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μ-CONSTANT DEFORMATIONS OF FUNCTIONS ON AN ICIS

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ABSTRACT. We study deformations of holomorphic function germs $f\colon (X,0) \to \mathbb{C}$, where $(X,0)$ is an ICIS. We present conditions, in terms of the integral closure of the ideal defining the singular set of $f|_X$, for these deformations to have constant Milnor number, Euler obstruction, and Bruce-Roberts number.

1. Introduction

Let $f\colon (\mathbb{C}^n,0) \to (\mathbb{C},0)$ be a holomorphic function germ with an isolated singularity and $F\colon (\mathbb{C}\times\mathbb{C}^n,0) \to (\mathbb{C},0)$ a deformation of f. For each $t \in \mathbb{C}$, we denote by $f_t\colon (\mathbb{C}^n,0) \to (\mathbb{C},0)$ the germ defined by $f_t(x) = F(t,x)$; hence F defines a family of function germs f_t. Many authors have studied the properties of such a family and a very important result is to know when the family has constant topological type. In this direction, we have the Milnor number ([18]), which is a well-known number related to a function germ. We know that, if $n \neq 3$, a family f_t has constant topological type if and only if it has constant Milnor number ([15], [27]). The $n = 3$ case is still an open problem.

However, the problem of determining whether a family has constant Milnor number is not easy. Greuel [12] presents methods to solve it. More specifically, he shows that the constancy of the Milnor number in F is equivalent to all the following assertions

(1) $\frac{\partial F}{\partial t} \in \overline{J}$, where $\overline{J}$ is the integral closure of the Jacobian ideal $J = \left\langle \frac{\partial F}{\partial x_1}, \dots, \frac{\partial F}{\partial x_n} \right\rangle$;
(2) $\frac{\partial F}{\partial t} \in \sqrt{J}$, where $\sqrt{J}$ denotes the radical of J;
(3) $v(J) = \{(t,x) \in \mathbb{C}\times\mathbb{C}^n \mid \frac{\partial F}{\partial x_i}(t,x) = 0,\ i = 1,\dots,n\} = \mathbb{C}\times\{0\}$ near $(0,0)$.

Let $(X,0)$ be a germ of an analytic variety (possibly singular) and $f\colon (\mathbb{C}^n,0) \to (\mathbb{C},0)$ a function germ. In [3], Bruce and Roberts generalize the Milnor number taking into account the variety $(X,0)$. Many authors call this new invariant the Bruce-Roberts number and denote it by $\mu_{BR}(X,f)$. The Bruce-Roberts number generalizes the Milnor number in the sense that the germ is $\mathcal{R}_X$-finitely determined if and only if the Bruce-Roberts number is finite, where $\mathcal{R}_X$ is the group of the diffeomorphisms which preserves $(X,0)$. In [1], Ahmed, Ruas, and Tomazella study how the results in [12] work for the Bruce-Roberts number.

In [13], Hamm introduces the Milnor number of an isolated complete intersection singularity (ICIS). The constancy of the Milnor number in a family of ICIS implies the constancy of the topological type if each member in the family has dimension $d \neq 2$ (see [23]). If $(X,0) \subset (\mathbb{C}^n,0)$ is an ICIS and $f\colon (X,0) \to (\mathbb{C},0)$ is a holomorphic function germ with isolated singularity, we can study the Milnor number of f, $\mu(f|_X,0)$. By the Lê-Greuel formula (see [16]),

$$\mu(f|_X,0) = \mu(X,0) + \mu(X \cap f^{-1}(0),0),$$

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where $\mu(X,0)$ and $\mu(X\cap f^{-1}(0),0)$ denote the Milnor number of the ICIS defined by Hamm ([13]).

The main goal of this work is to study how Greuel's result adapts to this singular case, that is, to deformations of function germs $f\colon (X,0)\to(\mathbb{C},0)$, where $(X,0)\subset(\mathbb{C}^n,0)$ is an ICIS. We characterize μ-constant deformations of f in terms of the integral closure of the ideal defining the singular set of $f|_X$.

We present conditions for the constancy of the local Euler obstruction and of the Bruce-Roberts number. Also, we present a sufficient condition for a family f_t to be C^0-$\mathcal{R}_X$-trivial.

Finally, we analyze the constancy of the Milnor number in a family $f_t\colon (X_t,0)\to(\mathbb{C},0)$, where $(X_t,0)$ is a deformation of an ICIS $(X,0)$. In order to do this, we use the strict integral closure of the module defined by the Jacobian matrix of the map defining the deformation.

2. Preliminary concepts

Let $(X,0)\subset(\mathbb{C}^n,0)$ be the isolated complete intersection singularity (ICIS) defined by a holomorphic map germ $\phi\colon(\mathbb{C}^n,0)\to(\mathbb{C}^p,0)$ and $f\colon(X,0)\to(\mathbb{C},0)$ a holomorphic function germ with isolated singularity. That is, in a sufficiently small neighborhood of 0, the zero set of the coordinate functions of ϕ intersected by the zero set of the maximal minors of the Jacobian matrix of (f,ϕ) is just 0.

Let $\Phi\colon(\mathbb{C}\times\mathbb{C}^n,0)\to(\mathbb{C}^p,0)$ be a holomorphic deformation of ϕ such that $\phi_0=\phi$, where $\phi_s(x)=\Phi(s,x)$ and $x=(x_1,\ldots,x_n)$. We write $\mathcal{X}=\Phi^{-1}(0)$, $(X_s,0):=(\phi_s^{-1}(0),0)$ and assume that $(X_s,0)$ is smooth for $s\neq 0$ sufficiently small. We say that $(\mathcal{X},0)$ is a smoothing of $(X,0)$.

Let

$$\tilde{f}\colon(\mathcal{X},0)\to(\mathbb{C},0)$$

be a holomorphic function germ such that for all $s\neq 0$ sufficiently small, the germ

$$\begin{array}{rccc} f_s\colon & (X_s,0) & \to & (\mathbb{C},0)\\ & x & \mapsto & \tilde{f}(s,x)\end{array}$$

is a Morse function germ.

Inspired by [18], in order to define the Milnor number of f, we take a representative of $(\mathcal{X},0)$, $\mathcal{X}=\Phi^{-1}(0)$, where $\Phi\colon B\to\mathbb{C}^p$ is defined in a small enough ball $B=B_\epsilon$ centered at the origin in $\mathbb{C}\times\mathbb{C}^n$, and the representative of $\tilde{f}$, $\tilde{f}\colon\mathcal{X}\to\mathbb{C}$. We define

$$\mu(f|_X,0)=\sharp S(f_s),$$

where $S(f_s)$ is the set of critical points of the representative of f_s defined by the representative of $\tilde{f}$ above.

We remark that this definition of the Milnor number is not original, but we did not find a good reference for it. One can show that

$$\mu(f|_X,0)=\mu(X,0)+\mu(X\cap f^{-1}(0),0),$$

where the numbers on the right side are the Milnor numbers of the ICIS's as defined in [13]. In fact, if $D=\{(s,x)\in\mathcal{X}\mid x \text{ is a singular point of } f_s\}$ and $\pi\colon D\to\mathbb{C}$ is the restriction of the projection on the first coordinate then

$$\mu(f|_X,0)=\deg(\pi).$$

Therefore

$$\mu(f|_X,0)=e\left(\langle s\rangle,\frac{\mathcal{O}_{\mathcal{X},0}}{J(f_s,\phi_s)}\right),$$

where $\mathcal{O}_{\mathcal{X},0}$ is the local ring of $(\mathcal{X},0)$, $J(f_s,\phi_s)$ is the ideal generated by the maximal order minors of the Jacobian matrix of (f_s,ϕ_s) (partial derivatives with respect to x only) and $e(I,R)$ denotes the Hilbert-Samuel multiplicity of the ideal I with respect to the ring R (see [19]).

Since $\frac{\mathcal{O}_{\mathcal{X},0}}{J(f_s,\phi_s)}$ is a determinantal ring, it is Cohen-Macaulay. Hence (see [17])

$$e\left(\langle s\rangle, \frac{\mathcal{O}_{\mathcal{X},0}}{J(f_s,\phi_s)}\right) = \dim_{\mathbb{C}} \frac{\mathcal{O}_{\mathcal{X},0}}{\langle s\rangle + J(f_s,\phi_s)}.$$

Thus

$$\begin{aligned}
\mu(f|_X,0) &= \dim_{\mathbb{C}} \frac{\mathcal{O}_{\mathcal{X},0}}{\langle s\rangle + J(f_s,\phi_s)} \\
&= \dim_{\mathbb{C}} \frac{\frac{\mathcal{O}_{n+1}}{\langle \Phi\rangle}}{\langle s\rangle + J(f_s,\phi_s)} \\
&= \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\langle \phi\rangle + J(f,\phi)} \\
&= \mu(X,0) + \mu(X\cap f^{-1}(0),0),
\end{aligned}$$

where $\mathcal{O}_m$ denotes the ring of holomorphic function germs from $(\mathbb{C}^m,0)$ to $\mathbb{C}$ and the last equality follows from the well-known Lê-Greuel formula.

We consider now a family of function germs on the ICIS $(X,0)$. That is, let

$$\begin{array}{cccc} F\colon & (\mathbb{C}\times X,0) & \to & (\mathbb{C},0) \\ & (t,x) & \mapsto & f_t(x) \end{array}$$

be a (flat) deformation of f such that $f_t(0)=0$ for t sufficiently small. We say that F is μ-constant if $\mu(f_t|_X,0)=\mu(f|_X,0)$ for t sufficiently small.

In the case where $(X,0)$ is regular, the constancy of the Milnor number of f_t is proved to be related to the integral closure of the Jacobian ideal of F, not considering the derivative with relation to the parameter t, see [12, Theorem 1.1]. In fact, Greuel shows that this family is μ-constant if and only if

$$\frac{\partial F}{\partial t} \in \overline{\left\langle \frac{\partial F}{\partial x_1},\ldots,\frac{\partial F}{\partial x_n}\right\rangle},$$

where the bar denotes the integral closure of the ideal in $\mathcal{O}_{\mathbb{C}\times X}$ (the local ring of $\mathbb{C}\times X$). We remember that the integral closure of an ideal I in a ring R is equal to

$$\overline{I} := \{h\in R \mid \exists\, a_i\in I^i \text{ with } h^k + a_1h^{k-1} + \ldots + a_{k-1}h + a_k = 0\}.$$

One of the main goals of this work is to generalize Greuel's result to the case where $(X,0)$ is a general ICIS. In order to do this, the next theorem of Teissier, which gives different characterizations for the integral closure of an ideal, will be very useful.

Theorem 2.1. [26, Proposition 0.4] *Let $(Y,0)\subset(\mathbb{C}^n,0)$ be a germ of an analytic variety and $\mathcal{O}_Y$ the local ring of $(Y,0)$. If I is an ideal in $\mathcal{O}_Y$, the following conditions are equivalent*

(i) $h\in\overline{I}$;

(ii) *For each system of generators $h_1,\ldots,h_r$ of I, there is a neighborhood U of 0 in Y and a constant $c>0$ such that $|h(x)|\le c\sup\{|h_1(x)|,\ldots,|h_r(x)|\}$, $\forall\ x\in U$;*

(iii) *For each analytic curve $\gamma\colon(\mathbb{C},0)\to(Y,0)$, $h\circ\gamma\in(\gamma^*(I))\mathcal{O}_1$, where $(\gamma^*(I))\mathcal{O}_1$ is the ideal generated by $h_i\circ\gamma$, $i=1,\ldots,r$;*

(iv) *$\nu(h\circ\gamma)\ge\inf\{\nu(h_1\circ\gamma),\ldots,\nu(h_r\circ\gamma)\}$, for ν being the usual valuation of the complex curve.*

The item (iii) of the previous theorem is usually called the *valuation criterion for integral dependence.*

Throughout this paper, we also need to work with integral closure of modules over a ring, as defined by Gaffney:

Definition 2.2. [5, Definition 1.3] Suppose $(Y,0)$ is a complex analytic germ, M a submodule of $\mathcal{O}^p_{Y,0}$. Then $h \in \mathcal{O}^p_{Y,0}$ is in the integral closure of M, denoted by $\overline{M}_{\mathcal{O}^p_{Y,0}}$, if and only if for all $\gamma\colon (\mathbb{C},0) \to (Y,0)$, $h \circ \gamma \in (\gamma^*(M))\mathcal{O}_1$.

Replacing $\mathcal{O}_1$ by its maximal ideal $\mathcal{M}_1$, we get the definition of strict integral closure of M, which is denoted by $\overline{M}^{\dagger}_{\mathcal{O}^p_{Y,0}}$ (see [4, Definition 1.1]). In this case $h \in \overline{M}^{\dagger}_{\mathcal{O}^p_{Y,0}}$ is said to be strictly dependent on M.

3. Main results

In [12], Greuel presents the following result.

Theorem 3.1. [12, Theorem 1.1] *Let $f\colon (\mathbb{C}^n,0) \to (\mathbb{C},0)$ be a holomorphic function germ with isolated singularity. For any deformation $F\colon (\mathbb{C}\times\mathbb{C}^n) \to (\mathbb{C},0)$ of f the following statements are equivalent*

(1) *F is μ-constant;*

(2) *For every holomorphic curve $\gamma\colon (\mathbb{C},0) \to (\mathbb{C}\times\mathbb{C}^n,0)$*

$$\nu\left(\frac{\partial F}{\partial t}\circ\gamma\right) > \inf\left\{\nu\left(\frac{\partial F}{\partial x_i}\circ\gamma\right) \mid i=1,\ldots,n\right\},$$

where ν denotes the usual valuation of a complex curve;

(3) *Same statement as in* (2) *with "$>$" replaced by "$\geq$";*

(4) *$\frac{\partial F}{\partial t} \in \overline{J}$, where $\overline{J}$ is the integral closure of the Jacobian ideal $J = \left\langle \frac{\partial F}{\partial x_1},\ldots,\frac{\partial F}{\partial x_n}\right\rangle$ as an ideal in $\mathcal{O}_{n+1}$;*

(5) *$\frac{\partial F}{\partial t} \in \sqrt{J}$, where $\sqrt{J}$ denotes the radical of J;*

(6) *$v(J) = \{(t,x) \in \mathbb{C}\times\mathbb{C}^n \mid \frac{\partial F}{\partial x_i}(t,x) = 0,\ i=1,\ldots,n\} = \mathbb{C}\times\{0\}$ near $(0,0)$.*

Let $(X,0) \subset (\mathbb{C}^n,0)$ be the ICIS defined by a holomorphic map germ $\phi\colon (\mathbb{C}^n,0) \to (\mathbb{C}^p,0)$ and $f\colon (X,0) \to (\mathbb{C},0)$ a holomorphic function germ with isolated singularity. We consider

$$\begin{array}{rccc} F\colon & (\mathbb{C}\times X,0) & \to & (\mathbb{C},0) \\ & (t,x) & \mapsto & f_t(x) \end{array}$$

a (flat) deformation of f, as defined in the previous section.

We want to study how Greuel's theorem would work for this case. The ideal J which appears in Theorem 3.1 is the ideal defining the singular set of each germ f_t if we consider t as a constant. It is therefore natural to look for the ideal defining the singular set of each germ $f_t\colon (X,0) \to \mathbb{C}$, that is, the ideal J_X generated by the maximal minors of the Jacobian matrix of (F,ϕ) (with respect to x only) as an ideal in $\mathcal{O}_{\mathbb{C}\times X}$.

Here, given a matrix A of size $k\times l$, for each pair of vectors $u = (i_1,\ldots,i_r) \in \{1,\ldots,k\}^r$, $v = (j_1,\ldots,j_r) \in \{1,\ldots,l\}^r$, with $i_1 < \cdots < i_r$ and $j_1 < \cdots < j_r$, we denote by $A_{u,v}$ the determinant of the submatrix obtained by taking the lines $i_1,\ldots,i_r$ and the columns $j_1,\ldots,j_r$ of A.

Hence, if M is the Jacobian matrix of (F,ϕ) (with respect to x only) then J_X is the ideal generated by $M_{u,v}$, with $u = (1,\ldots,p+1)$ and

$$v = (j_1,\ldots,j_{p+1}),\ \text{with}\, j_1 < \cdots < j_{p+1}\, \text{and}\, j_1,\ldots,j_{p+1} \in \{1,\ldots,n\}. \tag{1}$$

The assertions of the Theorem 3.1 in this context would be

(1_X) F is μ-constant;

(2_X) For every holomorphic curve $\gamma\colon (\mathbb{C},0) \to (\mathbb{C}\times X,0)$

$$\nu\left(\frac{\partial F}{\partial t}\circ\gamma\right) > \inf\{\nu(M_{u,v}\circ\gamma),\ \text{for all}\, v\, \text{in}\, (1)\},$$

where ν denotes the usual valuation of a complex curve;

(3_X) Same statement as in (2_X) with " $>$ " replaced by " $\geq$ ";

(4_X) $\frac{\partial F}{\partial t} \in \overline{J_X}$ as an ideal in $\mathcal{O}_{\mathbb{C}\times X}$;

(5_X) $\frac{\partial F}{\partial t} \in \sqrt{J_X}$ as an ideal in $\mathcal{O}_{\mathbb{C}\times X}$;

(6_X) $v(J_X) = \{(t,x) \in \mathbb{C}\times\mathbb{C}^n \mid M_{u,v}(t,x) = 0$, for all v in (1)$\} = \mathbb{C}\times\{0\}$ near $(0,0)$ in $\mathcal{O}_{\mathbb{C}\times X}$.

Unfortunately, these assertions are not equivalent in this singular context. But in this section we show that

$$\begin{gathered}(2_X) \Rightarrow (3_X) \Leftrightarrow (4_X) \Rightarrow (5_X),\\ (4_X) \Rightarrow (1_X) \Leftrightarrow (6_X) \Rightarrow (5_X),\end{gathered}$$

and present a counterexample for each of the other implications.

By the Lê-Greuel formula,

$$\mu(f_t|_X, 0) = \mu(X,0) + \mu(X \cap f_t^{-1}(0), 0);$$

hence in order to decide whether F is a μ-constant family, we only need to check whether or not $\mu(X \cap f_t^{-1}(0), 0)$ is constant, as we see in the following theorem.

Theorem 3.2. *Let $(X,0) \subset (\mathbb{C}^n,0)$ be an ICIS defined by $\phi\colon (\mathbb{C}^n,0) \to (\mathbb{C}^p,0)$ (not smooth at 0) and $f\colon (X,0) \to (\mathbb{C},0)$ a holomorphic function germ with isolated singularity. Let $F\colon (\mathbb{C}\times X,0) \to (\mathbb{C},0)$ be a (flat) deformation of f and $G\colon \mathbb{C}\times\mathbb{C}^n \to (\mathbb{C}\times\mathbb{C}^p,0)$ the map defined by $G(t,x) = (F(t,x),\phi(x))$. If $\frac{\partial F}{\partial t} \in \overline{J_X}$ as an ideal in $\mathcal{O}_{\mathbb{C}\times X}$ then*

(1) $\frac{\partial G}{\partial t} \in \overline{\{x_i \frac{\partial G}{\partial x_j} \mid i,j = 1,\dots,n\}}_{\mathcal{O}_{\mathbb{C}\times X}}$, where $\overline{\{x_i \frac{\partial G}{\partial x_j} \mid i,j = 1,\dots,n\}}_{\mathcal{O}_{\mathbb{C}\times X}}$ denotes the integral closure of the $\mathcal{O}_{\mathbb{C}\times X}$-module generated by $x_i \frac{\partial G}{\partial x_j}$;

(2) $X \cap f_t^{-1}(0)$ is an ICIS and $\mu(X \cap f_t^{-1}(0),0)$ is constant.

Proof. Again, denoting the Jacobian matrix of (F,ϕ) (with relation to x only) by M and calculating each minor in J_X by the first line, we get, for $u = (1,\dots,p+1)$ and $v = (j_1,\dots,j_{p+1})$,

$$M_{u,v} = \frac{\partial F}{\partial x_{j_1}} M_{u_1,v_1} - \frac{\partial F}{\partial x_{j_2}} M_{u_1,v_2} + \cdots + (-1)^{p+2} \frac{\partial F}{\partial x_{j_{p+1}}} M_{u_1,v_{p+1}},$$

where $u_1 = (2,\dots,p+1)$ and $v_k = (j_1,\dots,\hat{j_k},\dots,j_{p+1})$.

Then

$$(M_{u,v},0,\dots,0) = \left(\sum_{l=1}^{p+1}(-1)^{1+l}\frac{\partial F}{\partial x_{j_l}} M_{u_1,v_l}, \sum_{l=1}^{p+1}(-1)^{1+l}\frac{\partial \phi_1}{\partial x_{j_l}} M_{u_1,v_l},\dots,\sum_{l=1}^{p+1}(-1)^{1+l}\frac{\partial \phi_p}{\partial x_{j_l}} M_{u_1,v_l}\right).$$

Hence

$$(M_{u,v},0,\dots,0) \in \left\{x_i \frac{\partial G}{\partial x_j} \mid i,j = 1,\dots,n\right\}_{\mathcal{O}_{\mathbb{C}\times X}}.$$

In fact, this is shown in the proof of [6, Lemma 2.8] but we include it here for the sake of completeness.

By the hypothesis, $\frac{\partial F}{\partial t} \in \overline{J_X}$ as an ideal in $\mathcal{O}_{\mathbb{C}\times X}$ then, by Theorem 2.1, for all curve $\gamma\colon (\mathbb{C},0) \to (\mathbb{C}\times X,0)$, $\frac{\partial F}{\partial t}\circ\gamma \in \langle M_{u,v}\circ\gamma\rangle$ and therefore

$$\left(\frac{\partial F}{\partial t}\circ\gamma,0,\dots,0\right) \in \{(M_{u,v}\circ\gamma,0,\dots,0)\} \subset \left\{x_i \frac{\partial G}{\partial x_j}\circ\gamma \mid i,j = 1,\dots,n\right\}_{\mathcal{O}_{\mathbb{C}\times X}}.$$

Thus $\frac{\partial G}{\partial t} \in \overline{\{x_i \frac{\partial G}{\partial x_j} \mid i,j = 1,\dots,n\}}_{\mathcal{O}_{\mathbb{C}\times X}}$, which concludes (1).

We prove now (2). By (1),

$$\frac{\partial G}{\partial t} \in \overline{\{x_i \frac{\partial G}{\partial x_j} \mid i,j = 1,\dots,n\}}_{\mathcal{O}_{\mathbb{C}\times X}}.$$

Hence, since $X \cap f_t^{-1}(0) \subset \mathbb{C} \times X$, we can assume that $\frac{\partial G}{\partial t} \in \overline{\{x_i \frac{\partial G}{\partial x_j} \mid i,j = 1,\dots,n\}}_{\mathcal{O}_{X\cap f_t^{-1}(0)}}$.

Then $\{X \cap f_t^{-1}(0)\}$ is a Whitney regular family (see [5, Theorem 2.5]). In particular, $\mu(X \cap f_t^{-1}(0), 0)$ is constant (see [7, Theorem 6.1]).

□

Remark 3.3. The first item of Theorem 3.2 means that the family satisfies the W_f condition, [9, Proposition 2.1], and therefore the family has a rugose trivialization [28, Proposition 4.6].

We are now ready to see how Greuel's result works in this singular context. In the following theorem, $(1_X), (2_X), (3_X), (4_X), (5_X)$ and (6_X) are the adaptations of Greuel's assertions for the singular context as we described before Theorem 3.2.

Theorem 3.4. *Let $(X, 0) \subset (\mathbb{C}^n, 0)$ be an ICIS defined by $\phi\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ and let $f\colon (X, 0) \to (\mathbb{C}, 0)$ be a germ with isolated singularity. Let $F\colon (\mathbb{C} \times X, 0) \to (\mathbb{C}, 0)$ be a (flat) deformation of f. Then*

$$(2_X) \Rightarrow (3_X) \Leftrightarrow (4_X) \Rightarrow (5_X),$$
$$(4_X) \Rightarrow (1_X) \Leftrightarrow (6_X) \Rightarrow (5_X).$$

Proof. $(2_X) \Rightarrow (3_X)$ is trivial. The equivalence $(3_X) \Leftrightarrow (4_X)$ is the valuation criterion (Theorem 2.1). Moreover, since $\overline{J_X} \subset \sqrt{J_X}$, (5_X) follows directly from (4_X). We work now on the other implications.

We take a representative of $(X, 0)$, $X = \phi^{-1}(0)$, where $\phi\colon B \to \mathbb{C}^p$ is defined in a small enough ball $B = B_\epsilon$ centered at the origin in $\mathbb{C}^n$, and one representative of F, $\tilde{F}\colon D \times X \to \mathbb{C}$, where D is a small disc around the origin in $\mathbb{C}$. We denote by f_t the representative of $f_t\colon (X, 0) \to \mathbb{C}$ determined by $\tilde{F}$. We write f instead of f_0.

$(1_X) \Rightarrow (6_X)$:

By the principle of conservation of number, since $\frac{\mathcal{O}_{n+1}}{\langle\phi\rangle + J_X}$ is Cohen-Macaulay,

$$\mu(f|_X, 0) = \sum_{(t,x)\in v(J_X)\cap(\{t\}\times\mathbb{C}^n)} \mu(f_t|_X, x),$$

for t sufficiently small.

From the hypothesis, $\mu(f_t|_X, 0) = \mu(f|_X, 0)$, therefore $\mu(f_t|_X, x) = 0$ for all $x \neq 0$, that is, $v(J_X) = \mathbb{C} \times \{0\}$ near $(0, 0)$.

$(6_X) \Rightarrow (1_X)$:

Again, it follows from the principle of conservation of number.

$(6_X) \Rightarrow (5_X)$:

By the hypothesis, $f_t \in \mathcal{M}_n$, where $\mathcal{M}_n$ is the maximal ideal in $\mathcal{O}_n$. Hence

$$\frac{\partial F}{\partial t}|_{v(J_X)} = \frac{\partial F}{\partial t}|_{\mathbb{C}\times\{0\}} \equiv 0,$$

that is $v(J_X) \subset v\left(\frac{\partial F}{\partial t}\right)$. Thus $\frac{\partial F}{\partial t} \in \sqrt{J_X}$ as an ideal in $\mathcal{O}_{\mathbb{C}\times X}$ by Hilbert's Nullstellensatz Theorem.

$(4_X) \Rightarrow (1_X)$:

By the Lê-Greuel formula, $\mu(f_t|_X, 0) = \mu(X, 0) + \mu(X \cap f_t^{-1}(0), 0)$. Moreover, since $\mu(X \cap f_t^{-1}(0), 0)$ is constant by Theorem 3.2, $\mu(f_t|_X, 0)$ is also constant.

□

From here to the end of this section, our goal is to present counterexamples for the other implications.

A good idea is to look for known examples of families with constant Milnor number. For this, we recall the results of [21] on deformations of weighted homogeneous germs.

We say that a map germ $f = (f_1, \dots, f_p)\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ is weighted homogeneous of type $(w_1, \dots, w_n; d_1, \dots, d_p)$ with $w_i, d_j \in \mathbb{Q}^+$ if for all $\lambda \in \mathbb{C} - \{0\}$,

$$f(\lambda^{w_1} x_1, \dots, \lambda^{w_n} x_n) = (\lambda^{d_1} f_1(x), \dots, \lambda^{d_p} f_p(x)).$$

We call d_j the weighted degree of f_j, which is denoted by $wt(f_j)$ and we call w_i the weight of the variable x_i. Moreover, if $(X, 0) \subset (\mathbb{C}^n, 0)$ is the germ of an analytic variety defined by the zero set of a weighted homogeneous germ $\phi\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ of type $(w_1, \dots, w_n; d_1, \dots, d_p)$, we say that $(X, 0)$ is weighted homogeneous of type $(w_1, \dots, w_n; d_1, \dots, d_p)$.

If $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is a weighted homogeneous function germ and

$$f_t(x) = f(x) + \sum_{i=1}^{k} \sigma_i(t)\alpha_i(x),$$

with $\alpha_i\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$, we say that the deformation f_t is non-negative, if the monomials which appear in each α_i have weighted degrees higher than or equal to the one of f.

Theorem 3.5. [21, Theorem 4.4] *Let $(X, 0) \subset (\mathbb{C}^n, 0)$ be a weighted homogeneous ICIS and $f\colon (\mathbb{C}^n, 0) \to \mathbb{C}$ a weighted homogeneous germ with an isolated singularity with the same weights of $(X, 0)$. Let f_t be a deformation of f. If f_t is a non-negative deformation, then $\mu(f_t|_X, 0)$ is constant.*

We use this result in the following example to show that (1_X) implies neither (2_X) nor (3_X). Moreover, this example shows that (5_X) implies neither (1_X) nor (3_X).

Example 3.6. Let $(X, 0) \subset (\mathbb{C}^2, 0)$ be defined by $\phi(x, y) = x^p - y^q$, with $q \geq 3$, and let $f\colon (X, 0) \to (\mathbb{C}, 0)$ be defined by $f(x, y) = x$. We consider the deformation of f defined by $F(t, (x, y)) = x + ty$. In this case $J_X = \langle -qy^{q-1} - ptx^{p-1} \rangle$.

Let $\gamma\colon (\mathbb{C}, 0) \to (\mathbb{C} \times X, 0)$ be the curve defined by $\gamma(s) = (0, s^q, s^p)$. It is easy to see that

$$\nu\left(\frac{\partial F}{\partial t} \circ \gamma\right) = p \text{ and } \nu((-qy^{q-1} - ptx^{p-1}) \circ \gamma) = (q-1)p.$$

Therefore (2_X) and (3_X) are not true.

On the other hand

(a) If $p > q$, then ϕ and f are weighted homogeneous of type $(q, p; pq)$ and $(q, p; q)$, respectively, and f_t is a non-negative deformation of f. Therefore $\mu(f_t|_X, 0)$ is constant by Theorem 3.5. That is, (1_X) is true.

(b) If $p < q$, then it is not hard to see that $\mu(f|_X, 0) = pq - p$ and $\mu(f_t|_X, 0) = pq - q$. Therefore (1_X) is not true. Moreover, $\frac{\partial F}{\partial t} \in \sqrt{J_X}$ as an ideal in $\mathcal{O}_{\mathbb{C}\times X}$, that is, (5_X) is true.

We show in the next example that (3_X) and (5_X) do not imply (2_X).

Example 3.7. Let $(X, 0) \subset (\mathbb{C}^2, 0)$ be defined by the zero set of $\phi(x, y) = x^{2q} - y^q$ with $q \geq 2$ and $f\colon (X, 0) \to (\mathbb{C}, 0)$ defined by $f(x, y) = x^{2q} + y^q$. Let F be the deformation of f defined by $F(t, (x, y)) = x^{2q} + y^q + tx^{4q-3}$. In this case

$$J_X = \langle -4q^2 x^{2q-1} y^{q-1} - q(4q-3)tx^{4q-4} y^{q-1} \rangle.$$

Let $\gamma\colon (\mathbb{C},0) \to (\mathbb{C}\times X,0)$ be any curve $\gamma(s) = (\gamma_1(s), \gamma_2(s), \gamma_3(s))$ such that $\gamma_2^{2q} - \gamma_3^q = 0$. Thus $\gamma_2^{2q} = \gamma_3^q$.

Since $\frac{\partial F}{\partial t}(x,y) = x^{4q-3}$, it follows that $\nu\left(\frac{\partial F}{\partial t}\circ\gamma\right) = (4q-3)\nu(\gamma_2)$. Furthermore,

$$\begin{aligned}\nu((-4q^2x^{2q-1}y^{q-1} - q(4q-3)tx^{4q-4}y^{q-1})\circ\gamma) &= \nu(-4q^2\gamma_2^{2q-1}\gamma_3^{q-1} - q(4q-3)\gamma_1\gamma_2^{4q-4}\gamma_3^{q-1})\\ &= (2q-1)\nu(\gamma_2) + 2(q-1)\nu(\gamma_2)\\ &= (4q-3)\nu(\gamma_2).\end{aligned}$$

Thus (3_X) is true. Consequently, (4_X) and (5_X) are also true.

On the other hand, let $\gamma\colon (\mathbb{C},0) \to (\mathbb{C}\times X,0)$ be the curve defined by $\gamma(s) = (0,s,s^2)$. In this case, $\nu\left(\frac{\partial F}{\partial t}\circ\gamma\right) = 4q-3$ and $\nu((-4q^2x^{2q-1}y^{q-1} - q(4q-3)tx^{4q-4}y^{q-1})\circ\gamma) = 4q-3$. That is, (2_X) is not true.

4. The Newton Polyhedron and the invariants

We now apply our results to produce examples of families of functions on an ICIS which have constant Milnor number, although they do not satisfy the hypotheses of Theorem 3.5. For this, we refer to the results about Newton polyhedron (see [25]).

Let

$$g(x) = \sum_{\alpha\in\mathbb{Z}^n} a_\alpha x^\alpha \in \mathcal{O}_n,$$

where if $\alpha = (\alpha_1,\dots,\alpha_n)$ we write $x^\alpha = x_1^{\alpha_1}\dots x_n^{\alpha_n}$. We define the support of g by

$$\text{supp }\ g := \{\alpha\in\mathbb{Z}^n \mid a_\alpha \neq 0\}$$

and for I an ideal in $\mathcal{O}_n$, we define supp $I := \bigcup\{\text{supp } g \mid g\in I\}$.

The convex hull in $\mathbb{R}^n_+$ of the set $\bigcup\{\alpha+v \mid \alpha\in \text{supp } I, v\in\mathbb{R}^n_+\}$ is called Newton polyhedron of I and is denoted by $\Gamma_+(I)$. We denote by $\Gamma(I)$ the union of all compact faces of $\Gamma_+(I)$.

Let $\Delta\subset\Gamma_+(I)$ be a finite set and $f(x) = \sum a_\alpha x^\alpha$; we define $f_\Delta = \sum_{\alpha\in\Delta} a_\alpha x^\alpha$.

If Δ is a face of $\Gamma_+(I)$, we denote by $C(\Delta)$ the cone of half-rays emanating from 0 and passing through Δ. We define $C[[\Delta]]$, the ring of power series with non-zero monomials $x^\alpha = x_1^{\alpha_1}x_2^{\alpha_2}\cdot\ldots\cdot x_n^{\alpha_n}$ such that $\alpha = (\alpha_1,\dots,\alpha_n)\in C(\Delta)$. When the ideal generated by $g_{1\Delta}, g_{2\Delta},\dots,g_{s\Delta}$ has finite codimension in $C[[\Delta]]$, we say that the compact face $\Delta\subset\Gamma(I)$ is Newton non-degenerate. Furthermore, if all compact faces of $\Gamma(I)$ are Newton non-degenerate, then the ideal I is said to be Newton non-degenerate.

Equivalently, in [25, p.2]: I is Newton non-degenerate if for each compact face $\Delta\subset\Gamma(I)$, the equations $g_{1\Delta}(x) = g_{2\Delta}(x) = \ldots = g_{s\Delta}(x) = 0$ have no common solution in $(\mathbb{C}-\{0\})^n$.

Theorem 4.1. [25, Theorem 3.4] *Let $I = \langle g_1, g_2,\dots,g_s\rangle$ be an ideal of finite codimension in $\mathcal{O}_n$. Then I is Newton non-degenerate if and only if $\Gamma_+(I) = C(\overline{I})$, where $C(\overline{I})$ is the convex hull in $\mathbb{R}^n_+$ of the set $\bigcup\{m \mid x^m\in\overline{I}\}$.*

Let $(X,0)\subset(\mathbb{C}^n,0)$ be the ICIS defined by a map germ $\phi\colon(\mathbb{C}^n,0)\to(\mathbb{C}^p,0)$ and let $f\colon(X,0)\to(\mathbb{C},0)$ be a holomorphic function germ with isolated singularity.

Let $F\colon(\mathbb{C}\times X,0)\to(\mathbb{C},0)$ be a deformation of f defined by $F(t,x) = f(x)+tg(x)$, where g is a holomorphic function germ such that $g(0)=0$. Throughout this section, we use this notation.

Let $u = (1,\dots,p+1)$ and for each

(2) $$v = (j_1,\dots,j_{p+1}),\ \text{with}\ j_1<\cdots<j_{p+1}\ \text{ and }\ j_1,\dots,j_{p+1}\in\{1,\dots,n\},$$

we denote by $M^f_{u,v}$ and $M^g_{u,v}$ the minors of the Jacobian matrix of the map (f,ϕ) and of the map (g,ϕ), respectively, obtained by taking the columns $j_1,\dots,j_{p+1}$. By the multilinearity of the determinant, $M_{u,v} = M^f_{u,v} + tM^g_{u,v}$, where $M_{u,v}$ is the minor of the Jacobian matrix of (F, ϕ) (derivatives with respect to x only).

Since (4_X) implies that the family has constant Milnor number, we are interested in more practical ways to decide when (4_X) is true. With the deformation above, it means that $g \in \overline{J_X}$ as an ideal in $\mathcal{O}_{\mathbb{C}\times X}$.

Lemma 4.2. *If* $g, M^g_{u,v} \in \overline{\langle\phi\rangle + J(f,\phi)}$ *as an ideal in* $\mathcal{O}_{n+1}$*, for all* v *defined in* (2), then $g \in \overline{J_X}$ as an ideal in $\mathcal{O}_{\mathbb{C}\times X}$.

Proof. Since $M^g_{u,v} \in \overline{\langle\phi\rangle + J(f,\phi)}$, we assume by Theorem 2.1 that there is a neighborhood U of 0 and a constant $c > 0$ such that

$$|t||M^g_{u,v}| \leq |t| c \sup_{u,v}\{|\phi|, |M^f_{u,v}|\}.$$

Moreover,

$$\begin{aligned}\sup_{u,v}\{|\phi|, |M_{u,v}|\} &= \sup_{u,v}\{|\phi|, |M^f_{u,v} + tM^g_{u,v}|\} \\ &\geq \sup_{u,v}\{|\phi|, |M^f_{u,v}|\} - |t| \sup_{u,v}\{|\phi|, |M^g_{u,v}|\} \\ &\geq \sup_{u,v}\{|\phi|, |M^f_{u,v}|\} - |t| c \sup_{u,v}\{|\phi|, |M^f_{u,v}|\} \\ &\geq (1-\alpha) \sup_{u,v}\{|\phi|, |M^f_{u,v}|\},\end{aligned}$$

where $0 < \alpha < 1$ and $|t| \leq \frac{\alpha}{c}$.

Therefore there is a constant $K > 0$ such that

$$\sup_{u,v}\{|\phi|, |M_{u,v}|\} \geq K \sup_{u,v}\{|\phi|, |M^f_{u,v}|\},$$

for t sufficiently small. Hence, by Theorem 2.1, $\overline{\langle\phi\rangle + J(f,\phi)} \subseteq \overline{\langle\phi\rangle + J_X}$ as an ideal in $\mathcal{O}_{n+1}$. Then, by the hypothesis, $g \in \overline{\langle\phi\rangle + J_X}$ as an ideal in $\mathcal{O}_{n+1}$. Thus, by Theorem 2.1 for all $\gamma\colon (\mathbb{C}, 0) \to (\mathbb{C}\times\mathbb{C}^n, 0)$, $g\circ\gamma \in \langle\phi\circ\gamma, M_{u,v}\circ\gamma\rangle$. Then, for all $\gamma\colon (\mathbb{C}, 0) \to (\mathbb{C}\times X, 0) \subset (\mathbb{C}\times\mathbb{C}^n, 0)$, $g \circ \gamma \in \langle M_{u,v} \circ \gamma\rangle$. Again, follows from Theorem 2.1 that $g \in \overline{J_X}$ as an ideal in $\mathcal{O}_{\mathbb{C}\times X}$. □

With this, we can see how to know if (4_X) is true by looking at Newton polyhedron.

Corollary 4.3. *If* $\Gamma_+(g), \Gamma_+(M^g_{u,v}) \subset C(\overline{\langle\phi\rangle + J(f,\phi)})$*, for all* v *defined in* (2), then $g \in \overline{J_X}$ as an ideal in $\mathcal{O}_{\mathbb{C}\times X}$.

Proof. Since $\Gamma_+(g), \Gamma_+(M^g_{u,v}) \subset C(\overline{\langle\phi\rangle + J(f,\phi)})$, we assume that $g, M^g_{u,v} \in \overline{\langle\phi\rangle + J(f,\phi)}$ as an ideal in $\mathcal{O}_{n+1}$.

Therefore, by Lemma 4.2, $g \in \overline{J_X}$ as an ideal in $\mathcal{O}_{\mathbb{C}\times X}$. □

Corollary 4.4. *If* $\Gamma_+(g), \Gamma_+(M^g_{u,v}) \subset \Gamma_+(\langle\phi\rangle + J(f,\phi))$*, for all* v *as in* (2) and $\langle\phi\rangle + J(f,\phi)$ is Newton non-degenerate, then $g \in \overline{J_X}$ as an ideal in $\mathcal{O}_{\mathbb{C}\times X}$.

Proof. Since $\langle\phi\rangle + J(f,\phi)$ is Newton non-degenerate then, by Theorem 4.1, we have

$$\Gamma_+(\langle\phi\rangle + J(f,\phi)) = C(\overline{\langle\phi\rangle + J(f,\phi)}).$$

Therefore, by Corollary 4.3, $g \in \overline{J_X}$ as an ideal in $\mathcal{O}_{\mathbb{C}\times X}$. □

We apply the results of this section to the construction of families which have constant Milnor number but do not satisfy the hypothesis of Theorem 3.5.

Example 4.5. Let $(X,0) \subset (\mathbb{C}^3,0)$ be defined as the zero set of $\phi(x,y,z) = (xy, x^{15}+y^{10}+z^6)$ and $f\colon (X,0) \to (\mathbb{C},0)$ defined by $f(x,y,z) = x+z$. Let F be the deformation of f defined by $F(t,(x,y,z)) = x+z+tg(x,y,z)$, where $g(x,y,z) = xy$.

The ideal $\langle \phi \rangle + J(f,\phi) = \langle xy, x^{15}+y^{10}+z^6, 6xz^5+10y^{10}-15x^{15}\rangle$ is Newton non-degenerate, $\Gamma_+(g) \subset \Gamma_+(\langle\phi\rangle + J(f,\phi))$, and $\Gamma_+(M^g_{u,v}) \subset \Gamma_+(\langle\phi\rangle + J(f,\phi))$ as ideals in $\mathcal{O}_4$. Thus, by Corollary 4.4, $\frac{\partial F}{\partial t} \in \overline{J_X}$ as an ideal in $\mathcal{O}_{\mathbb{C}\times X}$. Therefore, by Theorem 3.4, F is μ-constant.

Example 4.6. Let $(X,0) \subset (\mathbb{C}^3,0)$ be defined as the zero set of $\phi(x,y,z) = x^3+y^3+z^4+xyz$ and $f\colon (X,0) \to (\mathbb{C},0)$ defined by $f(x,y,z) = xy+z^2$. We consider the deformation of f defined by $F(t,(x,y,z)) = f(x,y,z)+tg(x,y,z)$, where g is a polynomial with degree higher than or equal to 3 different from z^3.

We have $\frac{\partial F}{\partial t} = g$ and

$$\langle\phi\rangle + J(f,\phi) = \langle x^3+y^3+z^4+xyz, -x^2y+6y^2z+2xz^2-4xz^3, -xy^2+6x^2z+2yz^2-4yz^3, -3x^3+3y^3\rangle$$

is Newton non-degenerate. In addition,

$$\Gamma_+(g) \subset \Gamma_+(\langle\phi\rangle + J(f,\phi)) \quad \text{and} \quad \Gamma_+(M^g_{u,v}) \subset \Gamma_+(\langle\phi\rangle + J(f,\phi)).$$

Thus, by Corollary 4.4, $\frac{\partial F}{\partial t} \in \overline{J_X}$ as an ideal in $\mathcal{O}_{\mathbb{C}\times X}$. Therefore F is μ-constant. Moreover, the deformation with $g = z^3$ is also μ-constant (we see this in Section 6).

5. Other invariants

Let $(X,0) \subset (\mathbb{C}^n,0)$ be a germ of an analytic variety. We consider two important subgroups of the $\mathcal{R}$ group of diffeomorphisms from $(\mathbb{C}^n,0)$ to $(\mathbb{C}^n,0)$: one is the group $\mathcal{R}_X$ of the diffeomorphisms which preserve $(X,0)$ and the other is $\mathcal{R}(X)$, the group of diffeomorphisms of X. We know that if the germs $f,g\colon (X,0) \to (\mathbb{C},0)$ are $\mathcal{R}_X$-equivalent, then they are $\mathcal{R}(X)$-equivalent, but the converse is not true.

In the smooth case, we know that a germ $f\colon (\mathbb{C}^n,0) \to (\mathbb{C},0)$ is finitely determined if and only if $\mu(f)$ is finite. There is a generalization of this result for the $\mathcal{R}_X$-group: $f\colon (\mathbb{C}^n,0) \to (\mathbb{C},0)$ is $\mathcal{R}_X$-finitely determined if and only if $\mu_{BR}(X,f)$ is finite. Here $\mu_{BR}(X,f)$ is the Bruce-Roberts number defined in [3] by

$$\mu_{BR}(X,f) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{J_f(\Theta_X)},$$

where Θ_X is the $\mathcal{O}_n$-module of vector fields in $(\mathbb{C}^n,0)$ which are tangent to $(X,0)$ and

$$J_f(\Theta_X) = \langle df(\xi) \mid \xi \in \Theta_X\rangle.$$

Because of this, it is important to know when a family has constant Bruce-Roberts number. In [1], Ahmed, Ruas, and Tomazella study the analogue of Theorem 3.1 for the Bruce-Roberts number assuming that $(X,0) \subset (\mathbb{C}^n,0)$ is a germ of an analytic variety and

$$F\colon (\mathbb{C}\times\mathbb{C}^n,0) \to (\mathbb{C},0), \quad F(t,x) := f_t(x)$$

is a family of function germs such that $\mu_{BR}(X,f_t)$ is finite. In this case the assertions of Theorem 3.1 would be our (1_X),…,(6_X) changing J_X by $J_{f_t}(\Theta_X)$ and the minors in (2_X) by $dF(\xi_i)$ where Θ_X is generated by $\xi_1,\dots,\xi_p$. They denote the assertions by (1_r),…,(6_r)

In [1], it is proved that $(2_r) \Rightarrow (3_r) \Leftrightarrow (4_r) \Rightarrow (5_r)$, $(1_r) \Rightarrow (6_r)$ and if the polar curve C is a Cohen-Macaulay variety, then $(6_r) \Rightarrow (1_r)$. Moreover, we can see in [1] that $(4_r) \Rightarrow (1_r)$ if $(X,0)$ is an isolated hypersurface singularity whose logarithmic characteristic variety $LC(X)$ is

Cohen-Macaulay (see [3] for the definition of $LC(X)$). Recently, it was proved that, if $(X,0)$ is an isolated hypersurface singularity, then $LC(X)$ is Cohen-Macaulay, see [20].

Another important number related to $f\colon (X,0)\to(\mathbb{C},0)$ is its local Euler obstruction $Eu_{f,X}(0)$. This number is very studied, for instance in [2] and [11]. In [11], Grulha presents the following theorem, which relates the constancy of the Euler obstruction in a family to the constancy of the Milnor number (of ICIS) in a family.

Theorem 5.1. [11, Proposition 5.17] *Assume that $(X,0)\subset(\mathbb{C}^n,0)$ is an ICIS, and let $F\colon(\mathbb{C}\times X,0)\to\mathbb{C}$ be a family of functions with isolated singularity. We write $F(t,x)=f_t(x)$. Then the following statements are equivalent*

(1) *$Eu_{f_t,X}(0)$ is constant for the family;*

(2) *$\mu(X\cap f_t^{-1}(0),0)$ is constant for the family.*

By using this result, we present in the next theorem several applications of our main theorem.

Theorem 5.2. *Let $(X,0)\subset(\mathbb{C}^n,0)$ be an ICIS and $f\colon(X,0)\to(\mathbb{C},0)$ a germ with isolated singularity $\mathcal{R}_X$-finitely determined. Let $F\colon(\mathbb{C}\times X,0)\to(\mathbb{C},0)$ be a deformation of f. If $\frac{\partial F}{\partial t}\in\overline{J_X}$ as an ideal in $\mathcal{O}_{\mathbb{C}\times X}$, then*

(i) *$\tilde{F}$ is C^0-$\mathcal{R}_X$-trivial (and hence $\tilde{F}$ is C^0-$\mathcal{R}(X)$-trivial), where $\tilde{F}\colon(\mathbb{C}\times\mathbb{C}^n,0)\to(\mathbb{C},0)$ is such that $F=\tilde{F}|_{\mathbb{C}\times X}$;*

(ii) *$Eu_{\tilde{f}_t,X}(0)$ is constant, where $\tilde{f}_t\colon(\mathbb{C}^n,0)\to(\mathbb{C},0)$ is such that $f_t=\tilde{f}_t|_X$;*

(iii) *If $(X,0)$ is a hypersurface with isolated singularity, then $\mu_{BR}(X,\tilde{f}_t)$ is constant;*

(iv) *If $(X,0)$ is a weighted homogeneous hypersurface with isolated singularity, then $\mu_{BR}(X,\tilde{f}_t)$ is constant and $m(\tilde{f}_t)$ is constant, where $m(\tilde{f}_t)$ is the multiplicity of $\tilde{f}_t$.*

Proof. (i) In [24, Theorem 4.3] it is shown that, if $(X,0)$ is an ICIS and $\frac{\partial F}{\partial t}\in\overline{J_F(\Theta_X)}$, then $\tilde{F}$ is C^0-$\mathcal{R}_X$-trivial. Therefore, since $J_X\subseteq J_F(\Theta_X)$, we have the desired result.

(ii) Follows directly from Theorem 3.4 and Theorem 5.1.

(iii) Just uses $J_X\subseteq J_F(\Theta_X)$ and the $(4_r)\Rightarrow(1_r)$ in [1].

(iv) In [22, Theorem 4.2], it is shown that, if $(X,0)$ is a weighted homogeneous hypersurface with isolated singularity, then $LC(X)$ is Cohen-Macaulay. Therefore it follows by (iii) that $\mu_{BR}(X,\tilde{f}_t)$ is constant. Thus $m(\tilde{f}_t)$ is also constant (see [1, Theorem 4.3]).

□

As an application of (ii) of the Theorem 5.2, the families of Examples 4.5 and 4.6 have constant Euler obstruction.

6. Deformation of the ICIS

Our final goal in this paper is to study the constancy of the Milnor number if we deform both the analytic variety and the function germ. That is, let $\phi\colon(\mathbb{C}^n,0)\to(\mathbb{C}^p,0)$ be a holomorphic map germ and $(X,0)\subset(\mathbb{C}^n,0)$ the ICIS defined by the zero set of ϕ. Let $\Phi\colon(\mathbb{C}\times\mathbb{C}^n,0)\to(\mathbb{C}^p,0)$ be a (flat) deformation of ϕ, defined by $\Phi(t,x)=\phi_t(x)$, such that $\phi_0=\phi$ and $(X_t,0):=(\phi_t^{-1}(0),0)$ is an ICIS for t sufficiently small. We write $\mathcal{X}=\Phi^{-1}(0)$. Moreover, let $f\colon(X,0)\to(\mathbb{C},0)$ be a holomorphic function germ with isolated singularity and

$$\begin{array}{rccc} F\colon & (\mathcal{X},0) & \to & (\mathbb{C},0)\\ & (t,x) & \mapsto & F(t,x)=f_t(x)\end{array}$$

a (flat) deformation of f. Here we study the constancy of the Milnor number $\mu(f_t|_{X_t},0)$. For this, let $M_{u,v}$ be the minor of the Jacobian matrix of (F,Φ) (with respect to x only) and J_X the ideal generated by $M_{u,v}$, with $u=(1,\dots,p+1)$ and

$$v=(j_1,\dots,j_{p+1}),\ \text{with}\ j_1<\cdots<j_{p+1}\ \text{and}\ j_1,\dots,j_{p+1}\in\{1,\dots,n\}. \tag{3}$$

We consider $G\colon \mathbb{C}\times\mathbb{C}^n \to (\mathbb{C}\times\mathbb{C}^p,0)$ defined by $G(t,x)=(\tilde{F}(t,x),\Phi(t,x))$, where $\tilde{F}:\mathbb{C}\times\mathbb{C}^N\to\mathbb{C}$ is such that $\tilde{F}|_{\mathcal{X}}=F$.

If $\frac{\partial G}{\partial t}\in\overline{\{x_i\frac{\partial G}{\partial x_j}\mid i,j=1,\dots,n\}}_{\mathcal{O}_{\mathcal{X}\cap F^{-1}(0)}}$, then $\{X_t\cap {f_t}^{-1}(0)\}$ is Whitney regular (see [5, Theorem 2.5]). Moreover, we know that if $\frac{\partial G}{\partial t}\in\overline{\{\frac{\partial G}{\partial x_j}\mid j=1,\dots,n\}}^{\dagger}_{\mathcal{O}_{\mathcal{X}}}$, then A_F (see [10]) holds for the pair $(\mathcal{X}_0,Y)$, where $\mathcal{X}_0=\mathcal{X}-\mathbb{C}\times\{0\}$ and $Y=\mathbb{C}\times\{0\}$ (see [8, Lemma 5.1]). Hence we can ask if $\frac{\partial G}{\partial t}\in\overline{\{\frac{\partial G}{\partial x_j}\mid j=1,\dots,n\}}^{\dagger}_{\mathcal{O}_{\mathcal{X}}}$ is related to the constancy of the Milnor number in the family F.

Theorem 6.1. *The following statements are equivalent*

(1_{Xt}) *F is μ-constant;*

(2_{Xt}) $\frac{\partial G}{\partial t}\in\overline{\{\frac{\partial G}{\partial x_j}\mid j=1,\dots,n\}}^{\dagger}_{\mathcal{O}_{\mathcal{X}}}$;

(3_{Xt}) $v(J_{X_t})=\{(t,x)\in\mathbb{C}\times\mathbb{C}^n\mid M_{u,v}(t,x)=0,$ *for all* v *in* (3)$\}=\mathbb{C}\times\{0\}$ *near* $(0,0)$ *in* $\mathcal{O}_{\mathcal{X}}$.

Proof. $(1_{X_t})\Leftrightarrow(3_{X_t})$:

This is the proof of $(1_X)\Leftrightarrow(6_X)$ in Theorem 3.4.

$(1_{X_t})\Rightarrow(2_{X_t})$:

By the Lê-Greuel formula, $\mu(f_t|_{X_t},0)=\mu(X_t,0)+\mu(X_t\cap f_t^{-1}(0),0)$. Thus $\mu(X_t,0)$ and $\mu(X_t\cap f_t^{-1}(0),0)$ are constant. Hence A_F holds for the pair $(\mathcal{X}_0,Y)$ (see [10, Theorem 5.8]). Therefore $\frac{\partial G}{\partial t}\in\overline{\{\frac{\partial G}{\partial x_j}\mid j=1,\dots,n\}}^{\dagger}_{\mathcal{O}_{\mathcal{X}}}$ (see [8, Lemma 5.1]).

$(2_{X_t})\Rightarrow(1_{X_t})$:

Since $\frac{\partial G}{\partial t}\in\overline{\{\frac{\partial G}{\partial x_j}\mid j=1,\dots,n\}}^{\dagger}_{\mathcal{O}_{\mathcal{X}}}$, A_F holds for the pair $(\mathcal{X}_0,Y)$ (see [8, Lemma 5.1]). Hence the Buchsbaum-Rim multiplicity of the module $\{\frac{\partial G}{\partial x_j}\mid j=1,\dots,n\}_{\mathcal{O}_{\mathcal{X}}}$ is constant (see [14, Theorem 3.2]). Thus F is μ-constant (see [14, Lemma 3.3]).

□

Remark 6.2. We remark that, with our hypothesis, $\frac{\partial G}{\partial t}\in\overline{\{\frac{\partial G}{\partial x_j}\mid j=1,\dots,n\}}_{\mathcal{O}_{\mathcal{X}}}$ if and only if $\frac{\partial G}{\partial t}\in\overline{\{\frac{\partial G}{\partial x_j}\mid j=1,\dots,n\}}^{\dagger}_{\mathcal{O}_{\mathcal{X}}}$; see [10, proof of Theorem 5.8] and [8, Lemma 5.1].

We return now to Example 4.6. In that case, if $g=z^3$, then $g\in\overline{J_X}$ as an ideal in $\mathcal{O}_{\mathbb{C}\times X}$ and therefore, by Theorem 3.2, $\frac{\partial G}{\partial t}\in\overline{\{x_i\frac{\partial G}{\partial x_j}\mid i,j=1,\dots,n\}}_{\mathcal{O}_{\mathbb{C}\times X}}$ with $G=(F,\phi)$. Hence $\frac{\partial G}{\partial t}\in\overline{\{\frac{\partial G}{\partial x_j}\mid j=1,\dots,n\}}_{\mathcal{O}_{\mathbb{C}\times X}}$. Thus $\frac{\partial G}{\partial t}\in\overline{\{\frac{\partial G}{\partial x_j}\mid j=1,\dots,n\}}^{\dagger}_{\mathcal{O}_{\mathbb{C}\times X}}$, by the previous remark. Therefore, by Theorem 6.1, F is μ-constant.

Example 6.3. Let $(X,0)\subset(\mathbb{C}^3,0)$ be defined as the zero set of

$$\phi(x,y,z)=(z-xy,y^2-x^3)$$

and $f\colon(X,0)\to(\mathbb{C},0)$ defined by $f(x,y,z)=x$. Let Φ be the deformation of ϕ defined by

$$\Phi(t,(x,y,z))=(z-xy+tx^2,y^2-x^3)$$

and F the deformation of f defined by $F(t,(x,y,z))=x+tz$. Let $\mathcal{X}=\Phi^{-1}(0)$.

We consider $G(t,(x,y,z))=(F(t,(x,y,z)),\Phi(t,(x,y,z)))$.

It is easy to see that $I_3((\frac{\partial G}{\partial t},\frac{\partial G}{\partial x},\frac{\partial G}{\partial y},\frac{\partial G}{\partial z}))\subset\overline{I_3((\frac{\partial G}{\partial x},\frac{\partial G}{\partial y},\frac{\partial G}{\partial z}))}$, where $I_3((\frac{\partial G}{\partial t},\frac{\partial G}{\partial x},\frac{\partial G}{\partial y},\frac{\partial G}{\partial z}))$ denotes the minors of order 3 of the matrix whose columns are formed by the vectors $\frac{\partial G}{\partial t},\frac{\partial G}{\partial x},\frac{\partial G}{\partial y},\frac{\partial G}{\partial z}$.

Hence $\frac{\partial G}{\partial t} \in \overline{\{\frac{\partial G}{\partial x}, \frac{\partial G}{\partial y}, \frac{\partial G}{\partial z}\}_{\mathcal{O}_{\mathcal{X}}}}$ (see [5, Proposition 1.7]). Therefore, by Remark 6.2 and Theorem 6.1, F is μ-constant.

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