

ON THE COLENGTH OF FRACTIONAL IDEALS

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ABSTRACT. The main goal of this paper is to give a recursive formula for the colength of a fractional ideal in terms of some maximal points of its value set and of its projections. The fractional ideals are relative to a class of rings called admissible, a more general class of one dimensional local rings that contains those of algebroid curves. For fractional ideals of such rings with two or three minimal primes, a closed formula for the colength is provided.

1. INTRODUCTION

The computation of the colength of a fractional ideal of a ring of an irreducible algebroid plane curve in terms of its value set was known since the work of Gorenstein in the fifties of last century, at least (cf. [6]). Such computation was performed for a larger class of analytically reduced but reducible rings by D'Anna in [2, §2], where colengths of fractional ideals and lengths of maximal saturated chains in their sets of values are related. D'Anna's method requires the knowledge of many elements in the set of values, a disadvantage that would be desirable to overcome to increase computational efficiency. In fact, in the particular case of an algebroid curve with two branches, Barucci, D'Anna and Fröberg, in [1], were able to give an explicit formula for the colength of a given fractional ideal in terms of some maximal points of its value set.

Local rings of algebroid curves and the class studied by D'Anna in [2] belong to the larger class of *admissible rings* considered in this paper. By such a ring, we mean a one dimensional, local, noetherian, Cohen-Macaulay, analytically reduced and residually rational ring such that the cardinality of its residue field is sufficiently large (see [8] for more details). For simplicity and without loss of generality (cf. [2, §1]), we will also assume that our rings are complete with respect to the topology induced by the maximal ideal. In such case, a sufficiently large residue field means that its cardinality is greater than or equal to the number r of minimal primes of the ring.

One of our main results, Theorem 10, gives a recursive formula on the number r for the colength of a fractional ideal in a complete admissible ring. The important feature is that the computation requires only few special points of the value set, namely, its relative maximal points and those of its projections. The other main result is Corollary 20 that provides a closed formula for the colength in the case of three minimal primes. It is worth noting that such a closed formula for three minimal primes is not a straightforward consequence of the recursive formula established in Theorem 10, since its proof demands a careful analysis of the geometry of the maximal points of the value set.

The outline of the paper is as follows. Section 2 collects some preliminaries and notation regarding the general background of the article. Section 3 is concerned with the definition of

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value sets, recalling three useful analog properties to ones obtained for semigroups of values by Delgado and Garcia (cf. [3] and [5]). Section 4 introduces and analyzes different kinds of maximal points in the value set to get enough tools to pass to Section 5 that is eventually concerned with the announced recursive formula for the colength of fractional ideals in admissible rings. To ease the comparison with the previous results due to Barucci, D'Anna and Fröberg, we first analyze their recipe for $r = 2$, while we devote Section 5.2 to the case $r \geq 3$. The closed formula for $r = 3$ is finally dealt with in Section 6 where a fine detailed analysis of the geometry of the maximal points is offered in a series of lemmas, culminating with Lemma 18 that unavoidably leads, after the case by case analysis, the statement and proof of Theorem 19 that confirms a conjectural formula by M. Hernandez (cf. [7]).

2. GENERAL BACKGROUND

In this section we refer to [2] for our unproved statements. Let \wp_1, \dots, \wp_r be the minimal primes of an admissible complete ring R . We will use the notation $I = \{1, \dots, r\}$. We set $R_i = R/\wp_i$ and will denote by $\pi_i: R \rightarrow R_i$ the canonical surjection. Since R is reduced, we have an injective homomorphism

$$\begin{aligned} \pi: R &\hookrightarrow R_1 \times \cdots \times R_r \\ h &\mapsto (\pi_1(h), \dots, \pi_r(h)). \end{aligned}$$

More generally, if $J = \{j_1 < \cdots < j_s\}$ is any subset of I , we may consider $R_J = R/\cap_{i=1}^s \wp_{j_i}$ and will denote by $\pi_J: R \rightarrow R_J$ the natural surjection.

We will denote by \mathcal{K} the total ring of fractions of R and when $J \subset I$ we denote by \mathcal{K}_J the total ring of fractions of the ring R_J . Notice that $R_I = R$ and $\mathcal{K}_I = \mathcal{K}$. If $J = \{i\}$, then $R_{\{i\}}$ is equal to the above defined domain R_i whose field of fractions will be denoted by \mathcal{K}_i . Let \tilde{R} be the integral closure of R in \mathcal{K} and \tilde{R}_J be that of R_J in \mathcal{K}_J . One has that $\tilde{R}_J \simeq \tilde{R}_{j_1} \times \cdots \times \tilde{R}_{j_s}$, which in turn is the integral closure of $R_{j_1} \times \cdots \times R_{j_s}$ in its total ring of fractions. We have the following diagram:

$$\begin{array}{ccc} \mathcal{K}_J & \simeq & \mathcal{K}_{j_1} \times \cdots \times \mathcal{K}_{j_s} \\ \uparrow & & \uparrow \\ \tilde{R}_J & \simeq & \tilde{R}_{j_1} \times \cdots \times \tilde{R}_{j_s} \\ \uparrow & & \uparrow \\ R_J & \hookrightarrow & R_{j_1} \times \cdots \times R_{j_s} \end{array}$$

Since each \tilde{R}_i is a DVR, with a valuation denoted by v_i , one has that \mathcal{K}_i is a valuated field with the extension of the valuation v_i which is denoted by the same symbol. This allows one to define the value map

$$\begin{aligned} v: \mathcal{K} \setminus Z(\mathcal{K}) &\rightarrow \mathbb{Z}^r \\ h &\mapsto (v_1(\pi_1(h)), \dots, v_r(\pi_r(h))), \end{aligned}$$

where π_i here denotes the projection $\mathcal{K} \rightarrow \mathcal{K}_i$, which is the extension of the previously defined projection map $\pi_i: R \rightarrow R_i$ and $Z(\mathcal{K})$ stands for the set of zero divisors of \mathcal{K} .

An R -submodule \mathcal{I} of \mathcal{K} will be called a *regular fractional ideal* of R if it contains a regular element of R and there is a regular element d in R such that $d\mathcal{I} \subset R$.

Since $d\mathcal{I}$ is an ideal of R , which is a noetherian ring, one has that $\mathcal{I} \subset \mathcal{K}$ is a nontrivial fractional ideal if and only if it contains a regular element of R and it is a finitely generated R -module.

Examples of fractional ideals of R are R itself, \tilde{R} , the conductor \mathcal{C} of \tilde{R} in R , or any ideal of R or of \tilde{R} that contains a regular element. Also, if \mathcal{I} is a regular fractional ideal of R , then for all

$\emptyset \neq J \subset I$ one has that $\pi_J(\mathcal{I})$ is a regular fractional ideal of R_J , where, this time, $\pi_J: \mathcal{K} \rightarrow \mathcal{K}_J$ denotes the natural projection.

3. VALUE SETS

If \mathcal{I} is a regular fractional ideal of R , we define the *value set* of \mathcal{I} as being

$$E = v(\mathcal{I} \setminus Z(\mathcal{K})) \subset \mathbb{Z}^r.$$

If $J = \{j_1 < \dots < j_s\} \subset I$, then we denote by pr_J the projection $\mathbb{Z}^r \rightarrow \mathbb{Z}^s$,

$$(\alpha_1, \dots, \alpha_r) \mapsto (\alpha_{j_1}, \dots, \alpha_{j_s}).$$

Let us define

$$E_J = v(\pi_J(\mathcal{I}) \setminus Z(\mathcal{K}_J)).$$

If $j \in J = \{j_1, \dots, j_t, \dots, j_s\} \subset I$, with $j_t = j$, for $\alpha = (\alpha_{j_1}, \dots, \alpha_{j_s}) \in E_J$, then we define

$$\tilde{\text{pr}}_j(\alpha) = \alpha_{j_t} = \alpha_j.$$

We will consider on \mathbb{Z}^r the product order \leq and will write $(a_1, \dots, a_r) < (b_1, \dots, b_r)$ when $a_i < b_i$, for all $i = 1, \dots, r$.

Value sets of fractional ideals have the following fundamental analog properties to those of semigroups of values described by Garcia for $r = 2$ in [5] and by Delgado for $r > 2$ in [3] (see also [2] or [1]):

Property (A). *If $\alpha = (\alpha_1, \dots, \alpha_r)$ and $\beta = (\beta_1, \dots, \beta_r)$ belong to E , then*

$$\min(\alpha, \beta) = (\min(\alpha_1, \beta_1), \dots, \min(\alpha_r, \beta_r)) \in E.$$

Property (B). *If $\alpha = (\alpha_1, \dots, \alpha_r), \beta = (\beta_1, \dots, \beta_r)$ belong to E , $\alpha \neq \beta$ and $\alpha_i = \beta_i$ for some $i \in \{1, \dots, r\}$, then there exists $\gamma \in E$ such that $\gamma_i > \alpha_i = \beta_i$ and $\gamma_j \geq \min\{\alpha_j, \beta_j\}$ for each $j \neq i$, with equality holding if $\alpha_j \neq \beta_j$.*

Property (C). *There exist $\alpha \in \mathbb{Z}^r$ and $\gamma \in \mathbb{N}^r$ such that*

$$\gamma + \mathbb{N}^r \subset E \subset \alpha + \mathbb{Z}^r.$$

Properties (A) and (C) allow one to conclude that there exist a unique $m_E = (m_1, \dots, m_r)$ such that $\beta_i \geq m_i$, $i = 1, \dots, r$, for all $(\beta_1, \dots, \beta_r) \in E$ and a unique least element $\gamma \in E$ with the property that $\gamma + \mathbb{N}^r \subset E$. This element is what we call the conductor of E and will denote it by $c(E)$.

Observe that one always has

$$c(E_J) \leq \text{pr}_J(c(E)), \quad \forall J \subset I.$$

One has the following result:

Lemma 1. *If \mathcal{I} is a fractional ideal of R and $\emptyset \neq J \subset I$, then $\text{pr}_J(E) = E_J$.*

Proof. One has obviously that $\text{pr}_J(E) \subset E_J$. On the other hand, let $\alpha_J \in E_J$. Take $h \in \mathcal{I}$ such that $v_J(\pi_J(h)) = \alpha_J$. If $h \notin Z(K)$ we are done. Otherwise, choose any $h' \in \mathcal{I} \setminus Z(K)$ such that $\text{pr}_J(v(h')) > \alpha_J$, which exists since E has a conductor. Hence, $v_J(h + h') = \alpha_J$, proving the other inclusion. \square

4. MAXIMAL POINTS

We now introduce the important notion of a *fiber* of an element $\alpha \in E$ with respect to a subset $J \subset I$ that will play a central role in what follows.

Definition 1. Given $A \subset \mathbb{Z}^r$, $\alpha \in \mathbb{Z}^r$ and $\emptyset \neq J \subset I$, we define

$$F_J(A, \alpha) = \{\beta \in A; \text{pr}_J(\beta) = \text{pr}_J(\alpha) \text{ and } \text{pr}_{I \setminus J}(\beta) > \text{pr}_{I \setminus J}(\alpha)\},$$

$$\overline{F}_J(A, \alpha) = \{\beta \in A; \text{pr}_J(\beta) = \text{pr}_J(\alpha), \text{ and } \text{pr}_{I \setminus J}(\beta) \geq \text{pr}_{I \setminus J}(\alpha)\},$$

The set $F(A, \alpha) = \bigcup_{i=1}^r F_{\{i\}}(A, \alpha)$ will be called the *fiber* of α in A .

The sets $F_{\{i\}}(A, \alpha)$ and $\overline{F}_{\{i\}}(A, \alpha)$ will be denoted simply by $F_i(A, \alpha)$ and $\overline{F}_i(A, \alpha)$. Notice that $F_I(\mathbb{Z}^r, \alpha) = \overline{F}_I(\mathbb{Z}^r, \alpha) = \{\alpha\}$.

Definition 2. $\alpha \in A$ is called a *maximal point* of A , if $F(A, \alpha) = \emptyset$.

This means that there is no element in A with one coordinate equal to the corresponding coordinate of α and the others bigger.

From now on, E will denote the value set of the regular fractional ideal \mathcal{I} of R . From the fact that E has a minimum m and a conductor $\gamma = c(E)$, one has immediately that all maximal points of E are in the limited region $\{(x_1, \dots, x_r) \in \mathbb{Z}^r; m_i \leq x_i < \gamma_i, i = 1, \dots, r\}$. This implies that E has finitely many maximal points.

Definition 3. We will say that a maximal point α of E is an *absolute maximal* if $F_J(E, \alpha) = \emptyset$ for every $J \subset I, J \neq I$. If a maximal point α of E is such that $F_J(E, \alpha) \neq \emptyset$, for every $J \subset I$ with $\#J \geq 2$, then α will be called a *relative maximal* of E .

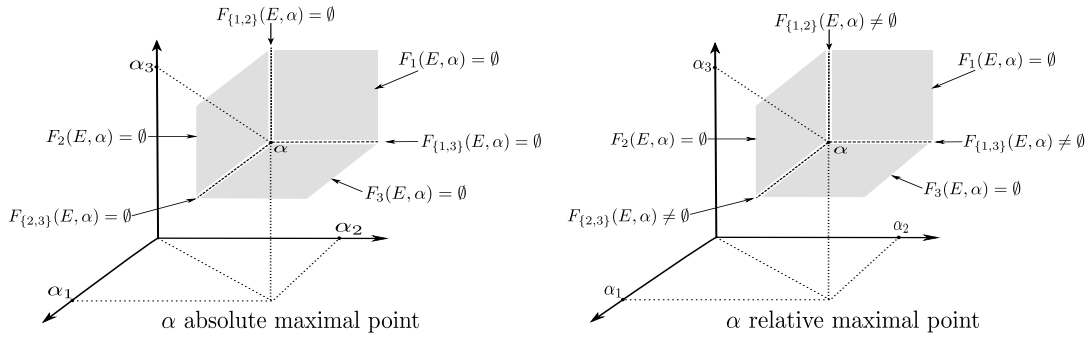


FIGURE 1. Maximal points

In the case where $r = 2$, the notions of maximal, relative maximal and absolute maximal coincide. For $r = 3$ we may only have relative maximals or absolute maximals, but in general there will be several types of maximals.

We will denote by $M(E)$, $RM(E)$ and $AM(E)$ the sets of maximals, of relative maximals and absolute maximals of the set E , respectively.

The importance of the relative maximals is attested by the theorem below that says that the set $RM(E)$ determines E in a combinatorial sense as follows:

Theorem 2 (generation). Let $\alpha \in \mathbb{Z}^r$ be such that $p_J(\alpha) \in E_J$ for all $J \subset I$ with $\#J = r - 1$. Then

$$\alpha \in E \iff \alpha \notin F(\mathbb{Z}^r, \beta), \forall \beta \in RM(E).$$

We will omit the proof since this result is a slight modification of [3, Theorem 1.5] with essentially the same proof.

The following two lemmas give us characterizations of the relative and absolute maximal points that will be useful in Section 4.

Lemma 3. *Given a value set $E \subset \mathbb{Z}^r$ and $\alpha \in \mathbb{Z}^r$ with the following properties:*

- i) *there is $i \in I$ such that $F_i(E, \alpha) = \emptyset$,*
- ii) *$F_{i,j}(E, \alpha) \neq \emptyset$ for all $j \in I \setminus \{i\}$.*

Then α is a relative maximal of E .

Proof. Follows the same steps as the proof of [3, Lemma 1.3] □

Lemma 4. *Given a value set $E \subset \mathbb{Z}^r$ and $\alpha \in E$, assume that there exists an index $i \in I$ such that $F_J(E, \alpha) = \emptyset$ for every $J \subsetneq I$ with $i \in J$. Then α is an absolute maximal of E .*

Proof. We have to prove that $F_K(E, \alpha) = \emptyset$ for all $K \subset I$ with $i \notin K$.

Assume, by reductio ad absurdum, that there exists some $K \subset I$ with $i \notin K$ such that $F_K(E, \alpha) \neq \emptyset$. Let β be an element in $F_K(E, \alpha)$, then $\beta_k = \alpha_k, \forall k \in K$ and $\beta_j > \alpha_j$, for all $j \notin K$. Applying Property (B) for α, β and any index $k' \in K$, we have that there exists $\theta \in E$ such that $\theta_{k'} > \beta_{k'} = \alpha_{k'}, \theta_l \geq \min\{\alpha_l, \beta_l\}, \forall l \neq k'$ and $\theta_j = \alpha_j$ for all $j \notin K$. If $B = (I \setminus K) \cup \{l \in K, \theta_l = \alpha_l\}$, then we have $\theta \in F_B(E, \alpha) (\neq \emptyset)$, with $i \in B$, which is a contradiction. □

5. COLENGTHS OF FRACTIONAL IDEALS

Let R be a complete admissible ring and let $\mathcal{J} \subset \mathcal{I}$ two regular fractional ideals of R with value sets D and E , respectively. Since $\mathcal{J} \subset \mathcal{I}$, one has that $D \subset E$, hence $c(E) \leq c(D)$. Our aim in this section is to find a formula for the length $\ell_R(\mathcal{I}/\mathcal{J})$ of \mathcal{I}/\mathcal{J} as R -modules, called the colength of \mathcal{J} with respect to \mathcal{I} , in terms of the value sets D and E .

The motivation comes from the case $r = 1$, that is, when R is a domain. In this case, as observed by Gorenstein [6], one can easily show that

$$\ell_R(\mathcal{I}/\mathcal{J}) = \#(E \setminus D).$$

When $r > 1$, then $E \setminus D$ is not finite anymore.

For $\alpha \in \mathbb{Z}^r$ and \mathcal{I} a fractional ideal of R , with value set E , we define

$$\mathcal{I}(\alpha) = \{h \in \mathcal{I}; v(h) \geq \alpha\}.$$

It is clear that if $m_E = \min E$, then $\mathcal{I}(m_E) = \mathcal{I}$.

One has the following result:

Proposition 5. ([1, Proposition 2.7]) *Let $\mathcal{J} \subseteq \mathcal{I}$ be two fractional ideals of R , with value sets D and E , respectively, then*

$$\ell_R\left(\frac{\mathcal{I}}{\mathcal{J}}\right) = \ell_R\left(\frac{\mathcal{I}}{\mathcal{I}(\gamma)}\right) - \ell_R\left(\frac{\mathcal{J}}{\mathcal{J}(\gamma)}\right),$$

for sufficiently large $\gamma \in \mathbb{N}^r$ (for instance, if $\gamma \geq c(D)$).

If $e_i \in \mathbb{Z}^r$ denotes the vector with zero entries except the i -th entry which is equal to 1, then the following result will give us an effective way to calculate colengths of ideals.

Proposition 6. [2, Proposition 2.2] *If $\alpha \in \mathbb{Z}^r$, then we have*

$$\ell_R\left(\frac{\mathcal{I}(\alpha)}{\mathcal{I}(\alpha + e_i)}\right) = \begin{cases} 1, & \text{if } \overline{F}_i(E, \alpha) \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases}$$

So, to compute, for instance, $\ell_R\left(\frac{\mathcal{I}}{\mathcal{I}(\gamma)}\right)$, one may take a chain

$$m_E = \alpha^0 \leq \alpha^1 \leq \cdots \leq \alpha^m = \gamma,$$

where $\alpha^j \in \mathbb{Z}^r$ and $\alpha^j - \alpha^{j-1} \in \{e_i, i = 1, \dots, r\}$, and then using Proposition 6 by observing that

$$\ell_R\left(\frac{\mathcal{I}}{\mathcal{I}(\gamma)}\right) = \ell_R\left(\frac{\mathcal{I}(\alpha^0)}{\mathcal{I}(\gamma)}\right) = \sum_{j=1}^m \ell_R\left(\frac{\mathcal{I}(\alpha^{j-1})}{\mathcal{I}(\alpha^j)}\right).$$

D'Anna in [2] showed that $\ell_R\left(\frac{\mathcal{I}}{\mathcal{I}(\gamma)}\right)$ is equal to the length n of a saturated chain

$$m_E < \alpha^0 < \alpha^1 < \cdots < \alpha^n = \gamma$$

in E . The drawback of this result is that one has to know all points of E in the hypercube with opposite vertices m_E and γ .

The fact that E is determined by its projections E_J and its relative maximal points, suggests that $\ell_R\left(\frac{\mathcal{I}}{\mathcal{I}(\gamma)}\right)$ can be computed in terms of these data. In fact, this will be done in Theorem 1 below.

In what follows we will denote ℓ_R simply by ℓ .

5.1. Case $r=2$. This simplest case was studied by Barucci, D'Anna and Fröberg in [1] and we reproduce it here because it gives a clue on how to proceed in general.

Let $\alpha^0 = m_E$ and consider the chain in \mathbb{Z}^2

$$\alpha^0 \leq \cdots \leq \alpha^m = \gamma = (\gamma_1, \gamma_2) \geq c(E)$$

such that

$$\alpha^0 = (\alpha_1^0, \alpha_2^0), \alpha^1 = (\alpha_1^0 + 1, \alpha_2^0), \dots, \alpha^s = (\gamma_1, \alpha_2^0),$$

$$\alpha^{s+1} = (\gamma_1, \alpha_2^0 + 1), \alpha^{s+2} = (\gamma_1, \alpha_2^0 + 2), \dots, \alpha^m = (\gamma_1, \gamma_2),$$

and consider the following sets

$$L_1 = \{\alpha^0, \alpha^1, \dots, \alpha^s\} \quad \text{and} \quad L_2 = \{\alpha^s, \alpha^{s+1}, \dots, \alpha^m\}.$$

By Proposition 6, we have

$$\begin{aligned} \ell\left(\frac{\mathcal{I}}{\mathcal{I}(\gamma)}\right) &= \#L_1 - \#\{\alpha \in L_1; \overline{F}_1(E, \alpha) = \emptyset\} + \\ &\quad \#L_2 - \#\{\alpha \in L_2; \overline{F}_2(E, \alpha) = \emptyset\}. \end{aligned}$$

Now, because of our choice of L_1 , denoting by $\mathcal{G}(E_i)$ the set of gaps of E_i in the interval $(\min(E_i), +\infty)$, we have that

$$\forall \alpha \in L_1, \overline{F}_1(E, \alpha) = \emptyset \iff \text{pr}_1(\alpha) \in \mathcal{G}(E_1),$$

hence

$$\#\{\alpha \in L_1; \overline{F}_1(E, \alpha) = \emptyset\} = \#\mathcal{G}(E_1).$$

Observe that not all $\alpha \in L_2$ with $\overline{F}_2(E, \alpha) = \emptyset$ are such that $\text{pr}_2(\alpha) \in \mathcal{G}(E_2)$, hence

$$\#\{\alpha \in L_2; \overline{F}_2(E, \alpha) = \emptyset\} = \#\mathcal{G}(E_2) - \xi,$$

where ξ is the number of α in L_2 with $\text{pr}_2(\alpha) \in E_2$ and $\overline{F}_2(E, \alpha) = \emptyset$. But, such α are in one-to-one correspondence with the maximal points of E , hence $\xi = \#M(E)$.

Putting all this together, we get

Proposition 7. *If $\gamma \geq c(E)$, then*

$$(1) \quad \ell \left(\frac{\mathcal{I}}{\mathcal{I}(\gamma)} \right) = (\gamma_1 - \alpha_1^0) + (\gamma_2 - \alpha_2^0) - \#\mathcal{G}(E_1) - \#\mathcal{G}(E_2) - \#M(E).$$

5.2. **Case $r \geq 3$.** Let us assume that \mathcal{I} is a fractional ideal of R , where R has r minimal primes.

Let

$$m_E = \alpha^0 \leq \alpha^1 \leq \dots \leq \alpha^m = \gamma \geq c(E),$$

be the chain in \mathbb{Z}^r , given by the union of the following paths (see Figure 2, for $r = 3$):

$$L_1: \alpha^0, \alpha^1 = \alpha^0 + e_1, \dots, \alpha^{s_1} = \alpha^0 + (\gamma_1 - \alpha_1^0)e_1 = (\gamma_1, \alpha_2^0, \dots, \alpha_r^0),$$

...

$$L_r: \alpha^{s_{r-1}} = (\gamma_1, \dots, \gamma_{r-1}, \alpha_r^0), \alpha^{s_{r-1}+1} = \alpha^{s_{r-1}} + e_r, \dots, \alpha^m = \gamma.$$

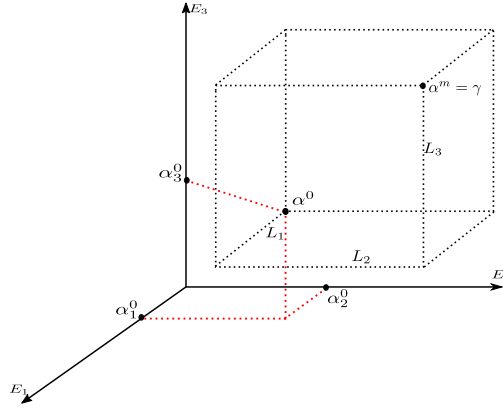


FIGURE 2. The chain for $r = 3$

For $i \in I$, let us define $[1, i] = [1, i + 1) = \{1, \dots, i\}$. We will need the following result:

Lemma 8. *For any $\alpha \in L_1 \cup \dots \cup L_{r-1}$, and for $i \in [1, r)$, one has*

$$\overline{F}_i(E, \alpha) \neq \emptyset \iff \overline{F}_i(E_{[1,r)}, \text{pr}_{[1,r)}(\alpha)) \neq \emptyset.$$

Proof. (\Rightarrow) This is obvious.

(\Leftarrow) Suppose that

$$(\theta_1, \dots, \theta_{r-1}) \in \overline{F}_i(E_{[1,r)}, \text{pr}_{[1,r)}(\alpha)) \neq \emptyset.$$

Since by Lemma 1 one has that $\text{pr}_{[1,r)}(E) = E_{[1,r)}$, then there exists $\theta = (\theta_1, \dots, \theta_{r-1}, \theta_r) \in E$. Since $\alpha \in L_i$ for some $i = 1, \dots, r - 1$, it follows that $\alpha_r = \alpha_r^0$. Then one cannot have $\theta_r < \alpha_r = \alpha_r^0$, because otherwise

$$(\alpha_1^0, \dots, \alpha_{r-1}^0, \theta_r) = \min(\alpha^0, \theta) \in E,$$

which is contradiction, since α^0 is the minimum of E . Hence $\theta_r \geq \alpha_r$, so $\theta \in \overline{F}_i(E, \alpha)$, and the result follows. \square

Lemma 8 allows us to write:

$$(2) \quad \ell \left(\frac{\mathcal{I}}{\mathcal{I}(\gamma)} \right) = \ell \left(\frac{\pi_{[1,r]}(\mathcal{I})}{\pi_{[1,r]}(\mathcal{I})(\text{pr}_{[1,r]}(\gamma))} \right) + (\gamma_r - \alpha_r^0) - \#\{\alpha \in L_r; \overline{F}_r(E, \alpha) = \emptyset\}.$$

Hence to get an inductive formula for $\ell \left(\frac{\mathcal{I}}{\mathcal{I}(\gamma)} \right)$, we only have to compute

$$\#\{\alpha \in L_r; \overline{F}_r(E, \alpha) = \emptyset\},$$

and for this we will need the following lemma.

Lemma 9. *Let $\alpha \in \mathbb{Z}^r$, then $\overline{F}_j(E, \alpha) = \emptyset$ if and only if either $\alpha_j \in \mathcal{G}(E_j)$ or there exist some $J \subseteq I$ with $\{j\} \subsetneq J$ and a relative maximal β of E_J such that $\tilde{\text{pr}}_j(\beta) = \alpha_j$ and $\tilde{\text{pr}}_i(\beta) < \alpha_i$, for all $i \in J, i \neq j$.*

Proof. (\Leftarrow) (We prove more, since it is enough to assume β is any maximal of E_J) It is obvious that if $\alpha_j \in \mathcal{G}(E_j)$, then $\overline{F}_j(E, \alpha) = \emptyset$. Let us now assume that there exist $J \subset I$, with $\{j\} \subsetneq J$ and $\beta \in M(E_J)$, such that $\tilde{\text{pr}}_j(\beta) = \alpha_j$ and $\tilde{\text{pr}}_i(\beta) < \alpha_i$, for all $i \in J, i \neq j$.

Suppose by reductio ad absurdum that $\overline{F}_j(E, \alpha) \neq \emptyset$. Let $\theta \in \overline{F}_j(E, \alpha)$, that is, $\theta_j = \alpha_j$ and $\theta_i \geq \alpha_i, \forall i \in J \setminus \{j\}$. Now since, $\forall i \in J, i \neq j$,

$$\tilde{\text{pr}}_j(\text{pr}_J(\theta)) = \theta_j = \alpha_j = \tilde{\text{pr}}_j(\beta) \text{ and } \tilde{\text{pr}}_i(\text{pr}_J(\theta)) = \theta_i \geq \alpha_i > \tilde{\text{pr}}_i(\beta),$$

then $\text{pr}_J(\theta) \in F_j(E_J, \beta)$, which contradicts the assumption that $\beta \in M(E_J)$.

(\Rightarrow) Since $\overline{F}_j(E, \alpha) = \emptyset$ implies $F_j(E, \alpha) = \emptyset$, the proof follows the same lines as the proof of [4, Theorem 1.5]. \square

Going back to our main calculation, by Lemma 9, if $\alpha \in L_r$ is such that $\overline{F}_r(E, \alpha) = \emptyset$, then either $\alpha_r \in \mathcal{G}(E_r)$, or there exist a subset J of $I = \{1, \dots, r\}$, with $\{r\} \subsetneq J$, and $\beta \in RM(E_J)$, with $\tilde{\text{pr}}_r(\beta) = \alpha_r$ and $\tilde{\text{pr}}_i(\beta) < \alpha_i$ for $i \in J, i \neq r$.

Notice that for $\alpha \in L_r$ one has $\alpha_i = \gamma_i$ for $i \neq r$, so the condition $\tilde{\text{pr}}_i(\beta) < \alpha_i$ for $i \in J, i \neq r$ is satisfied, since $\beta \in M(E_J)$. So, we have a bijection

$$\{\alpha \in L_r; \overline{F}_r(E, \alpha) = \emptyset\} \longleftrightarrow \mathcal{G}(E_r) \cup \bigcup_{\{r\} \subsetneq J \subseteq I} \tilde{\text{pr}}_r(RM(E_J)).$$

Since for all J , with $\{r\} \subsetneq J \subseteq I$, the sets $\mathcal{G}(E_r)$ and $\tilde{\text{pr}}_r(RM(E_J))$ are disjoint, it follows that

$$(3) \quad \#\{\alpha \in L_r; \overline{F}_r(E, \alpha) = \emptyset\} = \#\mathcal{G}(E_r) + \#\left(\bigcup_{\{r\} \subsetneq J \subseteq I} \tilde{\text{pr}}_r(RM(E_J)) \right).$$

Let us define

$$\Theta_1 = 0, \quad \text{and} \quad \Theta_i = \#\bigcup_{\{i\} \subsetneq J \subseteq [1, i]} \tilde{\text{pr}}_i(RM(E_J)), \quad 2 \leq i \leq r.$$

Now, putting together Equations (2) and (3), we get the following recursive formula:

Theorem 10. *Let \mathcal{I} be a fractional ideal of a ring R with r minimal primes with values set E . If $\gamma \geq c(E)$, then*

$$(4) \quad \ell \left(\frac{\mathcal{I}}{\mathcal{I}(\gamma)} \right) = \ell \left(\frac{\pi_{[1,r]}(\mathcal{I})}{\pi_{[1,r]}(\mathcal{I})(\text{pr}_{[1,r]}(\gamma))} \right) + (\gamma_r - \alpha_r^0) - \#\mathcal{G}(E_r) - \Theta_r$$

Corollary 11. *With the same hypotheses as in Theorem 10, one has the formula*

$$\ell\left(\frac{\mathcal{I}}{\mathcal{I}(\gamma)}\right) = \sum_{i=1}^r (\gamma_i - \alpha_i^0 - \#\mathcal{G}(E_i) - \Theta_i).$$

6. A CLOSED FORMULA FOR $r = 3$

In this section, we provide a nicer formula than Equation (4), when $r = 3$. To simplify notation, for any $J \subset I = \{1, 2, 3\}$, we will denote by RM_J , AM_J and M_J the sets $RM(E_J)$, $AM(E_J)$ and $M(E_J)$, respectively. Notice also that if $\#J = 2$, then $RM_J = AM_J = M_J$.

From Formulas (1) and (2), for $\gamma \geq c(E)$, one has

$$\begin{aligned} \ell\left(\frac{\mathcal{I}}{\mathcal{I}(\gamma)}\right) &= (\gamma_1 - \alpha_1^0) - \#\mathcal{G}(E_1) + (\gamma_2 - \alpha_2^0) - \#\mathcal{G}(E_2) - \#M_{\{1,2\}} + \\ &\quad (\gamma_3 - \alpha_3^0) - \#\{\alpha \in L_3; \overline{F}_3(E, \alpha) = \emptyset\}. \end{aligned}$$

We will use the following notation:

$$L'_3 = \{\alpha \in L_3; \overline{F}_3(E, \alpha) = \emptyset\}.$$

Now, from Lemma 9, the points $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in L'_3$ are such that $\alpha_3 \in \mathcal{G}(E_3)$ or they are associated to maximal points of either $E_{\{1,3\}}$, $E_{\{2,3\}}$, or E with last coordinate equal to α_3 . So, we have

$$(5) \quad \#L'_3 = \#\mathcal{G}(E_3) + \#M_{\{1,3\}} + \#M_{\{2,3\}} + \#RM - \eta,$$

where η is some correcting term which will take into account the eventual multiple counting of maximals having the same last coordinate.

To compute η we will analyze in greater detail the geometry of maximal points.

If $\alpha, \beta \in M$ with $\alpha_3 = \beta_3$, then $\alpha_1 \neq \beta_1$ and $\alpha_2 \neq \beta_2$. If $\alpha_1 < \beta_1$, then necessarily $\beta_2 < \alpha_2$.

We say that two relative (respectively, absolute) maximals α and β of E with $\alpha_3 = \beta_3$ and $\alpha_1 < \beta_1$ are *adjacent*, if there is no $(\theta_1, \theta_2, \alpha_3)$ in RM (respectively, in AM) with $\alpha_1 < \theta_1 < \beta_1$ and $\beta_2 < \theta_2 < \alpha_2$.

We will describe below the geometry of the maximal points of E

Lemma 12. *If $\alpha \in AM$, then one of the following three conditions is verified:*

- (i) *there exist two adjacent relative maximals β and θ of E such that $\text{pr}_{\{1,3\}}(\beta) = \text{pr}_{\{1,3\}}(\alpha)$ and $\text{pr}_{\{2,3\}}(\theta) = \text{pr}_{\{2,3\}}(\alpha)$;*
- (ii) *there exists $\beta \in RM$ such that $\text{pr}_{\{1,3\}}(\beta) = \text{pr}_{\{1,3\}}(\alpha)$ and $\text{pr}_{\{2,3\}}(\alpha) \in M_{\{2,3\}}$, or $\text{pr}_{\{2,3\}}(\beta) = \text{pr}_{\{2,3\}}(\alpha)$ and $\text{pr}_{\{1,3\}}(\alpha) \in M_{\{1,3\}}$;*
- (iii) *$\text{pr}_{\{1,3\}}(\alpha) \in M_{\{1,3\}}$ and $\text{pr}_{\{2,3\}}(\alpha) \in M_{\{2,3\}}$.*

Proof. Let $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in AM$, then $F(E, \alpha) = \emptyset$. We consider the following sets:

$$R_1 = \{\beta \in \mathbb{Z}^3; \beta_3 = \alpha_3, \beta_1 > \alpha_1, \beta_2 < \alpha_2\}$$

and

$$R_2 = \{\theta \in \mathbb{Z}^3; \theta_3 = \alpha_3, \theta_1 < \alpha_1, \theta_2 > \alpha_2\}.$$

Then there are four possibilities:

$$R_1 \cap E \neq \emptyset \text{ and } R_2 \cap E \neq \emptyset, \quad R_1 \cap E \neq \emptyset \text{ and } R_2 \cap E = \emptyset.$$

$$R_1 \cap E = \emptyset \text{ and } R_2 \cap E \neq \emptyset, \quad R_1 \cap E = \emptyset \text{ and } R_2 \cap E = \emptyset.$$

Suppose $R_1 \cap E \neq \emptyset$ and $R_2 \cap E \neq \emptyset$. Choose $\beta \in R_1 \cap E$ and $\theta \in R_2 \cap E$, such that $\alpha_2 - \beta_2$ and $\alpha_1 - \theta_1$ are as small as possible. Then by Property (A), we have $\min(\alpha, \beta), \min(\alpha, \theta) \in E$. Obviously $\text{pr}_{\{1,3\}}(\beta) = \text{pr}_{\{1,3\}}(\alpha)$ and $\text{pr}_{\{2,3\}}(\theta) = \text{pr}_{\{2,3\}}(\alpha)$. Moreover, according to Lemma 3, these are relative maximals because $F_3(E, \min(\alpha, \beta))$ and $F_3(E, \min(\alpha, \theta))$ are empty and the sets $F_{\{1,3\}}(E, \min(\alpha, \beta)), F_{\{1,3\}}(E, \min(\alpha, \theta)), F_{\{2,3\}}(E, \min(\alpha, \beta))$ and $F_{\{2,3\}}(E, \min(\alpha, \theta))$ are nonempty. It follows that $\min(\alpha, \beta)$ and $\min(\alpha, \theta)$ are adjacent relative maximals.

Suppose $R_1 \cap E \neq \emptyset$ and $R_2 \cap E = \emptyset$. Choose $\beta \in R_1 \cap E$ such that $\alpha_2 - \beta_2$ is as small as possible, then, as we argued above, we have that $\min(\alpha, \beta) \in RM$ and $\text{pr}_{\{1,3\}}(\beta) = \text{pr}_{\{1,3\}}(\alpha)$. Moreover, as $R_2 \cap E = \emptyset$, it follows that $\text{pr}_{\{2,3\}}(\alpha) \in M_{\{2,3\}}$.

The case $R_1 \cap E = \emptyset$ and $R_2 \cap E \neq \emptyset$ is similar to the above one, giving us the second possibility in (ii).

Suppose $R_1 \cap E = \emptyset$ and $R_2 \cap E = \emptyset$. It is obvious that

$$\text{pr}_{\{1,3\}}(\alpha) \in M_{\{1,3\}} \quad \text{and} \quad \text{pr}_{\{2,3\}}(\alpha) \in M_{\{2,3\}}.$$

□

Given two points $\theta^1, \theta^2 \in \mathbb{Z}^3$ such that $\text{pr}_3(\theta^1) = \text{pr}_3(\theta^2)$, we will denote by $\mathcal{R}(\theta^1, \theta^2)$ the parallelogram determined by the coplanar points $\theta^1, \theta^2, \min(\theta^1, \theta^2)$ and $\max(\theta^1, \theta^2)$. We have the following result:

Corollary 13. *Let $\theta^1, \theta^2 \in AM$ be such that $\text{pr}_3(\theta^1) = \text{pr}_3(\theta^2)$. Then one has*

$$\mathcal{R}(\theta^1, \theta^2) \cap RM \neq \emptyset.$$

Proof. Because $\theta^1, \theta^2 \in AM$, it follows immediately that (iii) of Lemma 12 cannot happen, therefore, the existence of the relative maximal is ensured by (i) or (ii). □

Lemma 14. *If β and β' are adjacent relative maximals, with $\beta_3 = \beta'_3$, then $\max(\beta, \beta')$ is an absolute maximal of E .*

Proof. We may suppose that $\beta_1 > \beta'_1$ and $\beta_2 < \beta'_2$. As β and β' are adjacent, we have that $F_{\{1,3\}}(E, \beta) \cap F_{\{2,3\}}(E, \beta') \neq \emptyset$, because otherwise, take $\alpha^1 \in F_{\{1,3\}}(E, \beta)$, with α_2^1 the greatest possible and $\alpha^2 \in F_{\{2,3\}}(E, \beta')$, with α_1^2 the greatest possible. From Lemma 4 it follows that α^1 and α^2 are absolute maximals of E , then by Corollary 13 there exists a relative maximal in the region $\mathcal{R}(\alpha^1, \alpha^2)$, this contradicts the fact that β and β' are adjacent relative maximals.

Then, effectively, $F_{\{1,3\}}(E, \beta) \cap F_{\{2,3\}}(E, \beta') = \{\max(\beta, \beta')\}$, which is an absolute maximal. □

Recall that the elements in L'_3 are of the form $(\gamma_1, \gamma_2, \alpha_3)$, with $\alpha_3^0 \leq \alpha_3 \leq \gamma_3$.

Lemma 15. *Let $\alpha \in L'_3$ be such that*

$$\alpha_3 \in (\tilde{\text{pr}}_3(M_{\{1,3\}}) \setminus \tilde{\text{pr}}_3(M_{\{2,3\}})) \cap \text{pr}_3(RM) \quad \text{or} \quad \alpha_3 \in (\tilde{\text{pr}}_3(M_{\{2,3\}}) \setminus \tilde{\text{pr}}_3(M_{\{1,3\}})) \cap \text{pr}_3(RM).$$

Then there are the same number of relative as absolute maximals in E with third coordinate equal to α_3 .

Proof. We assume that $\alpha_3 \in (\tilde{\text{pr}}_3(M_{\{1,3\}}) \setminus \tilde{\text{pr}}_3(M_{\{2,3\}})) \cap \text{pr}_3(RM)$, since the other case is analogous.

Since $\alpha_3 \in \text{pr}_3(RM)$, we may assume that there are $s (\geq 1)$ relative maximals β^1, \dots, β^s in E with third coordinate equal to α_3 . We may suppose that $\beta_1^1 < \beta_1^2 < \dots < \beta_1^s$, so the β^i 's are successively adjacent relative maximals, hence, by lemma 14, we have that

$$\max(\beta^1, \beta^2), \dots, \max(\beta^{s-1}, \beta^s) \in AM.$$

This shows that there are at least $s - 1$ absolute maximals in E with third coordinate α_3 .

Now as $\text{pr}_3(\alpha) \in \tilde{\text{pr}}_3(M_{\{1,3\}})$, then there is a $(\eta_1^1, \alpha_3) \in M_{\{1,3\}}$ with $\eta_1^1 \leq \alpha_1 (= \gamma_1)$, because $c(E_{\{1,3\}}) \leq \text{pr}_{\{1,3\}}(c(E)) = (\gamma_1, \gamma_3)$. Because of our hypothesis, the elements δ in the fiber $F_{\{1,3\}}(E, \beta^s)$ are such that $\beta_1^s < \delta_1 \leq \eta_1^1$. But we must have $\delta_1 = \eta_1^1$, because, otherwise, there would be a point $\eta^1 = (\eta_1^1, \eta_2^1, \alpha_3) \in \text{pr}_{\{1,3\}}^{-1}(\eta_1^1, \alpha_3)$, with $\eta_2^1 < \beta_2^s$, and a point $\eta^2 \in F_{\{2,3\}}(E, \beta^s)$ with $\eta_1^2 < \eta_1^1$ and $\eta_2^2 = \beta_2^s$. These η^1 and η^2 are absolute maximals, due to Lemma 4, then from Corollary 13, there would exist a relative maximal in the region $\mathcal{R}(\eta^1, \eta^2)$, which contradicts the fact that we have s relative maximals. This implies that $(\beta_1^s, \eta_2^1, \alpha_3)$ is an absolute maximal of E .

We have to show that there are no other absolute maximals. If such maximal existed, then one of the three conditions in Lemma 12 would be satisfied. Obviously conditions (i) and (iii) cannot be satisfied, but neither condition (ii) can be satisfied, because otherwise $\alpha_3 \in \tilde{\text{pr}}_3(M_{\{2,3\}})$, which is a contradiction. \square

Lemma 16. *Let $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in L'_3$ be such that $\alpha_3 \in (\tilde{\text{pr}}_3(M_{\{1,3\}}) \cap \tilde{\text{pr}}_3(M_{\{2,3\}})) \setminus \text{pr}_3(RM)$, then there exists one and only one absolute maximal of E with third coordinate equal to α_3 .*

Proof. As

$$\alpha_3 \in \tilde{\text{pr}}_3(M_{\{1,3\}}) \cap \tilde{\text{pr}}_3(M_{\{2,3\}}),$$

then there exist $(\beta_1^1, \alpha_3) \in M_{\{1,3\}}$ and $(\beta_2^2, \alpha_3) \in M_{\{2,3\}}$ such that $\beta_1^1 < \alpha_1 (= \gamma_1)$ and $\beta_2^2 < \alpha_2 (= \gamma_2)$, because one always has that $c(E_{\{i,j\}}) \leq \text{pr}_{\{i,j\}}(c(E))$.

Consider the element $\theta = (\beta_1^1, \beta_2^2, \alpha_3)$. If $\theta \in E$, since it is easy to verify that $F_J(E, \theta) = \emptyset$ for $3 \in J \subsetneq \{1, 2, 3\}$, it follows by Lemma 4 that θ is an absolute maximal of E , which is unique in view of Corollary 13 and the hypothesis that $\alpha_3 \notin \text{pr}_3(RM)$.

If $\theta \notin E$, then take $\theta_1 = (\beta_1^1, \delta_2^1, \alpha_3) \in \text{pr}_{\{1,3\}}^{-1}(\beta_1^1, \alpha_3) \cap E$, and

$$\theta_2 = (\delta_1^2, \beta_2^2, \alpha_3) \in \text{pr}_{\{2,3\}}^{-1}(\beta_2^2, \alpha_3) \cap E.$$

We have that $\delta_2^1 < \beta_1^1$ and $\delta_1^2 < \beta_2^2$, because otherwise $\theta \in E$ or, (β_1^1, α_3) and/or (β_2^2, α_3) would not be maximals of $E_{\{1,3\}}$ and/or $E_{\{2,3\}}$. Choose δ_2^1 and δ_1^2 the greatest possible, then it is easy to verify that $F_J(E, \theta_i) = \emptyset$ for $i = 1, 2$ and $3 \in J \subsetneq \{1, 2, 3\}$. Hence from Lemma 4, θ_1 and θ_2 are absolute maximals of E , therefore from Corollary 13 there would be a relative maximal of E with third coordinate equal to α_3 , which is a contradiction. \square

Lemma 17. *Let $\alpha \in L'_3$ be such that $\alpha_3 \in \tilde{\text{pr}}_3(M_{\{1,3\}}) \cap \tilde{\text{pr}}_3(M_{\{2,3\}}) \cap \text{pr}_3(RM)$. If there exist s relative maximals with third coordinate equal to α_3 , then there exist $s + 1$ absolute maximals with third coordinate equal to α_3 .*

Proof. Following the proof of Lemma 15, we have $s - 1$ absolute maximals obtained by taking the maximum of each pair of adjacent relative maximals. The conditions $\alpha_3 \in \tilde{\text{pr}}_3(M_{\{1,3\}})$ and $\alpha_3 \in \tilde{\text{pr}}_3(M_{\{2,3\}})$ give us two extra absolute maximals, and the same argument used there, shows that there are no other. \square

Lemma 18. *Let $\alpha \in L'_3$ be such that $\alpha_3 \in \text{pr}_3(RM) \setminus (\tilde{\text{pr}}_3(M_{\{1,3\}}) \cup \tilde{\text{pr}}_3(M_{\{2,3\}}))$. If there exist s relative maximals with third coordinate equal to α_3 , then we have $s - 1$ absolute maximals with third coordinate equal to α_3 .*

Proof. The arguments used in the proofs of the last two lemmas give us the result. \square

Going back to Formula (5), we want to calculate η . From Lemma 9 we can ensure that $\alpha \in L'_3 = \{\alpha \in L_3; \bar{F}_3(E, \alpha) = \emptyset\} \setminus \mathcal{G}(E_3)$, only if α falls into one of the following five cases:

- (i) $\alpha_3 \in (\tilde{\text{pr}}_3(M_{\{1,3\}}) \setminus \tilde{\text{pr}}_3(M_{\{2,3\}})) \cap \text{pr}_3(RM)$.

If there exist such α , then they are related to a unique element of $M_{\{1,3\}}$ and if there are s_1 relative maximals with third coordinate α_3 , then in our formula α was counted $s_1 + 1$ times. By Lemma 15 we know that there exist s_1 absolute maximals of E with third coordinate α_3 . So, we subtract s_1 from our counting to partially correct the formula.

- (ii) $\alpha_3 \in (\tilde{\text{pr}}_3(M_{\{2,3\}}) \setminus \tilde{\text{pr}}_3(M_{\{1,3\}})) \cap \text{pr}_3(RM)$.

Analogously to (i), α is related to a unique element of $M_{\{2,3\}}$ and if there are s_2 relative maximals with third coordinate α_3 , then α was counted $s_2 + 1$ times in the formula. Again, by Lemma 15 we know that there are s_2 absolute maximals of E with third coordinate α_3 . So, we subtract s_2 from our counting to partially correct the formula.

- (iii) $\alpha_3 \in (\tilde{\text{pr}}_3(M_{\{1,3\}}) \cap \tilde{\text{pr}}_3(M_{\{2,3\}})) \setminus \text{pr}_3(RM)$.

In this case, α is related to a unique elements in $M_{\{1,3\}}$ and in $M_{\{2,3\}}$, so in the formula we are counting α twice. By Lemma 16 there is a unique absolute maximal of E with third coordinate α_3 such that its projections $\text{pr}_{\{1,3\}}$ and $\text{pr}_{\{2,3\}}$ are in $M_{\{1,3\}}$ and $M_{\{1,3\}}$, respectively. So, we correct partially the formula by subtracting 1, which corresponds to this unique absolute maximal.

- (iv) $\alpha_3 \in \tilde{\text{pr}}_3(M_{\{1,3\}}) \cap \tilde{\text{pr}}_3(M_{\{2,3\}}) \cap \text{pr}_3(RM)$.

In this case, α is related to a unique element of $M_{\{1,3\}}$, to a unique element of $M_{\{2,3\}}$ and, let us say, s_3 elements of RM , so in our counting, α was counted $s_3 + 2$ times. By Lemma 17 there exist $s_3 + 1$ absolute maximals of E with third coordinate α_3 . In this case, the correcting term is $s_3 + 1$, equal to the number of these absolute maximals.

- (v) $\alpha_3 \in \text{pr}_3(RM) \setminus (\tilde{\text{pr}}_3(M_{\{1,3\}}) \cup \tilde{\text{pr}}_3(M_{\{2,3\}}))$.

In this case, α is related with, let us say, s_4 elements of RM with third coordinate equal to α_3 , so we are counting it s_4 times. By Lemma 18 there exist $s_4 - 1$ absolute maximals with third coordinate α_3 . This is exactly the correcting term we must apply to our formula.

Observe that the above cases exhaust all absolute maximals of E , implying the following result conjectured by M. E. Hernandez after having analyzed several examples (cf. [7]):

Theorem 19. *Let R be an admissible ring with three minimal primes and let \mathcal{I} be a fractional ideal of R with values set E . If $\gamma \geq c(E)$, then*

$$\ell\left(\frac{\mathcal{I}}{\overline{\mathcal{I}(\gamma)}}\right) = \sum_{i=1}^r ((\gamma_i - \alpha_i^0) - \#\mathcal{G}(E_i)) - \sum_{1 \leq i < j \leq 3} \#M_{\{i,j\}} - \#RM + \#AM.$$

Corollary 20. *Let $\mathcal{J} \subseteq \mathcal{I}$ be two fractional ideals of an admissible ring R , with three minimal primes. Denote by E and D , respectively, the value sets of \mathcal{I} and \mathcal{J} . Then*

$$\begin{aligned} \ell_R\left(\frac{\mathcal{I}}{\mathcal{J}}\right) &= \sum_{i=1}^3 ((\beta_i^0 - \alpha_i^0) + (\#\mathcal{G}(D_i) - \#\mathcal{G}(E_i))) + \\ &\quad \sum_{1 \leq i < j \leq 3} \#M_{\{i,j\}}(D) - \sum_{1 \leq i < j \leq 3} \#M_{\{i,j\}}(E) + \\ &\quad \#RM(D) - \#RM(E) + \#AM(E) - \#AM(D), \end{aligned}$$

where $\alpha^0 = \min(E)$ and $\beta^0 = \min(D)$.

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