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WHITNEY STRATIFIED MAPPING CYLINDERS

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To Andrew du Plessis for his 60th birthday.

Abstract. In this paper we investigate (b)-regularity for stratified mapping cylinders $C_{\mathcal{W}'}(\mathcal{W})$ of a stratified submersion $f : \mathcal{W} \to \mathcal{W}'$ between two Whitney stratifications. We show how Goresky's condition (D) for f is sufficient to obtain (b)-regularity of $C_{\mathcal{W}'}(\mathcal{W})$.

Revisiting some ideas of Goresky we give different proofs, a finer analysis and new equivalent properties.

1. Introduction.

Let $\mathcal{X} = (A, \Sigma)$ be a stratified set of support A and stratification Σ (see §2 for the definition) contained in a Euclidean space $\mathbb{R}^N$. A substratified object of $\mathcal{X}$ is a stratified space $\mathcal{W} = (W, \Sigma_{\mathcal{W}})$, where W is a subset of A, such that each stratum in $\Sigma_{\mathcal{W}}$ is contained in a single stratum of $\mathcal{X}$. In this paper we study the (b)-regularity of the stratified mapping cylinder $M(f_{\mathcal{W}})$ of a stratified surjective submersion $f_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}'$ when $\mathcal{W}$ and $\mathcal{W}'$ are (b)-regular.

Since $f_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}'$ is surjective, $M(f_{\mathcal{W}})$ will be a cone that we will denote by $C_{\mathcal{W}'}(\mathcal{W})$.

Our motivation comes from the works of Goresky **[6, 7]** which followed his thesis **[5]**.

In 1976 and 1978 Goresky **[5, 6]** proved an important triangulation theorem for Thom-Mather abstract stratified sets $\mathcal{X}$. The proof was obtained by a double induction on $k \leq \dim \mathcal{X}$, first by triangulating, for each k-stratum X of $\mathcal{X}$, a boundary k-manifold $X_d^o \subseteq X$, and then using a stratified mapping cylinder $C_{\mathcal{W}'}(\mathcal{W})$ to glue a triangulation of X_d^o with a triangulation of a submanifold of the singular part $\partial X = \overline{X} - X = \sqcup_{X'<X} X'$ of $\overline{X}$. This method allowed one to extend the triangulation to the part $X - X_d^o$ of X near the singularity ∂X of $\overline{X}$.

Such mapping cylinders produce cellular (but not necessarily triangulated) stratified sets.

In this context to know how to obtain Whitney (i.e. (b)-) regularity of such mapping cylinders would be very useful in order to obtain a proof of the following:

Conjecture 1. 1. *Every compact Whitney stratified space $\mathcal{X}$ admits a Whitney cellularisation.*

This would be also a first important step of a possible proof of the celebrated Thom conjecture:

Conjecture 1. 2. *Every compact Whitney stratified space $\mathcal{X}$ admits a Whitney triangulation.*

Let us recall that in 2005 M. Shiota proved that semi-algebraic sets admit a Whitney triangulation **[16]** and more recently M. Czapla announced a new proof of this result **[2]** as a corollary of a more general triangulation theorem for definable sets. On the other hand, our motivation being the applications to Goresky's geometric homology theory, we are interested in the stronger Conjectures 1.1 and 1.2 for stratifications having C^1-strata.

In 1981 Goresky defines for a Whitney stratification $\mathcal{X}$, two geometric homology and cohomology theories $WH_k(\mathcal{X})$ and $WH^k(\mathcal{X})$ whose cycles and cocycles are substratified Whitney objects of $\mathcal{X}$ and proves the following representation theorems (**[7]**, Theorems 3.4. and 4.7) :

Key words and phrases. Stratified sets and maps, Whitney Condtions (a) and (b), regular cellularisations.

Theorem 1. 1. *If $\mathcal{X} = (M, \{M\})$ is the trivial stratification of a compact C^1-manifold M, the homology representation map $R_k : WH_k(\mathcal{X}) \to H_k(M)$ is a bijection.*

Theorem 1. 2. *If $\mathcal{X} = (A, \Sigma)$ is a compact Whitney stratification, the cohomology representation map $R^k : WH^k(\mathcal{X}) \to H^k(A)$ is a bijection.*

Here the Goresky maps R_k and R^k are the analogues of the Thom-Steenrod representation maps between the differential bordism of a space and its singular homology.

In 1994 such theories were improved by the author of this paper by introducing a sum operation in $WH_k(\mathcal{X})$ and $WH^k(\mathcal{X})$, geometrically meaning transverse union of stratified cycles [**12, 13**], with which the bijections R_k and R^k become group isomorphisms.

The possibility of constructing Whitney cellularisations of Whitney cycles and cocycles using mapping cylinders ([**7**], Appendices 1,2,3) was the main tool of Goresky to obtain two such important representation theorems.

We underline here that in the homology case the main result $R_k : WH_k(\mathcal{X}) \to H_k(M)$ was established *only* when $\mathcal{X} = (M, \{M\})$ is a trivial stratification of a compact manifold M and that the complete homology statement for $\mathcal{X}$ an arbitrary compact (b)-regular stratification remains a famous problem of Goresky, still unsolved ([**7**] p.178) :

Conjecture 1. 3. *If $\mathcal{X} = (A, \Sigma)$ is a compact <u>Whitney stratification</u>, the homology representation map $R_k : WH_k(\mathcal{X}) \to H_k(A)$ is a bijection.*

Hovewer, the proof of Conjecture 1.3 would follow as a corollary if one proves Conjecture 1.1.

In conclusion Whitney regularity of the mapping cylinders of stratified submersions would play an extremely important role in answering affirmatively the Conjectures 1.1, 1.2 and 1.3.

The content of the paper is the following.

In §2 we review the most important classes of regular stratifications concerned by our analysis: the abstract stratified sets of Thom-Mather [**17, 8, 9**], and the Whitney (b)-regular stratifications [**19**], and we briefly recall the relation between them.

Then we recall the definition of condition (D), introduced by Goresky in his thesis [**5, 6**] for stratified submersions $f_{|\mathcal{W}} : \mathcal{W} \subseteq M \to \mathcal{W}' \subseteq M$, as a technical tool to obtain (b)-regularity of stratified mapping cylinders, and recall the results of Goresky of 1976-81 [**5, 7**] about it.

In §3 we study relations between condition (D) and stratified mapping cylinders.

The section is an exploration of some ideas of Goresky [**5, 7**] of which we give a finer analysis, different proofs, and some new equivalent properties.

For $\mathcal{X} = (A, \Sigma)$ a Whitney stratification, we consider the important case in which the stratified submersion $f_{|\mathcal{W}} : \mathcal{W} \subseteq M \to \mathcal{W}' \subseteq M$ is the restriction of a projection $\pi_X : T_X \to X$ on a stratum X of an system of control data $\mathcal{F} = \{(\pi_X, \rho_X) : T_X \to X \times \mathbb{R}\}_{X \in \Sigma}$ of $\mathcal{X}$ [**8, 9**].

The stratified mapping cylinder of $\pi_{X\,|\mathcal{W}}$ has then as embedded model the cone $C_{\mathcal{W}'}(\mathcal{W})$ equipped with its natural stratification $\bigsqcup_{S \subseteq \mathcal{W}\ ,\ S' = \pi_X(S)} \left[S \sqcup C^o_{S'}(S) \sqcup S' \right]$ (Proposition 3.4).

First, in Proposition 3.5 we explain what incidence relations in $C_{\mathcal{W}'}(\mathcal{W})$ are always (b)-regular, then using a convenient horizontal distribution $\{\mathcal{D}(y)\}_y$ in Theorem 3.3 and in Corollary 3.1.3) we prove that, if $\pi_{X\,|\mathcal{W}} : \mathcal{W} \to \pi_X(\mathcal{W})$ satisfies Condition (D), all remaining incidence relations $R' < C^o_{S'}(S)$ (with $R < S$ in $\mathcal{W}$) are (a)-regular, and thanks to this in Proposition 3.6 and Theorem 3.4 we prove that the naturally stratified cone $C_{\mathcal{W}'}(\mathcal{W})$ is a Whitney (b)-regular stratification.

In Corollary 3.2 we conclude that if $\mathcal{W}$ is a Whitney cellularisation of a compact subset $W \subseteq S_X(1) \subseteq T_X(1)$ such that $\pi_{\mathcal{W}}$ is cellular then $C_{\mathcal{W}'}(\mathcal{W})$ is a Whitney cellularisation too.

2. Stratified Spaces and Maps and Condition (D).

We recall that a *stratification* of a topological space A is a locally finite partition Σ of A into C^1 connected manifolds (called the *strata* of Σ) satisfying the *frontier condition* : if X and Y are disjoint strata such that X intersects the closure of Y, then X is contained in the closure of Y. We write then $X < Y$ and $\partial Y = \sqcup_{X<Y} X$ so that $\overline{Y} = Y \sqcup \left(\sqcup_{X<Y} X\right) = Y \sqcup \partial Y$ and $\partial Y = \overline{Y} - Y$ ($\sqcup$ = disjoint union).

The pair $\mathcal{X} = (A, \Sigma)$ is called a *stratified space* with *support* A and *stratification* Σ.

The k-*skeleton* of $\mathcal{X}$ is the stratified space $\mathcal{X}_k = (A_k, \Sigma_{|A_k})$ of support $A_k = \sqcup_{\dim X \le k} X$.

A *stratified map* $f : \mathcal{X} \to \mathcal{X}'$ between stratified spaces $\mathcal{X} = (A, \Sigma)$ and $\mathcal{X}' = (B, \Sigma')$ is a continuous map $f : A \to B$ which sends each stratum X of $\mathcal{X}$ into a unique stratum X' of $\mathcal{X}'$, such that the restriction $f_X : X \to X'$ is C^1.

A *stratified submersion* is a stratified map f such that each $f_X : X \to X'$ is a C^1-submersion.

2.1. Regular Stratified Spaces and Maps.

Extra conditions may be imposed on the stratification Σ, such as to be an *abstract stratified set* in the sense of Thom-Mather **[17, 8, 9]** or, when A is a subset of a C^1 manifold, to satisfy conditions (a) or (b) of Whitney **[19]**, or (c) of K. Bekka **[1]** or, when A is a subset of a C^2 manifold, to satisfy conditions (w) of Kuo-Verdier **[20]**, or (L) of Mostowski **[15]**.

In this paper we will consider essentially Whitney (i.e. (b)-regular) stratifications :

Definition 2. 1. Let Σ be a stratification of a subset $A \subseteq \mathbb{R}^N$, $X < Y$ strata of Σ and $x \in$.

One says that $X < Y$ *is* (b)-*regular* (or that it satisfies *Condition* (b) *of Whitney*) at x if for every pair of sequences $\{y_i\}_i \subseteq Y$ and $\{x_i\}_i \subseteq X$ such that $\lim_i y_i = x \in X$ and $\lim_i x_i = x$ and moreover $\lim_i T_{y_i} Y = \tau$ and $\lim_i [y_i - x_i] = L$ in the appropriate Grassmann manifolds (here $[v]$ denotes the vector space spanned by v) then $L \subseteq \tau$.

The pair $X < Y$ *is called* (b)-*regular* if it is (b)-regular at every $x \in X$.

Σ is called a (b)-*regular* (or a *Whitney)* stratification if all $X < Y$ in Σ are (b)-regular.

For a C^1-retraction $\pi : U \to X$ defined on a neighbourhood U of x, one says that $X < Y$ is (b^π)-*regular at* x (or that it satisfies *Condition* (b^π) *at* x) if $L = \lim_i [y_i - \pi(y_i)]$ implies $L \subseteq \tau$.

One says that $X < Y$ is (a)-*regular at* x (or that it satisfies *Condition* (a) *at* x) if $T_x X \subseteq \tau$.

We recall that $X < Y$ is (b)-regular (at x) if and only if it is (a)- and (b^π)-regular (at x) for some C^1-retraction $\pi : U_x \to X$ defined in a neighbourhood U of x **[18]**.

Most important properties of Whitney stratifications follow because they are in particular abstract stratified sets **[8, 9]**. It is then helpful to recall the definition below.

Definition 2. 2. (Thom-Mather 1970) Let $\mathcal{X} = (A, \Sigma)$ be a stratified space.

A family $\mathcal{F} = \{(\pi_X, \rho_X) : T_X \to X \times [0, \infty[)\}_{X \in \Sigma}$ is called *a system of control data (SCD)* of $\mathcal{X}$ if for each stratum $X \in \Sigma$ we have that:

1) T_X is a neighbourhood of X in A (called a *tubular neighbourhood of* X);

2) $\pi_X : T_X \to X$ is a continuous retraction of T_X onto X (called *projection on* X);

3) $\rho_X : T_X \to [0, \infty[$ is a continuous function : $X = \rho_X^{-1}(0)$ (called *distance function from* X)

and, furthermore, for every pair of adjacent strata $X < Y$, by considering the restriction maps $\pi_{XY} = \pi_{X|T_{XY}}$ and $\rho_{XY} = \rho_{X|T_{XY}}$, on the subset $T_{XY} = T_X \cap Y$, we have that :

5) the map $(\pi_{XY}, \rho_{XY}) : T_{XY} \to X \times]0, \infty[$ is a C^1-submersion (it follows in particular that :

$\dim X < \dim Y$);

6) for every stratum Z of $\mathcal{X}$ such that $Z > Y > X$ and for every $z \in T_{YZ} \cap T_{XZ}$ the following *control conditions* are satisfied :
i) $\pi_{XY}\pi_{YZ}(z) = \pi_{XZ}(z)$ (called the π*-control condition*)
ii) $\rho_{XY}\pi_{YZ}(z) = \rho_{XZ}(z)$ (called the ρ*-control condition*).

In what follows we will pose $T_X(\epsilon) = \rho_X^{-1}([0,\epsilon[)\,, \forall\, \epsilon \geq 0$, and without loss of generality will assume $T_X = T_X(1)$ **[8, 9]**.

The pair $(\mathcal{X}, \mathcal{F})$ is called an *abstract stratified set* if A is Hausdorff, locally compact and admits a countable basis for its topology.

Since one usually works with a unique SCD $\mathcal{F}$ of $\mathcal{X}$, in what follows we will omit $\mathcal{F}$.

If $\mathcal{X}$ is an abstract stratified set, then A is metrizable and the tubular neighbourhoods $\{T_X\}_{X\in\Sigma}$ may (and will always) be chosen such that: "$T_{XY} \neq \emptyset \Leftrightarrow X \leq Y$" and "$T_X \cap T_Y \neq \emptyset \Leftrightarrow X \leq Y$ or $X \geq Y$" (where both implications $\Leftarrow$ automatically hold for each $\{T_X\}_X$) as in **[8, 9]**, pp. 41-46.

The notion of system of control data of $\mathcal{X}$, introduced by Mather, is very important because it allows one to obtain good extensions of (stratified) vector fields **[8, 9]** which are the fundamental tool in showing that a stratified (controlled) submersion $f : \mathcal{X} \to M$ into a manifold, satisfies Thom's First Isotopy Theorem : the stratified version to Ehresmann's fibration theorem **[17, 8, 9, 3]**. Moreover by applying it to the projections $\pi_X : T_X \to X$ it follows in particular that $\mathcal{X}$ has a *locally trivial structure* and so also a locally trivial topologically conical structure.

Since Whitney (b)-regular) stratification are abstract stratified sets **[8, 9]**, they are locally trivial.

2.2. **Condition** (D) **and Goresky's results.** The following definition was introduced by Goresky first in **[5]** (1976) and **[7]** (1981).

Definition 2. 3. Let $f : M \to M'$ be a C^1 map between C^1-manifolds and $\mathcal{W} \subseteq M$ and $\mathcal{W}' \subseteq M'$ Whitney stratifications such that the restriction $f_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}'$ is a surjective stratified submersion (so f takes each stratum Y of $\mathcal{W}$ to only one stratum $Y' = f(Y)$ of $\mathcal{W}' = f(\mathcal{W})$).

One says that $f : M \to M'$ *satisfies condition* (D) *with respect to* $\mathcal{W}$ *and* $\mathcal{W}'$ and we will say for short that the restriction $f_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}'$ *satisfies the condition* (D) if the following holds :

for every pair of adjacent strata $X < Y$ of $\mathcal{W}$ and every point $x \in X$ and every sequence $\{y_i\}_i \subseteq Y$ such that $\lim_i y_i = x \in X$ and moreover $\lim_i T_{y_i}Y = \tau$ and $\lim_i T_{f(y_i)}Y' = \tau'$ in the appropriate Grassmann manifolds then $f_{*x}(\tau) \supseteq \tau'$.

Later on we will also consider given, with the obvious restricted meaning of the definition 2.3, what one intends by : *"$f : M \to M'$ satisfies condition (D) with respect to $X < Y$"* and *"$f : M \to M'$ satisfies condition (D) with respect to $X < Y$ at $x \in X$"* (*"at $x \in X < Y$"*).

In the whole of the paper we will denote $Y' = f(Y)$ and $y' = f(y)\,,\ \forall\, y \in Y$.

Two simple examples of f satisfying and not-satisfying the condition (D) are the following.

Example 2. 1. Let M be the horizontal plane $M = \{z = 1\} \subseteq \mathbb{R}^3$, $M' = L(0,1,0) = y$-axis $\subseteq \mathbb{R}^3$ and $f : M \to M'$ the standard projection $f(x,y,z) = y$.

Let $\mathcal{W}$ be the stratified space of support the half parabola $W = \{y = x^2, x \geq 0\} \cap M$ in M and stratification $\Sigma_{\mathcal{W}} = \{R, S\}$ where $R = \{(0,0,1)\}$ and $S = W \cap \{x > 0\}$. Then $R < S$.

Let $\mathcal{W}'$ be the stratfied space of support the half y-axis, $W' = M' \cap \{y \geq 0\}$ in M' and stratification $\Sigma_{\mathcal{W}'} = \{R', S'\}$ where $R' = \{(0,0,0)\}$ and $S' = M' \cap \{y > 0\}$. Then $R' < S'$.

Then for every sequence $\{s_n\}_n \subseteq S$ such that $\lim_n s_n = (0,0,1) \in R$ one has :
$\tau = \lim_n T_{s_n} S = x$-axis $\subseteq ker\, f_*$ and $\tau' = \lim_n T_{s'_n} S' = y$-axis. Thus $f_*(\tau) \not\supseteq \tau'$.
Hence $f_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}'$ does not satisfies the condition (D) at $(0,0,1) \in R < S$. $\square$

Example 2. 2. Let consider the same stratified spaces of the example 2.1 but using now $W = \{y = tan(x), x \geq 0\} \cap M$ the half graph of the tangent map in M.

Then for every sequence $\{s_n\}_n \subseteq S$ such that $\lim_n s_n = (0,0,1) \in R$ one has :
$\tau = \lim_n T_{s_n} S = L(1,1,0) \not\subseteq \ker f_*$ and $\tau' = \lim_n T_{s'_n} S' =$ the y-axis line. Thus $f_*(\tau) \supseteq \tau'$
Hence $f_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}'$ satisfies the condition (D) at $(0,0,1) \in R < S$. $\square$

Below Figure 1a represents the case of Example 2.1 while Figure 1b the case of Example 2.2

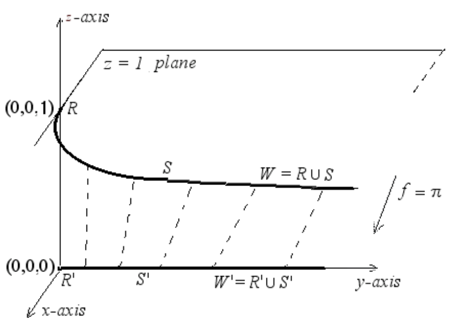

Figure 1a of Example 2.1

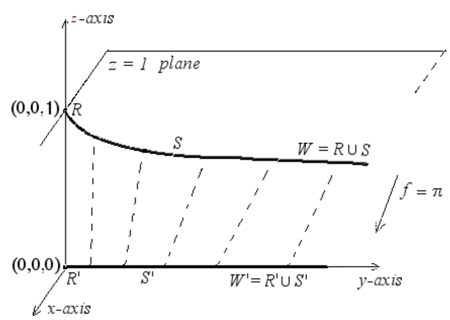

Figure 1b of Example 2.2

An important case in which condition (D) is satisfied is given by the following ([**5**] 3.7.4):

Example 2. 3. *Let $h : \mathbb{R}^N \to \mathbb{R}^l \times 0^k$ be a surjective submersion and $\mathcal{H} \subseteq \mathbb{R}^N$ and $\mathcal{H}' \subseteq \mathbb{R}^l \times 0^k$ linear cellular complexes such that the restriction $h_{\mathcal{H}} : \mathcal{H} \to \mathcal{H}' = f(\mathcal{H})$ is a cellular map.*
Then $h_{\mathcal{H}} : \mathcal{H} \to \mathcal{H}'$ satisfies the condition (D).

Proof. Obviously, $\mathcal{H}$ and $\mathcal{H}'$ are Whitney stratifications whose strata are their linear cells.
Let $R < S$ be cells of $\mathcal{H}$, $\{s_i\}_i \subseteq S$ a sequence such that $\lim_i s_i = r \in R \subseteq \overline{S}$, and let us denote $R' = h(R)$, $S' = h(S)$ and $s'_i = h(s_i)$ and $r' = h(r)$.
Since S and S' are linear cells, then $T_{s_i} S$ and $T_{s'_i} S'$ are always the same two vector subspaces independently of $i \in \mathbb{N}$: namely $[S] \subseteq \mathbb{R}^N$ and $[S'] \subseteq \mathbb{R}^l \times 0^k$.
So $\lim_i T_{s_i} S = [S]$ and $\lim_i T_{s'_i} S' = [S']$.
Similarly since $h : \overline{S} \to \overline{S'}$ is a cellular map, it is the restriction of a linear affine map and then $h_{*s_i} : T_{s_i} S \to T_{s'_i} S'$ is independently of $i \in \mathbb{N}$ always the same linear surjective map $H : [S] \to [S']$.
Thus

$$h_{*r}(\lim_i T_{s_i} S) = h_{*r}([S]) = H([S]) = [S'] = \lim_i T_{s'_i} S' = \lim_i h_{*s_i}([s]) \,. \quad \square$$

Example 2. 4. *Let $f : M \to M'$ be a surjective C^1-submersion and h and h' two C^1 cellularisations of two subsets $\mathcal{K} \subseteq M$ and $\mathcal{K}' \subseteq M'$ making the following diagram*

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{h} & \mathcal{K} \subseteq M \\ g \downarrow & & \downarrow f \\ \mathcal{H}' & \xrightarrow{h'} & \mathcal{K}' \subseteq M' . \end{array}$$

commutative where $g : \mathcal{H} \to \mathcal{H}'$ is a cellular map of cellular complexes.

Then $f_{\mathcal{K}} : \mathcal{K} \to \mathcal{K}'$ satisfies the condition (D).

Proof. Since h is a C^1 cellularisation of $\mathcal{K}$, then by definition **[6]**, $\forall p \in \tau$ in a simplex $\tau < \sigma$ of $\mathcal{H}$, the map h admits a C^1 extension $\tilde{h}$, a diffeomorphism on a neighbourhood U_p of p in the affine plane spanned by the linear cell σ.

Similarly, h' being a C^1 cellularisation of $\mathcal{K}'$ it admits a C^1 extension $\tilde{h}'$, a diffeomorphism on a neighbourhood $U_{p'}$ of $p' = g(p)$ in the affine plane spanned by the linear cell $\sigma' = g(\sigma)$.

Therefore, $\forall q = h(p) \in \mathcal{K}$ (h a bijection), with the two isomorphisms $(\tilde{h}_{*p})^{-1}$ and $\tilde{h}'_{*p'}$ one has:

$$f_{*q} = \tilde{h}'_{*p'} \circ g_{*p} \circ (\tilde{h}_{*p})^{-1} .$$

Finally, since by Example 2.3 g satisfies Condition (D) at $p \in \tau < \sigma$, then f satisfies Condition (D) at $q = f(p) \in f(\tau) < f(\sigma)$. $\square$

The main reason for which Goresky introduced Condition (D) is that it provided the (b)-regularity for the natural stratifications on the mapping cylinder of a stratified submersion.

Proposition 2. 1. *Let $\pi : E \to M'$ be a C^1 riemannian vector bundle and $M = S^{\epsilon}_{M'}$ the ϵ-sphere bundle of E. If $\mathcal{W} \subseteq M$, $\mathcal{W}' = \pi(\mathcal{W}) \subseteq M'$ are two Whitney stratifications such that $\pi_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}'$ is a stratified submersion which satisfies condition (D), then the closed stratified mapping cylinder*

$$\overline{C_{M'}(\mathcal{W})} = \bigsqcup_{Y \subseteq \mathcal{W}} \Big[(C_{M'}(Y) - \pi_{M'}(Y)) \sqcup \pi_{M'}(Y) \sqcup Y \Big]$$

is a Whitney (i.e. (b)-regular) stratified space.

Proof. **[7]** Appendix A.1. Lemma (i). $\square$

Our work in §3 will be essentially to give a new proof, together with a finer analysis, of the following important statement which is the key property in proving the Proposition below :

Proposition 2. 2. *Every Whitney stratification $\mathcal{W}$ with conical singularities and conical control data admits a Whitney cellularisation.*

Proof. **[7]** Appendix A.2. Proposition. $\square$

Propositions 2.1 and 2.2 are the main properties which allowed Goresky to prove Proposition below and, thanks to this, his two homology representation theorems, Theorem 1.1 and Theorem 1.2, recalled in the introduction.

Proposition 2. 3. *Every Whitney stratification $\mathcal{W}$ in a manifold M is "cobordant" in M to one $\mathcal{W}'$ having conical singularities and control data, and which is hence (b)-regular.*

Proof. [**7**] Appendix A.3. Proposition. $\square$

We end this section recalling that a detailed account of condition (D) including new analytic sufficient conditions in terms of limits of a new distance function between tangent spaces is given in [**14**].

3. Condition (D) and stratified mapping cylinders.

Let $\mathcal{X} = (A, \Sigma)$ be a Whitney stratified space with stratification Σ and support A closed in $\mathbb{R}^N$.

In this section we consider the important case in which $f_{|\mathcal{W}} : \mathcal{W} \subseteq M \to \mathcal{W}' \subseteq M$ is obtained as the restriction of a projection $\pi_X : T_X \to X$ on a stratum X of an SCD $\mathcal{F} = \{(\pi_X, \rho_X) : T_X \to X \times \mathbb{R}\}_{X \in \Sigma}$ of $\mathcal{X}$.

For our analysis it will be convenient to add to the stratification $\mathcal{X}$ all strata of $\mathbb{R}^N - A$.

The connected components of $\mathbb{R}^N - A$ being N-manifolds this will again give a Whitney stratification, namely again $\mathcal{X}$ of $A \cup (\mathbb{R}^N - A) = \mathbb{R}^N$ and then we will not lose generality.

It is well known that each neighbourhood T_X of an SCD of $\mathcal{X}$ can be obtained as a tubular neighbourhood of X in $\mathbb{R}^N$ and $\pi_X : T_X \to X$ as a C^1 map [**8**].

On the other hand T_X remains equipped with the induced Whitney stratification by its intersections with all strata $Y > X$ of $\mathcal{X}$; that is : $T_X = \sqcup_{Y \geq X} T_{XY}$ (as usual $T_{XY} = T_X \cap Y$).

Similarly the ϵ-sphere bundle $S^\epsilon_X = \rho_X^{-1}(\epsilon)$ of T_X, remains equipped with a natural induced Whitney stratification $S^\epsilon_X = \sqcup_{Y>X} S^\epsilon_{XY}$ where $S^\epsilon_{XY} = S^\epsilon_X \cap Y$.

Let consider then for $f : M \to M'$ the restriction map $f = \pi_{X|S^\epsilon_X} : S^\epsilon_X \to X$ between the C^1-manifolds $M = S^\epsilon_X$ and $M' = X$ which is a C^1-submersion [**8**].

We will consider for $\mathcal{W}$ a Whitney stratification of a compact set $W \subseteq S^\epsilon_X$ *stratifying* π_X as defined below.

Definition 3. 4. Let $\mathcal{W} = (W, \Sigma')$ be a Whitney stratification of a compact set $W \subseteq S^\epsilon_X$.

We will say that $\mathcal{W}$ *stratifies* π_X if the image $W' = \pi_X(W)$ has a natural Whitney stratification $\mathcal{W}' = \sqcup_{S'} S'$ (where $S' = \pi_X(S)$, and S ranges over all strata of $\mathcal{W}$) which makes $\pi_{X|\mathcal{W}} : \mathcal{W} \to \mathcal{W}'$ a stratified surjective submersion (denoted $\pi_\mathcal{W}$).

We will investigate the condition (D) for the restriction $f_\mathcal{W} = \pi_\mathcal{W} : \mathcal{W} \subseteq S^\epsilon_X \to \mathcal{W}' \subseteq X$.

A very important example occurs when $\mathcal{W}$ is a Whitney triangulation of $S^\epsilon_X - \cup_{X'<X} T^\epsilon_{X'}$ for which the restriction $\pi_{X|} : S^\epsilon_X - \cup_{X'<X} T^\epsilon_{X'} \to X - \cup_{X'<X} T^\epsilon_{X'}$ is a PL map [**5**] : this case will be treated in Corollary 3.2.

Let $l = \dim X$. The analysis of condition (D) at a point $x \in R$ for every stratum R of $\mathcal{W}$ is local and invariant by C^1-diffeomorphisms, hence starting from now we will suppose [**18**] that $\epsilon = 1$, $X = \mathbb{R}^l \times 0^k$ $(l + k = N)$ and $\pi_X = \pi$, $\rho_X = \rho$ are the standard data :

$$\rho(z) = \left(z_{l+1}^2 + \cdots + z_N^2\right)^{\frac{1}{2}} \ , \quad \pi(z) = (z_1, \ldots, z_l, 0^k) \quad \text{where} \quad z = (z_1, \ldots, z_N) \in \mathbb{R}^N .$$

Thus $S^\epsilon_X = S^1_X = \{z \in \mathbb{R}^N \mid z_{l+1}^2 + \cdots + z_N^2 = 1\} = \mathbb{R}^l \times S^{k-1}$ and the C^1-submersion $f = \pi_{X|S^\epsilon_X}$ is the canonical projection : $\mathbb{R}^l \times S^{k-1} \to \mathbb{R}^l \times 0^k$ (also denoted π_X).

In particular $\mathcal{W}$ will be a Whitney stratification $\subseteq S^1_X = \mathbb{R}^l \times S^{k-1}$ stratifying π_X.

With these hypotheses the closed cone with straight lines in $\mathbb{R}^N$:

$$C_{\mathcal{W}'}(\mathcal{W}) = \{tp + (1-t)\pi(p) \mid p \in \mathcal{W} \ , \ t \in [0,1]\} ,$$

with its natural stratification, gives a differential model of the stratified mapping cylinder of the stratified submersion $\pi_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}'$ as follows.

For every subset $H \subseteq S^1_X$, written $H' = \pi_X(H)$ let us denote by :

$$C_{H'}(H) = \big\{ tp + (1-t)\pi(p) \,|\, p \in H\,,\ \ t \in [0,1] \big\}$$

$$C^o_{H'}(H) = \big\{ tp + (1-t)\pi(p) \,|\, p \in H\,,\ \ t \in]0,1[\big\}$$

respectively the closed and the open cone of H induced by π.

The natural stratification of $C_{\mathcal{W}'}(\mathcal{W})$ is then given by :

$$C_{\mathcal{W}'}(\mathcal{W}) = \bigsqcup_{S \subseteq \mathcal{W}} \Big[S \sqcup C^o_{S'}(S) \sqcup S' \Big]\,.$$

Proposition below says that $C_{\mathcal{W}'}(\mathcal{W})$ can be stratified as the stratified image of an appropriate globally C^1 stratified map F which makes it into a differential model of the stratified mapping cylinder $M(\pi_{\mathcal{W}}) = (\mathcal{W} \times [0,1] \sqcup \mathcal{W}') \,/ \text{“}(z,0) \ \sim \pi(z)\text{”}$.

Proposition 3. 4. *Let F be the map*

$$F \ : S^1_X \times [0,1] \to C_X(S^1_X) \quad , \quad F(z,t) = tz + (1-t)z' \quad , \quad z' = \pi_X(z)\,.$$

1) F is a homotopy satisfying $F_0(z) = 1_{S^1_X}(z)$ and $F_1(z) = \pi_{X|S^1_X}(z)$ whose restriction off $F(S^1_X \times \{0\}) = X$, that is $F_| : S^1_X \times]0,1] \to C_X(S^1_X) - X = C^o_X(S^1_X) \sqcup S^1_X$, is a C^1-isotopy.

2) $C_{\mathcal{W}'}(\mathcal{W}) = F(\mathcal{W}' \times [0,1])$.

Proof. Immediate. $\square$

Looking at the regularity of the incidence relations in $C_{\mathcal{W}'}(\mathcal{W})$ we have :

Proposition 3. 5. *Let $\mathcal{W}$ be a Whitney stratification in $S^1_X = \mathbb{R}^l \times S^{k-1}$ which stratifies the canonical projection $\pi_X : S^1_X \to X = \mathbb{R}^l \times 0^k$ and let $\mathcal{W}' = \pi_X(\mathcal{W})$.*

For every pair of strata $R < S$ of $\mathcal{W}$, by denoting $S' = \pi_X(S)$, $R' = \pi_X(R)$, the cone

$$C_{R' \cup S'}(R \cup S) = \big(R \sqcup C^o_{R'}(R) \sqcup R' \big) \sqcup \big(S \sqcup C^o_{S'}(S) \sqcup S' \big)$$

satisfies (b)-regularity for all incidence relations $<$ below :

$$\begin{array}{ccccccc}
R & < & S & & \subseteq \ \mathcal{W} & \subseteq & S^1_X \\
\wedge & \nwarrow & \wedge & & & & \\
C^o_{R'}(R) & < & C^o_{S'}(S) & & \subseteq C_{\mathcal{W}'}(\mathcal{W}) & \subseteq & \mathbb{R}^N \\
\vee & & \vee & & & & \\
R' & < & S' & & \subseteq \ \mathcal{W}' & \subseteq & X\,.
\end{array}$$

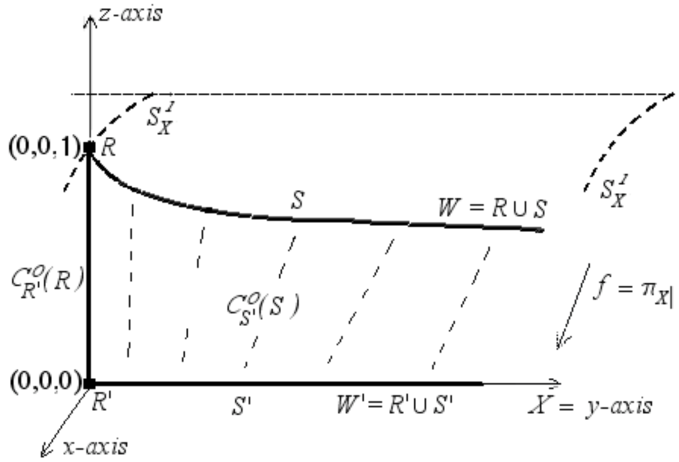

Figure 2

Proof. Since $\mathcal{W}$ and $\mathcal{W}'$ are Whitney (b)-regular stratifications the pair of strata $R < S$ in $\mathcal{W}$ and $R' < S'$ in $\mathcal{W}'$ are trivially (b)-regular.

Since the proofs of (b)-regularity for the pairs $R' < C^o_{R'}(R)$ and $S' < C^o_{S'}(S)$ are obviously the same and this also holds for the pairs $R < C^o_{R'}(R)$ and $S < C^o_{S'}(S)$ it will be sufficient to prove the (b)-regularity of the following adjacent pairs of strata :

$$\begin{array}{ccc} R & & S \\ & \nwarrow & \wedge \\ C^o_{R'}(R) & < & C^o_{S'}(S) \\ & & \vee \\ & & S'\,. \end{array}$$

The restriction of the C^1-homotopy F to $S^1_X\times]0,1]$ (namely again F) :

$$F \;:\; S^1_X\times]0,1] \to C_X(S^1_X) - X \quad , \quad F(z,t) = \pi(z) + t(z - \pi(z))$$

is a C^1 diffeomorphism of manifolds with boundary such that :

$$C^o_{S'}(S) = F(S\times]0,1[) \quad , \quad S = F(S \;\times \{1\}) \quad \text{and} \quad C^o_{R'}(R) = F(R\times]0,1[)\,.$$

Hence the (b)-regularity of

$$R < C^o_{S'}(S)\,, \qquad S < C^o_{S'}(S) \qquad \text{and} \qquad C^o_{R'}(R) < C^o_{S'}(S)$$

follows via F respectively by the (b)-regularity in $\mathbb{R}^N$ of

$$R < S\;\times]0,1[\,, \qquad S < S\;\times]0,1[\qquad \text{and} \qquad R\;\times]0,1[< S\times]0,1[.$$

Then, it only remains to prove that $S' < C^o_{S'}(S)$ is (b)-regular.

It is well known that (b)-regularity is satisfied for a pair of strata $S' < Y$ if and only if (a)-regularity and $(b^{\pi_{S'Y}})$-regularity are satisfied for the restriction $\pi_{S'Y} : T_{S'} \cap Y \to S'$ of a C^1-retraction $\pi_{S'} : T_{S'} \to S'$ defined on a neighbourhood $T_{S'}$ of S' **[18]**.

We will show then that $S' < Y = C^o_{S'}(S)$ is (a)- and $(b^{\pi_{S'Y}})$-regular.

<u>*(a)-regularity.*</u> For every point $z \in S^1_X$, by denoting $z = (x, x')$ with $x \in \mathbb{R}^l$ and $x' \in \mathbb{R}^k$ then $\pi(z) = (x, 0^k)$ and $z - \pi(z) = (0^l, x')$ and $F(x, x', t) = (x, tx')$. Similarly for every $v \in \mathbb{R}^{l+k}$,

$v=(u,u')$, and at every point $(z,t)=(x,x',t)\in S^1_X\times]0,1[$ the image of the differential map F

$$F_{*(z,t)} \quad : \quad T_{(z,t)}(S^1_X\times[0,1]) \quad \to \quad T_{F(z,t)}C_X(S^1_X)$$

is given by :

$$F_{*(z,t)}(v,\lambda)=\begin{pmatrix}1_{\mathbb{R}^l} & 0 & 0\\ & & \\ 0 & t\cdot 1_{\mathbb{R}^k} & x'\end{pmatrix}\cdot\begin{pmatrix}u\\ u'\\ \lambda\end{pmatrix}=(u,tu')+\lambda(0,x')=$$

$$=\pi(v)+t(v-\pi(v))+\lambda(z-\pi(z))\,.$$

By considering the submanifold $Y_t=F(S\times\{t\})$ of $Y=C^o_{S'}(S)=F(S\times]0,1[)$ and a point $y=F(s,t)\in Y_t\subseteq Y$ one finds :

$$T_{F(s,t)}Y_t \;=\; F_{*(s,t)}\big(T_{(s,t)}(S\times\{t\})\big) \;=\; F_{*(s,t)}(T_sS\times\{0\}) \;=\; \big\{F_{*(s,t)}(v,0)\mid v\in T_sS\big\}$$

with

$$F_{*(s,t)}(v,0)=(tu,u')=\pi(v)+t(v-\pi(v))$$

and so for every $s_0\in S$, if $s'_0=\pi(s_0)$, F being a C^1 map at $(s_0,0)$ one has :

$$\lim_{(s,t)\to(s_0,0)}T_{F(s,t)}Y_t=\lim_{(s,t)\to(s_0,0)}F_{*(s,t)}(T_sS\times\{0\})=F_{*(s_0,0)}(T_sS\times\{0\})=\pi_{*s_0}(T_{s_0}S)=T_{s'_0}S'.$$

Consequently, for each point $s_0\in S$:

$$\lim_{(s,t)\to(s_0,0)}T_{(s,t)}\,C^o_{S'}(S) \quad\supseteq\quad \lim_{(s,t)\to(s_0,0)}T_{F(s,t)}Y_t \quad=\quad T_{s'_0}S'$$

which proves the (a)-regularity $S'<C^o_{S'}(S)$.

<u>*$(b^{\pi_{S'Y}})$-regularity.*</u> To prove that $S'<C^o_{S'}(S)$ is $(b^{\pi_{S'Y}})$-regular, it is natural to take for $\pi_{S'}$ the restriction of the canonical projection $\pi:\mathbb{R}^N\to\mathbb{R}^l\times 0^k$, and denote it again by π.

Let us consider a sequence $\{F(s_n,t_n)\}_n\subseteq C^o_{S'}(S)$ such that $\lim_n F(s_n,t_n)=s'_0\in S'$ and there exist both limits of lines and tangent spaces :

$$L=\lim_n\overline{F(s_n,t_n)\,\pi(F(s_n,t_n))}\in G^1_n \quad\text{and}\quad \tau=\lim_n T_{F(s_n,t_n)}\,C^o_{S'}(S)\in\mathbb{G}^h_n\,,\quad (h=\dim S+1)\,.$$

Then $\{s_n\}\subseteq S$ is a convergent sequence, $\lim_n s_n=s_0\in S$, such that if $s'_n=\pi(s_n)$ then $\lim_n s'_n=s'_0=\pi(s_0)$ and $\lim_n t_n=0$.

Since $C^o_{S'}(S)=F(S\times]0,1[)=C_{S'}(S)-S\cup S'$, with $S'=\pi(S)$ and $\pi(F(s_n,t_n))=\pi(s_n)=s'_n$, then for every line $L_n=\overline{F(s_n,t_n)\,\pi(F(s_n,t_n))}$ we have :

$$L_n \;=\; \overline{F(s_n,t_n)\,\pi(F(s_n,t_n))} \;=\; \overline{s_n\,s'_n} \;=\; [s_n-s'_n]\,,$$

where $[v]$ denotes the vector subspace spanned by $v\in\mathbb{R}^N$, so that

$$L \;=\; \lim_n L_n \;=\; \lim_n\,[s_n-s'_n] \;=\; [s_0-s'_0]\,.$$

On the other hand, for every $n\in\mathbb{N}$, by decomposing in a direct sum

$$T_{(s_n,t_n)}\;S\times]0,1[\;=\; T_{s_n}S\;\times\;\mathbb{R} \;=\; T_{s_n}S\times\{0\} \quad+\quad \{0^h\}\times\mathbb{R}$$

one also has :

$$F_{*(s_n,t_n)}\big(T_{(s_n,t_n)}S\times]0,1[\big) \;=\; F_{*(s_n,t_n)}\big(T_{s_n}S\times\{0\}\big) \;+\; F_{*(s_n,t_n)}\big(\{0^h\}\times\mathbb{R}\big) =$$

$$\big\{\pi(v)+t_n(v-\pi(v)) \mid v\in T_{s_n}S\big\} \;+\; \big\{\lambda(s_n-s'_n)\;|\lambda\in\mathbb{R}\big\} \;=$$

as in the previous proof of (a)-regularity :

$$= \; T_{F(s_n,t_n)}Y_{t_n} \;+\; [s_n-s'_n]\,.$$

Finally, since

$$\lim_n(T_{F(s_n,t_n)}Y_{t_n} \;+\; [s_n-s'_n]) \;\supseteq\; \lim_n T_{F(s_n,t_n)}Y_{t_n} \;+\; \lim_n\;[s_n-s'_n]\,,$$

one finds :

$$\tau \;=\; \lim_n\; T_{F(s_n,t_n)}\,C^o_{S'}(S) \;=\; \lim_n\; F_{*(s_n,t_n)}\big(T_{(s_n,t_n))}S\times]0,1)[\big) \;=$$

$$= \; \lim_n\;\Big(T_{F(s_n,t_n)}Y_{t_n} \;+\; [s_n-s'_n]\Big) \;\supseteq\; T_{s'_0}S' \;+\; [s_0-s'_0]\,.$$

This proves $\tau\supseteq L$ and concludes the proof of (b^π)-regularity of $S'<C^o_{S'}(S)$. $\square$

If we consider as in Proposition 3.5 for $\pi_{S'\cup Y}:Y\cup S'\to S'$ the restriction of $\pi:\mathbb{R}^N\to\mathbb{R}^l\times 0$ and similarly for the distance function to S' the restriction of the standard distance $\rho(z_1,\dots,z_N)=\big(z^2_{l+1}+\dots+z^2_N\big)^{\frac{1}{2}}$, then the stratification of only two strata $S'<C^o_{S'}(S)=Y$ remains equipped with an SCD $\{(\pi_{S'},\rho_{S'})\}$. With such an SCD one can consider the canonical distribution $\mathcal{D}_{S'Y}:S'\cup Y\to\mathbb{G}^{\dim S'}_N$ relative to the (a)-regular pair of strata $S'<Y=C^o_{S'}(S)=F(S\times]0,1[)$ as defined in **[10, 11]**, by the subspace of T_yY closest to $T_{s'}S'$:

$$\mathcal{D}_{S'Y}(y)=\perp(\ker(\pi_{S'Y},\rho_{S'Y})_{*y}\,;\,\ker\,\rho_{S'Y*y})$$

where the notation $\perp(U,V)$ means the orthogonal complement of a vector subspace V in a vector space U and $V\subseteq U\subseteq\mathbb{R}^N$ are considered with the standard Euclidian scalar product.

Remark 3. 1. *By Proposition 3.5, $S'<C^o_{S'}(S)$ is (a)-regular, hence the canonical distribution $\mathcal{D}_{S'Y}(y)$ relative to $S'<C^o_{S'}(S)=Y$ satisfies : $\lim_{y\to s'\in S'}\mathcal{D}_{S'Y}(y)\supseteq T_{s'}S'$* **[10, 11]**. $\square$

Now, for every $t\in]0,1]$, the diffeomorphism

$$F_t:S=Y_1\to Y_t=F(S\times\{t\}) \quad,\quad y=F_t(s)=F(s,t)=\pi(s)+t(s-\pi(s))$$

induces (as in the proof of 3.3) an isomorphism between the tangent spaces and their subspaces

$$F_{t*s}:T_sS\to T_yY_t \quad,\quad F_{t*s}(v)=\pi(v)+t(v-\pi(v))\,.$$

By considering for the Whitney stratification $\mathcal{W}\subseteq S^1_X=\mathbb{R}^l\times S^{k-1}$ stratifying the canonical projection $\pi_X:S^1_X=\mathbb{R}^l\times S^{k-1}\to X=\mathbb{R}^l\times 0^k$ (i.e. such that the map $\pi_{\mathcal{W}}:\mathcal{W}\to\mathcal{W}'=\pi_X(\mathcal{W})$ is a stratified surjective submersion) and for each stratum S of $\mathcal{W}$ the canonical distribution $\{\mathcal{D}(s)\}_s$ of $\pi_{\mathcal{W}|S}$ (see also **[14]** §3) defined by, $\mathcal{D}(s)=\perp(\ker\pi_{X|S_{*s}},T_sS)$, we have :

Lemma 3. 1. *The stratification $S' < Y = C_{S'}(S)$, with the SCD $\{(\pi_{S'Y}, \rho_{S'Y})\}$, satisfies:*

1) Each hypersurface $Y_t = F_t(S)$ of Y, coincides with the hypersurface $\rho^{-1}_{S'Y}(t)$: $Y_t = \rho^{-1}_{S'Y}(t)$.

2) If $y = F(s, 1)$, so that $y = s \in Y_1 = S$ the distributions $\mathcal{D}(s) = \mathcal{D}_{S'Y}(y)$ coincide.

3) $F_t : S \to Y_t$, carries the distribution $\mathcal{D}(s)$ into $\mathcal{D}_{S'Y}(y)$:

$$F_{t*s}(\mathcal{D}(s)) = \mathcal{D}_{S'Y}(y)\,.$$

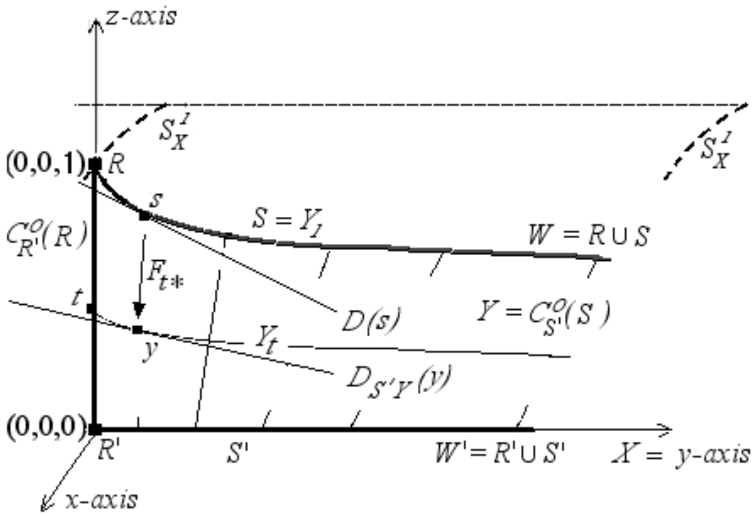

Figure 3

Proof 1). If $y = F(s, \tau) \in Y$, being $y - \pi(y) = \tau(s - \pi(s))$ and $||s - \pi(s)|| = 1$ one has :

$$\rho_{S'Y}(y) = ||y - \pi(y)|| = ||\tau(s - \pi(s))|| = \tau \cdot ||s - \pi(s)|| = \tau \qquad \text{and so :}$$

$$y \in Y_t \quad \Leftrightarrow \quad \tau = t \quad \Leftrightarrow \quad \rho_{S'Y}(y) = t \quad \Leftrightarrow \quad y \in \rho^{-1}_{S'Y}(t)\,.$$

Proof 2). If $y = F(s, 1)$, so $s = y$ and $S = Y_1 = \rho^{-1}_{S'Y}(1) \subseteq Y$ (by i)) one has :

$$T_s S = T_y Y_1 = T_y \rho^{-1}_{S'Y}(1) = \ker \rho_{S'Y*y} \subseteq T_y Y$$

and since $\pi_{X|S} = \pi_{S'Y|Y_1}$ we also have

$$\ker \pi_{X|S*s} = \ker \pi_{S'Y|Y_1*y} = \ker \pi_{S'Y*y} \cap T_y Y_1 = \ker \pi_{S'Y*y} \cap \ker \rho_{S'Y*y}$$

so that, using again $T_s S = \ker \rho_{S'Y*y}$, one concludes :

$$\mathcal{D}(s) = \perp (\ker \pi_{X|S*s}, T_s S) = \perp (\ker \pi_{S'Y*y} \cap \ker \rho_{S'Y*y}\,;\ \ker \rho_{S'Y*y}) = \mathcal{D}_{S'Y}(y)\,.$$

Proof 3). First remark that, for every point $y = F(t, s)$ and vector $v \in \mathcal{D}(s)$, one has :

$$F_{t*s}(v) \in F_{t*s}(T_sS) = T_{F(t,s)}F_t(S) = T_yY_t = T_y\rho^{-1}_{S'Y}(t) = \ker\rho_{S'Y*y}$$

By $\ker\pi_{*s} \supseteq \ker\pi_{S'Y*y} \supseteq \ker\pi_{S'Y*y} \cap \ker\rho_{S'Y*y}$ it follows :

$$\pi(v) \in \mathbb{R}^l \times 0 = (\ker\pi_{*s})^{\perp} \subseteq (\ker\pi_{S'Y*y})^{\perp}$$

and since $v - \pi(v) \in \ker\pi_{S'Y*y} = (\ker\rho_{S'Y*y})^{\perp}$ we find :

$$F_{t*s}(v) = \pi(v) - t(v - \pi(v)) \in (\ker\pi_{S'Y*y})^{\perp} + (\ker\rho_{S'Y*y})^{\perp} = (\ker\pi_{X*y} \cap \ker\rho_{S'Y*y})^{\perp}$$

and finally thanks to $F_{t*s}(v) \in \ker\rho_{S'Y*y}$ we deduce that $F_{t*s}(v)$ also lies in :

$$[\ker\pi_{X*y} \cap \ker\rho_{S'Y*y}]^{\perp} \cap \ker\rho_{S'Y*y} \subseteq \perp(\ker\pi_{S'Y*y} \cap \ker\rho_{S'Y*y}, \ker\rho_{S'Y*y}) = \mathcal{D}_{S'Y}(y)\,.$$

In conclusion $F_{t*s}(\mathcal{D}(s)) \subseteq \mathcal{D}_{S'Y}(y)$ and having the same dimension (by 2)) they coincide. $\square$

Proposition 3.5 proves the (b)-regularity of each pair of adjacent strata of the cone $C_{R'\cup S'}(R\cup S)$ except for $R' < C^{o}_{S'}(S)$.

Therefore, to have finally the global (b)-regularity of a cone $C_{\mathcal{W}'}(\mathcal{W})$ one needs to obtain the (b)-regularity of the pair $R' < C^{o}_{S'}(S)$ for each stratum $R' = \pi_X(R)$ and $R < S$.

This property will be described in terms of condition (D) in Theorem below.

Theorem 3. 3. *Let $\mathcal{W}$ be a Whitney stratification in $S^1_X = \mathbb{R}^l \times S^{k-1}$ stratifying the canonical projection $\pi_X : S^1_X = \mathbb{R}^l \times S^{k-1} \to X = \mathbb{R}^l \times 0^k$ and let $\mathcal{W}' = \pi_X(\mathcal{W})$.*

Let $R < S$ be two strata of $\mathcal{W}$ and $r \in R$, $S' = \pi_X(S)$, $R' = \pi_X(R)$ and $s' = \pi_X(s)$, $\forall s \in S$.

The following conditions are equivalent :

1) $\pi_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}'$ satisfies the condition (D) at $r \in R < S$;

2) $\pi_{X*r}(\lim_i \mathcal{D}(s_i)) \supseteq \lim_i \pi_{X*s_i}(\mathcal{D}(s_i))$ for every sequence $\{s_i\}_i \subseteq S : \lim_i s_i = r \in R < S$.

3) The cone $C_{R'\cup S'}(R \cup S)$ has the strata $S' < Y = C^{o}_{S'}(S)$ such that the canonical distribution $\mathcal{D}_{S'Y}(y)$ satisfies : for every sequence $\{y_i = F(s_i, t_i)\}_i \subseteq Y$ such that $\lim_i y_i = r' \in R'$ $\lim_i \mathcal{D}_{S'Y}(y_i) \supseteq \lim_i \pi_{S'Y*y_i}(\mathcal{D}_{S'Y}(y_i))$.

Proof. Let $\{s_i\} \subseteq S$ be a sequence such that $\lim_i s_i = r \in R$ and both limits $\lim_i T_{s_i}S = \tau$ and $\lim_i \pi_{*s_i}(T_{s_i}S) = \tau'$ exist in the appropriate Grassmann manifolds.

Since $\mathcal{W}$ stratifies $\pi_X : \mathcal{W} \to \mathcal{W}'$ then the restriction $\pi_S : S \to \pi_X(S) = S'$ is a C^1 submersion and in particular $T_{s'_i}S' = \pi_{*s_i}(T_{s_i}S)$.

$(1 \Leftrightarrow 2)$. It is $(1 \Leftrightarrow 4)$ of Theorem 4.1 **[14]** for the stratified submersion $\pi_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}'$.

$(2 \Leftrightarrow 3)$. Statement 2) above is obviously intended for every sequence $\{s_i\} \subseteq S$ such that both limits $\lim_i \mathcal{D}(s_i) = \mathcal{D}$ and $\lim_i \pi_{*s_i}(\mathcal{D}(s_i)) = \mathcal{D}'$ exist in the appropriate Grassmann manifold and similarly for the limits in the statement 3).

By Lemma 3.1 $\mathcal{D}_{S'Y}(y_i) = F_{t_i*s_i}(\mathcal{D}(s_i))$ and because the homotopy $F : id \sim \pi$ is a C^1 map such that $F_0 = \pi_X$, if $(r, 0) = \lim_i(s_i, t_i)$ we have :

$$\lim_i \mathcal{D}_{S'Y}(y_i) = \lim_i F_{t_i*s_i}(\mathcal{D}(s_i)) = F_{0*r}(\lim_i \mathcal{D}(s_i)) = \pi_{X*r}(\lim_i \mathcal{D}(s_i))\,.$$

By the submersivity of $\pi_{X|S} : S \to S'$ and of $\pi_{S'Y} : Y \to S'$ (**[11]**), for every i we have both: $\pi_{X*s_i}(\mathcal{D}(s_i)) = T_{s'_i}S' = \pi_{S'Y*y_i}(\mathcal{D}_{S'Y}(y_i))$ and in conclusion :

$$\pi_{X*r}(\lim_i \mathcal{D}(s_i)) \supseteq \lim_i \pi_{X*s_i}(\mathcal{D}(s_i)) \iff \lim_i \mathcal{D}_{S'Y}(y_i) \supseteq \lim_i \pi_{S'Y*y_i}(\mathcal{D}_{S'Y}(y_i)) . \qquad \square$$

Corollary 3. 1. *Let $\mathcal{W}$ be a Whitney stratification in $S^1_X = \mathbb{R}^l \times S^{k-1}$ stratifying the canonical projection $\pi_X : S^1_X = \mathbb{R}^l \times S^{k-1} \to X = \mathbb{R}^l \times 0^k$ and let $\mathcal{W}\,' = \pi_X(\mathcal{W})$.*

Let $R < S$ be two strata of $\mathcal{W}$ and $r \in R$, $S' = \pi_X(S)$, $R' = \pi_X(R)$ and $s' = \pi_X(s)$, $\forall s \in S$.

If the stratified submersion $\pi_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}\,'$ satisfies condition (D) at $r \in R < S$ then :

1) The cone $C_{R'\cup S'}(R \cup S)$ has strata $Y = C^o_{S'}(S) > S'$ such that for every sequence of points $\{y_i = F(s_i, t_i)\} \subseteq Y$ such that $\lim_i y_i = r' \in R'$ one has $\lim_i \mathcal{D}_{S'Y}(y_i) \supseteq T_{r'}R'$.

2) The cone $C_{R'\cup S'}(R \cup S)$ has the strata $Y = C^o_{S'}(S) > S'$ such that for every sequence of points $\{y_i = F(s_i, t_i)\} \subseteq Y$ such that $\lim_i y_i = r' \in R'$ one has $\lim_i T_{y_i}Y \supseteq \lim_i T_{s'_i}S'$.

3) The cone $C_{R'\cup S'}(R \cup S)$ has the pair of strata $Y = C^o_{S'}(S) > R'$ which is (a)-regular.

Proof 1). By hypothesis the stratified submersion $\pi_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}\,'$ satisfies the condition (D) at $r \in R < S$ so by Theorem 3.3 :

$$\lim_i \mathcal{D}_{S'Y}(y_i) \quad \supseteq \quad \lim_i \pi_{S'Y*y_i}\mathcal{D}_{S'Y}(y_i) \quad = \quad \lim_i T_{s'_i}S'$$

and moreover $R' < S'$ being (a)-regular by hypothesis on $\mathcal{W}\,'$ one also has

$$\lim_i T_{s'_i}S' \ \supseteq \ T_{r'}R' \qquad \text{and so} \qquad \lim_i \mathcal{D}_{S'Y}(y_i) \ \supseteq \ T_{r'}R' \ .$$

Proof 2). From the proof of 1) one has : $\lim_i T_{y_i}Y \supseteq \lim_i \mathcal{D}_{S'Y}(y_i) \supseteq \lim_i T_{s'_i}S'$.

Proof 3). Thanks to 2) and 1), $\forall \, \{y_i = F(s_i, t_i)\} \subseteq Y$ such that $\lim_i y_i = r' \in R'$ one has :

$$\lim_{y_i \to r'} T_{y_i}Y \ \supseteq \ \lim_i \mathcal{D}_{S'Y}(y_i) \ \supseteq \ \lim_i T_{s'_i}S' \ \supseteq \ T_{r'}R' \ . \qquad \square$$

Proposition 3. 6. *Let $\mathcal{W}$ be a Whitney stratification in $S^1_X = \mathbb{R}^l \times S^{k-1}$ stratifying the canonical projection $\pi_X : S^1_X = \mathbb{R}^l \times S^{k-1} \to X = \mathbb{R}^l \times 0^k$ and let $\mathcal{W}\,' = \pi_X(\mathcal{W})$.*

Let $R < S$ be two strata of $\mathcal{W}$ and $r \in R$, $S' = \pi_X(S)$, $R' = \pi_X(R)$ and $s' = \pi_X(s)$, $\forall s \in S$.

If the stratified submersion $\pi_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}\,'$ satisfies the condition (D) at $r \in R < S$ then the following conditions are equivalent :

1) The cone $C_{R'\cup S'}(R \cup S)$ is (a)-regular at $r' \in R' < C^o_{S'}(S)$.

2) The cone $C_{R'\cup S'}(R \cup S)$ is (b)-regular at $r' \in R' < C^o_{S'}(S)$.

Proof. 1) $\Rightarrow$ 2). As in Proposition 3.5 we use that condition (b) holds if and only if the conditions (a) and $(b^{\pi_{R'}})$ hold for some C^1-retraction $\pi_{R'}$ defined on an open neighbourhood of R'.

The proof reduces then to proving that $(b^{\pi_{R'}})$ holds with respect to the pair $R' < Y = C^o_{S'}(S)$. As in Proposition 3.5 if $y = ts + (1-t)s' \in Y$, $\pi_{S'}(y) = s'$ since $C^o_{S'}(S)$ is a cone, then :

$$\overline{y\,\pi_{S'}(y)} = [s - s'] \, .$$

Let us fix a sequence $\{y_i = t_i s_i + (1-t)s'_i\}_i \subseteq Y$ converging to a point $r' \in R' < S'$ such that both limits exist in the appropriate Grassmann manifolds :

$$\tau = \lim_i T_{y_i} C^o_{S'}(S) \quad \text{ and } \quad L = \lim_i \overline{y_i\,\pi_{R'}(y_i)} = \lim_i \, [y_i - \pi_{R'}(y_i)] \, .$$

Splitting every vector $y_i - \pi_{R'}(y_i)$ in the following orthogonal sum :

$$y_i - \pi_{R'}(y_i) \ = \ (y_i - s'_i) \ + \ (s'_i - \pi_{R'}(y_i))$$

every 1-dimensional vector space $\overline{y_i\, \pi_{R'}(y_i)} = [y_i - \pi_{R'}(y_i)]$ is contained in the 2-dimensional vector space spanned by the two orthogonal 1-dimensional vector space as follows :

$$\overline{y_i\, \pi_{R'}(y_i)} \ = \ [y_i - \pi_{R'}(y_i)] \ \subseteq \ [y_i - s'_i] \ + \ [s'_i - \pi_{R'}(y_i)]\,.$$

Obviously $\lim_i y_i = r'$ if and only if $\lim_i t_i = 0$, $\lim_i s_i = r$ and so $\lim_i s'_i = r'$. Hence :

$$\lim_i\, [y_i - s'_i] \ = \ [r - r']\,.$$

By hypothesis, $R' < S'$ being (b)-regular the condition $(b^{\pi_{R'}})$ holds with respect to $R' < S'$, up to taking a subsequence if necessary, such that $\lim_i [s'_i - \pi_{R'}(y_i)]$ exists in $\mathbb{G}^1_N$, we have :

$$\lim_i\, [s'_i - \pi_{R'}(y'_i)] \ \subseteq \ \lim_i T_{s'_i} S'\,.$$

Every $[y_i - s'_i] \perp \ [s'_i - \pi_{R'}(y_i)]$ being orthogonal, then

$$\lim_i \ \big([y_i - s'_i] + [s'_i - \pi_{R'}(y_i)]\big) = \lim_i\, [y_i - s'_i] \ + \ \lim_i \ [s'_i - \pi_{R'}(y_i)]$$

and by Theorem 3.3, since the stratified submersion $\pi_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}\,'$ satisfies condition (D) at $r \in R < S$ then $\lim_i T_{s'_i} S' \subseteq \lim_i \mathcal{D}_{S'Y}(y_i)$. Therefore one finds :

$$\lim_i \overline{y_i\, \pi_{R'}(y_i)} \ \subseteq \ \lim_i \Big([y_i - s'_i] \ + \ [s'_i - \pi_{R'}(y_i)] \Big) =$$

$$= \ \lim_i\, [y_i - s'_i] \ + \ \lim_i\, [s'_i - \pi_{R'}(y_i)] \ \subseteq \ \lim_i\,, [y_i - s'_i] + \lim_i \ \mathcal{D}_{S'Y}(y_i) =$$

and finally, again since $[y_{ii} - s'_i] \perp \mathcal{D}_{S'Y}(y_i)$ are orthogonal for every i one concludes :

$$= \ \lim_i \ \Big([y_i - s'_i] \ + \ \ \mathcal{D}_{S'Y}(y_i) \Big) \ \subseteq \ \lim_i T_{y_i} Y\,.$$

That is $R' < Y = C^o_{S'}(S)$ satisfies the condition $(b^{\pi_{R'}})$ at $r' \in R'$.

Proof. 2) $\Rightarrow$ 1). The (b)-regularity always implies the (a)-regularity **[19, 3]**. $\square$

We find then the following equivalent version of Goresky's result Proposition 2.1 :

Theorem 3. 4. *Let $\mathcal{W}$ be a Whitney stratification in $S^1_X = \mathbb{R}^l \times S^{k-1}$ which stratifies the canonical projection $\pi_X : S^1_X = \mathbb{R}^l \times S^{k-1} \to X = \mathbb{R}^l \times 0^k$ and let $\mathcal{W}\,' = \pi_X(\mathcal{W}\,)$.*

If $\pi_{\mathcal{W}} : \mathcal{W} \subseteq S^1_X \to \mathcal{W}\,' = \pi_X(\mathcal{W}\,) \subseteq X$ satisfies the condition (D), then :

1) *The closed cone $C_{\mathcal{W}\,'}(\mathcal{W}\,) = \{tp + (1-t)\pi(p) \mid p \in \mathcal{W}\ ,\ t \in [0,1]\ \}$ is (a)-regular.*

2) *The closed cone $C_{\mathcal{W}\,'}(\mathcal{W}\,) = \{tp + (1-t)\pi(p) \mid p \in \mathcal{W}\ ,\ t \in [0,1]\ \}$ is (b)-regular.*

Proof. Every incidence relation in $C_{\mathcal{W}\,'}(\mathcal{W}\,)$ comes from some strata $R < S$ of $\mathcal{W}$ in a cone $C_{R' \cup S'}(R \cup S) \subseteq C_{\mathcal{W}\,'}(\mathcal{W}\,)$ as treated in Proposition 3.5, Corollary 3.1 and Proposition 3.6.

By Proposition 3.5, all incidence relations on $C_{R' \cup S'}(R \cup S)$ are (a)- and (b)-regular except possibly for the pairs $R' < C^o_{S'}(S)$.

Since by hypothesis $\pi_{\mathcal{W}} : \mathcal{W} \subseteq S^1_X \to \mathcal{W}' = \pi_X(\mathcal{W}) \subseteq X$ satisfies the condition (D), every pair $R' < C^o_{S'}(S)$ is (a)-regular by Corollary 3.1 and so also (b)-regular by Propostion 3.6. $\square$

We also find, when $\mathcal{W}$ and $\mathcal{W}'$ are Whitney triangulations (or cellularisations), the following important corollary which is helpful as an approach to Conjectures 1.1 and 1.2. :

Corollary 3. 2. *If $\mathcal{W}$ and $\mathcal{W}'$ are Whitney triangulations (resp. cellularisations) of compact sets $W \subseteq S_X(1)$ and $W' \subseteq X$ such that $\pi_{X|\mathcal{W}} : \mathcal{W} \to \mathcal{W}'$ is a simplicial (resp. cellular) map, then the stratified closed cone $C_{\mathcal{W}'}(\mathcal{W})$ is a Whitney cellularisation of $C_{W'}(W)$.*

Proof. Since $\pi_{X|\mathcal{W}} : \mathcal{W} \to \mathcal{W}'$ is a simplicial (resp. cellular) map, thanks to Example 2.4 it satisfies Condition (D) and so the closed cone $C_{\mathcal{W}'}(\mathcal{W})$ is (b)-regular thanks to Theorem 3.4. $\square$

Condition (D) for $\pi_{|\mathcal{W}} : \mathcal{W} \to \mathcal{W}'$ is however sufficient for (b) regularity but not necessary :

Example 3. 5. Let us consider a quarter of the Whitney umbrella :

$$C_{\mathcal{W}'}(\mathcal{W}) = \{(x, y, z) \in \mathbb{R}^3 \mid \quad yz^2 = x^2 \ , \ \ x \geq 0 \ , \ z \geq 0\}$$

where $\mathcal{W} = R \sqcup S$ and $\mathcal{W}' = R' \sqcup S'$ are stratified by :

$R = \{(0, 0, 1)\} < S =$ half parabola $\subseteq S_X(1)$;
$R' = \{(0, 0, 0)\} < S' = \{0\} \times [0, +\infty[\times \{0\} \subseteq X = \{0\} \times \mathbb{R} \times \{0\}$.

Then as in Example 2.1, $\pi_{|\mathcal{W}} : \mathcal{W} \to \mathcal{W}'$ does not satisfy condition (D), but $R' = \{0\}$ is a point, so $R' < Y = C^o_{S'}(S)$ is automatically (a)–regular and easily also (b)-regular. $\square$

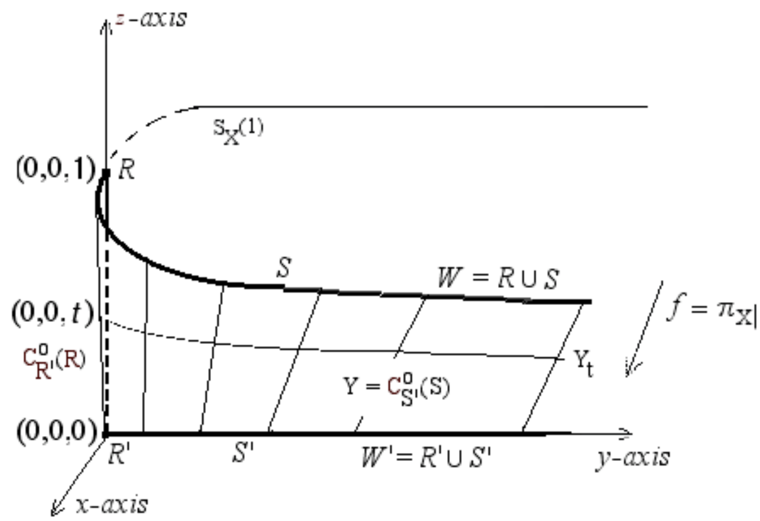

Figure 4

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