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FRONTS OF CONTROL-AFFINE SYSTEMS IN $\mathbb{R}^3$

ILYA BOGAEVSKY

To Goo Ishikawa on the occasion of his sixtieth birthday

Abstract. We consider a control-affine system in three-dimensional space with control parameters belonging to a two-dimensional disk and study its fronts evolving from a point for small times. We prove that generically the Legendrian lifts of such fronts have standard singularities and there are only two principally different typical cases — hyperbolic and elliptic.

Introduction

The ends of local time-optimal trajectories of a control system that start at a given point form its front depending on time. We consider control-affine systems in three-dimensional space with control parameters belonging to a two-dimensional disk and study singularities of their fronts for small times.

If our system is linear-control then it defines a sub-Riemannian structure and its fronts are described in [1] in the case that the sub-Riemannian structure is contact. For such a typical system the fronts have infinite number of swallowtails at any neighborhood of the initial point. Therefore their structure is complicated but it becomes much more simpler from the viewpoint of contact geometry. Namely, let us consider the Legendrian surface consisting of all contact elements being tangent to a considered front and cooriented outside. According to our result this submanifold is smooth except two points lying over the initial point. Moreover, these singularities are standard for all contact sub-Riemannian structures — not only for typical ones. It means that all of them have the same normal form with respect to contact diffeomorphisms of the ambient space.

A considered control-affine system can have hyperbolic and elliptic points introduced in [6]. The sets formed by them are open always and its union is dense for a typical system. In particular, a linear-control system cannot have hyperbolic points at all and is elliptic exactly at the points where the corresponding sub-Riemannian structure is contact.

According to the present paper the Legendrian surface consisting of all contact elements being tangent to a front and cooriented outside is homeomorphic to the two-dimensional sphere and has the following singularities.

If the initial point is elliptic then the considered Legendrian surface is smooth outside two points where it has singularities E_2. If the initial point is hyperbolic then the considered Legendrian surface is smooth outside two disjoint segments, where it has singularities H_1 at their inner points and H_2 at their four ends. All singularities with the same name (E_2, H_1, or H_2) are equivalent to each other with respect to contact diffeomorphisms of the ambient space. In particular, their normal forms do not contain continuous invariants.

Non-typical examples of instant fronts of elliptic (left) and hyperbolic (right) points are shown in Fig. 1. (These figures are published in [7] and [6] respectively.)

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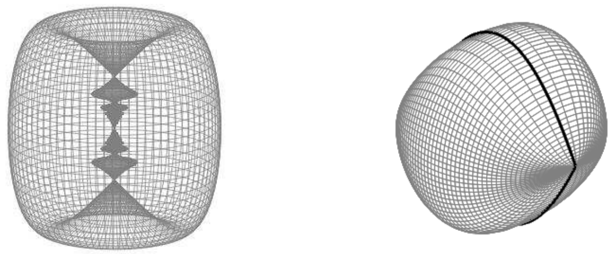

FIGURE 1. Non-typical examples of instant fronts of elliptic (left) and hyperbolic (right) points

1. Definitions

1.1. **Instant fronts of control-affine systems in $\mathbb{R}^3$.** We consider a control-affine system in $\mathbb{R}^3$ with control parameters $u = (u_1, u_2)$:

$$\dot{\mathbf{x}} = \xi_0(\mathbf{x}) + u_1\xi_1(\mathbf{x}) + u_2\xi_2(\mathbf{x}), \quad u_1^2 + u_2^2 \leq 1 \tag{1}$$

as a family of vector fields in $\mathbb{R}^3$ depending on u. Here $\mathbf{x} \in \mathbb{R}^3$, $(\mathbf{x}, \dot{\mathbf{x}}) \in T^*\mathbb{R}^3$, and ξ_0, ξ_1, ξ_2 are bounded smooth[1] vector fields on $\mathbb{R}^3$ such that the vectors $\xi_1(\mathbf{x})$ and $\xi_2(\mathbf{x})$ are linearly independent at any point $\mathbf{x} \in \mathbb{R}^3$.

DEFINITION. A Lipschitzian mapping $\varphi : [0, T] \to \mathbb{R}^3$, $T > 0$ is called a *trajectory* of the control-affine system (1) if there exist measurable functions $\tilde{u}_1, \tilde{u}_2 : [0, T] \to \mathbb{R}$ such that the equations

$$\frac{d\varphi}{dt} = \xi_0(\varphi(t)) + \tilde{u}_1(t)\ \xi_1(\varphi(t)) + \tilde{u}_2(t)\ \xi_2(\varphi(t)), \quad \tilde{u}_1^2(t) + \tilde{u}_2^2(t) \leq 1$$

hold for almost all $t \in [0, T]$.

DEFINITION. The ends $\varphi(T)$ of all trajectories $\varphi : [0, T] \to \mathbb{R}^3$ of the system (1) starting at a given point $\varphi(0) = \mathbf{x}_0$ form the *attainable set* of the point $\mathbf{x}_0 \in \mathbb{R}^3$ for the time T:

$$\mathcal{A}_{\mathbf{x}_0}(T) = \left\{\mathbf{x} \in \mathbb{R}^3 \mid \exists\, \varphi \text{ s. t. } \varphi(0) = \mathbf{x}_0,\ \varphi(T) = \mathbf{x}\right\}.$$

Its boundary is denoted by $\partial\mathcal{A}_{\mathbf{x}_0}(T)$.

DEFINITION. If a trajectory $\varphi : [0, T] \to \mathbb{R}^3$ of the system (1) satisfies the condition

$$\varphi\,(T) \in \partial\mathcal{A}_{\varphi(0)}(T)$$

then it is called *geometrically optimal*.

REMARK. According to Filippov's theorem (Theorem 10.1 in [2]) the attainable set $\mathcal{A}_{\mathbf{x}_0}(T)$ is compact. Therefore its boundary $\partial\mathcal{A}_{\mathbf{x}_0}(T) \subseteq \mathcal{A}_{\mathbf{x}_0}(T)$ consists of the ends $\varphi(T)$ of all geometrically optimal trajectories $\varphi : [0, T] \to \mathbb{R}^3$ starting at the point $\varphi(0) = \mathbf{x}_0$.

[1] "Smooth" means "infinitely smooth" everywhere.

Definition. A trajectory $\varphi : [0,T] \to \mathbb{R}^3$ of the system (1) is called *locally geometrically optimal* if there exists $\delta > 0$ such that

$$\varphi(t) \in \partial\mathcal{A}_{\varphi(t_0)}(t - t_0) \quad \forall\, t_0, t \in [0,T] : t_0 < t < t_0 + \delta.$$

Remark. It is well known that any geometrically optimal trajectory $\varphi : [0,T] \to \mathbb{R}^3$ of the system (1) satisfies the condition

$$\varphi\,(t) \in \partial\mathcal{A}_{\varphi(t_0)}(t - t_0) \quad \forall\, t_0, t \in [0,T] : t_0 < t.$$

In particular, $\varphi : [0,T] \to \mathbb{R}^3$ is locally geometrically optimal.

Definition. The closure of the set formed by the ends $\varphi(T)$ of all locally geometrically optimal trajectories $\varphi : [0,T] \to \mathbb{R}^3$ starting at a given point $\varphi(0) = \mathbf{x}_0$ is called its *instant front* $\mathcal{F}_{\mathbf{x}_0}(T)$ for the time T.

Remark. By definition, $\mathcal{F}_{\mathbf{x}_0}(T) \supseteq \partial\mathcal{A}_{\mathbf{x}_0}(T)$.

1.2. **Relativistic viewpoint: hyperbolic and elliptic points.** Let us consider the space-time $\mathbb{R}^{3+1}$ and fix a point $m = (\mathbf{x}, 0) \in \mathbb{R}^{3+1}$. The control-affine system (1) defines a hyperplane

$$\Pi(m) = \langle \Xi_0(m), \Xi_1(m), \Xi_2(m) \rangle_{\mathbb{R}} \subset T_m\mathbb{R}^{3+1}$$

where

$$\Xi_0 = (\xi_0, 1), \quad \Xi_1 = (\xi_1, 0), \quad \Xi_2 = (\xi_2, 0)$$

are vector fields on $\mathbb{R}^{3+1}$. This hyperplane contains the cone

$$C(m) = \left\{ v_0\, \Xi_0(m) + v_1\, \Xi_1(m) + v_2\, \Xi_2(m) \mid v_0^2 - v_1^2 - v_2^2 = 0 \right\} \subset \Pi(m)$$

formed by all directions belonging to the control-affine system (1) such that $u_1^2 + u_2^2 = 1$.

Let Π be locally defined as the field of 0-spaces of some non-zero 1-form θ on $\mathbb{R}^{3+1}$. The restriction $d\theta|_{\Pi(m)}$ is an antisymmetric 2-form in the three-dimensional vector space $\Pi(m)$. Its kernel

$$k(m) = \ker d\theta|_{\Pi(m)} \subset \Pi(m)$$

has dimension 1 or 3 and is defined by the field Π, i.e. does not depend on the choice of a non-zero 1-form θ.

Definition. Let $m = (\mathbf{x}, 0)$ and the kernel $k(m)$ be one-dimensional. If the kernel $k(m)$ lies in the inner part of the complement of the cone $C(m)$, then the point $\mathbf{x}$ is called *elliptic*. If the kernel $k(m)$ lies in the outer part of the complement of the cone $C(m)$, then the point $\mathbf{x}$ is called *hyperbolic*. If the kernel $k(m)$ belongs to the cone $C(m)$ itself, then the point $\mathbf{x}$ is called *parabolic*. All these cases are shown in Fig. 2.

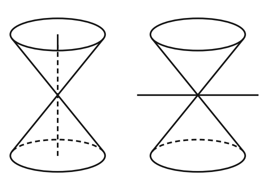

Figure 2. Elliptic, hyperbolic, and parabolic points

Remark. In the present paper parabolic points are not studied.

Example H. All points of the control-affine system

$$\dot{x} = u_1, \quad \dot{y} = u_2, \quad \dot{z} = y, \quad u_1^2 + u_2^2 \le 1$$

are hyperbolic. Here

- $\xi_0 = (0, 0, y)$, $\xi_1 = (1, 0, 0)$, $\xi_2 = (0, 1, 0)$;
- $\Pi = \{v_0(0, 0, y, 1) + v_1(1, 0, 0, 0) + v_2(0, 1, 0, 0)\}$;
- $\theta = y\,dt - dz$, $d\theta = dy \wedge dt$, $d\theta|_\Pi = dv_2 \wedge dv_0$;
- $k = \{v_0 = v_2 = 0\} \subset \Pi$;
- $C = \left\{v_0^2 - v_1^2 - v_2^2 = 0\right\}$.

The instant fronts of these control-affine system are diffeomorphic to the shown in Fig. 1 on the right.

Example E. All points of the control-affine system

$$\dot{x} = u_1, \quad \dot{y} = u_2, \quad \dot{z} = u_1 y, \quad u_1^2 + u_2^2 \leq 1$$

are elliptic. Here

- $\xi_0 = 0$, $\xi_1 = (1, 0, y)$, $\xi_2 = (0, 1, 0)$;
- $\Pi = \{v_0(0, 0, 0, 1) + v_1(1, 0, y, 0) + v_2(0, 1, 0, 0)\}$;
- $\theta = y\,dx - dz$, $d\theta = dy \wedge dx$, $d\theta|_\Pi = dv_2 \wedge dv_1$;
- $k = \{v_1 = v_2 = 0\} \subset \Pi$;
- $C = \left\{v_0^2 - v_1^2 - v_2^2 = 0\right\}$.

The instant fronts of these control-affine system are diffeomorphic to the shown in Fig. 1 on the left.

1.3. Stratified Legendrian submanifolds.

Definition. A stratified submanifold of a contact space is called *Legendrian* if it is the closure of the smooth Legendrian submanifold being the union of its strata of maximal dimension.

Let $\mathbb{R}^5$ be a contact space with coordinates (P_1, P_2, Q_1, Q_2, U), the origin

$$O = \{P_1 = P_2 = Q_1 = Q_2 = U = 0\},$$

and the contact structure defined as the field of 0-spaces of the contact form

$$\Theta = \frac{1}{2} P\,dQ - \frac{1}{2} Q\,dP - dU.$$

The following stratified submanifolds are Legendrian:

- $\mathcal{H}_1 = \left\{2P_1 \ln P_1^2 + Q_1 = Q_2 = U + P_1^2 = 0\right\}$ where $P_1 \ln P_1^2 = 0$ if $P_1 = 0$;
- $\mathcal{H}_2 = \left\{P_1 = A^2,\ P_2 = AB,\ Q_1 = B^2,\ Q_2 = 2AB \ln A^2,\ U = A^2B^2/2\right\}$ where $A, B \in \mathbb{R}$ are parameters and $A \ln A^2 = 0$ if $A = 0$;
- $\mathcal{E}_2 = \left\{P_1 + iQ_1 = Ue^{i\left(\psi - \frac{1}{U}\right)}, Q_2 + iP_2 = Ue^{i\left(\psi + \frac{1}{U}\right)}, U \geq 0\right\}$ where $i = \sqrt{-1}$, $\psi \in \mathbb{R} \bmod 2\pi\mathbb{Z}$ is a parameter, and $Ue^{i\left(\psi \pm \frac{1}{U}\right)} = 0$ if $U = 0$.

The submanifold $\mathcal{H}_1$ consists of three connected smooth strata: the two surfaces distinguished by the inequalities $P_1 \gtrless 0$ and the line $\mathcal{H}_1^1 = \{P_1 = Q_1 = Q_2 = U = 0\}$.

The submanifold $\mathcal{H}_2$ appears in [4] (Chapter 8) and consists of three connected smooth strata: the surface distinguished by the conditions $A \neq 0$, the open ray

$$\mathcal{H}_2^1 = \{P_1 = P_2 = Q_2 = U = 0, Q_1 > 0\}$$

distinguished by the conditions $A = 0$, $B \neq 0$, and the origin O distinguished by the conditions $A = B = 0$.

The submanifold $\mathcal{E}_2$ consists of two connected smooth strata: the cylinder distinguished by the conditions $U > 0$ and the origin O distinguished by the conditions $U = 0$.

Definition. We say that a two-dimensional stratified Legendrian submanifold Λ of a contact space has a singularity H_1, H_2, or E_2 at a point $\lambda \in \Lambda$ if its germ (Λ, λ) is contact diffeomorphic to the germ $(\mathcal{H}_1, O)$, $(\mathcal{H}_2, O)$, or $(\mathcal{E}_2, O)$ respectively.

For instance, it is clear that the stratified Legendrian submanifold $\mathcal{H}_1$ has a singularity H_1 not only at the origin O but at any point of its stratum $\mathcal{H}_1^1$ as well. Besides, the stratified Legendrian submanifold $\mathcal{H}_2$ has singularities H_1 at all points of its stratum $\mathcal{H}_2^1$ — it is shown in [5] .

2. Main result

Let $ST^*\mathbb{R}^n$ be the space of cooriented contact elements in $\mathbb{R}^n$ with the standard contact structure and $\pi : ST^*\mathbb{R}^n \to \mathbb{R}^n$ be the natural projection. (A cooriented contact element in $\mathbb{R}^n$ is a pair $([\mathbf{p}]; \mathbf{x})$ consisting of a point $\mathbf{x} \in \mathbb{R}^n$ and a ray $[\mathbf{p}] = \{\kappa\mathbf{p} \mid \kappa > 0\}$ generated by a non-zero covector $\mathbf{p} \in T^*_{\mathbf{x}}\mathbb{R}^n \cong \mathbb{R}^{n*}$).

Definition. The image $\pi(\Lambda)$ is called the *front* of a stratified Legendrian submanifold Λ.

Theorem 1. *Let $\mathbf{x}_0$ be any hyperbolic or elliptic point of the control-affine system (1). Then there exists $\delta > 0$ such that for any $T \in (0, \delta)$ the instant front $\mathcal{F}_{\mathbf{x}_0}(T)$ is the front of some stratified Legendrian submanifold of $ST^*\mathbb{R}^3$ denoted by $\mathcal{L}_{\mathbf{x}_0}(T)$ and satisfying the following conditions:*

- *$\mathcal{L}_{\mathbf{x}_0}(T)$ is homeomorphic to the two-dimensional sphere;*
- *in the hyperbolic case $\mathcal{L}_{\mathbf{x}_0}(T)$ is smooth outside two disjoint segments and has singularities H_1 at inner their points and H_2 at their four ends;*
- *in the elliptic case $\mathcal{L}_{\mathbf{x}_0}(T)$ is smooth outside two points where it has singularities E_2.*

Remark. Theorem 1 claims the existence of stratified Legendrian submanifolds $\mathcal{L}_{\mathbf{x}_0}(T)$ satisfying the indicated conditions. The submanifolds $\mathcal{L}_{\mathbf{x}_0}(T)$ themselves are explicitly constructed in Subsection 3.1.

3. Proofs

3.1. **Construction of $\mathcal{L}_{\mathbf{x}_0}(T)$.** Let $ST^*\mathbb{R}^{3+1}$ be the space of cooriented contact elements $([\mathbf{p}, s]; \mathbf{x}, t)$ in the space-time $\mathbb{R}^{3+1}$ with the standard contact structure and $\pi : ST^*\mathbb{R}^{3+1} \to \mathbb{R}^{3+1}$ be the natural projection where $[\mathbf{p}, s] = \{\kappa(\mathbf{p}, s) \mid \kappa > 0\}$ is the open ray generated by a non-zero covector $(\mathbf{p}, s) \in T^*_{\mathbf{x},t}\mathbb{R}^{3+1} \cong \mathbb{R}^{3+1*}$.

Following Section 12.1 in [2] let us construct the Hamiltonian

$$h(\mathbf{p}; \mathbf{x}) = \max_{u_1^2+u_2^2 \le 1} \langle \mathbf{p}, \xi_0(\mathbf{x}) + u_1\xi_1(\mathbf{x}) + u_2\xi_2(\mathbf{x})\rangle$$

$$= \langle \mathbf{p}, \xi_0(\mathbf{x})\rangle + \sqrt{\langle \mathbf{p}, \xi_1(\mathbf{x})\rangle^2 + \langle \mathbf{p}, \xi_2(\mathbf{x})\rangle^2}$$

associated with the control-affine system (1). The Hamiltonian h defines the singular hypersurface

$$\Sigma = \left\{([\mathbf{p}, s]; \mathbf{x}, t) \in ST^*\mathbb{R}^{3+1} \mid h(\mathbf{p}; \mathbf{x}) + s = 0\right\}$$

$$= \left\{\left(\langle \mathbf{p}, \xi_0(\mathbf{x})\rangle + s\right)^2 = \langle \mathbf{p}, \xi_1(\mathbf{x})\rangle^2 + \langle \mathbf{p}, \xi_2(\mathbf{x})\rangle^2,\ \langle \mathbf{p}, \xi_0(\mathbf{x})\rangle + s \le 0\right\},$$

its singularities form the smooth 4-dimensional submanifold:

$$\Sigma^4 = \left\{([\mathbf{p}, s]; \mathbf{x}, t) \in ST^*\mathbb{R}^{3+1} \mid \langle \mathbf{p}, \xi_0(\mathbf{x})\rangle + s = \langle \mathbf{p}, \xi_1(\mathbf{x})\rangle = \langle \mathbf{p}, \xi_2(\mathbf{x})\rangle = 0\right\}.$$

The smooth stratum $\Sigma \setminus \Sigma^4$ (as a hypersurface in a contact space) consists of its characteristics. Such a characteristic satisfies the equations

$$\frac{d\mathbf{p}}{dt} = -\partial_{\mathbf{x}} h(\mathbf{p};\mathbf{x}), \quad \frac{d\mathbf{x}}{dt} = \partial_{\mathbf{p}} h(\mathbf{p};\mathbf{x}), \quad h(\mathbf{p};\mathbf{x}) + s = 0$$

and its projection to the space-time is the graph of a locally geometrically optimal trajectory according to Proposition 12.1 and Section 17.1 in [2].

Definition. The *world* stratified Legendrian submanifold of a point $\mathbf{x}_0 \in \mathbb{R}^3$ is the closure of the union of all characteristics Γ of $\Sigma \setminus \Sigma^4$ passing through $\pi^{-1}(\mathbf{x}_0, 0)$:

$$\Lambda_{\mathbf{x}_0} = \bigcup_{\pi(\Gamma)\ni(\mathbf{x}_0,0)} \Gamma \subset ST^*\mathbb{R}^{3+1}.$$

Let $\tau : ST^*\mathbb{R}^{3+1} \to \mathbb{R}$ be the time function sending $([\mathbf{p}, s]; \mathbf{x}, t) \mapsto t$ and $\varrho : \Sigma \to ST^*\mathbb{R}^3$ be the projection sending $([\mathbf{p}, s]; \mathbf{x}, t) \mapsto ([\mathbf{p}]; \mathbf{x})$ which is correctly defined because Σ does not contain contact elements with $\mathbf{p} = 0$ and $s \neq 0$. The *instant* stratified Legendrian submanifold of the point $\mathbf{x}_0$ at a time T

$$\mathcal{L}_{\mathbf{x}_0}(T) = \varrho\left(\Lambda_{\mathbf{x}_0} \cap \tau^{-1}(T)\right) \subset ST^*\mathbb{R}^3$$

is the projection of the section of the world stratified Legendrian submanifold with the isochrone $\tau = T$.

3.2. **Arnold's singularities of Σ.** For any point $(\mathbf{x}_0, t_0) \in \mathbb{R}^{3+1}$ the fiber $\pi^{-1}(\mathbf{x}_0, t_0)$ contains exactly two singularities of Σ: the contact elements $([\mathbf{p}, s]; \mathbf{x}_0, t_0)$ distinguished by the conditions

$$\langle \mathbf{p}, \xi_0(\mathbf{x}_0)\rangle + s = \langle \mathbf{p}, \xi_1(\mathbf{x}_0)\rangle = \langle \mathbf{p}, \xi_2(\mathbf{x}_0)\rangle = 0.$$

In other words, they are exactly the hyperplane $\Pi(\mathbf{x}_0, t_0)$ introduced in Subsection 1.2 with two possible coorientations and denoted as $\Pi^+(\mathbf{x}_0, t_0)$ and $\Pi^-(\mathbf{x}_0, t_0)$.

Let $O = \Pi^+(\mathbf{x}_0, t_0)$ or $O = \Pi^-(\mathbf{x}_0, t_0)$. Then in a neighborhood of O there exist local coordinates $(P_1, P_2, P_3, Q_1, Q_2, Q_3, U)$ such that the contact structure is given as the field of 0-spaces of the contact form

$$\Theta = \frac{1}{2} P\, dQ - \frac{1}{2} Q\, dP - dU \tag{2}$$

and:

- $\Sigma = \left\{P_1 Q_1 - P_2^2 = 0,\ P_1 + Q_1 \geq 0\right\}$ if $\mathbf{x}_0$ is a hyperbolic point of the control-affine system (1);
- $\Sigma = \left\{P_1^2 + Q_1^2 - P_2^2 = 0,\ P_2 \geq 0\right\}$ if $\mathbf{x}_0$ is an elliptic point of the control-affine system (1).

This fact follows directly from [3] where the equations $P_1 Q_1 - P_2^2 = 0$ and $P_1^2 + Q_1^2 - P_2^2 = 0$ appear as normal forms of degeneracy hypersurfaces for symbols of systems of partial differential equations.

Example H. For the hyperbolic control-affine system

$$\dot{x} = u_1, \quad \dot{y} = u_2, \quad \dot{z} = y, \quad u_1^2 + u_2^2 \leq 1$$

from Example H of Subsection 1.2 we get

$$\langle \mathbf{p}, \xi_1(\mathbf{x})\rangle = p, \quad \langle \mathbf{p}, \xi_2(\mathbf{x})\rangle = q, \quad \langle \mathbf{p}, \xi_0(\mathbf{x})\rangle + s = ry + s.$$

Hence in the affine chart $r = -1$

$$\Sigma = \left\{p^2 + q^2 = (-y + s)^2,\ -y + s \leq 0\right\}$$

and
$$p\,dx + q\,dy - dz + s\,dt = 0$$
is the contact structure. Let
$$U = 2z - qy - px - st$$
and
$$\begin{array}{rclrclrcl} P_1 & = & q-s+y, & P_2 & = & p, & P_3 & = & -q-s+t, \\ Q_1 & = & -q-s+y, & Q_2 & = & 2x, & Q_3 & = & q-s-t. \end{array}$$
In these coordinates
$$\Sigma = \left\{P_1Q_1 - P_2^2 = 0,\ P_1 + Q_1 \geq 0\right\},$$
$$\pi^{-1}(0) = \{x = y = z = t = 0\} = \{Q_1 = P_3, Q_2 = 0, Q_3 = P_1, U = 0\},$$
and the contact structure is given by the equation $\Theta = 0$.

Example E. For the elliptic control-affine system
$$\dot{x} = u_1, \quad \dot{y} = u_2, \quad \dot{z} = u_1 y, \quad u_1^2 + u_2^2 \leq 1$$
from Example E of Subsection 1.2 we get
$$\langle \mathbf{p}, \xi_1(\mathbf{x})\rangle = p + ry, \quad \langle \mathbf{p}, \xi_2(\mathbf{x})\rangle = q, \quad \langle \mathbf{p}, \xi_0(\mathbf{x})\rangle + s = s.$$
Hence in the affine chart $r = -1$
$$\Sigma = \left\{(p-y)^2 + q^2 = s^2,\ s \leq 0\right\}$$
and
$$p\,dx + q\,dy - dz + s\,dt = 0$$
is the contact structure. Let
$$U = 2z - qy - px - st$$
and
$$\begin{array}{rclrclrcl} P_1 & = & p-y, & P_2 & = & -s, & P_3 & = & q-x, \\ Q_1 & = & q, & Q_2 & = & -t, & Q_3 & = & p. \end{array}$$
In these coordinates
$$\Sigma = \left\{P_1^2 + Q_1^2 - P_2^2 = 0,\ P_2 \geq 0\right\},$$
$$\pi^{-1}(0) = \{x = y = z = t = 0\} = \{Q_1 = P_3, Q_2 = 0, Q_3 = P_1, U = 0\},$$
and the contact structure is given by the equation $\Theta = 0$.

3.3. **Contact vector fields.** A vector field $\vec{K}$ in a contact space is called *contact* if it preserves the contact structure. If the contact structure is given as the field of 0-spaces of a contact form Θ then $K = \Theta(\vec{K})$ is called the *generating function* of $\vec{K}$. We will use the following well known facts:

- $\vec{K}$ is uniquely defined by its generating function $K = \Theta(\vec{K})$;
- $\vec{K}$ is tangent to the hypersurface $\{K = 0\}$ and its characteristics;
- $\vec{K}$ is tangent to a smooth Legendrian submanifold L if and only if $K|_L = 0$.

In our case (2)
$$\vec{K} = \left\{ \begin{array}{rcrcrcr} \dot{P} & = & -\partial_Q K & - & P\,\partial_U K/2 & & \\ \dot{Q} & = & \partial_P K & & & - & Q\,\partial_U K/2 \\ \dot{U} & = & -K & + & P\,\partial_P K/2 & + & Q\,\partial_Q K/2 \end{array} \right. . \tag{3}$$
In particular,
$$\vec{K}(O) = 0 \quad \Longleftrightarrow \quad K(O) = 0 \quad \text{and} \quad d_O K|_{\{dU=0\}} = 0 \tag{4}$$

where $d_O K$ is the differential of the generating function at O and $\{dU = 0\}$ is the contact hyperplane at O.

3.4. **Topology of $\Lambda_{\mathbf{x}_0}$.** If $K = P_1 Q_1 - P_2^2$ the formulas (3) give:

$$\dot{P}_1 = -P_1, \ \ \dot{Q}_1 = Q_1, \ \ \dot{P}_2 = 0, \ \ \dot{Q}_2 = -2P_2, \ \ \dot{P}_3 = \dot{Q}_3 = \dot{U} = 0.$$

According to Subsections 3.2 and 3.3 in the hyperbolic case a characteristic of the smooth stratum $\Sigma \setminus \Sigma^4$ is tangent to this contact vector field.

In particular, $P_2 = \text{const}$ along the characteristics. A characteristic with $P_2 \neq 0$ lies in the smooth stratum $\Sigma \setminus \Sigma^4$. In the limit case $P_2 = 0$ we get $P_1 Q_1 = 0$, $Q_2 = \text{const}$, $P_3 = \text{const}$, $Q_3 = \text{const}$, $U = \text{const}$, $P_1 + Q_1 \geq 0$. This curve intersects the stratum Σ^4 as $P_1 = Q_1 = 0$ and is not smooth at the intersection point. Such curves and characteristics of $\Sigma \setminus \Sigma^4$ with $P_2 \neq 0$ are called *characteristics of* Σ.

If $K = P_1^2/2 + Q_1^2/2 - P_2^2/2$ the formulas (3) give:

$$\dot{P}_1 = -Q_1, \ \ \dot{Q}_1 = P_1, \ \ \dot{P}_2 = 0, \ \ \dot{Q}_2 = -P_2, \ \ \dot{P}_3 = \dot{Q}_3 = \dot{U} = 0,$$

According to Subsections 3.2 and 3.3 in the elliptic case a characteristic of the smooth stratum $\Sigma \setminus \Sigma^4$ is tangent to this contact vector field.

In particular, $P_2 = \text{const}$. The characteristics with $P_2 > 0$ lie in the smooth stratum $\Sigma \setminus \Sigma^4$. In the limit case $P_2 = 0$ we get a line $P_1 = Q_1 = 0$, $P_3 = \text{const}$, $Q_3 = \text{const}$, $U = \text{const}$ which lies in the stratum Σ^4. Such lines and characteristics of $\Sigma \setminus \Sigma^4$ with $P_2 \neq 0$ are called *characteristics of* Σ.

Characteristics of Σ satisfy the existence–uniqueness–continuity property: any point of Σ belongs a locally unique characteristic which depends continuously on the point.

LEMMA 1. *The Legendrian submanifold $\Lambda_{\mathbf{x}_0}$ in some neighborhood of $(\mathbf{x}_0, 0)$ is homeomorphic to the cylinder over the two-dimensional sphere if $\mathbf{x}_0$ is hyperbolic or elliptic point of the control-affine system (1).*

Proof. The Legendrian submanifold is the union of all characteristics of Σ intersecting the set

$$\Sigma \cap \pi^{-1}(\mathbf{x}_0, 0) = \left\{ [\mathbf{p}, s] \in ST^*_{\mathbf{x}_0, 0}\mathbb{R}^{3+1} \mid h(\mathbf{p}; \mathbf{x}) + s = 0 \right\},$$

which is homeomorphic to the two-dimensional sphere. But in some neighborhood of $(\mathbf{x}_0, 0)$ characteristics of Σ satisfy the existence–uniqueness–continuity property. $\square$

3.5. **Basic Lemmas.** Let $\mathbb{R}^7$ be a contact space with coordinates $(P_1, P_2, P_3, Q_1, Q_2, Q_3, U)$, its contact structure be defined as the field of 0-spaces of the contact form (2), and Σ be one of the two hypersurfaces:

$$\Sigma = \left\{ P_1 Q_1 - P_2^2 = 0 \right\} \quad \text{or} \quad \Sigma = \left\{ P_1^2 + Q_1^2 - P_2^2 = 0 \right\}.$$

The hypersurface consists of the two smooth strata:

$$\Sigma^4 = \{P_1 = Q_1 = P_2 = 0\}$$

and $\Sigma \setminus \Sigma^4$. Let $O \in \Sigma^4$ be the origin $P = Q = U = 0$ and $\mathfrak{L}$ be the space of the germs (L, O) at the origin of all smooth Legendrian submanifolds L that pass through the origin and are transversal to Σ^4. In particular,

$$(L_0, O) \in \mathfrak{L}, \quad L_0 = \{Q_1 = P_3, Q_2 = 0, Q_3 = P_1, U = 0\}.$$

LEMMA 2. *The space $\mathfrak{L}$ is arcwise connected and P_2, P_3, Q_3 are coordinates on any $(L, O) \in \mathfrak{L}$.*

Proof. A germ (L,O) of a Legendrian submanifold at the origin is transversal to Σ^4 if and only if the restrictions of the differentials dP_1, dQ_1, and dP_2 to the tangent plane $T_O L$ are linearly independent. Hence:

$$T_O L = \begin{cases} dQ_2 &= a_{11}\,dP_1 + a_{12}\,dQ_1 + a_{13}\,dP_2 \\ dP_3 &= a_{21}\,dP_1 + a_{22}\,dQ_1 + a_{23}\,dP_2 \\ dQ_3 &= a_{31}\,dP_1 + a_{32}\,dQ_1 + a_{33}\,dP_2 \\ dU &= 0 \end{cases}.$$

But the tangent plane $T_O L$ is a Lagrangian subspace of the contact hyperplane $dU = 0$ endowed with a linear symplectic form $d\Theta|_{\Theta=0} = dP \wedge dQ$; and the condition

$$dP \wedge dQ\big|_{T_O L} = 0, \quad dP \wedge dQ\big|_{T_O L} = (1 + a_{21}a_{32} - a_{22}a_{31})\,dP_1 \wedge dQ_1$$
$$+ (-a_{11} + a_{21}a_{33} - a_{23}a_{31})\,dP_1 \wedge dP_2 + (-a_{12} + a_{22}a_{33} - a_{23}a_{32})\,dQ_1 \wedge dP_2$$

gives

$$a_{21}a_{32} - a_{22}a_{31} = -1, \quad a_{11} = a_{21}a_{33} - a_{23}a_{31}, \quad a_{12} = a_{22}a_{33} - a_{23}a_{32}.$$

These three equalities show that the space formed by all tangent planes $T_O L$ such that $(L,0) \in \mathfrak{L}$ is homotopically equivalent to a circle and, in particular, arcwise connected. But two germs of Legendrian submanifolds at the origin with the same tangent plane can be connected by a continuous path consisting of germs having the same tangent plane. Hence the space $\mathfrak{L}$ is arcwise connected.

The equality $a_{21}a_{32} - a_{22}a_{31} = -1$ implies that the restrictions of the differentials dP_2, dP_3, and dQ_3 to the tangent plane $T_O L$ are linearly independent. So P_2, P_3, Q_3 are coordinates on $(L,O) \in \mathfrak{L}$. □

Lemma 3. *For any $(L_1,O) \in \mathfrak{L}$ there exists a local contact diffeomorphism h_1 such that $(L_1,O) = h_1(L_0,O)$ and $h_1(\Sigma) = \Sigma$.*

Proof. According to Lemma 2 we can include the Legendrian germs (L_0,O) and (L_1,O) into a family $(L_\varepsilon,O) \in \mathfrak{L}$ where $\varepsilon \in [0,1]$, $L_\varepsilon = k_\varepsilon(L_0)$, and k_ε is a smooth family of contact diffeomorphisms such that $k_\varepsilon(O) = O$ for all $\varepsilon \in [0,1]$. Let

$$\vec{K}_\varepsilon(k_\varepsilon e) = \frac{d}{d\varepsilon}\,k_\varepsilon e, \quad e \in \mathbb{R}^7, \quad \vec{K}_\varepsilon(O) = 0$$

be a contact vector field which depends smoothly on ε.

Let $K_\varepsilon = \Theta(\vec{K}_\varepsilon)$. According to Lemma 2 in some neighborhood $\mathcal{U}_O$ of the origin P_2, P_3, and Q_3 are coordinates on L_ε for any $\varepsilon \in [0,1]$. Therefore there exists a unique function $H_\star : [0,1] \times \mathcal{U}_O \to \mathbb{R}$ depending only on ε, P_2, P_3, Q_3 such that

$$H_\varepsilon\big|_{L_\varepsilon} = K_\varepsilon\big|_{L_\varepsilon}. \tag{5}$$

Let $\vec{H}_\varepsilon$ be the contact vector field defined by the formulas (3) where $K = H_\varepsilon$.

First of all, let us show that $\vec{H}_\varepsilon(O) = 0$. Indeed, according to (4) $K_\varepsilon(O) = 0$ and $d_O K_\varepsilon|_{\{dU=0\}} = 0$ because $\vec{K}_\varepsilon(O) = 0$. Hence $H_\varepsilon(O) = 0$ and $d_O H_\varepsilon = 0$ because L_ε is tangent to the hyperplane $\{dU = 0\}$. So according to (4) $\vec{H}_\varepsilon(O) = 0$.

Now we can define a family of local contact diffeomorphisms h_ε depending on $\varepsilon \in [0,1]$ such that

$$\vec{H}_\varepsilon(h_\varepsilon e) = \frac{d}{d\varepsilon}\,h_\varepsilon e \quad \forall\, e \in \mathcal{V}_O,$$

where $\mathcal{V}_O$ is a neighborhood of the origin. Indeed, it is possible because $\vec{H}_\varepsilon(O) = 0$. Besides, the equality $\vec{H}_\varepsilon(O) = 0$ implies that $h_\varepsilon(O) = O$.

The formulas (3) imply that the coordinate functions P_1, P_2, and Q_1 are first integrals of the contact vector field $\vec{H}_\varepsilon$ because its generating function $\vec{H}_\varepsilon$ does not depend on P_1, Q_1, Q_2, and U. Hence the contact vector field $\vec{H}_\varepsilon$ is tangent to Σ^4 and $\Sigma \setminus \Sigma^4$. Therefore $h_\varepsilon(\Sigma) = \Sigma$ for all $\varepsilon \in [0,1]$.

The equality (5) implies that for any $\varepsilon \in [0,1]$ the vector field $\vec{H}_\varepsilon - \vec{K}_\varepsilon$ is tangent to $L_\varepsilon = k_\varepsilon(L_0)$. So $h_\varepsilon(L_0) = k_\varepsilon(L_0)$ for all $\varepsilon \in [0,1]$.

Therefore $(L_\varepsilon, O) = h_\varepsilon(L_0, O)$ and $h_\varepsilon(\Sigma) = \Sigma$ for all $\varepsilon \in [0,1]$. In particular, it holds for $\varepsilon = 1$. □

3.6. **Local normal forms of $\Lambda_{\mathbf{x}_0}$.** Lemma 3 implies the following

Lemma 4. *Let $O = \Pi^+(\mathbf{x}_0, 0)$ or $O = \Pi^-(\mathbf{x}_0, 0)$. Then in a neighborhood of O there exist local coordinates $(P_1, P_2, P_3, Q_1, Q_2, Q_3, U)$ such that:*

- *the contact structure is given as the field of 0-spaces of the contact form*
$$\Theta = \frac{1}{2} P\, dQ - \frac{1}{2} Q\, dP - dU;$$
- $\pi^{-1}(\mathbf{x}_0, 0) = \{Q_1 = P_3, Q_2 = 0, Q_3 = P_1, U = 0\}$;
- *if $\mathbf{x}_0$ is a hyperbolic point then*
$$\Sigma = \left\{P_1 Q_1 - P_2^2 = 0,\ P_1 + Q_1 \geq 0\right\} \quad \text{and} \quad \Lambda_{\mathbf{x}_0} = \Lambda^H_+ \cup \Lambda^H_-$$
where
$$\Lambda^H_+ = \begin{cases} P_1 = a^2 b^2, & Q_1 = c^2, \\ P_2 = abc, & Q_2 = 2abc \ln a^2, \quad U = 0, \\ P_3 = a^2 c^2, & Q_3 = b^2, \end{cases}$$
$$\Lambda^H_- = \begin{cases} P_1 = b^2, & Q_1 = a^2 c^2, \\ P_2 = abc, & Q_2 = -2abc \ln a^2, \quad U = 0, \\ P_3 = c^2, & Q_3 = a^2 b^2, \end{cases}$$
$a \in [0,1]$, $b, c \in \mathbb{R}$ are parameters, and $a \ln a^2 = 0$ if $a = 0$;
- *if $\mathbf{x}_0$ is an elliptic point then*
$$\Sigma = \left\{P_1^2 + Q_1^2 - P_2^2 = 0,\ P_2 \geq 0\right\} \quad \text{and} \quad \Lambda_{\mathbf{x}_0} = \Lambda^E$$
where
$$\Lambda^E = \begin{cases} P_2 \geq 0, \\ P_1 + iQ_1 = P_2 e^{i\left(\psi - \frac{Q_2}{2P_2}\right)}, \quad U = 0, \\ Q_3 + iP_3 = P_2 e^{i\left(\psi + \frac{Q_2}{2P_2}\right)}, \end{cases}$$
$i = \sqrt{-1}$, $\psi \in \mathbb{R} \mod 2\pi\mathbb{Z}$ is a parameter, and $P_2 e^{i\left(\psi \pm \frac{Q_2}{2P_2}\right)} = 0$ if $P_2 = 0$.

Remark. Examples H and E of coordinates from Lemma 4 are given in Subsection 3.2.

Proof. According to Subsection 3.2 and Lemma 3 in a neighborhood of O there exist local coordinates $(P_1, P_2, P_3, Q_1, Q_2, Q_3, U)$ such that:

- the contact structure is given by the equation $\Theta = 0$;
- $\pi^{-1}(\mathbf{x}_0, 0) = \{Q_1 = P_3, Q_2 = 0, Q_3 = P_1, U = 0\}$;
- if $\mathbf{x}_0$ is a hyperbolic point then $\Sigma = \left\{P_1 Q_1 - P_2^2 = 0,\ P_1 + Q_1 \geq 0\right\}$;
- if $\mathbf{x}_0$ is an elliptic point then $\Sigma = \left\{P_1^2 + Q_1^2 - P_2^2 = 0,\ P_2 \geq 0\right\}$.

Let us consider the following parameterizations of $\Sigma \cap \pi^{-1}(\mathbf{x}_0, 0)$:

- $P_1 = b^2$, $Q_1 = c^2$, $P_2 = bc$, $Q_2 = 0$, $P_3 = c^2$, $Q_3 = b^2$, $U = 0$ (where $b, c \in \mathbb{R}$ are parameters and $\mathbf{x}_0$ is hyperbolic);
- $P_2 \geq 0$, $P_1 + iQ_1 = P_2 e^{i\psi}$, $Q_2 = 0$, $Q_3 + iP_3 = P_2 e^{i\psi}$, $U = 0$ (where $\psi \in \mathbb{R} \mod 2\pi\mathbb{Z}$ is a parameter and $\mathbf{x}_0$ is elliptic).

According to Subsection 3.4 the characteristics of Σ have parameterizations (with a real parameter σ) satisfying the differential equations:

- $\frac{dP_1}{d\sigma} = -P_1$, $\frac{dQ_1}{d\sigma} = Q_1$, $\frac{dP_2}{d\sigma} = 0$, $\frac{dQ_2}{d\sigma} = -2P_2$, $\frac{dP_3}{d\sigma} = \frac{dQ_3}{d\sigma} = \frac{dU}{d\sigma} = 0$,
  if $\mathbf{x}_0$ is hyperbolic;
- $\frac{dP_1}{d\sigma} + i\frac{dQ_1}{d\sigma} = i(P_1 + iQ_1)$, $\frac{dP_2}{d\sigma} = 0$, $\frac{dQ_2}{d\sigma} = -P_2$, $\frac{dP_3}{d\sigma} = \frac{dQ_3}{d\sigma} = \frac{dU}{d\sigma} = 0$,
  if $\mathbf{x}_0$ is elliptic.

Therefore the characteristics passing through $\Sigma \cap \pi^{-1}(\mathbf{x}_0, 0)$ are given by the equations:

- $P_1 = b^2 e^{-\sigma}$, $Q_1 = c^2 e^{\sigma}$, $P_2 = bc$, $Q_2 = -2bc\sigma$, $P_3 = c^2$, $Q_3 = b^2$, $U = 0$,
  if $\mathbf{x}_0$ is hyperbolic;
- $P_2 \geq 0$, $P_1 + iQ_1 = P_2 e^{i(\psi+\sigma)}$, $Q_2 = -P_2\sigma$, $Q_3 + iP_3 = P_2 e^{i\psi}$, $U = 0$,
  if $\mathbf{x}_0$ is elliptic.

Here $\sigma \in \mathbb{R}$ is a parameter along the characteristics.

In the hyperbolic case for $\sigma \geq 0$ we get the formulas for Λ_+^H from Lemma 4 changing $c \mapsto ce^{-\sigma/2}$ and setting $a = e^{-\sigma/2}$.

In the hyperbolic case for $\sigma \leq 0$ we get the above formulas for Λ_-^H from Lemma 4 changing $b \mapsto be^{\sigma/2}$ and setting $a = e^{\sigma/2}$.

In the elliptic case we get the formulas for Λ^E changing $\psi \mapsto \psi - \sigma/2$ and setting $\sigma = -Q_2/P_2$. □

3.7. **Singularities of $\Lambda_{\mathbf{x}_0}$.**

Definition. We say that a three-dimensional stratified Legendrian submanifold Λ of a contact space has a singularity H_1, H_2, or E_2 at a point $\lambda \in \Lambda$ if its germ (Λ, λ) is contact diffeomorphic to the germ $(\mathcal{H}_1 \times \mathbb{R}, O)$, $(\mathcal{H}_2 \times \mathbb{R}, O)$, or $(\mathcal{E}_2 \times \mathbb{R}, O)$ respectively.

Lemma 5. *The Legendrian submanifold* $\Lambda_+^H \cup \Lambda_-^H$

(1) *has singularities* H_1 *if*
$$P_1 = P_2 = P_3 = Q_2 = U = 0, \quad Q_1 > 0, \quad Q_3 > 0,$$
or
$$P_2 = Q_1 = Q_2 = Q_3 = U = 0, \quad P_1 > 0, \quad P_3 > 0;$$
(2) *has singularities* H_2 *if*
$$P_1 = P_2 = P_3 = Q_2 = U = 0, \quad Q_1 = 0, \quad Q_3 > 0,$$
or
$$P_1 = P_2 = P_3 = Q_2 = U = 0, \quad Q_1 > 0, \quad Q_3 = 0,$$
or
$$P_2 = Q_1 = Q_2 = Q_3 = U = 0, \quad P_1 = 0, \quad P_3 > 0,$$
or
$$P_2 = Q_1 = Q_2 = Q_3 = U = 0, \quad P_1 > 0, \quad P_3 = 0;$$
(3) *has more complicated singularity if*
$$P_1 = P_2 = P_3 = Q_1 = Q_2 = Q_3 = U = 0;$$
(4) *is smooth at the other points.*

Proof. The Legendrian submanifold $\Lambda^H_+ \cup \Lambda^H_-$ has singularities if and only if $a = 0$ in the formulas of Lemma 4. It gives the set of singularities of Λ^H_+:

$$P_1 = P_2 = P_3 = Q_2 = U = 0, \quad Q_1 \geq 0, \quad Q_3 \geq 0;$$

and the set of singularities of Λ^H_-:

$$P_2 = Q_1 = Q_2 = Q_3 = U = 0, \quad P_1 \geq 0, \quad P_3 \geq 0;$$

proving the item 4 from Lemma 5. Let us consider the following transformations:

- $a \mapsto a$, $b \mapsto \kappa b$, $c \mapsto c$, $\kappa > 0$,
  $(P_1, P_2, P_3, Q_1, Q_2, Q_3, U) \mapsto (\kappa^2 P_1, \kappa P_2, P_3, Q_1, \kappa Q_2, \kappa^2 Q_3, \kappa^2 U)$;
- $a \mapsto a$, $b \mapsto b$, $c \mapsto \kappa c$, $\kappa > 0$,
  $(P_1, P_2, P_3, Q_1, Q_2, Q_3, U) \mapsto (P_1, \kappa P_2, \kappa^2 P_3, \kappa^2 Q_1, \kappa Q_2, Q_3, \kappa^2 U)$;
- $a \mapsto a$, $b \mapsto c$, $c \mapsto b$,
  $(P_1, P_2, P_3, Q_1, Q_2, Q_3, U) \mapsto (P_3, P_2, P_1, Q_3, Q_2, Q_1, U)$;
- $a \mapsto a$, $b \mapsto b$, $c \mapsto c$,
  $(P_1, P_2, P_3, Q_1, Q_2, Q_3, U) \mapsto (Q_3, P_2, Q_1, P_3, -Q_2, P_1, -U)$.

All of them preserve the contact structure and the Legendrian submanifold $\Lambda^H_+ \cup \Lambda^H_-$. Besides, these transformations divide the set of singularities of $\Lambda^H_+ \cup \Lambda^H_-$ into the three orbits mentioned in the items 1–3 of Lemma 5. In particular, we prove its item 3.

The point $P_1 = P_2 = P_3 = Q_1 = Q_2 = U = 0$, $Q_3 = 1$ belongs to Λ^H_+. Let us consider its section with $Q_3 = 1$. Then $b = 1$ or $b = -1$ but these conditions define the same submanifold:

$$\begin{cases} P_1 = a^2, & Q_1 = c^2, \\ P_2 = ac, & Q_2 = 2ac \ln a^2, \quad U = 0. \\ P_3 = a^2 c^2, & Q_3 = 1, \end{cases}$$

The form Θ defines the contact structure

$$\frac{1}{2}\left(P_1\, dQ_1 + P_2\, dQ_2 - Q_1\, dP_1 - Q_2\, dP_2\right) - d\frac{P_3}{2} = 0$$

in the plane $Q_3 = 1$, $U = 0$ and our section is Legendrian. Denoting $A = a$, $B = c$, $U = P_3/2$ we get the Legendrian submanifold $\mathcal{H}_2$ from Subsection 1.3 and prove the item 2 of Lemma 5.

But the stratified Legendrian submanifold $\mathcal{H}_2$ has singularities H_1 if $A = 0$ and $B \neq 0$ that is shown in [5]. It proves the item 1 of Lemma 5. ∎

Lemma 6. *The Legendrian submanifold* Λ^E

(1) *has singularities* E_2 *if*

$$P_1 = P_2 = P_3 = Q_1 = Q_3 = U = 0, \quad Q_2 \neq 0;$$

(2) *has more complicated singularity if*

$$P_1 = P_2 = P_3 = Q_1 = Q_2 = Q_3 = U = 0;$$

(3) *is smooth at the other points.*

Proof. The Legendrian submanifold Λ^E has singularities if and only if $P_2 = 0$ in the formulas of Lemma 4. It gives the set of singularities of Λ^E:

$$P_1 = P_2 = P_3 = Q_1 = Q_3 = U = 0;$$

and proves the item 3 from Lemma 6. Let us consider the following transformations:

- $(P_1, P_2, P_3, Q_1, Q_2, Q_3, U) \mapsto (\kappa P_1, \kappa P_2, \kappa P_3, \kappa Q_1, \kappa Q_2, \kappa Q_3, \kappa^2 U)$, $\kappa > 0$;
- $(P_1, P_2, P_3, Q_1, Q_2, Q_3, U) \mapsto (Q_3, P_2, Q_1, P_3, -Q_2, P_1, -U)$.

All of them preserve the contact structure and the Legendrian submanifold Λ^E. Besides, these transformations divide the set of singularities of Λ^E into the two orbits mentioned in the items 1, 2 of Lemma 6. In particular, we prove its item 2.

Let us consider the section of Λ^E with $Q_2 = 2$:

$$\Lambda^E = \begin{cases} \quad P_2 & \geq \quad 0, \\ P_1 + iQ_1 & = \quad P_2 e^{i\left(\psi - \frac{1}{P_2}\right)}, \quad U = 0, \\ Q_3 + iP_3 & = \quad P_2 e^{i\left(\psi + \frac{1}{P_2}\right)}, \end{cases}$$

The form Θ defines the contact structure

$$\frac{1}{2}\left(P_1\, dQ_1 + P_3\, dQ_3 - Q_1\, dP_1 - Q_3\, dP_3\right) - dP_2 = 0$$

in the plane $Q_3 = 1$, $U = 0$ and our section is Legendrian. After obvious renaming $P_2 \mapsto U$, $P_3 \mapsto P_2$, $Q_3 \mapsto Q_2$ we get the Legendrian submanifold $\mathcal{E}_2$ from Subsection 1.3 and prove the item 1 of Lemma 6. □

3.8. **Time function τ.** Here we prove some conditions which have to be satisfied by the time function τ in the coordinates from Lemma 4.

Lemma 7. *Let $O = \Pi^+(\mathbf{x}_0, 0)$ or $O = \Pi^-(\mathbf{x}_0, 0)$ and $d_O\tau$ be the differential of the time function τ at O. Then in the coordinates from Lemma 4*

$$d_O\tau = \gamma_1(dQ_1 - dP_3) + \gamma_2\, dQ_2 + \gamma_3(dQ_3 - dP_1) + \gamma_0\, dU$$

where

- $\gamma_1\gamma_3 > \gamma_2^2$ *if* $\mathbf{x}_0$ *is hyperbolic;*
- $\gamma_2^2 > \gamma_1^2 + \gamma_3^2$ *if* $\mathbf{x}_0$ *is elliptic.*

Proof. The equality

$$d_O\tau = \gamma_1(dQ_1 - dP_3) + \gamma_2\, dQ_2 + \gamma_3(dQ_3 - dP_1) + \gamma_0\, dU$$

follows from the conditions

$$\pi^{-1}(\mathbf{x}_0, 0) = \{Q_1 = P_3, Q_2 = 0, Q_3 = P_1, U = 0\} \subset \tau^{-1}(0).$$

Let us prove the inequalities $\gamma_1\gamma_3 > \gamma_2^2$ and $\gamma_2^2 > \gamma_1^2 + \gamma_3^2$.

The Legendrian submanifold $\pi^{-1}(\mathbf{x}_0, 0) \subset ST^*\mathbb{R}^{3+1}$ is situated in the isochrone $\tau^{-1}(0)$ and consists of its characteristics: the lines $([\mathbf{p}, \cdot]; \mathbf{x}_0, 0)$ with $\mathbf{p} \neq 0$ and the two points $([0, \pm 1]; \mathbf{x}_0, 0)$. In an affine neighborhood of O the hypersurface $\Sigma \cap \pi^{-1}(\mathbf{x}_0, 0)$ is a half-cone. It turns out that one of the two half-characteristics of the isochrone $\tau^{-1}(0)$ starting at O lies inside of this half-cone.

Indeed, let us choose local coordinates (x, y, z) in a neighborhood of $\mathbf{x}_0 \in \mathbb{R}^3$ such that $\xi_1(\mathbf{x}_0) = (1, 0, 0)$, $\xi_2(\mathbf{x}_0) = (0, 1, 0)$, and $\xi_0(\mathbf{x}_0) = (a_0, b_0, c_0)$. Then according to Subsection 3.1 we get that in the coordinates (p, q, r, s) that are dual to (x, y, z, t):

$$\Sigma \cap \pi^{-1}(\mathbf{x}_0, 0) = \left\{[p, q, r, s] \;\middle|\; a_0 p + b_0 q + c_0 r + s + \sqrt{p^2 + q^2} = 0\right\}$$

and $O = [0, 0, 1, -c_0]$ or $O = [0, 0, -1, c_0]$. So, we can take the affine neighborhood $r = 1$ or $r = -1$ respectively. It is clear that in each case the ray

$$p = q = 0, \quad \pm c_0 + s < 0$$

is situated inside of the half-cone $\left\{a_0 p + b_0 q \pm c_0 + s + \sqrt{p^2 + q^2} = 0\right\}$.

But according to Subsection 3.3 the characteristics of $\tau^{-1}(0)$ are tangent to the contact vector field $\vec{\tau}$ defined by the formulas (3) for $K = \tau$. Hence one of the two vectors $\pm\vec{\tau}(0)$ where

$$\vec{\tau}(0) = \left\{\dot{P}_1 = \gamma_1,\ \dot{P}_2 = \gamma_2,\ \dot{P}_3 = \gamma_3,\ \dot{Q}_1 = \gamma_3,\ \dot{Q}_2 = 0,\ \dot{Q}_3 = \gamma_1,\ \dot{U} = 0\right\}$$

must lie inside of the half-cone

$$\Sigma \cap \pi^{-1}(\mathbf{x}_0, 0) = \Sigma \cap \{Q_1 = P_3, Q_2 = 0, Q_3 = P_1, U = 0\}.$$

It means that

- $\dot{P}_1\dot{Q}_1 - \dot{P}_2{}^2 = \gamma_1\gamma_3 - \gamma_2^2 > 0$ if $\mathbf{x}_0$ is hyperbolic and
- $\dot{P}_1{}^2 + \dot{Q}_1{}^2 - \dot{P}_2{}^2 = \gamma_1^2 + \gamma_3^2 - \gamma_2^2 < 0$ if $\mathbf{x}_0$ is elliptic.

□

3.9. **Proof of Theorem 1.** According to Lemma 1 in some neighborhood of $(\mathbf{x}_0, 0)$ the Legendrian submanifold $\Lambda_{\mathbf{x}_0}$ is homeomorphic to the cylinder over the two-dimensional sphere, the elements of the cylinder are characteristics of Σ. But an isochrone $\tau^{-1}(T)$ is transversal to these characteristics because their projections are the graphs of trajectories of the control-affine system (1). It proves that $\mathcal{L}_{\mathbf{x}_0}(T)$ is homeomorphic to the two-dimensional sphere.

In neighborhoods of two contact elements $\Pi^+(\mathbf{x}_0, 0)$ or $\Pi^-(\mathbf{x}_0, 0)$ the Legendrian submanifold $\Lambda_{\mathbf{x}_0}$ has singularities described in Lemmas 4, 5, and 6.

In the hyperbolic case Theorem 1 follows from Lemma 5. Namely, singularities H_1 form two quadrants described in Lemma 5. But one and only one of them lies in the domain $\tau > 0$ according to Lemma 7.

In the elliptic case Theorem 1 follows from Lemma 6. Namely, singularities E_2 form two rays described in Lemma 6. But one and only one of them lies in the domain $\tau > 0$ according to Lemma 7.

4. Appendix

Theorem 1 implies that for enough small $T > 0$ the stratified Legendrian submanifolds $\mathcal{L}_{\mathbf{x}_0}(T)$ are reduced to a normal form $\mathcal{L}^H$ in the hyperbolic case and to a normal form $\mathcal{L}^E$ in the elliptic case. Here we give explicit formulas for $\mathcal{L}^H$ based on [6] and for $\mathcal{L}^E$ based on [7]. The fronts of the stratified Legendrian submanifolds $\mathcal{L}^E$ and $\mathcal{L}^H$ are shown in Fig. 1 on the left and the right respectively.

Normal form $\mathcal{L}^H$:

$$\mathcal{L}^H = \left\{(p : q : r; x, y, z) \in ST^*\mathbb{R}^3 \;\middle|\; p = \frac{4\alpha\beta}{(1+\alpha^2)(1+\beta^2)}, \quad q = \frac{1-\beta^2}{1+\beta^2}, \right.$$
$$\left. r = \frac{1-\alpha^2}{1+\alpha^2}, \quad x = \Phi(\alpha)\frac{2\beta}{1+\beta^2}, \quad y = \frac{1-\beta^2}{1+\beta^2}, \quad z = \Psi(\alpha)\frac{2\beta^2}{(1+\beta^2)^2}\right\}$$

where $\alpha, \beta \in \mathbb{R} \cup \{\infty\}$ are parameters,

$$\Phi(\alpha) = -\frac{\alpha \ln \alpha^2}{1-\alpha^2}, \quad \Psi(\alpha) = \frac{1-\alpha^4+2\alpha^2\ln\alpha^2}{(1-\alpha^2)^2},$$

$$\Phi(0) = \Phi(\infty) = \Psi(1) = \Psi(-1) = 0, \quad \Phi(1) = -\Phi(-1) = \Psi(0) = -\Psi(\infty) = 1.$$

NORMAL FORM $\mathcal{L}^E$:

$$\mathcal{L}^E = L^E \cup \{\mathcal{P}^+, \mathcal{P}^-\}, \quad \mathcal{P}^\pm = (0:0:\pm 1; 0,0,0) \in ST^*\mathbb{R}^3,$$

$$L^E = \left\{ (p:q:r;x,y,z) \in ST^*\mathbb{R}^3 \;\middle|\; p = \cos r\, \cos\phi, \quad q = \cos r\, \sin\phi, \right.$$
$$\left. x = \frac{2\,\sin r\, \cos\phi}{r}, \quad y = \frac{2\,\sin r\, \sin\phi}{r}, \quad z = \frac{2r - \sin 2r}{2r^2} \right\}$$

where $\phi \in \mathbb{R} \mod 2\pi\mathbb{Z}$ is a parameter.

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LOMONOSOV MOSCOW STATE UNIVERSITY, FACULTY OF MECHANICS AND MATHEMATICS, LENINSKIE GORY, MOSCOW GSP-1, 119991, RUSSIA

SCIENTIFIC RESEARCH INSTITUTE FOR SYSTEM ANALYSIS OF RAS, MOSCOW, RUSSIA