \cdot [\lambda_0^{(123)}, \lambda_0^{(12)^{-1}}\lambda_0^{(2)}\lambda_0^{(12)}] \cdot (\lambda_3\lambda_0^{(12)})^{-1} \\
&= \lambda_3\lambda_0^{(12)} \cdot [\lambda_0^{(123)}, \lambda_1\lambda_0^{(2)}\lambda_1^{-1}] \cdot (\lambda_3\lambda_0^{(12)})^{-1} = \lambda_3\lambda_0^{(12)} \cdot 1 \cdot (\lambda_3\lambda_0^{(12)})^{-1} = 1.
\end{aligned}$$

In similar ways, the case of $(i,j) = (1,3)$ and $(i,j) = (2,1)$ follow from the second and third relations of (5.59), respectively. Finally, we consider the first relation of (5.54). By (5.60), we have

$$\lambda_0^{(23)}\lambda_0^{(123)}\lambda_1 = \lambda_0^{(123)}\lambda_1\lambda_0^{(23)}, \quad \lambda_0^{(13)}\lambda_0^{(123)}\lambda_2 = \lambda_0^{(123)}\lambda_2\lambda_0^{(13)}.$$

Then the third relation of (5.57) implies

$$\begin{aligned}
&[\lambda_1\lambda_0^{(23)}\lambda_1^{-1}, \lambda_2\lambda_0^{(13)}\lambda_2^{-1}] = [\lambda_0^{(123)^{-1}}\lambda_0^{(23)}\lambda_0^{(123)}, \lambda_0^{(123)^{-1}}\lambda_0^{(13)}\lambda_0^{(123)}] \\
&= \lambda_0^{(123)^{-1}} \cdot [\lambda_0^{(23)}, \lambda_0^{(13)}] \cdot \lambda_0^{(123)} = \lambda_0^{(123)^{-1}} \cdot 1 \cdot \lambda_0^{(123)} = 1.
\end{aligned}$$

Similarly, the second and third relations of (5.54) follow from the second and first ones of (5.57), respectively. □

## Appendix A. Induction step of the proof of Theorem 1.2 for small $n$

Since the proof of Theorem 1.2 in Section 3 is complicated, we give the proof for the case $n = 6$, $p = 3$, $q = 2$ as an example. (In fact, the proof for $p = 2$, $q = 3$ is easier than that for this case. Since this example uses all claims in our proof, we explain it.) We show

$$[(\gamma_1\gamma_2\gamma_5)\gamma_0(\gamma_1\gamma_2\gamma_5)^{-1}, (\gamma_3\gamma_4)\gamma_0(\gamma_3\gamma_4)^{-1}] = 1,$$

by using the induction hypothesis:

$$\begin{aligned}
[\gamma_i\gamma_0\gamma_i^{-1}, \gamma_j\gamma_0\gamma_j^{-1}] &= 1, \\
[(\gamma_i\gamma_j)\gamma_0(\gamma_i\gamma_j)^{-1}, \gamma_k\gamma_0\gamma_k^{-1}] &= 1, \\
[(\gamma_i\gamma_j\gamma_k)\gamma_0(\gamma_i\gamma_j\gamma_k)^{-1}, \gamma_l\gamma_0\gamma_l^{-1}] &= 1, \\
[(\gamma_i\gamma_j)\gamma_0(\gamma_i\gamma_j)^{-1}, (\gamma_k\gamma_l)\gamma_0(\gamma_k\gamma_l)^{-1}] &= 1
\end{aligned}$$

(the cardinality of $\{i,j,k,l\} \subset \{1,\ldots,6\}$ is 4), and relations obtained from them by conjugation.

Since the loop $\ell_{11000}$ changes into

$$(\ell_{00000}\ell_{10000}\ell_{01000}\ell_{11000}\ell_{00100}\ell_{10100}\ell_{01100}\ell_{11100}\ell_{00010}\ell_{10010}\ell_{01010}\ell_{11010}\ell_{00110}\ell_{10110}\ell_{01110}\ell_{11110}) \\ \cdot \ell_{11001} \cdot (\cdots)^{-1},$$

by moving of $L(\theta)$ for $k = 5$ (see Figures 23 and 24). This implies the relation

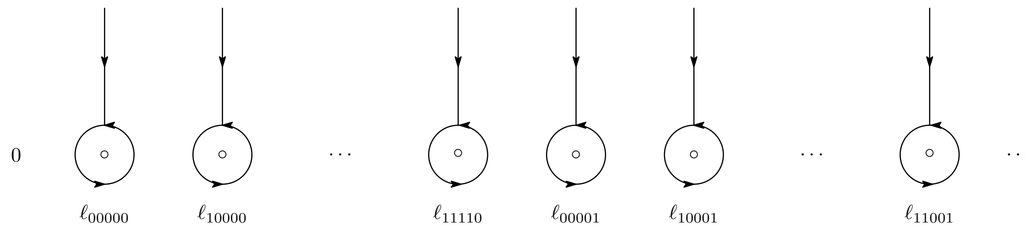


FIGURE 23. Loops in $P(0)$

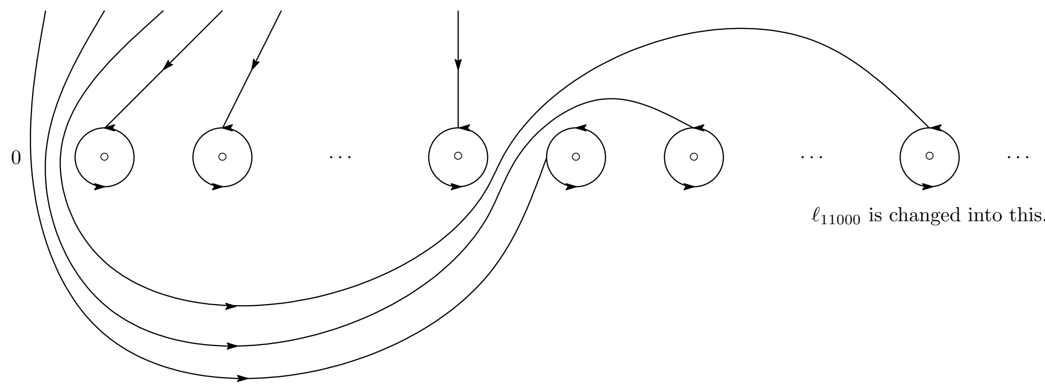


FIGURE 24. Loops in $P(1)$

$$\begin{aligned}
&(\gamma_1\gamma_2)\gamma_0(\gamma_1\gamma_2)^{-1}\\
&= \gamma_5[\gamma_0 \cdot \gamma_1\gamma_0\gamma_1^{-1} \cdot \gamma_2\gamma_0\gamma_2^{-1} \cdot (\gamma_1\gamma_2)\gamma_0(\gamma_1\gamma_2)^{-1}\\
&\cdot \gamma_3\gamma_0\gamma_3^{-1} \cdot (\gamma_1\gamma_3)\gamma_0(\gamma_1\gamma_3)^{-1} \cdot (\gamma_2\gamma_3)\gamma_0(\gamma_2\gamma_3)^{-1} \cdot (\gamma_1\gamma_2\gamma_3)\gamma_0(\gamma_1\gamma_2\gamma_3)^{-1}\\
&\cdot \gamma_4\gamma_0\gamma_4^{-1} \cdot (\gamma_1\gamma_4)\gamma_0(\gamma_1\gamma_4)^{-1} \cdot (\gamma_2\gamma_4)\gamma_0(\gamma_2\gamma_4)^{-1} \cdot (\gamma_1\gamma_2\gamma_4)\gamma_0(\gamma_1\gamma_2\gamma_4)^{-1}\\
&\cdot (\gamma_3\gamma_4)\gamma_0(\gamma_3\gamma_4)^{-1} \cdot (\gamma_1\gamma_3\gamma_4)\gamma_0(\gamma_1\gamma_3\gamma_4)^{-1} \cdot (\gamma_2\gamma_3\gamma_4)\gamma_0(\gamma_2\gamma_3\gamma_4)^{-1} \cdot (\gamma_1\gamma_2\gamma_3\gamma_4)\gamma_0(\gamma_1\gamma_2\gamma_3\gamma_4)^{-1}]\\
&\cdot (\gamma_1\gamma_2\gamma_5)\gamma_0(\gamma_1\gamma_2\gamma_5)^{-1} \cdot [\cdots]^{-1}\gamma_5^{-1}.
\end{aligned}$$

It is equivalent to

$$\begin{aligned}
&\gamma_0^{-1}\gamma_5^{-1}(\gamma_1\gamma_2)\gamma_0(\gamma_1\gamma_2)^{-1}\gamma_5\gamma_0[\gamma_1\gamma_0\gamma_1^{-1} \cdot \gamma_2\gamma_0\gamma_2^{-1} \cdot (\gamma_1\gamma_2)\gamma_0(\gamma_1\gamma_2)^{-1} \cdots]\\
&= [\gamma_1\gamma_0\gamma_1^{-1} \cdot \gamma_2\gamma_0\gamma_2^{-1} \cdot (\gamma_1\gamma_2)\gamma_0(\gamma_1\gamma_2)^{-1}\\
&\cdot \gamma_3\gamma_0\gamma_3^{-1} \cdot (\gamma_1\gamma_3)\gamma_0(\gamma_1\gamma_3)^{-1} \cdot (\gamma_2\gamma_3)\gamma_0(\gamma_2\gamma_3)^{-1} \cdot (\gamma_1\gamma_2\gamma_3)\gamma_0(\gamma_1\gamma_2\gamma_3)^{-1}\\
&\cdot \gamma_4\gamma_0\gamma_4^{-1} \cdot (\gamma_1\gamma_4)\gamma_0(\gamma_1\gamma_4)^{-1} \cdot (\gamma_2\gamma_4)\gamma_0(\gamma_2\gamma_4)^{-1} \cdot (\gamma_1\gamma_2\gamma_4)\gamma_0(\gamma_1\gamma_2\gamma_4)^{-1}\\
&\cdot (\gamma_3\gamma_4)\gamma_0(\gamma_3\gamma_4)^{-1} \cdot (\gamma_1\gamma_3\gamma_4)\gamma_0(\gamma_1\gamma_3\gamma_4)^{-1} \cdot (\gamma_2\gamma_3\gamma_4)\gamma_0(\gamma_2\gamma_3\gamma_4)^{-1} \cdot (\gamma_1\gamma_2\gamma_3\gamma_4)\gamma_0(\gamma_1\gamma_2\gamma_3\gamma_4)^{-1}]\\
&\cdot (\gamma_1\gamma_2\gamma_5)\gamma_0(\gamma_1\gamma_2\gamma_5)^{-1}.
\end{aligned}$$

Since

$$[\gamma_5\gamma_0\gamma_5^{-1}, (\gamma_3\gamma_4)\gamma_0(\gamma_3\gamma_4)^{-1}] = 1,\ [(\gamma_1\gamma_5)\gamma_0(\gamma_1\gamma_5)^{-1}, (\gamma_3\gamma_4)\gamma_0(\gamma_3\gamma_4)^{-1}] = 1,$$
$$[(\gamma_2\gamma_5)\gamma_0(\gamma_2\gamma_5)^{-1}, (\gamma_3\gamma_4)\gamma_0(\gamma_3\gamma_4)^{-1}] = 1$$

imply that $(\gamma_1\gamma_2\gamma_5)\gamma_0(\gamma_1\gamma_2\gamma_5)^{-1}$ commutes with the last three factors in $[\cdots]$, we obtain

(A.1)

$$
\begin{aligned}
&\gamma_0^{-1}\gamma_5^{-1}(\gamma_1\gamma_2)\gamma_0(\gamma_1\gamma_2)^{-1}\gamma_5\gamma_0\\
&\cdot[\gamma_1\gamma_0\gamma_1^{-1}\cdot\gamma_2\gamma_0\gamma_2^{-1}\cdot(\gamma_1\gamma_2)\gamma_0(\gamma_1\gamma_2)^{-1}\\
&\cdot\gamma_3\gamma_0\gamma_3^{-1}\cdot(\gamma_1\gamma_3)\gamma_0(\gamma_1\gamma_3)^{-1}\cdot(\gamma_2\gamma_3)\gamma_0(\gamma_2\gamma_3)^{-1}\cdot(\gamma_1\gamma_2\gamma_3)\gamma_0(\gamma_1\gamma_2\gamma_3)^{-1}\\
&\cdot\gamma_4\gamma_0\gamma_4^{-1}\cdot(\gamma_1\gamma_4)\gamma_0(\gamma_1\gamma_4)^{-1}\cdot(\gamma_2\gamma_4)\gamma_0(\gamma_2\gamma_4)^{-1}\cdot(\gamma_1\gamma_2\gamma_4)\gamma_0(\gamma_1\gamma_2\gamma_4)^{-1}\cdot(\gamma_3\gamma_4)\gamma_0(\gamma_3\gamma_4)^{-1}]\\
&=[\gamma_1\gamma_0\gamma_1^{-1}\cdot\gamma_2\gamma_0\gamma_2^{-1}\cdot(\gamma_1\gamma_2)\gamma_0(\gamma_1\gamma_2)^{-1}\\
&\quad\cdot\gamma_3\gamma_0\gamma_3^{-1}\cdot(\gamma_1\gamma_3)\gamma_0(\gamma_1\gamma_3)^{-1}\cdot(\gamma_2\gamma_3)\gamma_0(\gamma_2\gamma_3)^{-1}\cdot(\gamma_1\gamma_2\gamma_3)\gamma_0(\gamma_1\gamma_2\gamma_3)^{-1}\\
&\quad\cdot\gamma_4\gamma_0\gamma_4^{-1}\cdot(\gamma_1\gamma_4)\gamma_0(\gamma_1\gamma_4)^{-1}\cdot(\gamma_2\gamma_4)\gamma_0(\gamma_2\gamma_4)^{-1}\cdot(\gamma_1\gamma_2\gamma_4)\gamma_0(\gamma_1\gamma_2\gamma_4)^{-1}\\
&\quad\cdot(\gamma_3\gamma_4)\gamma_0(\gamma_3\gamma_4)^{-1}]\cdot(\gamma_1\gamma_2\gamma_5)\gamma_0(\gamma_1\gamma_2\gamma_5)^{-1}
\end{aligned}
$$

(cf. Claim 3.6). On the other hand,

$$
\begin{aligned}
&\gamma_0^{-1}\gamma_5^{-1}(\gamma_1\gamma_2)\gamma_0(\gamma_1\gamma_2)^{-1}\gamma_5\gamma_0\\
&=\gamma_0^{-1}(\gamma_1\gamma_2)\cdot\gamma_5^{-1}\gamma_0\gamma_5\cdot(\gamma_1\gamma_2)^{-1}\gamma_0(\gamma_1\gamma_2)\cdot(\gamma_1\gamma_2)^{-1}\\
&=\gamma_0^{-1}(\gamma_1\gamma_2)\cdot(\gamma_1\gamma_2)^{-1}\gamma_0(\gamma_1\gamma_2)\cdot\gamma_5^{-1}\gamma_0\gamma_5\cdot(\gamma_1\gamma_2)^{-1}\\
&=(\gamma_1\gamma_2)\cdot\gamma_5^{-1}\gamma_0\gamma_5\cdot(\gamma_1\gamma_2)^{-1}
\end{aligned}
$$

commutes with

$$\gamma_1\gamma_0\gamma_1^{-1}=(\gamma_1\gamma_2)\gamma_2^{-1}\gamma_0\gamma_2(\gamma_1\gamma_2)^{-1},\quad \gamma_2\gamma_0\gamma_2^{-1}=(\gamma_1\gamma_2)\gamma_1^{-1}\gamma_0\gamma_1(\gamma_1\gamma_2)^{-1}$$

(cf. Claim 3.7). Thus, (A.1) is

(A.2)

$$
\begin{aligned}
&(\gamma_1\gamma_2)\cdot\gamma_5^{-1}\gamma_0\gamma_5\cdot(\gamma_1\gamma_2)^{-1}\cdot(\gamma_1\gamma_2)\gamma_0(\gamma_1\gamma_2)^{-1}\\
&\cdot\gamma_3\gamma_0\gamma_3^{-1}\cdot(\gamma_1\gamma_3)\gamma_0(\gamma_1\gamma_3)^{-1}\cdot(\gamma_2\gamma_3)\gamma_0(\gamma_2\gamma_3)^{-1}\cdot(\gamma_1\gamma_2\gamma_3)\gamma_0(\gamma_1\gamma_2\gamma_3)^{-1}\\
&\cdot\gamma_4\gamma_0\gamma_4^{-1}\cdot(\gamma_1\gamma_4)\gamma_0(\gamma_1\gamma_4)^{-1}\cdot(\gamma_2\gamma_4)\gamma_0(\gamma_2\gamma_4)^{-1}\cdot(\gamma_1\gamma_2\gamma_4)\gamma_0(\gamma_1\gamma_2\gamma_4)^{-1}\cdot(\gamma_3\gamma_4)\gamma_0(\gamma_3\gamma_4)^{-1}\\
&=(\gamma_1\gamma_2)\gamma_0(\gamma_1\gamma_2)^{-1}\\
&\quad\cdot\gamma_3\gamma_0\gamma_3^{-1}\cdot(\gamma_1\gamma_3)\gamma_0(\gamma_1\gamma_3)^{-1}\cdot(\gamma_2\gamma_3)\gamma_0(\gamma_2\gamma_3)^{-1}\cdot(\gamma_1\gamma_2\gamma_3)\gamma_0(\gamma_1\gamma_2\gamma_3)^{-1}\\
&\quad\cdot\gamma_4\gamma_0\gamma_4^{-1}\cdot(\gamma_1\gamma_4)\gamma_0(\gamma_1\gamma_4)^{-1}\cdot(\gamma_2\gamma_4)\gamma_0(\gamma_2\gamma_4)^{-1}\cdot(\gamma_1\gamma_2\gamma_4)\gamma_0(\gamma_1\gamma_2\gamma_4)^{-1}\\
&\quad\cdot(\gamma_3\gamma_4)\gamma_0(\gamma_3\gamma_4)^{-1}\cdot(\gamma_1\gamma_2\gamma_5)\gamma_0(\gamma_1\gamma_2\gamma_5)^{-1}.
\end{aligned}
$$

Further, by

$$
\begin{aligned}
&(\gamma_1\gamma_2)\cdot\gamma_5^{-1}\gamma_0\gamma_5\cdot(\gamma_1\gamma_2)^{-1}\\
&=(\gamma_1\gamma_2)\cdot\gamma_5^{-1}\gamma_0\gamma_5\gamma_0\cdot\gamma_0^{-1}\cdot(\gamma_1\gamma_2)^{-1}\\
&=(\gamma_1\gamma_2)\cdot\gamma_0\gamma_5\gamma_0\gamma_5^{-1}\cdot\gamma_0^{-1}\cdot(\gamma_1\gamma_2)^{-1}\\
&=(\gamma_1\gamma_2)\gamma_0(\gamma_1\gamma_2)^{-1}\cdot(\gamma_1\gamma_2\gamma_5)\gamma_0(\gamma_1\gamma_2\gamma_5)^{-1}\cdot(\gamma_1\gamma_2)\gamma_0^{-1}(\gamma_1\gamma_2)^{-1},
\end{aligned}
$$

the first line of (A.2) equals to

$$(\gamma_1\gamma_2)\gamma_0(\gamma_1\gamma_2)^{-1}\cdot(\gamma_1\gamma_2\gamma_5)\gamma_0(\gamma_1\gamma_2\gamma_5)^{-1}.$$

Since $(\gamma_1\gamma_2\gamma_5)\gamma_0(\gamma_1\gamma_2\gamma_5)^{-1}$ commutes with

$$\gamma_3\gamma_0\gamma_3^{-1}\cdot(\gamma_1\gamma_3)\gamma_0(\gamma_1\gamma_3)^{-1}\cdot(\gamma_2\gamma_3)\gamma_0(\gamma_2\gamma_3)^{-1}\cdot(\gamma_1\gamma_2\gamma_3)\gamma_0(\gamma_1\gamma_2\gamma_3)^{-1}$$
$$\cdot\gamma_4\gamma_0\gamma_4^{-1}\cdot(\gamma_1\gamma_4)\gamma_0(\gamma_1\gamma_4)^{-1}\cdot(\gamma_2\gamma_4)\gamma_0(\gamma_2\gamma_4)^{-1}\cdot(\gamma_1\gamma_2\gamma_4)\gamma_0(\gamma_1\gamma_2\gamma_4)^{-1}$$

(cf. Claim 3.6), the left-hand side of (A.2) is equal to

$$(\gamma_1\gamma_2)\gamma_0(\gamma_1\gamma_2)^{-1}$$
$$\cdot\gamma_3\gamma_0\gamma_3^{-1}\cdot(\gamma_1\gamma_3)\gamma_0(\gamma_1\gamma_3)^{-1}\cdot(\gamma_2\gamma_3)\gamma_0(\gamma_2\gamma_3)^{-1}\cdot(\gamma_1\gamma_2\gamma_3)\gamma_0(\gamma_1\gamma_2\gamma_3)^{-1}$$
$$\cdot\gamma_4\gamma_0\gamma_4^{-1}\cdot(\gamma_1\gamma_4)\gamma_0(\gamma_1\gamma_4)^{-1}\cdot(\gamma_2\gamma_4)\gamma_0(\gamma_2\gamma_4)^{-1}\cdot(\gamma_1\gamma_2\gamma_4)\gamma_0(\gamma_1\gamma_2\gamma_4)^{-1}$$
$$\cdot(\gamma_1\gamma_2\gamma_5)\gamma_0(\gamma_1\gamma_2\gamma_5)^{-1}\cdot(\gamma_3\gamma_4)\gamma_0(\gamma_3\gamma_4)^{-1}.$$

Therefore, the equality (A.2) implies

$$(\gamma_1\gamma_2\gamma_5)\gamma_0(\gamma_1\gamma_2\gamma_5)^{-1}\cdot(\gamma_3\gamma_4)\gamma_0(\gamma_3\gamma_4)^{-1}=(\gamma_3\gamma_4)\gamma_0(\gamma_3\gamma_4)^{-1}\cdot(\gamma_1\gamma_2\gamma_5)\gamma_0(\gamma_1\gamma_2\gamma_5)^{-1}.$$

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