

IRREDUCIBLE HOLONOMY GROUPS AND RICCATI FOLIATIONS IN HIGHER COMPLEX DIMENSION

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ABSTRACT. We study groups of germs of complex diffeomorphisms having a property called *irreducibility*. The notion is motivated by a similar property of the fundamental group of the complement of an irreducible hypersurface in the complex projective space. Natural examples of such groups of germ maps are given by holonomy groups and monodromy groups of integrable systems (foliations) under certain conditions on the singular or ramification set. The case of complex dimension one is studied in [15] where finiteness is proved for irreducible groups under certain arithmetic hypothesis on the linear part. In dimension $n \geq 2$ the picture changes since linear groups are not always abelian in dimension two or bigger. Nevertheless, we still obtain a finiteness result under some conditions in the linear part of the group, for instance if the linear part is abelian. Examples are given illustrating the role of our hypotheses. Applications are given to the framework of holomorphic foliations and analytic deformations of rational fibrations by Riccati foliations.

1. INTRODUCTION

In [15] we introduced the notion of irreducible group of germs of diffeomorphisms in dimension 1. In that work we give conditions under which such a group is finite and prove some applications of this to the problem of existence of holomorphic first integrals for codimension one foliations. In this work we investigate the extension of this to the case of any dimension. We make the following definition:

Definition 1.1. A group G is *irreducible* if it admits a finite set of generators $g_1, \dots, g_{\nu+1}$ such that:

- (a) $g_1 \circ \dots \circ g_{\nu+1} = e_G$
- (b) g_i and g_j are conjugate in G for all i, j .

We shall refer to $\{g_1, \dots, g_{\nu+1}\}$ as a *basic set of generators*. The above definition does not exclude the possibility that $g_i = g_j$. An irreducible abelian group is finite cyclic: indeed, since the group is abelian we have $g_i = g_j$, for all i, j . Therefore the group is generated by an element g_1 such that $g_1^{\nu+1} = e_G$.

We shall denote by $\text{Diff}(\mathbb{C}^n, 0)$ the group of germs of complex diffeomorphisms fixing the origin $0 \in \mathbb{C}^n$. In this work we shall focus on irreducible subgroups $G \subset \text{Diff}(\mathbb{C}^n, 0)$. It is important to point out that conditions (a) and (b) in Definition 1.1 are independent and therefore not equivalent (cf. Proposition 5.1). Every cyclic subgroup of *finite order* $G \subset \text{Diff}(\mathbb{C}^n, 0)$ is irreducible. A first question would be whether finite subgroups of $\text{Diff}(\mathbb{C}^n, 0)$ are also irreducible. A second, more challenging, question is whether irreducible subgroups of $\text{Diff}(\mathbb{C}^n, 0)$ are always finite. The above questions have negative response even in the linear case (group of matrices) (cf. Proposition 5.1).

1.1. Irreducible groups of germs of complex diffeomorphisms. From now on we shall consider $\mathbb{N} = \{0, 1, 2, \dots\}$ and $\mathbb{N}^* = \{1, 2, \dots\}$. Our basic result is the following:

Theorem 1.1. Let $G \subset \text{Diff}(\mathbb{C}^n, 0)$ be an irreducible group having a basic set of generators $\{f_1, \dots, f_{\nu+1}\}$ with the same linear part $Df_j(0) = A \in \text{GL}(n, \mathbb{C})$. Assume that the eigenvalues of A are $\lambda_1, \dots, \lambda_n$ such that λ_j is a root of the unit of order p^{s_j} , where p is prime and $s_j \in \mathbb{N}$. Then G is finite cyclic. Indeed, G is analytically conjugate to a cyclic group generated by a diagonal matrix of the form $A = \text{diag}(\xi_1, \dots, \xi_n)$ where ξ_j is a root of the unit of order p^{r_j} .

We point out that Theorem 1.1 holds with an analogous statement for the case of groups of formal diffeomorphisms. Indeed, the proofs are based on some normal forms for the resonant case and on the Taylor series expansion, so the proofs apply *ipsis litteris* to the formal case.

The *linear part* of a group $G \subset \text{Diff}(\mathbb{C}^n, 0)$ is the subgroup of $\text{GL}(n, \mathbb{C})$ of the linear maps $Df(0)$ where $f \in G$ and the coordinates are the canonical affine coordinates $(z_1, \dots, z_n) \in \mathbb{C}^n$.

Corollary 1.1. Let $G \subset \text{Diff}(\mathbb{C}^n, 0)$ be an irreducible group with abelian linear part, having any (not necessarily basic) set of generators $g_1, g_2, \dots, g_{p^s} \in G$ such that $g_1 \circ \dots \circ g_{p^s} = \text{Id}$ for some prime number p and some $s \in \mathbb{N}$. Then G is finite cyclic of order p^ℓ for some $\ell \leq s$.

1.2. Holonomy of holomorphic foliations. As for an application of the preceding results to the framework of foliations we have:

Theorem 1.2. Let \mathcal{F} be a codimension n holomorphic foliation with singularities on a complex manifold M^{n+2} . Assume that there is a leaf $L_0 \in \mathcal{F}$ which is homeomorphic to $\mathbb{P}^2 \setminus C$ where $C \subset \mathbb{P}^2$ is an irreducible algebraic curve of degree p^s for some prime number p and some $s \in \mathbb{N}$. Assume that the linear holonomy of L_0 is abelian. Then the holonomy group of the leaf L_0 is a finite cyclic analytically linearizable group.

1.3. A stability result for groups of germs of complex diffeomorphisms. Next we state a sort of Reeb stability theorem ([4]) for groups of germs of complex diffeomorphisms. Given a subgroup $G \subset \text{Diff}(\mathbb{C}^n, 0)$ with a finite set of generators f_1, \dots, f_r , by an *analytic deformation* of G we shall mean a family $\{G_t\}_{t \in D}$ of subgroups $G_t \subset \text{Diff}(\mathbb{C}^n, 0)$, parametrized by $t \in D \subset \mathbb{C}$, where each G_t is generated by maps $f_{j,t} \in \text{Diff}(\mathbb{C}^n, 0)$, depending analytically on t , of the form $f_{j,t} = f_j + \sum_{k=1}^{\infty} a_{j,k} t^k$ where each $a_{j,k}$ is holomorphic with a zero of order ≥ 2 at the origin (i.e., the linear part of $a_{j,k}$ at the origin is zero) for all j, k .

Then we can state the following stability theorem for groups of germs of complex diffeomorphisms:

Theorem 1.3. Let $G \subset \text{Diff}(\mathbb{C}^n, 0)$ be a cyclic finite subgroup of order p^s for some prime number p and $s \in \mathbb{N}$. Given an analytic deformation $\{G_t\}_{t \in D}$ of G we have the following equivalences:

- (1) G_t is irreducible for all t close to 0.
- (2) G_t is finite cyclic for all t close to 0.

Furthermore, if G is trivial then any analytic deformation $\{G_t\}$ of G by irreducible groups G_t is such that G_t is trivial for all t close to 0.

1.4. Applications to Riccati foliations. Theorems 1.2 and 1.3 apply to the study of *Riccati foliations* in a general setting (cf. § 4). Let us give a brief description of the notion of Riccati foliation we deal with in this paper. More details are found in § 4. We shall consider a complex manifold M admitting a locally trivial holomorphic fibration $\pi: M \rightarrow B$, onto a complex manifold B , with fiber F . A singular holomorphic foliation \mathcal{F} on M will be called a *Riccati*

foliation on M if (i) $\text{codim } \mathcal{F} = \dim F$ and (ii) there is a set $\sigma \subset B$, analytic of codimension ≥ 1 , such that $\pi^{-1}(\sigma)$ is invariant (a union of \mathcal{F} -invariant fibers), and the restriction of \mathcal{F} to $M \setminus \pi^{-1}(\sigma)$ is a foliation transverse to the fibre bundle $\pi|_{M \setminus \pi^{-1}(\sigma)}: M \setminus \pi^{-1}(\sigma)$ in the classical sense of Ehresmann (see for instance [4] Chapter V). In particular we have $\text{sing}(\mathcal{F}) \subset \pi^{-1}(\sigma)$. Most relevant is the fact, well-known for foliations transverse to fibre bundles, that (iii) in case the fundamental group of $B \setminus \sigma$ is finitely generated, $\mathcal{F}|_{M \setminus \pi^{-1}(\sigma)}$ is conjugate to the suspension of a subgroup of holomorphic diffeomorphisms $G \subset \text{Diff}(F)$, given by a (so called *holonomy or monodromy*) homomorphism $\varphi: \pi_1(B \setminus \sigma) \rightarrow \text{Diff}(F)$.

In view of (iii) our notion of Riccati foliation is quite general. Indeed, in our framework, a Riccati foliation on a fibered space $\pi: M \rightarrow B$, with fiber F , may be seen as an extension, holomorphic with singularities, of a foliation given by a suspension of a group of complex diffeomorphisms $G \subset \text{Diff}(F)$, obtained as a representation $\pi_1(B \setminus \sigma) \rightarrow \text{Diff}(F)$, where σ is a codimension ≥ 1 analytic subset of B . This idea is reinforced by the following. When $M = \mathbb{P} \times \mathbb{P}$, a holomorphic foliation with singularities, is of Riccati type $\frac{dy}{dx} = \frac{a(x)y^2 + b(x)y + c(x)}{p(x)}$, if and only if, it is transverse to a generic vertical line $\mathbb{P}_{x_0} = \{x_0\} \times \mathbb{P} \subset \mathbb{P} \times \mathbb{P}$. This fact, widely used by Paul Painlevé in his memoir ([14]), has been extended in a natural way in [16]. Indeed, in [16] the notion of Riccati foliation adopted is pretty much the same we use here, by considering suitable (natural) fibrations. We shall resume this discussion in §4 (cf. Remark 4.2).

As a sample of how our results apply in this framework we give:

Theorem 1.4. Let \mathcal{F} be the foliation by levels of a rational function $R: \mathbb{P}^m \times \mathbb{P}^n \rightarrow \mathbb{P}^n$. Assume that the codimension one component $\sigma_1 \subset \sigma$ of the ramification set $\sigma \subset \mathbb{P}^m$ of R is empty or irreducible (not necessarily smooth nor normal crossing type) of degree p^s for some prime number p and some $s \in \mathbb{N}$. Let now $\{\mathcal{F}_t\}_{t \in \mathbb{D}}$ be an analytic deformation of $\mathcal{F} = \mathcal{F}_0$ by Riccati foliations on $\mathbb{P}^m \times \mathbb{P}^n$ leaving invariant the basis $\mathbb{P}^m \times \{0\}$ (for some point $0 \in \mathbb{P}^n$). Then the global holonomy of \mathcal{F}_t is finite cyclic for each t close to 0. In particular, the leaves of \mathcal{F}_t are closed in $(\mathbb{P}^m \setminus \sigma_1(t)) \times \mathbb{P}^n$, i.e., $\lim(\mathcal{F}_t) \subset \sigma_1(t) \times \mathbb{P}^n$, for all t close to 0. If R is the second projection $\mathbb{P}^m \times \mathbb{P}^n \rightarrow \mathbb{P}^n$ then \mathcal{F}_t is analytically conjugate to \mathcal{F} in $(\mathbb{P}^m \setminus \sigma(t)) \times \mathbb{P}^n$.

Roughly speaking, the ramification set of \mathcal{F} is the set $\sigma \subset \mathbb{P}^m$ of base points $x \in \mathbb{P}^m$ for which the fiber $\{x\} \times \mathbb{P}^n$ is not transverse to \mathcal{F} . In the above statement $\sigma(t)$ denotes the ramification set of \mathcal{F}_t .

We refer to §4 for the details and further recent references on Riccati foliations. In Remark 4.2 one finds how our results compare and contrast with other work on the topology of Riccati foliations.

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2. DIMENSION n

According to Definition 1.1 we have:

Definition 2.1 (irreducible group). A subgroup $G \subset \text{Diff}(\mathbb{C}^n, 0)$ is *irreducible* if it admits a finite *basic set of generators* $f_1, f_2, \dots, f_{\nu+1} \in G$ such that:

- (a) $f_1 \circ f_2 \circ \dots \circ f_{\nu+1} = \text{Id}$.
- (b) f_i and f_j are conjugate in G for all i, j .

In order to prove Theorem 1.1 we will make use of the Taylor expansion. Given $f \in \text{Diff}(\mathbb{C}^n, 0)$, since $f(0) = 0$, for all $Z \in \mathbb{C}^n$ close to 0 we have:

$$f(Z) = Df(0) \cdot Z + \frac{1}{2}f''(0) \cdot Z^2 + \dots + \frac{1}{p!}f^{(p)}(0) \cdot Z^p + \dots$$

where

$$f^{(p)}(0) \cdot Z^p = \frac{\partial^p f}{\partial Z^p}(0) = \frac{\partial}{\partial Z} \left(\frac{\partial^{p-1} f}{\partial Z^{p-1}} \right) (0) = \sum_{k_1, \dots, k_p=1}^n \frac{\partial^p f}{\partial Z_{k_r p}}(0) z_{k_1} \cdots z_{k_p}$$

here $Z = (z_1, \dots, z_n)$ and $\frac{\partial^p}{\partial Z_{k_r p}} = \frac{\partial^p}{\partial z_{k_1} \partial z_{k_2} \cdots \partial z_{k_p}}$. We shall need the following expression for the n -th derivative of the function composition of maps:

Lemma 2.1. For any $\varphi : \mathbb{C}^n \rightarrow \mathbb{C}$, $\psi : \mathbb{C}^n \rightarrow \mathbb{C}^n$ holomorphic and $m \in \mathbb{N}$, $m \geq 3$ we have that

$$(1) \quad \frac{\partial^m(\varphi \circ \psi)}{\partial z_{r_m} \cdots \partial z_{r_1}} = \sum_{k_1, \dots, k_m=1}^n \frac{\partial^m \varphi(\psi)}{\partial Z_{k_r m}} \frac{\partial \psi_{k_m}}{\partial z_{r_m}} \cdots \frac{\partial \psi_{k_1}}{\partial z_{r_1}} + \sum_{k=1}^n \frac{\partial \varphi(\psi)}{\partial z_k} \frac{\partial^m \psi_k}{\partial z_{r_m} \cdots \partial z_{r_1}} + \\ + \sum_{p=2}^{m-1} \sum_{k_1, \dots, k_p=1}^n \frac{\partial^p \varphi(\psi)}{\partial Z_{k_r p}} \cdot R_{k_r p}(\cdot),$$

where ψ_{k_j} is the k_j -th coordinate of ψ and $R_{k_r p}$ is a polynomial expression as a function of the derivatives of ψ from order 1 to order $m-1$ and has no terms containing only derivatives of order 1.

Proof. Applying $\frac{\partial}{\partial z_{r_1}}$ to $\varphi \circ \psi$, we have

$$\frac{\partial(\varphi \circ \psi)}{\partial z_{r_1}} = \sum_{k=1}^n \frac{\partial \varphi(\psi)}{\partial z_k} \frac{\partial \psi_k}{\partial z_{r_1}}$$

applying $\frac{\partial}{\partial z_{r_2}}$ we have

$$\frac{\partial^2(\varphi \circ \psi)}{\partial z_{r_2} \partial z_{r_1}} = \sum_{k_1, k_2=1}^n \frac{\partial^2 \varphi(\psi)}{\partial z_{k_2} \partial z_{k_1}} \frac{\partial \psi_{k_2}}{\partial z_{r_2}} \frac{\partial \psi_{k_1}}{\partial z_{r_1}} + \sum_{k=1}^n \frac{\partial \varphi(\psi)}{\partial z_k} \frac{\partial^2 \psi_k}{\partial z_{r_2} \partial z_{r_1}}$$

applying $\frac{\partial}{\partial z_{r_3}}$ we have

$$\frac{\partial^3(\varphi \circ \psi)}{\partial z_{r_3} \partial z_{r_2} \partial z_{r_1}} = \sum_{k_1, k_2, k_3=1}^n \frac{\partial^3 \varphi(\psi)}{\partial z_{k_3} \partial z_{k_2} \partial z_{k_1}} \frac{\partial \psi_{k_3}}{\partial z_{r_3}} \frac{\partial \psi_{k_2}}{\partial z_{r_2}} \frac{\partial \psi_{k_1}}{\partial z_{r_1}} + \sum_{k=1}^n \frac{\partial \varphi(\psi)}{\partial z_k} \frac{\partial^3 \psi_k}{\partial z_{r_3} \partial z_{r_2} \partial z_{r_1}} + \\ + \sum_{k_1, k_2=1}^n \frac{\partial^2 \varphi(\psi)}{\partial z_{k_2} \partial z_{k_1}} \left[\frac{\partial^2 \psi_{k_2}}{\partial z_{r_3} \partial z_{r_2}} \frac{\partial \psi_{k_1}}{\partial z_{r_1}} + \frac{\partial^2 \psi_{k_2}}{\partial z_{r_3} \partial z_{r_1}} \frac{\partial \psi_{k_1}}{\partial z_{r_2}} + \frac{\partial^2 \psi_{k_2}}{\partial z_{r_2} \partial z_{r_1}} \frac{\partial \psi_{k_1}}{\partial z_{r_3}} \right]$$

then

$$R_{k_r p}(\cdot) = \frac{\partial^2 \psi_{k_2}}{\partial z_{r_3} \partial z_{r_2}} \frac{\partial \psi_{k_1}}{\partial z_{r_1}} + \frac{\partial^2 \psi_{k_2}}{\partial z_{r_3} \partial z_{r_1}} \frac{\partial \psi_{k_1}}{\partial z_{r_2}} + \frac{\partial^2 \psi_{k_2}}{\partial z_{r_2} \partial z_{r_1}} \frac{\partial \psi_{k_1}}{\partial z_{r_3}}.$$

Thus $R_{k_r p}(\cdot)$ is a polynomial expression as a function of the derivatives of ψ from order 1 and 2 and has no terms containing only derivatives of order 1.

Let us assume that equation (1) is satisfied for m by showing that it is valid for $m+1$. By the hypothesis of induction (1) is valid. Applying $\frac{\partial}{\partial z_{r_{m+1}}}$ to (1) we have

$$\begin{aligned}
 \frac{\partial^{m+1}(\varphi \circ \psi)}{\partial z_{r_{m+1}} \cdots \partial z_{r_1}} &= \sum_{k_1, \dots, k_{m+1}=1}^n \frac{\partial^{m+1} \varphi(\psi)}{\partial Z_{k_{r_{m+1}}}} \frac{\partial \psi_{k_{m+1}}}{\partial z_{r_{m+1}}} \cdots \frac{\partial \psi_{k_1}}{\partial z_{r_1}} + \sum_{k=1}^n \frac{\partial \varphi(\psi)}{\partial z_k} \frac{\partial^{m+1} \psi_k}{\partial z_{r_{m+1}} \cdots \partial z_{r_1}} + \\
 &+ \sum_{k_1, \dots, k_m=1}^n \frac{\partial^m \varphi(\psi)}{\partial Z_{k_{r_m}}} \frac{\partial^2 \psi_{k_m}}{\partial z_{r_{m+1}} \partial z_{r_m}} \frac{\partial \psi_{k_{m-1}}}{\partial z_{r_{m-1}}} \cdots \frac{\partial \psi_{k_1}}{\partial z_{r_1}} + \sum_{k_1, \dots, k_m=1}^n \frac{\partial^m \varphi(\psi)}{\partial Z_{k_{r_m}}} \frac{\partial \psi_{k_m}}{\partial z_{r_m}} \frac{\partial^2 \psi_{k_{m-1}}}{\partial z_{r_{m+1}} \partial z_{r_{m-1}}} \cdots \frac{\partial \psi_{k_1}}{\partial z_{r_1}} \\
 &+ \cdots + \sum_{k_1, \dots, k_m=1}^n \frac{\partial^m \varphi(\psi)}{\partial Z_{k_{r_m}}} \frac{\partial \psi_{k_m}}{\partial z_{r_m}} \frac{\partial \psi_{k_{m-1}}}{\partial z_{r_{m-1}}} \cdots \frac{\partial^2 \psi_{k_1}}{\partial z_{r_{m+1}} \partial z_{r_1}} + \sum_{p=2}^{m-1} \sum_{k_1, \dots, k_{p+1}=1}^n \frac{\partial^{p+1} \varphi(\psi)}{\partial Z_{k_{r_{p+1}}}} \frac{\partial \psi_{k_{p+1}}}{\partial z_{r_{m+1}}} \cdot R_{k_p}(\cdot) \\
 &+ \sum_{p=2}^{m-1} \sum_{k_1, \dots, k_p=1}^n \frac{\partial^p \varphi(\psi)}{\partial Z_{k_{r_p}}} \frac{\partial [R_{k_{r_p}}(\cdot)]}{\partial z_{r_{m+1}}} + \sum_{k_1, k_2=1}^n \frac{\partial^2 \varphi(\psi)}{\partial z_{k_2} \partial z_{k_1}} \frac{\partial \psi_{k_2}}{\partial z_{r_{m+1}}} \frac{\partial^m \psi_{k_1}}{\partial z_{r_m} \cdots \partial z_{r_1}}.
 \end{aligned}$$

Notice that $R_{k_{r_p}}$ is a polynomial expression as a function of the derivatives of ψ from order 1 to order $m-1$ and has no terms containing only derivatives of order 1. Thus by the chain rule we have that $\frac{\partial [R_{k_{r_p}}(\cdot)]}{\partial z_{r_{m+1}}}$ is a polynomial expression as a function of the derivatives of ψ from order 1 to order m and has no terms containing only derivatives of order 1. Putting $\frac{\partial^p \varphi(\psi)}{\partial Z_{k_{r_p}}}$ in evidence in the above expression we conclude. \square

The very first case in Theorem 1.1 is the following:

Proposition 2.1. Let $G \subset \text{Diff}(\mathbb{C}^n, 0)$ be an irreducible group such that G has a generator tangent to identity. Then $G = \{\text{Id}\}$.

Proof. Since G is an irreducible group that has a generator tangent to identity there exists a finite set of generators $f_{(1)}, f_{(2)}, \dots, f_{(\nu+1)} \in G$ such that

- (a) $f_{(1)} \circ f_{(2)} \circ \dots \circ f_{(\nu+1)} = \text{Id}$.
- (b) $f_{(i)}$ and $f_{(j)}$ are conjugate in G for all i, j .
- (c) $Df_{(r)}(0) = \text{Id}$ for some r .

From (b) given $f_{(i)} \in G$ there is $h \in G$, such that $f_{(i)} \circ h(Z) = h \circ f_{(r)}(Z)$. Now from (c) we have

$$Df_{(i)}(0)Dh(0) = Dh(0)Df_{(r)}(0) = Dh(0)\text{Id} = Dh(0).$$

Hence $Df_{(i)}(0) = \text{Id}$ for all i and consequently for all $g \in G$, $Dg(0) = \text{Id}$. Note that if $g(Z) = Z + P_k(Z) + h.o.t \in G$ then

$$g^{(m)}(Z) = g \circ g \circ \dots \circ g(Z) = Z + mP_k(Z) + h.o.t.$$

Now for simplicity consider $f = f_{(i)}$ and $g = f_{(j)}$, we will show that

$$\frac{\partial^m f(0)}{\partial z_{r_m} \cdots \partial z_{r_1}} = \frac{\partial^m g(0)}{\partial z_{r_m} \cdots \partial z_{r_1}} = 0 \text{ for all } m \in \mathbb{N}, m \geq 2.$$

Indeed, we will prove for each coordinate of f and g using induction. Now from (b) there is $h \in G$, such that $f \circ h(Z) = h \circ g(Z)$. We consider $f_s : \mathbb{C}^n \rightarrow \mathbb{C}$ the s -coordinate of f and $h_s : \mathbb{C}^n \rightarrow \mathbb{C}$ the s -coordinate of h . Then $f_s \circ h = h_s \circ g$ applying $\frac{\partial}{\partial z_{r_1}}$ on both sides we have

$$\sum_{k=1}^n \frac{\partial f_s(h)}{\partial z_k} \frac{\partial h_k}{\partial z_{r_1}} = \sum_{k=1}^n \frac{\partial h_s(g)}{\partial z_k} \frac{\partial g_k}{\partial z_{r_1}}$$

applying $\frac{\partial}{\partial z_{r_2}}$ on both sides we have

$$\sum_{k_1, k_2=1}^n \frac{\partial^2 f_s(h)}{\partial z_{k_2} \partial z_{k_1}} \frac{\partial h_{k_2}}{\partial z_{r_2}} \frac{\partial h_{k_1}}{\partial z_{r_1}} + \sum_{k=1}^n \frac{\partial f_s(h)}{\partial z_k} \frac{\partial^2 h_k}{\partial z_{r_2} \partial z_{r_1}} =$$

$$\sum_{k_1, k_2=1}^n \frac{\partial^2 h_s(g)}{\partial z_{k_2} \partial z_{k_1}} \frac{\partial g_{k_2}}{\partial z_{r_2}} \frac{\partial g_{k_1}}{\partial z_{r_1}} + \sum_{k=1}^n \frac{\partial h_s(g)}{\partial z_k} \frac{\partial^2 g_k}{\partial z_{r_2} \partial z_{r_1}}.$$

As $Df(0) = Dg(0) = Dh(0) = \text{Id}$, we have that

- $\frac{\partial f_s}{\partial z_k}(0) = 0$ for all $k \neq s$ and $\frac{\partial f_s}{\partial z_s}(0) = 1$,
- $\frac{\partial h_k}{\partial z_r}(0) = 0$ for all $k \neq r$ and $\frac{\partial h_r}{\partial z_r}(0) = 1$,
- $\frac{\partial g_k}{\partial z_r}(0) = 0$ for all $k \neq r$ and $\frac{\partial g_r}{\partial z_r}(0) = 1$.

Then

$$\frac{\partial^2 f_s(0)}{\partial z_{r_2} \partial z_{r_1}} + \frac{\partial^2 h_s(0)}{\partial z_{r_2} \partial z_{r_1}} = \frac{\partial^2 h_s(0)}{\partial z_{r_2} \partial z_{r_1}} + \frac{\partial^2 g_s(0)}{\partial z_{r_2} \partial z_{r_1}}$$

thus

$$\frac{\partial^2 f_s(0)}{\partial z_{r_2} \partial z_{r_1}} = \frac{\partial^2 g_s(0)}{\partial z_{r_2} \partial z_{r_1}}.$$

Now using Taylor theorem we have that

$$f_{(i)}(Z) = Z + P_2(Z) + h.o.t., \text{ for all } 1 \leq i \leq \nu + 1.$$

so

$$f_{(i)}^{(\nu+1)}(Z) = Z + (\nu + 1)P_2(Z) + h.o.t.$$

From (a) we have that $P_2(0) = 0$. Then

$$\frac{\partial^2 f_s(0)}{\partial z_{r_2} \partial z_{r_1}} = \frac{\partial^2 g_s(0)}{\partial z_{r_2} \partial z_{r_1}} = 0.$$

Now suppose the statement below is satisfied for $3 \leq l < m$. We will be showing that it is valid for m . Then

$$\frac{\partial^l f(0)}{\partial z_{r_l} \cdots \partial z_{r_1}} = \frac{\partial^l g(0)}{\partial z_{r_l} \cdots \partial z_{r_1}} = 0.$$

As $f_s \circ h = h_s \circ g$ we have

$$\frac{\partial^m (f_s \circ h)}{\partial z_{r_m} \cdots \partial z_{r_1}} = \frac{\partial^m (h_s \circ g)}{\partial z_{r_m} \cdots \partial z_{r_1}}.$$

From Lemma 2.1 we have

$$\sum_{k_1, \dots, k_m=1}^n \frac{\partial^m f_s(h)}{\partial Z_{k_r m}} \frac{\partial h_{k_m}}{\partial z_{r_m}} \cdots \frac{\partial h_{k_1}}{\partial z_{r_1}} + \sum_{k=1}^n \frac{\partial f_s(h)}{\partial z_k} \frac{\partial^m h_k}{\partial z_{r_m} \cdots \partial z_{r_1}} + \sum_{p=2}^{m-1} \sum_{k_1, \dots, k_p=1}^n \frac{\partial^p f_s(h)}{\partial Z_{k_r p}} \cdot R_{k_r p}(\cdot)$$

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$$\sum_{k_1, \dots, k_m=1}^n \frac{\partial^m h_s(g)}{\partial Z^{k_{r_m}} \partial z_{r_m} \dots \partial z_{r_1}} + \sum_{k=1}^n \frac{\partial h_s(g)}{\partial z_k} \frac{\partial^m g_k}{\partial z_{r_m} \dots \partial z_{r_1}} + \sum_{p=2}^{m-1} \sum_{k_1, \dots, k_p=1}^n \frac{\partial^p h_s(g)}{\partial Z^{k_{r_p}}} \cdot R_{k_{r_p}}(\cdot).$$

As $R_{k_{r_p}}$ is a polynomial expression as a function of the derivatives of f_s (respectively h_s) from order 1 to order $m-1$ and has no terms containing only derivatives of order 1, by the induction hypothesis when $Z = 0$ we have that $R_{k_{r_p}}(\cdot) = 0$. Now as $Df(0) = Dg(0) = Dh(0) = \text{Id}$, we have that

$$\frac{\partial^m f_s(0)}{\partial z_{r_m} \dots \partial z_{r_1}} + \frac{\partial^m h_s(0)}{\partial z_{r_m} \dots \partial z_{r_1}} = \frac{\partial^m h_s(0)}{\partial z_{r_m} \dots \partial z_{r_1}} + \frac{\partial^m g_s(0)}{\partial z_{r_m} \dots \partial z_{r_1}}$$

then

$$\frac{\partial^m f_s(0)}{\partial z_{r_m} \dots \partial z_{r_1}} = \frac{\partial^m g_s(0)}{\partial z_{r_m} \dots \partial z_{r_1}}.$$

Now using Taylor development we have that

$$f_{(i)}(Z) = Z + P_m(Z) + h.o.t., \text{ for all } 1 \leq i \leq \nu + 1$$

so

$$f_{(i)}^{(\nu+1)}(Z) = Z + (\nu + 1)P_m(Z) + h.o.t.$$

From (a) we have that $P_m(0) = 0$. Then

$$\frac{\partial^m f_s(0)}{\partial z_{r_m} \dots \partial z_{r_1}} = \frac{\partial^m g_s(0)}{\partial z_{r_m} \dots \partial z_{r_1}} = 0 \text{ for all } m \in \mathbb{N}.$$

Consequently

$$f_{(i)}(Z) = Z, \text{ for all } 1 \leq i \leq \nu + 1.$$

therefore $G = \{\text{Id}\}$. □

Now let us investigate what happens when the linear part of the diffeomorphism is different from the identity. We will now show the case where all the diffeomorphisms have the same linear part, for this we will use the definition of resonance ([10] and [2]):

Definition 2.2. A *multiplicative resonance* between non zero complex numbers $\lambda_1, \dots, \lambda_n$ is an identity of form

$$\lambda_s = \lambda_1^{m_1} \dots \lambda_n^{m_n}$$

where $s \in \{1, \dots, n\}$, $m_1, \dots, m_n \in \mathbb{N}$ and $m_1 + \dots + m_n \geq 2$.

The vector $M = (m_1, \dots, m_n) \in \mathbb{N}^n$ is called the *order* of resonance. For simplicity we can say that λ_s is *resonant* and that $\lambda^M = \lambda_1^{m_1} \dots \lambda_n^{m_n}$.

We are interested in matrices that have resonant eigenvalues and their relation with polynomial functions:

Definition 2.3. If λ_s is a resonant eigenvalue with order of resonance (m_1, \dots, m_n) we call

$$z_1^{m_1} \dots z_n^{m_n} \cdot e_s$$

a *resonant monomial*. Here $e_1 = (1, \dots, 0), \dots, e_n = (0, \dots, 1)$ defines the canonical basis of \mathbb{C}^n .

With these definitions we have:

Theorem 2.1 (*Poincaré-Dulac normal form* [2], [10]). Let $f \in \text{Diff}(\mathbb{C}^n, 0)$ be a germ of complex diffeomorphism. If $Df(0)$ is diagonalizable then f is formally conjugate to a formal series $\hat{F} = df(0) + \hat{P}_2 + \dots$, where the \hat{P}_j are complex polynomial made only of resonant monomials of f . In particular if $df(0)$ has no resonances then f is formally linearizable.

Now we can finish our proof:

Proof of Theorem 1.1. In short, there exists a finite set of generators $f_{(1)}, f_{(2)}, \dots, f_{(\nu+1)} \in G$ such that

- (a) $f_{(1)} \circ f_{(2)} \circ \dots \circ f_{(\nu+1)} = \text{Id}$.
- (b) $f_{(i)}$ and $f_{(j)}$ are conjugate in G for all i, j .
- (c) All maps $f_{(i)}$ have the same linear part, $Df_{(i)}(0) = A$, for all i .

If $A = \text{Id}$ by Proposition 2.1 $G = \{\text{Id}\}$ is a finite group. Suppose that $A \neq \text{Id}$ from (a) we have that $A^{\nu+1} = \text{Id}$. An easy computation with Jordan blocks shows that A is diagonalizable. Let us then assume that A is already in diagonal form with eigenvalues $\lambda_1, \dots, \lambda_n$. Then $\lambda_j^{\nu+1} = 1$ and the eigenvalues of A are in resonance. Now by Theorem 2.1 we can write $f_{(1)}$ in the normal form, that is, there is $\varphi \in \widehat{\text{Diff}}(\mathbb{C}^n, 0)$ such that

$$(2) \quad \varphi \circ f_{(1)} \circ \varphi^{-1}(Z) = \tilde{f}_{(1)}(Z) = (\lambda_1 z_1, \dots, \lambda_n z_n) + \hat{P}_2(Z) + \hat{P}_3(Z) + \dots$$

where $\hat{P}_k(Z)$ contains only resonant monomials with complex coefficients.

Now we take $\tilde{G} = \varphi \circ G \circ \varphi^{-1}$. Thus \tilde{G} is isomorphic to G and satisfies all properties of G . Thus we consider G as \tilde{G} , then we can write the generators as

$$f_j(Z) = \left(\lambda_1 z_1 + \sum_{|Q|=k+1} a_{1,Q}^{(j)} Z^Q + h.o.t., \dots, \lambda_n z_n + \sum_{|Q|=k+1} a_{n,Q}^{(j)} Z^Q + h.o.t. \right)$$

For $j = 1, \dots, \nu + 1$, and f_1 have the nonlinear part contains only resonant monomials with complex coefficients. Moreover if $g \in G$ then exists $\ell \in \mathbb{Z}$ such that

$$Dg(0) = D\varphi(0)A^\ell(D\varphi(0))^{-1} = \begin{pmatrix} \lambda_1^\ell & 0 & \dots & 0 \\ 0 & \lambda_2^\ell & \dots & 0 \\ 0 & \dots & \ddots & 0 \\ 0 & 0 & \dots & \lambda_n^\ell \end{pmatrix}.$$

The idea is to show by a formal algorithm that G is formally linearizable. For this, suppose that G has no terms of order k , we will prove that the same is true for the terms of order $k + 1$. First, note that given $f, g \in \text{Diff}_{k+1}(\mathbb{C}^n, 0) \cap G$ we have

$$f(Z) = \left(a_1 z_1 + \sum_{|Q|=k+1} a_{1,Q} Z^Q + h.o.t., \dots, a_n z_n + \sum_{|Q|=k+1} a_{n,Q} Z^Q + h.o.t. \right)$$

and

$$g(Z) = \left(b_1 z_1 + \sum_{|Q|=k+1} b_{1,Q} Z^Q + h.o.t., \dots, b_n z_n + \sum_{|Q|=k+1} b_{n,Q} Z^Q + h.o.t. \right),$$

where $Q = (q_1, \dots, q_n) \in \mathbb{N}^n$ with $|Q| = k + 1$, $a_r = \lambda_r^m$ and $b_r = \lambda_r^l$ for all $r = 1, \dots, n$ and some $m, l \in \mathbb{N}$. So we have

$$f \circ g(Z) = \left(a_1 b_1 z_1 + \sum_{|Q|=k+1} [a_1 b_{1,Q} + a_{1,Q} B^Q] Z^Q + h.o.t., \dots, \right. \\ \left. a_n b_n z_n + \sum_{|Q|=k+1} [a_n b_{n,Q} + a_{n,Q} B^Q] Z^Q + h.o.t. \right)$$

where $B^Q = b_1^{q_1} \dots b_n^{q_n}$. We will study two cases:

Claim 2.1. Let $Q \in \mathbb{N}^n$ be an order of resonance for some λ_r , $|Q| = k + 1$. Then $a_{r,Q}^{(j)} = 0$.

Proof of the claim. As $Q \in \mathbb{N}^n$ is an order of resonance for λ_r , we have that $\lambda^Q = \lambda_r$. Then $B^Q = (\lambda^Q)^l = b_r$. Now we define the following application

$$\varphi_{r,Q} : \begin{array}{ccc} G & & \rightarrow \mathbb{C} \\ \left(a_1 z_1 + \sum_{|Q|=k+1} a_{1,Q} Z^Q + h.o.t., \dots, a_n z_n + \sum_{|Q|=k+1} a_{n,Q} Z^Q + h.o.t. \right) & \mapsto & \frac{a_{r,Q}}{a_r} \end{array}$$

which defines a morphism between (G, \circ) and $(\mathbb{C}, +)$, indeed

$$\varphi_{r,Q}(f \circ g) = \frac{a_r b_{r,Q} + a_{r,Q} B^Q}{a_r b_r} = \frac{a_r b_{r,Q} + a_{r,Q} b_r}{a_r b_r} = \varphi_{r,Q}(f) + \varphi_{r,Q}(g)$$

Now for $i \neq j$ from (b) we have $h \in G$ such that

$$f_i \circ h = h \circ f_j$$

applying the morphism we have

$$\begin{aligned} \varphi_{r,Q}(f_i \circ h) &= \varphi_{r,Q}(h \circ f_j) \\ \varphi_{r,Q}(f_i) + \varphi_{r,Q}(h) &= \varphi_{r,Q}(h) + \varphi_{r,Q}(f_j) \\ \varphi_{r,Q}(f_i) &= \varphi_{r,Q}(f_j) \\ \frac{a_{r,Q}^{(i)}}{\lambda_r} &= \frac{a_{r,Q}^{(j)}}{\lambda_r} \\ a_{r,Q}^{(i)} &= a_{r,Q}^{(j)}. \end{aligned}$$

Lets denote $a_{r,Q} := a_{r,Q}^{(1)} = \dots = a_{r,Q}^{(\nu+1)}$. Now from (a) we have

$$\varphi_{r,Q}(f_1 \circ f_2 \circ \dots \circ f_{\nu+1}) = \varphi_{r,Q}(\text{Id}) = 0$$

$$\varphi_{r,Q}(f_1) + \varphi_{r,Q}(f_2) + \dots + \varphi_{r,Q}(f_{\nu+1}) = 0$$

$$\frac{a_{r,Q}^{(1)}}{\lambda_r} + \frac{a_{r,Q}^{(2)}}{\lambda_r} + \dots + \frac{a_{r,Q}^{(\nu+1)}}{\lambda_r} = 0$$

$$(\nu + 1) \left(\frac{a_{r,Q}}{\lambda_r} \right) = 0.$$

Therefore $a_{r,Q} = 0$. □

Claim 2.2. If Q is not the order of resonance of any λ_r , $|Q| = k + 1$ then $a_{r,Q}^{(j)} = a_{r,Q}^{(1)} = 0$.

Proof of the claim. Note that f_1 is written in its normal form, then $a_{r,Q}^{(1)} = 0$. Now we define the following application $\psi_Q: G \rightarrow \text{Aff}(\mathbb{C})^n$,

$$\psi_Q \left(a_1 z_1 + \sum_{|Q|=k+1} a_{1,Q} Z^Q + h.o.t., \dots, a_n z_n + \sum_{|Q|=k+1} a_{n,Q} Z^Q + h.o.t. \right) = \left(\frac{a_1 z_1 + a_{1,Q}}{A^Q}, \dots, \frac{a_n z_n + a_{n,Q}}{A^Q} \right)$$

where $A^Q = a_1^{q_1} \cdots a_n^{q_n}$, which defines a morphism from (G, \circ) into $(\text{Aff}(\mathbb{C})^n, \circ)$. Indeed,

$$\psi_Q(f \circ g) = \left(\frac{a_1 b_1 z_1 + a_1 b_{1,Q} + a_{1,Q} B^Q}{A^Q B^Q}, \dots, \frac{a_n b_n z_n + a_n b_{n,Q} + a_{n,Q} B^Q}{A^Q B^Q} \right)$$

on the other hand

$$\begin{aligned} \psi_Q(f) \circ \psi_Q(g) &= \left(\frac{a_1 \left(\frac{b_1 z_1 + b_{1,Q}}{B^Q} \right) + a_{1,Q}}{A^Q}, \dots, \frac{a_n \left(\frac{b_n z_n + b_{n,Q}}{B^Q} \right) + a_{n,Q}}{A^Q} \right) \\ &= \left(\frac{a_1 b_1 z_1 + a_1 b_{1,Q} + a_{1,Q} B^Q}{A^Q B^Q}, \dots, \frac{a_n b_n z_n + a_n b_{n,Q} + a_{n,Q} B^Q}{A^Q B^Q} \right) \\ &= \psi_Q(f \circ g) \end{aligned}$$

Denote by G_Q the image of G by ψ_Q . Therefore G_Q is a product affine group generated by the transformations $g_{(1)}^Q, g_{(2)}^Q, \dots, g_{(\nu+1)}^Q$, pairwise conjugate in G_Q , where

$$g_{(i)}^Q(Z) = \left(\frac{\lambda_1 z_1 + a_{1,Q}^{(i)}}{\lambda_1^{q_1} \cdots \lambda_n^{q_n}}, \dots, \frac{\lambda_n z_n + a_{n,Q}^{(i)}}{\lambda_1^{q_1} \cdots \lambda_n^{q_n}} \right) = (g_{i,1}^Q(z_1), \dots, g_{i,n}^Q(z_n)).$$

Denote by $\lambda^Q = \lambda_1^{q_1} \cdots \lambda_n^{q_n}$ and G_r^Q is an affine group generated by the transformations $g_{1,r}^Q, \dots, g_{\nu+1,r}^Q$, pairwise conjugate in G_r^Q , where

$$g_{i,r}^Q(w) = \frac{\lambda_r w + a_{r,Q}^{(i)}}{\lambda^Q}.$$

We now apply the following lemma whose proof is found in [5] (page 222):

Lemma 2.2. Let η be a l -th root of the unit, $l > 1$, $\beta_1, \beta_2, \dots, \beta_{r+1} \in \mathbb{C}$ and Γ an affine group generated by the transformations $h_i(z) = \eta z + \beta_i$, $i = 1, 2, \dots, r + 1$. Then the h_i 's are pairwise conjugate in Γ if and only if either l has two distinct prime divisors or $l = q^m$, for some prime q and some $m \in \mathbb{N}^*$ and $\beta_1 = \beta_2 = \dots = \beta_{r+1}$.

Taking $\eta = \frac{\lambda_r}{\lambda^Q}$, $l = p^{s_1 + \dots + s_n}$, $r = \nu$ and $\beta_i = \frac{a_{r,Q}^{(i)}}{\lambda^Q}$ ($i = 1, \dots, \nu + 1$) by Lemma 2.2 we have $p^{s_1 + \dots + s_n} = q^m$, q prime, $m \in \mathbb{N}^*$ and

$$\frac{a_{r,Q}^{(1)}}{\lambda^Q} = \dots = \frac{a_{r,Q}^{(\nu+1)}}{\lambda^Q}.$$

Therefore $q = p$, $m = s_1 + \dots + s_n$ and

$$a_{r,Q}^{(1)} = \dots = a_{r,Q}^{(\nu+1)}.$$

□

This produces a convergent algorithm in the Krull topology. This already shows that G is formally linearizable. Since there is a set of generators all with the same linear part A which is a finite order matrix, this implies that G is cyclic generated by A . The proof of Theorem 1.1 is now complete. □

Proof of Corollary 1.1. Since the linear part of G is abelian, given a basic set of generators f_j of G in the definition of irreducible group, all the maps f_j have the same linear part, say $A \in \text{GL}(n, \mathbb{C})$. In particular A satisfies $A^{\nu+1} = \text{Id}$. On the other hand, there is a set of generators g_1, \dots, g_{p^s} with $g_1 \circ \dots \circ g_{p^s} = \text{Id}$. This implies that $A^{p^s} = \text{Id}$. Since p is prime this implies that the order of A is p^r for some $r \in \{0, \dots, s\}$ and therefore A satisfies the hypothesis of Theorem 1.1. The group G is therefore finite and cyclic. □

The hypothesis that the eigenvalues of the linear part are roots of the unit of order power of a *same prime number* cannot be dropped, as shown in the following examples.

Example 2.1. Let $G \subset \text{Diff}(\mathbb{C}^2, 0)$ be the subgroup generated by the maps

$$f_1(Z) = f_2(Z) = f_3(Z) = f_4(Z) = (-z_1, \lambda z_2), f_5(Z) = (-z_1 + z_2^2, \lambda z_2)$$

and $f_6(Z) = (-z_1 + \lambda^2 z_2^2, \lambda z_2)$ where $\lambda^3 = 1$ so that $\lambda^2 + \lambda + 1 = 0$. Note that the generators have the same linear part with eigenvalues roots of order 2 and 3. We claim that G is irreducible and not finite (not linearizable). The first condition is satisfied

$$\begin{aligned} f_1 \circ f_2 \circ f_3 \circ f_4 \circ f_5 \circ f_6(Z) &= f_1 \circ f_2 \circ f_3 \circ f_4 \circ f_5(-z_1 + \lambda^2 z_2^2, \lambda z_2) \\ &= f_1 \circ f_2 \circ f_3 \circ f_4(z_1 - \lambda^2 z_2^2 + \lambda^2 z_2^2, \lambda^2 z_2) \\ &= f_1 \circ f_2 \circ f_3 \circ f_4(z_1, \lambda^2 z_2) = (z_1, \lambda^6 z_2) = Z \end{aligned}$$

To check the second condition take $g_1(Z) = f_1^4 \circ f_5 \circ f_1(Z) = (z_1 + \lambda^2 z_2^2, z_2)$ then $g_1 \in G$ and note that

$$\begin{aligned} f_1 \circ g_1(Z) &= f_1(z_1 + \lambda^2 z_2^2, z_2) = (-z_1 - \lambda^2 z_2^2, \lambda z_2) \\ g_1 \circ f_5(Z) &= g_1(-z_1 + z_2^2, \lambda z_2) = (-z_1 + z_2^2 + \lambda^2(\lambda^2 z_2^2), \lambda z_2) = (-z_1 + (1 + \lambda)z_2^2, \lambda z_2) \end{aligned}$$

Since $\lambda^2 + \lambda + 1 = 0$ we have $1 + \lambda = -\lambda^2$ therefore $f_1 \circ g_1 = g_1 \circ f_5$.

Take $g_2(Z) = f_5 \circ f_1^5(Z) = (z_1 + \lambda z_2^2, z_2)$ then $g_2 \in G$ and note that

$$\begin{aligned} f_1 \circ g_2(Z) &= f_1(z_1 + \lambda z_2^2, z_2) = (-z_1 - \lambda z_2^2, \lambda z_2) \\ g_2 \circ f_6(Z) &= g_2(-z_1 + \lambda^2 z_2^2, \lambda z_2) = (-z_1 + \lambda^2 z_2^2 + \lambda(\lambda^2 z_2^2), \lambda z_2) = (-z_1 + (\lambda^2 + 1)z_2^2, \lambda z_2) \end{aligned}$$

Since $\lambda^2 + \lambda + 1 = 0$ we have $1 + \lambda^2 = -\lambda$ therefore $f_1 \circ g_2 = g_2 \circ f_6$.

Take $g_3(Z) = f_5^2 \circ f_1^4(Z) = (z_1 + (\lambda - \lambda^2)z_2^2, z_2)$ then $g_3 \in G$ and note that

$$f_5 \circ g_3(Z) = f_5(z_1 + (\lambda - \lambda^2)z_2^2, z_2) = (-z_1 + (\lambda^2 - \lambda)z_2^2 + z_2^2, \lambda z_2) = (-z_1 + (\lambda^2 - \lambda + 1)z_2^2, \lambda z_2)$$

$$g_3 \circ f_6(Z) = g_3(-z_1 + \lambda^2 z_2^2, \lambda z_2) = (-z_1 + \lambda^2 z_2^2 - (\lambda^2 - \lambda)(\lambda^2 z_2^2), \lambda z_2) = (-z_1 + (\lambda^2 - \lambda + 1)z_2^2, \lambda z_2)$$

therefore $f_5 \circ g_3 = g_3 \circ f_6$.

Consequently the f_j are pairwise conjugate in the group G . Note that also

$$g_1^n(Z) = (z_1 + n\lambda^2 z_2^2, z_2) \neq Z$$

for all $n \in \mathbb{N}^*$. Therefore G is not finite. Finally, we observe that G is not abelian and in particular, it is not analytically linearizable: indeed, if g linearizes G , then $g \circ f_1 \circ g^{-1} = f_1$. On the other hand $g \circ f_5 \circ g^{-1} = f_1$, so

$$f_5 = g^{-1} \circ f_1 \circ g = f_1$$

which is a contradiction.

Example 2.2. Consider $G \subset \text{Diff}(\mathbb{C}^2, 0)$ the subgroup generated by the maps

$$f_1(Z) = \dots = f_{10}(Z) = (iz_1, \lambda z_2), f_{11}(Z) = (iz_1, \lambda z_2 + z_1^2) \quad \text{and} \quad f_{12}(Z) = (iz_1, \lambda z_2 + \lambda^2 z_1^2)$$

where $\lambda^3 = 1$ so that $\lambda^2 + \lambda + 1 = 0$. Note that the generators have the same linear part with eigenvalues roots of order 4 and 3. G is irreducible and not finite (not linearizable).

Example 2.3. Consider $G \subset \text{Diff}(\mathbb{C}^2, 0)$ the subgroup generated by the maps

$$f_1(Z) = \dots = f_{34}(Z) = (-z_1, \lambda z_2), f_{35}(Z) = (-z_1 + z_2^3, \lambda z_2) \quad \text{and} \quad f_{36}(Z) = (-z_1 + \lambda^3 z_2^3, \lambda z_2)$$

where $\lambda^9 = 1$ so that $\lambda^6 + \lambda^3 + 1 = 0$. Note that the generators have the same linear part with eigenvalues roots of order 4 and 9. G is irreducible and not finite (not linearizable).

3. APPLICATIONS

As a first application we prove

Proof of Theorem 1.2. By hypothesis the linear part of the holonomy group

$$\text{Hol}(L_0) \hookrightarrow \text{Diff}(\mathbb{C}^n, 0)$$

is abelian. By Deligne's theorem the fundamental group $\pi_1(L_0)$ is irreducible. Indeed, it is generated by a small simple loop γ around C and its conjugacy homotopy classes $\gamma_j, j \in J$. Choose a linear embedding $\ell: \mathbb{P}^1 \rightarrow \mathbb{P}^2$ in general position with respect to C . This means that $\ell(\mathbb{P}^1)$ meets C transversely and only at nonsingular points. In particular the intersection $\ell(\mathbb{P}^1) \cap C$ is a set of $\nu + 1 = p^s$ points say $\{p_1, \dots, p_{\nu+1}\}$. Given a base point $p_0 \in \ell^{-1}(\mathbb{P}^2 \setminus \{p_1, \dots, p_{\nu+1}\})$ by Lefschetz hyperplane section theorem, there is a surjective morphism

$$\pi_1(\ell^{-1}(\mathbb{P}^2 \setminus \{p_1, \dots, p_{\nu+1}\}), p_0) \rightarrow \pi_1(\mathbb{P}^2 \setminus C).$$

Thus we may take the small loop $\gamma = \gamma_1$ contained in a small disc in $\ell(\mathbb{P}^1)$ centered at p_1 and the other homotopy classes as given by small loops γ_j contained in small discs in $\ell(\mathbb{P}^1)$ and centered at the points $p_j, j = 2, \dots, \nu + 1$. In particular, $\pi(L_0)$ is irreducible with a set of generators $[\gamma_1], \dots, [\gamma_{\nu+1}]$ as in Definition 1.1. The corresponding holonomy maps $f_{[\gamma_j]} \in \text{Hol}(\mathcal{F}, L_0)$ are such that $f_{[\gamma_1]}, \dots, f_{[\gamma_{\nu+1}]}$ is a set of generators for holonomy group as an irreducible subgroup of $\text{Diff}(\mathbb{C}^n, 0)$. By hypothesis this group has an abelian linear part. Since $\nu + 1 = p^s$, by Theorem 1.1 this holonomy group is finite. □

Proof of Theorem 1.3. If G_t is finite and cyclic then it is irreducible. Thus we shall prove that (1) implies (2). Assume that G_t is irreducible for all t close to 0. By hypothesis G_t is generated by the maps $f_{j,t}$ above. If G is trivial then clearly any map $f_{j,t}$ is tangent to the identity. In this case, by Theorem 1.1 the group G is also trivial. Assume now that $s > 0$. Since $G = G_0$ is cyclic of order p^s , any set of non-trivial generators $\{f_j, j = 1, \dots, r\}$ is of the form $f_j = f^{n_j}$ for some $n_j \in \{1, \dots, p^s - 1\}$, where f is a generator of G as a cyclic group. Thus we have

$f'_{j,t}(0) = (f'(0))^{n_j}$ and therefore the linear part of the group G_t satisfies the conditions of Theorem 1.1. By this same theorem the group G_t is finite cyclic. \square

Definition 3.1. Given a map germ $f \in \text{Diff}(\mathbb{C}^n, 0)$ and a hypersurface germ $H \subset \mathbb{C}^n$ through the origin $0 \in \mathbb{C}^n$ we say that H is *f-invariant at order $k \in \mathbb{N}$* if:

- (1) $f(H) \subset H$.
- (2) We have $f^k|_H = \text{Id}$.

We shall also say that H is *infinitesimally f-invariant at order $k \in \mathbb{N}$* if:

- (1) the tangent space $T_0(H) \subset \mathbb{C}_0^n$ is invariant by the derivative $f'(0)$, i.e.,

$$f'(0) \cdot T_0(H) = T_0(H) \subset \mathbb{C}_0^n.$$

- (2) We have $f^k|_H = \text{Id}$.

Clearly, if H is f -invariant at order k then it is f -infinitesimally invariant at order k .

Corollary 3.1. Let $G \subset \text{Diff}(\mathbb{C}^n, 0)$ be an irreducible group, $p \in \mathbb{N}$ a prime number. Assume that there are analytic hypersurface germs $H_1, \dots, H_n \subset \mathbb{C}^n$ meeting transversely at the origin such that each H_j is infinitesimally invariant at order p^{s_j} by each element of G . Then G is a finite group.

Proof. Up to a change of coordinates we may assume that $H_j = \{z_j = 0\}$, $j = 1, \dots, n$. Thus, G admits a finite set of generators $f_1, f_2, \dots, f_{\nu+1} \in G$ such that:

- (a) $f_1 \circ f_2 \circ \dots \circ f_{\nu+1} = \text{Id}$.
- (b) f_i and f_j are conjugate in G for all i, j .

By hypothesis for each i, j we have $f'_j(0) \cdot T_0(H_i) \subset T_0(H_i)$. This implies that

- (c) For each $j = 1, \dots, n$ we have

$$f_j(Z) = T_j Z + P_{j2}(Z) + \dots + P_{jk}(Z) + \dots$$

where P_{jk} is homogeneous of degree $k \geq 2$ and

$$T_j = Df_j(0) = \begin{pmatrix} \lambda_{j1} & 0 & \dots & 0 \\ 0 & \lambda_{j2} & \dots & 0 \\ 0 & \dots & \ddots & 0 \\ 0 & 0 & \dots & \lambda_{jn} \end{pmatrix}.$$

Also by hypothesis we have $f_j^{p^{s_i}}|_{H_i} = \text{Id}$, for all $j = 1, \dots, \nu + 1$. This implies

- (d) $\lambda_{ji}^{p^{s_i}} = 1$ for each $j \in \{1, \dots, \nu + 1\}$ and each $i \in \{1, \dots, n\}$.

From (b) we have that for $i \neq j$ there exists $g \in G$ such that

$$f_i \circ g = g \circ f_j$$

hence we obtain

$$Df_i(0)Dg(0) = Dg(0)Df_j(0).$$

Since the generators f_j have a diagonal linear part in the chosen coordinates, the same holds for any element of G . Hence $Dg(0)$ is a diagonal matrix. Then $Df_i(0) = Df_j(0)$ for all i, j and their eigenvalues can be listed as $\lambda_1, \dots, \lambda_n$ where λ_j is a p^{s_j} -th root of the unit, with p prime and $s_j \in \mathbb{N}$. Therefore by Theorem 1.1 G is a finite group. \square

4. RICCATI FOLIATIONS

Let us state the exact notion of Riccati foliation we consider. Let $\pi: E \rightarrow B$ be a holomorphic fiber space with fiber F . A holomorphic foliation \mathcal{F} on E with singular set $\text{sing}(\mathcal{F}) \subset E$ is a *Riccati foliation* if there is a subset $\sigma \subset B$ such that:

- (1) $\pi^{-1}(\sigma) \subset E$ is a union of invariant fibers.
- (2) $\mathcal{F}|_{E \setminus \pi^{-1}(\sigma)}$ is transverse to the fibers of the fiber space $\pi|_{E \setminus \pi^{-1}(\sigma)}: E \setminus \pi^{-1}(\sigma) \rightarrow B \setminus \sigma$ in the sense of Ehresmann ([4] Chapter V).

In particular, we have:

- (3) $\dim E = \dim F + \dim B$ and $\dim \mathcal{F} = \dim B$;
- (4) $\text{sing}(\mathcal{F}) \subset \pi^{-1}(\sigma)$.

The set $\sigma \subset B$ is called *ramification set* of \mathcal{F} . Since the restriction $\mathcal{F}|_{E \setminus \pi^{-1}(\sigma)}$ is a foliation transverse to the fibers of the fiber space $\pi: E \setminus \pi^{-1}(\sigma) \rightarrow B \setminus \sigma$ in the ordinary Ehresmann sense, it is completely described by its *global holonomy* ([4] Chapter V). This is a lifting paths homomorphism $\phi: \pi_1(B \setminus \sigma) \rightarrow \text{Aut}(F)$. The very basic example is given by the compactification of the foliation \mathcal{F} on $\mathbb{P}^1 \times \mathbb{P}^1$ given in affine coordinates $(x, y) \in \mathbb{C} \times \mathbb{C}$ by a *Riccati differential equation* $\frac{dy}{dx} = \frac{a(x)y^2 + b(x)y + c(x)}{p(x)}$ where the coefficients a, b, c, p are polynomials. In this case the fiber space structure is given by the product and the projection $\pi(x, y) = x$. Using this notion a *Riccati foliation* on $\mathbb{P}^m \times \mathbb{P}^n$ is a codimension n holomorphic foliation with singularities, such that for some analytic codimension \geq one subset $\sigma \subset \mathbb{P}^m$, the foliation is transverse to the vertical fibers $\{x\} \times \mathbb{P}^n$, $x \in \mathbb{P}^m \setminus \sigma$ while $\sigma \times \mathbb{P}^n$ is a union of invariant fibers. A Riccati foliation will be called a *Bernoulli foliation* if there is an invariant horizontal fiber $\mathbb{P}^m \times \{q\}$, for some $q \in \mathbb{P}^n$.

We investigate the connection between the geometry of the ramification set with the dynamics of a given Riccati foliation. We first we make a basic remark: if the ramification set is empty (or, more generally if it has codimension ≥ 2) then $\pi_1(\mathbb{P}^m \setminus \sigma) = \{0\}$. This implies that all leaves are compact diffeomorphic to \mathbb{P}^m and the foliation is equivalent to the second projection $\mathbb{P}^n \times \mathbb{P}^m \rightarrow \mathbb{P}^n$, $(x, y) \mapsto y$. Thus we shall assume that $\sigma \subset \mathbb{P}^m$ is nonempty of codimension one.

4.1. Case $n = m = 1$. Let us begin with the dimension two case. More precisely we consider the case where \mathcal{F} is a Riccati foliation in $\mathbb{P}^1 \times \mathbb{P}^1$, assuming that \mathcal{F} has an irreducible ramification set $\sigma \subset \mathbb{P}^1$. This implies that σ is a single point and we may assume that in affine coordinates (x, y) the ramification point is the point $x = \infty, y = 0$. Then we may write \mathcal{F} as given by a polynomial differential equation $\frac{dy}{dx} = a(x)y^2 + b(x)y + c(x)$. The global holonomy of \mathcal{F} is given by an homomorphism $\phi: \pi_1(\mathbb{P}^1 \setminus \sigma) \rightarrow \text{Aut}(\mathbb{P}^1)$. Since σ is a single point we have $\mathbb{P}^1 \setminus \sigma = \mathbb{C}$ is simply-connected and therefore the global holonomy is trivial. By the classification of foliations transverse to fibrations ([4] Chapter V) there is a fibered biholomorphic map $\Phi: \mathbb{C} \times \mathbb{P}^1 \rightarrow \mathbb{C} \times \mathbb{P}^1$ that takes the foliation \mathcal{F} into the foliation \mathcal{H} given by the horizontal fibers $\mathbb{C} \times \{y\}$, $y \in \mathbb{P}^1$.

Lemma 4.1. A holomorphic diffeomorphism $\Phi: \mathbb{C} \times \mathbb{P}^1 \rightarrow \mathbb{C} \times \mathbb{P}^1$ preserving the vertical fibration writes in affine coordinates $(x, y) \in \mathbb{C}^2 \subset \mathbb{C} \times \mathbb{P}^1$ as $\Phi(x, y) = \left(Ax + B, \frac{a(x)y + b(x)}{c(x)y + d(x)} \right)$ where a, b, c, d are entire functions satisfying $ad - bc = 1$, $0 \neq A, B \in \mathbb{C}$.

Proof of Lemma 4.1. Picard's theorem and the fact that Φ preserves the fibration $x = \text{const}$ show that it is of the form $\Phi(x, y) = (f(x), g(x, y))$ where $f(x) = Ax + B$ is an affine map. Finally, for each fixed $x \in \mathbb{C}$ the map $\mathbb{P}^1 \ni y \mapsto g(x, y) \in \mathbb{P}^1$ is a diffeomorphism so it must write as $g(x, y) = \frac{a(x)y + b(x)}{c(x)y + d(x)}$ for some entire functions a, b, c, d satisfying $ad - bc = 1$. \square

In particular we conclude that the leaves of \mathcal{F} are diffeomorphic with \mathbb{C} (including the one contained in the invariant fiber $\{(0, \infty)\} \times \mathbb{P}^1$, and \mathcal{F} admits a holomorphic first integral

$$g: \mathbb{C} \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$$

of the above form $g(x, y) = \frac{a(x)y+b(x)}{c(x)y+d(x)}$.

4.2. Case $m = 2, n = 1$. Assume now that \mathcal{F} is a codimension one Riccati foliation in $\mathbb{P}^2 \times \mathbb{P}^1$. If the codimension one component $\sigma_1 \subset \sigma$ of the ramification set $\sigma \subset \mathbb{P}^2$ is irreducible, smooth or with (double ordinary) normal crossings, then the fundamental group $\pi_1(\mathbb{P}^2 \setminus \sigma)$ is finite cyclic of order $\deg(\sigma_1)$ (Zariski-Fulton-Deligne). In this case the global holonomy of \mathcal{F} is a finite cyclic subgroup of $\text{Aut}(\mathbb{P}^1)$ which corresponds to one of the following possibilities:

- (1) A cyclic subgroup generated by a map of the form $z \mapsto \xi z$ where ξ is a root of the unit of order $k \leq \deg \sigma$.
- (2) The group generated by the inversion $f(z) = \frac{1}{z}$.

Assume that we are in case (1) above. This gives a function $z \mapsto z^k$ in the fiber \mathbb{P}^1 which admits a holonomy extension to $(\mathbb{P}^2 \setminus \sigma) \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$ which is constant along the leaves of \mathcal{F} in $(\mathbb{P}^2 \setminus \sigma) \times \mathbb{P}^1$. This shows that \mathcal{F} admits a holomorphic first integral $\varphi: (\mathbb{P}^2 \setminus \sigma) \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$.

Assume now that we are in case (2). In this case we can take the holonomy invariant function $z \mapsto (\ln z)^2$ and extend it to a Liouvillian first integral φ for \mathcal{F} in $\mathbb{P}^2 \setminus \sigma$.

For a different framework we shall need the remark below:

Remark 4.1. It is well-known that the group of automorphisms $\text{Aut}(\mathbb{P}^n)$ is the projectivization of the linear group $\text{GL}(n + 1, \mathbb{C})$ of non-singular linear maps of \mathbb{C}^{n+1} and therefore isomorphic to $\text{PSL}(n, \mathbb{C})$.

Next we consider another situation:

Theorem 4.1. Let \mathcal{F} be a *Bernoulli* foliation on $\mathbb{P}^2 \times \mathbb{P}^1$. Assume that the ramification set $\sigma \subset \mathbb{P}^2$ is irreducible (not necessarily smooth nor normal crossing type) of degree p^s for some prime number p and some $s \in \mathbb{N}$. Then the global holonomy of \mathcal{F} is finite cyclic. In particular, the leaves of \mathcal{F} are closed in $(\mathbb{P}^2 \setminus \sigma) \times \mathbb{P}^1$, i.e., $\text{lim}(\mathcal{F}) \subset \sigma \times \mathbb{P}^1$. Moreover, \mathcal{F} admits a holomorphic first integral $\varphi: (\mathbb{P}^2 \setminus \sigma) \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$.

Proof. The global holonomy identifies with a subgroup $H \subset \text{Aut}(\mathbb{P}^1)$. Since $\sigma \subset \mathbb{P}^2$ is irreducible, H is irreducible. Since $\deg(\sigma) = p^s$ and σ is irreducible it follows from the same ideas in the proof of Theorem 1.2 that H admits a basic set of generators of the form $\{f_1, \dots, f_{\nu+1}\} \subset \text{Aut}(\mathbb{P}^1)$ with $\nu + 1 = p^s$. By hypothesis \mathcal{F} has an invariant horizontal fiber say $\mathbb{P}^2 \times \{q\}$. This implies that H has a fixed point at $\{q\}$. We denote by $H(q) \subset \text{Diff}(\mathbb{C}^1, 0)$ the subgroup induced by the germs at q of maps $h \in H$ (we may identify $q = 0$). This group is irreducible and has a basic set of generators consisting of the germs $f_{j,q}$ at q of the maps $f_j, j = 1, \dots, \nu + 1 = p^s$. By Theorem 1.1 for dimension $n = 1$ this implies that $H(q)$ is finite. In particular $H(q)$ is abelian and each map $f_{j,q}$ has finite order. By the identity principle the maps f_j commute and have finite order. This implies that H is finite cyclic analytically conjugate in \mathbb{P}^1 to the cyclic group generated by $z \mapsto e^{2\pi i/k} z$ for some $k \in \mathbb{N}^*$. As above we can extend the function z^k as a holomorphic first integral $\varphi: (\mathbb{P}^2 \setminus \sigma) \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$ for \mathcal{F} .

Now we proceed. Given a leaf L of \mathcal{F} not contained in $\pi^{-1}(\sigma)$ we claim that the closure $\bar{L} \subset \mathbb{P}^2 \times \mathbb{P}^1$ is contained in $\pi^{-1}(\sigma)$. Indeed, given a generic point $p \in \mathbb{P}^2 \setminus \sigma$ the fiber

$$F_p := \pi^{-1}(p) = \{p\} \times \mathbb{P}^1$$

is transverse to \mathcal{F} . Let us prove that the intersection $L \cap F_p$ is a discrete set. Given two points $z_1, z_2 \in F_p \cap L$ we choose a path $\gamma \subset L$ joining z_1 to z_2 and project this path into a path $\gamma_0 \subset \mathbb{P}^2 \setminus \sigma$

(recall that $\pi^{-1}(\sigma)$ is invariant). The path γ_0 is closed based at p . The corresponding global holonomy map h_{γ_0} to γ_0 is such that $h_{\gamma_0}(z_1) = z_2$. Since the global holonomy group H is finite and cyclic this implies that $\#(F_p \cap L) \leq |H| < \infty$. This already shows that $\text{lim}(\mathcal{F}) \subset \pi^{-1}(\sigma)$. \square

In the above theorem the ramification set is irreducible but we make no hypothesis on the type of singularities it may have. The price we pay is to assume that there is a non-vertical invariant algebraic hypersurface. This condition is natural in the following situation. Let R be a rational function $R: \mathbb{P}^m \times \mathbb{P}^n \rightarrow \mathbb{P}^n$. We shall define the *ramification set* of R (with respect to the vertical fibration) as the set $\sigma \subset \mathbb{P}^m$ of points p for which the fiber $\{p\} \times \mathbb{P}^n$ is not transverse to \mathcal{F} (this means that there is some point $q \in \mathbb{P}^n$ for which the leaf of \mathcal{F} through (p, q) is not transverse to the fiber $\{p\} \times \mathbb{P}^n$). In general σ is an algebraic subset of codimension ≥ 1 in the projective plane \mathbb{P}^m .

Let us consider an analytic family of Riccati foliations in $\mathbb{P}^1 \times \mathbb{P}^1$ given in affine coordinates by the 1-forms $\omega_t = (1 + tp(x))dy - t(a(x)y^2 + b(x)y)dx$ where $p(x), a(x), b(x)$ are polynomials. If \mathcal{F}_t denotes the foliation on $\mathbb{P}^1 \times \mathbb{P}^1$ induced by ω_t then $\mathcal{F}_0: \omega_0 = dy$ is the horizontal fibration, given by the second coordinate projection. The ramification set of \mathcal{F}_0 is irreducible empty, while for $t \neq 0$ the ramification set of \mathcal{F}_t is given by $p(x) = -1/t$ and possibly the point at the infinity $x = \infty$. This set is not irreducible for many choices of the coefficients a, b, p . In general \mathcal{F}_t is not analytically equivalent to a trivial foliation in $\mathbb{P}^1 \times \mathbb{P}^1$. Thus, an irreducible ramification set can deform into a reducible ramification set during a deformation by Riccati foliations.

Taking this into account we can state:

Theorem 4.2. Let \mathcal{F} be the foliation by level surfaces of a rational function $R: \mathbb{P}^2 \times \mathbb{P}^n \rightarrow \mathbb{P}^n$. Assume that codimension one component of the ramification set $\sigma \subset \mathbb{P}^2$ of R is empty or irreducible (not necessarily smooth nor normal crossing type) of degree p^s for some prime number p and some $s \in \mathbb{N}$. Let now $\{\mathcal{F}_t\}_{t \in \mathbb{D}}$ be an analytic deformation of $\mathcal{F} = \mathcal{F}_0$ by Riccati foliations on $\mathbb{P}^2 \times \mathbb{P}^n$ with irreducible ramification set $\sigma(t) \subset \mathbb{P}^2$. Assume that there is some level ($R = c$) of R which is invariant by each foliation \mathcal{F}_t . Then the global holonomy of \mathcal{F}_t is finite cyclic for each t close to 0. In particular, the leaves of \mathcal{F}_t are closed in $(\mathbb{P}^2 \setminus \sigma(t)) \times \mathbb{P}^n$, i.e., $\text{lim}(\mathcal{F}_t) \subset \sigma(t) \times \mathbb{P}^n$, for all t close to 0. If R is the second projection $\mathbb{P}^2 \times \mathbb{P}^n \rightarrow \mathbb{P}^n$ then \mathcal{F}_t is analytically conjugate to \mathcal{F} in $(\mathbb{P}^2 \setminus \sigma(t)) \times \mathbb{P}^n$.

Proof. First we consider the case where $\sigma(\mathcal{F})$ has codimension ≥ 2 . We denote by

$$P_1: \mathbb{P}^m \times \mathbb{P}^n \rightarrow \mathbb{P}^m$$

the first coordinate projection. Given a leaf $L \in \mathcal{F}$ the restriction $P_1|_L: L \rightarrow \mathbb{P}^m \setminus \sigma(\mathcal{F})$ is a covering map. The fundamental group $\pi_1(\mathbb{P}^m \setminus \sigma(\mathcal{F}))$ is trivial because $\text{codim } \sigma(\mathcal{F}) \geq 2$ in \mathbb{P}^m . This implies that $P_1|_L$ is a holomorphic diffeomorphism from L to $B_0 := \mathbb{P}^m \setminus \sigma(\mathcal{F})$. By Hartogs' extension theorem, applied to the inverse $(P_1|_L)^{-1}$, again using the fact that $\text{codim } \sigma(\mathcal{F}) \geq 2$, we can extend $P_1|_L$ to a holomorphic diffeomorphism between \bar{L} and \mathbb{P}^m . Moreover, by this extension we conclude that indeed $\sigma(\mathcal{F}) = \emptyset$. Thus the function R has levels that correspond to the horizontal fibration, i.e., it depends only on the second coordinate. If we take R as a primitive rational function then we may assume that $R(x, y) = y$ in coordinates $(x, y) \in \mathbb{P}^m \times \mathbb{P}^n$. Now we assume that $\sigma(\mathcal{F}) \neq \emptyset$ is irreducible of degree p^s . Given t close enough to 0, by hypothesis the ramification set $\sigma(t)$ of the Riccati foliation \mathcal{F}_t is still irreducible and therefore has degree p^s . Indeed, $\{\sigma(t)\}_{t \in D}$ defines an analytic family of irreducible algebraic curves in \mathbb{P}^2 . In particular, the fundamental groups $\pi_1(\mathbb{P}^2 \setminus \sigma(t))$ are the same. This implies that the holonomy group $\text{Hol}(\mathcal{F}_t, L_c)$ of the common leaf contained in $L_c \subset R_c$ is an analytic deformation of the holonomy group of $\mathcal{F}_0 = \mathcal{F}$. Let us be more precise. Given a non-invariant fiber $F_{x_0}: \{x_0\} \times \mathbb{P}^n$ and the

common invariant level $R_c : (R = c)$ there is a finite intersection set $R_c \cap F_{x_0} = \{y_1, \dots, y_r\}$. Denote by H_t the global holonomy group of \mathcal{F}_t given by the representation $H_t \subset \text{Aut}(F_{x_0})$. Given the point $y_1 \in F_{x_0}$ we consider the holonomy group $H_{t,y_1} := \text{Hol}(\mathcal{F}_t, L_{t,y_1})$ of the leaf L_{t,y_1} of \mathcal{F}_t passing through y_1 and calculated with respect to the transverse section contained in the fiber F_{x_0} . By hypothesis R_c is invariant by \mathcal{F}_t so that $L_{t,y_1} \subset R_c$. By Theorem 1.3 each group H_{t,y_1} is finite cyclic of uniformly bounded order for t close to 0. Given $t \approx 0$ and a global holonomy map $f \in H_t$ we have that $f(R_c \cap F_{x_0}) = R_c \cap F_{x_0} = \{y_1, \dots, y_r\}$. Thus $f^{r_1}(y_1) = y_1$. This implies that $f^{r_1} \in H_{t,y_1}$ and since this group is cyclic finite of uniformly bounded order, this shows that each map $f \in H_t$ has a uniformly bounded finite order. Indeed, since each holonomy map in H_{t,y_1} comes from a global holonomy map in H_t this shows that each global holonomy group H_t is finite cyclic of a uniformly bounded order. The limit set part is proved as before. Assume now that R is the second coordinate projection (for instance if $\sigma(\mathcal{F})$ has codimension ≥ 2). Then the global holonomy group H_0 is trivial. By Theorem 1.3 the holonomy groups H_{t,y_1} are trivial. Also note that $r = 1$ i.e. $R_c \cap F_{x_0}$ consists of a single point. Similarly to above we then conclude that H_t is trivial for each $t \approx 0$. This shows that the foliation \mathcal{F}_t is equivalent to \mathcal{F}_0 in $\mathbb{P}^2 \setminus \sigma(t) \times \mathbb{P}^n$. \square

4.3. **Case $m \geq 2, n \geq 2$.** Now we study Riccati foliations in $\mathbb{P}^m \times \mathbb{P}^n, m \geq 2, n \geq 2$ under some hypothesis on the ramification set $\sigma \subset \mathbb{P}^m$.

Proof of Theorem 1.4. The proof is pretty much the same given for the other cases $m = 2, n = 1$. For using the irreducibility of $\sigma_1 \subset \mathbb{P}^m$ it is enough to apply Lefschetz hyperplane section theorem together with Deligne’s theorem for $m = 2$. All the rest goes as in Theorem 4.2. \square

Remark 4.2 (Riccati foliations: other notions and recent works in their topology). The notion of Riccati foliation goes back to the Riccati differential equation $\frac{dy}{dx} = a(x)y^2 + b(x)y + c(x)$ in the classical literature. It is for sure one of the most interesting ordinary differential equations of first order. It appears in different areas as differential geometry and physics. One of its main aspects is the relation with the theory of conformal mapping in complex geometry. It was probably Paul Painlevé who first looked at these equations in the framework of foliations ([14]).

Indeed, in his pioneering works, Painlevé considered complex differential equations of the form $\frac{dy}{dx} = \frac{a(x)y^2 + b(x)y + c(x)}{p(x)}$, where a, b, c, p are complex polynomials. The introduction of projective coordinates in $\mathbb{P} \times \mathbb{P}$ allows us to consider a corresponding Riccati foliation \mathcal{F} in the rational manifold $\mathbb{P} \times \mathbb{P}$, where the fibre bundle structure is given by the first coordinate projection $\mathbb{P} \times \mathbb{P} \rightarrow \mathbb{P}, (x, y) \mapsto x$. Well, in this case the corresponding global holonomy group $G(\mathcal{F})$ is a (finitely generated) group of Moebius maps. Usually the global topology of the foliation \mathcal{F} is studied through the study of the dynamics of the Moebius maps group $G(\mathcal{F})$. This is the case of works in existence of invariant measures, topology of limits sets, and others. Recent nice references for this are the works of Nicolas Hussenot. For instance, in [8] the author studies the dynamics of Riccati foliations of the Painlevé form above, addressing basically two questions:(a) What is the asymptotic behavior of the holonomy map along a generic Brownian path in the leaves? (b) How are the analytic continuations of holonomy germs of the foliation along a generic Brownian path in a transversal? This is a quite interesting work, but quite transverse to our approach. Indeed, this is placed in complex dimension two. Some of the ideas relative to question (b) above have been further developed in [1]. As for question (a) one finds a recent study in [9].

In a different line of investigation, we find the work of H. Zoladek ([18]) where the author introduced a new method of investigation of periodic solutions of planar holomorphic polynomial differential equations with trigonometric coefficients. This method is based on analysis of

geometry of leaves of certain holomorphic foliations in the rational manifold $\mathbb{P} \times \mathbb{P}$, with a special focus on the case of Riccati equations.

Another line of study is followed by the authors in [13]. This line is more related to the complex and algebraic geometry, exploring therefore the connections between the existence of suitable differential forms (or more generally, certain connections and morphisms on an algebraic manifold) and the existence and classification of Riccati foliations in this manifold. It is indeed much more than this but, though placed in complex dimension ≥ 3 , we are in the codimension one case. In this line of investigation, usually the transverse structure plays an important role. More precisely, the fact that a Riccati foliation is a model for transversely homogeneous transverse structure in the complex case, is used in a very effective way. For this we shall refer to [17] and [12]. In this last, one finds a notion of Riccati foliation *over an algebraic projective manifold*, which is *slightly different* from ours. We refer to the original paper [12] for the precise notion.

There are also inspiring works with more precise description of the geometry of Kleinian groups in any dimension, suggesting possible applications to the framework of Riccati foliations. For instance we mention the very complete book [3]. As for a simple notion and construction of Riccati foliation in $\mathbb{P} \times \mathbb{P}^2$, of dimension one, we refer to [16].

The above is a non-exhaustive list since there are many works not mentioned, and some with hidden information on Riccati foliations, as [11], where the author proves that the any finitely generated group of Moebius maps is the global holonomy of some Riccati foliation in $\mathbb{P} \times \mathbb{P}$.

Our results are therefore new, for arbitrary codimension ≥ 1 , and we highlight the novelty that we transfer information from the geometry of the ramification set of the Riccati foliation, indeed from the complement of that hypersurface, to the geometry and classification of the foliation.

5. SOME EXAMPLES AND COMMENTS

Let us address the questions mentioned in the Introduction. We summarize our conclusions as follows:

Proposition 5.1. Regarding irreducible groups of germs we have:

- (1) Conditions (a) and (b) in Definition 2.1 are not equivalent.
- (2) A finite abelian subgroup $G \subset \text{Diff}(\mathbb{C}^n, 0)$ is not necessarily irreducible.
- (3) An irreducible subgroup $G \subset \text{Diff}(\mathbb{C}^n, 0)$ is not necessarily finite.
- (4) A finite irreducible subgroup $G \subset \text{Diff}(\mathbb{C}^n, 0)$ is not necessarily cyclic.

Proof. We start with (1). We look at the linear case. Let

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$$

and $G = \langle A, B \rangle \subset \text{GL}(2, \mathbb{C})$. Let us now see that A and B are conjugate in G , since

$$H = B^{-1}A = \begin{pmatrix} \frac{1}{2} & 0 \\ -\frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \in G$$

and

$$AH = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} = HB.$$

Therefore G satisfies (b) but not (a).

Now we consider

$$\tilde{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad \tilde{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix}$$

and $\tilde{G} = \langle \tilde{A}, \tilde{B}, \tilde{C} \rangle \subset \mathrm{GL}(3, \mathbb{C})$. Note that \tilde{G} satisfies (a). Let us see that it does not satisfy (b). Indeed, since the generators of \tilde{G} are upper triangular matrices then any element of \tilde{G} must be an upper triangular matrix. Then it is not difficult to see that there is no conjugation in \tilde{G} between \tilde{A} and \tilde{B} .

Now we show (2). Put

$$A = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and consider $G = \langle A, B \rangle \subset \mathrm{GL}(2, \mathbb{C})$. Note that

$$A^4 = B^2 = (AB)^2 = (BA)^2 = (BA^2)^2 = \mathrm{Id}.$$

Also $AB^2 = A$, $ABA = B$, $BAB = A^3$, $BA^2B = A^2$ and $BA^3B = A$. Thus G is finite:

$$G = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \right. \\ \left. \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \right\}.$$

But G is not irreducible because A and B are not conjugate since A and B have different orders ($A^4 = \mathrm{Id}$ and $B^2 = \mathrm{Id}$).

The group G above is not abelian. We may ask then what happens in the abelian case. Again the answer is negative: let $\lambda \in \mathbb{C}$ be such that $\lambda^n = 1$ for some $n \in \mathbb{N}$ $n > 4$ and let

$$A = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}.$$

We take

$$G = \langle A, B \rangle \subset \mathrm{GL}(2, \mathbb{C})$$

then G is abelian and finite but not irreducible ($A^4 = \mathrm{Id} = B^n$, $n \neq 4$). Note that G is finite abelian, not generated by a single element, and it is not irreducible.

Let us now show (3). Put

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} -\frac{1}{3^{\frac{1}{2}}} & 1 \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} -\frac{1}{3^{\frac{1}{2}}} & 2 \\ \frac{1}{8} & \frac{1}{2} \end{pmatrix}.$$

Note that $A^2 = B^2 = C^2 = \mathrm{Id}$, now take

$$H = B^{-1}A = \begin{pmatrix} -\frac{1}{3^{\frac{1}{2}}} & -1 \\ \frac{1}{4} & -\frac{1}{2} \end{pmatrix} \quad \text{and} \quad T = C^{-1}A = \begin{pmatrix} -\frac{1}{3^{\frac{1}{2}}} & -2 \\ \frac{1}{8} & -\frac{1}{2} \end{pmatrix}.$$

Then $H, T \in G$, $HBH^{-1} = A$, $TCT^{-1} = A$ and $T^{-1}HB(T^{-1}H)^{-1} = C$. Thus, the group:

$$G = \langle A, B, B, C, C, A \rangle$$

is irreducible and not cyclic. Now we will verify that the group G is not finite. Indeed, we take

$$BC = \begin{pmatrix} -\frac{1}{2} & 1 \\ \frac{3}{4} & \frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} -\frac{1}{2} & 2 \\ \frac{3}{8} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{5}{8} & -\frac{1}{4} \\ -\frac{3}{16} & \frac{7}{4} \end{pmatrix}.$$

Now we will study the signal of entries in the matrix BC without importing its values. So we can represent the matrix $(BC)^2$:

$$(BC)^2 = \begin{pmatrix} + & - \\ - & + \end{pmatrix} \cdot \begin{pmatrix} + & - \\ - & + \end{pmatrix} = \begin{pmatrix} (+\cdot+) + (-\cdot-) & (+\cdot-) + (-\cdot+) \\ (-\cdot+) + (+\cdot-) & (-\cdot-) + (+\cdot+) \end{pmatrix} = \begin{pmatrix} + & - \\ - & + \end{pmatrix}.$$

Thus each entries in the matrix increases in module. Therefore, there is no $n \in \mathbb{N}$ such that $(BC)^n = \text{Id}$.

Finally, we address (4). Let

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} -\frac{1}{2} & 1 \\ \frac{3}{4} & \frac{1}{2} \end{pmatrix} \quad \text{and} \quad H = B^{-1}A = \begin{pmatrix} -\frac{1}{2} & -1 \\ \frac{3}{4} & -\frac{1}{2} \end{pmatrix}.$$

Note that $A^2 = B^2 = \text{Id}$, $HBH^{-1} = A$ and $H^{-1}AH = B$. Now consider the 4×4 matrices formed by 2×2 diagonal blocks. We denote by $D_{A,A}$ the matrix with diagonal blocks A . Now consider

$$G = \langle D_{A,A}, D_{A,B}, D_{B,B}, D_{B,A} \rangle \subset \text{GL}(4, \mathbb{C}).$$

We have that $D_{A,A} \cdot D_{A,B} \cdot D_{B,B} \cdot D_{B,A} = D_{A^2B^2, AB^2A} = \text{Id}$. The generators are conjugates 2 to 2 in the groups, without loss of generality we verify that $D_{A,A}$ is conjugates to $D_{A,B}$ in the groups. Indeed, take

$$T = D_{A,B}^{-1} \cdot D_{A,A} = D_{A^{-1}A, B^{-1}A} = D_{\text{Id}, H} \in G.$$

Then

$$T \cdot D_{A,B} \cdot T^{-1} = D_{\text{Id}, H} \cdot D_{A,B} \cdot D_{\text{Id}, H}^{-1} = D_{\text{Id} \cdot A, \text{Id} \cdot HBH^{-1}} = D_{A,A}.$$

This ends the proof. \square

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