

ON THE TOPOLOGY OF NON-ISOLATED REAL SINGULARITIES

NICOLAS DUTERTRE

Dedicated to Terence Gaffney and Maria Ruas, on the occasions of their 70th birthdays, and to Marcelo Saia, on the occasion of his 60th birthday

ABSTRACT. Khimshiashvili proved a topological degree formula for the Euler characteristic of the Milnor fibres of a real function-germ with an isolated singularity. We give two generalizations of this result for non-isolated singularities. As corollaries we obtain an algebraic formula for the Euler characteristic of the fibres of a real weighted-homogeneous polynomial and a real version of the Lê-Iomdine formula. We have also included some results of the same flavor on the local topology of locally closed definable sets.

1. INTRODUCTION

Let $f : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}, 0)$ be an analytic function-germ with an isolated critical point at the origin. Khimshiashvili [16] proved the following formula for the Euler characteristic of the real Milnor fibres of f :

$$\chi(f^{-1}(\delta) \cap B_\epsilon) = 1 - \text{sign}(-\delta)^n \text{deg}_0 \nabla f,$$

where $0 < |\delta| \ll \epsilon \ll 1$, B_ϵ is the closed ball centered at the origin of radius ϵ and $\text{deg}_0 \nabla f$ is the topological degree of the mapping $\frac{\nabla f}{|\nabla f|} : S_\epsilon \rightarrow S^{n-1}$ (here S_ϵ is the boundary of B_ϵ). Later Fukui [15] generalized this result for the fibres of a one-parameter deformation of f . A corollary of the Khimshiashvili formula due to Arnol'd [1] and Wall [37] states that

$$\begin{aligned} \chi(\{f \leq 0\} \cap S_\epsilon) &= 1 - \text{deg}_0 \nabla f, \\ \chi(\{f \geq 0\} \cap S_\epsilon) &= 1 + (-1)^{n-1} \text{deg}_0 \nabla f, \end{aligned}$$

and if n is even,

$$\chi(\{f = 0\} \cap S_\epsilon) = 2 - 2 \text{deg}_0 \nabla f.$$

In [31] Szafraniec extended the results of Arnold and Wall to the case of an analytic function-germ $f : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}, 0)$ with non-isolated singularities. Namely he constructed two function-germs g_- and g_+ with isolated critical points and proved that

$$\chi(\{f \leq 0\} \cap S_\epsilon) = 1 - \text{deg}_0 \nabla g_+ \text{ and } \chi(\{f \geq 0\} \cap S_\epsilon) = 1 - \text{deg}_0 \nabla g_-.$$

In [32] he improved this result for weighted homogeneous polynomials. If $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is a weighted homogeneous polynomial then he constructed two polynomials g_1 and g_2 with an algebraically isolated critical point at 0 such that

$$\chi(\{f \leq 0\} \cap S^{n-1}) = 1 - \text{deg}_0 \nabla g_1 \text{ and } \chi(\{f \geq 0\} \cap S^{n-1}) = 1 - \text{deg}_0 \nabla g_2.$$

Thanks to the Eisenbud-Levine-Khimshiashvili formula [14, 16],

$$\chi(\{f \leq 0\} \cap S^{n-1}) \text{ and } \chi(\{f \geq 0\} \cap S^{n-1})$$

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can be computed algebraically.

The aim of this paper is to extend the Khimshiashvili formula for function-germs with arbitrary singularities. We will work in the more general framework of definable functions. Let $f : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}, 0)$ be a definable function-germ of class C^r , $r \geq 2$. Our first new result is Lemma 2.5 where we give a relation between the Euler characteristic of $f^{-1}(\delta) \cap B_\epsilon$ (resp. $f^{-1}(-\delta) \cap B_\epsilon$), with $0 < \delta \ll \epsilon \ll 1$, and the Euler characteristic of the link at the origin of $\{f \leq 0\}$ (resp. $\{f \geq 0\}$). Applying the results of Szafraniec, we obtain our first generalization of the Khimshiashvili formula (Corollary 2.6) for polynomially bounded structures and an algebraic formula for the Euler characteristic of a regular fibre of a weighted homogeneous polynomial (Corollary 2.7). We note that the paper [7] presents a different approach for the computation of this Euler characteristic.

Our second generalization of the Khimshiashvili formula is an adaptation to the real case of the methods based on the generic polar curve, introduced in the complex case by Lê [19] and Teissier [34, 35] and developed later by Massey [24, 25, 26]. For $v \in S^{n-1}$, we denote by Γ_v the following relative polar set:

$$\Gamma_v = \{x \in \mathbb{R}^n \setminus \Sigma_f \mid \text{rank}(\nabla f(x), v) < 2\},$$

where $\Sigma_f = \{x \in \mathbb{R}^n \mid \nabla f(x) = 0\}$ is the critical locus of f . For v generic in S^{n-1} , Γ_v is a curve. Let \mathcal{B} be the set of its connected components. For each $\mathbf{b} \in \mathcal{B}$, we denote by $\sigma(\mathbf{b})$ the sign of $\det[\nabla f_{x_1}, \dots, \nabla f_{x_n}]$ on \mathbf{b} , where for $i = 1, \dots, n$, f_{x_i} denotes the partial derivative $\frac{\partial f}{\partial x_i}$. Moreover on \mathbf{b} the partial derivative $\frac{\partial f}{\partial v}$ does not vanish so we can decompose \mathcal{B} into the disjoint union $\mathcal{B}^+ \sqcup \mathcal{B}^-$, where \mathcal{B}^+ (resp. \mathcal{B}^-) is the set of half-branches on which $\frac{\partial f}{\partial v} > 0$ (resp. $\frac{\partial f}{\partial v} < 0$). This enables to define the following indices (Definition 4.8):

$$\lambda^+ = \sum_{\mathbf{b} \in \mathcal{B}^+} \sigma(\mathbf{b}) \text{ and } \lambda^- = \sum_{\mathbf{b} \in \mathcal{B}^-} \sigma(\mathbf{b}).$$

Then we define the following four indices (Definition 4.11):

$$\begin{aligned} \gamma^{+,+} &= \chi(f^{-1}(0) \cap \{x_1 = a\} \cap B_\epsilon) - \chi(f^{-1}(\alpha) \cap \{x_1 = a\} \cap B_\epsilon), \\ \gamma^{+,-} &= \chi(f^{-1}(0) \cap \{x_1 = -a\} \cap B_\epsilon) - \chi(f^{-1}(\alpha) \cap \{x_1 = -a\} \cap B_\epsilon), \\ \gamma^{-,+} &= \chi(f^{-1}(0) \cap \{x_1 = a\} \cap B_\epsilon) - \chi(f^{-1}(-\alpha) \cap \{x_1 = a\} \cap B_\epsilon), \\ \gamma^{-,-} &= \chi(f^{-1}(0) \cap \{x_1 = -a\} \cap B_\epsilon) - \chi(f^{-1}(-\alpha) \cap \{x_1 = -a\} \cap B_\epsilon), \end{aligned}$$

where $0 < \alpha \ll a \ll \epsilon$. Our second generalization of the Khimshiashvili formula relates the Euler characteristic of the real Milnor fibres to these new indices. Namely in Theorem 4.12 we show that

$$\chi(f^{-1}(-\delta) \cap B_\epsilon) = 1 - \lambda^- - \gamma^{-,-} = 1 - \lambda^+ - \gamma^{-,+},$$

and that

$$\chi(f^{-1}(\delta) \cap B_\epsilon) = 1 - (-1)^n \lambda^- - \gamma^{+,+} = 1 - (-1)^n \lambda^+ - \gamma^{+,-},$$

where $0 < \delta \ll \epsilon \ll 1$. Then we apply this result to the case where Σ_f has dimension one. In this case, we denote by \mathcal{C} the set of connected components of $\Sigma_f \setminus \{0\}$. For $v \in S^{n-1}$ generic, the function v^* does not vanish on any half-branch $\mathbf{c} \in \mathcal{C}$, so we can decompose \mathcal{C} into the disjoint union $\mathcal{C}^+ \sqcup \mathcal{C}^-$, where \mathcal{C}^+ (resp. \mathcal{C}^-) is the set of half-branches on which $v^* > 0$ (resp. $v^* < 0$). For each $\mathbf{c} \in \mathcal{C}$, let $\tau(\mathbf{c})$ be the value that the function $a \mapsto \deg_q \nabla f|_{x_1^{-1}(a)}$, $\{q\} = \mathbf{c} \cap \{x_1 = a\}$, takes close to the origin. Then we set $\gamma^+ = \sum_{\mathbf{c} \in \mathcal{C}^+} \tau(\mathbf{c})$ and $\gamma^- = \sum_{\mathbf{c} \in \mathcal{C}^-} \tau(\mathbf{c})$. In this situation, Theorem 4.12 takes the following form (Theorem 5.4):

$$\begin{aligned} \chi(f^{-1}(-\delta) \cap B_\epsilon) &= 1 - \lambda^- - \gamma^- = 1 - \lambda^+ - \gamma^+, \\ \chi(f^{-1}(\delta) \cap B_\epsilon) &= 1 - (-1)^n (\lambda^+ - \gamma^-) = 1 - (-1)^n (\lambda^- - \gamma^+), \end{aligned}$$

where $0 < \delta \ll \epsilon \ll 1$. Hence the indices λ^+ , λ^- , γ^+ and γ^- appear to be real versions of the first two Lê numbers defined by Massey in [24]. We note that the paper [36] contains also formulas for the Euler characteristic of the real Milnor fibres of a function-germ with a one-dimensional critical locus.

In the complex case, the Lê-Iomdine formula ([20, 18], see also [24, 27, 28, 30] for improved versions) relates the Euler characteristic of the Milnor fibre of an analytic function-germ with one-dimensional singular set to the Milnor fibre of an analytic function-germ with an isolated singularity, given as the sum of the initial function and a sufficiently big power of a generic linear form. As a corollary of Theorem 5.4, we establish a real version of this formula (Theorem 5.12), i.e., a relation between the Euler characteristic of the real Milnor fibres of f and the real Milnor fibres of a function of the type $f + v^{*k}$, for $v \in S^{n-1}$ generic and $k \in \mathbb{N}$ big enough.

We have also included some results on the local topology of locally closed definable sets. More precisely, we consider a locally closed definable set X equipped with a Whitney stratification such that $0 \in X$, and a definable function $g : (X, 0) \rightarrow (\mathbb{R}, 0)$ with an isolated critical point at the origin. In Lemma 3.1 we extend to this setting the results of Arnold and Wall mentioned above, i.e., we give relations between the Euler characteristics of the sets $X \cap \{g \ ? \ \pm \delta\} \cap B_\epsilon$, where $0 < \delta \ll \epsilon \ll 1$ and $? \in \{\leq, \geq\}$, and the Euler characteristics of the sets $X \cap \{g \ ? \ 0\} \cap S_\epsilon$, where $0 < \epsilon \ll 1$ and $? \in \{\leq, \geq\}$. We give two corollaries (Corollaries 3.3 and 3.4) when the stratum that contains 0 has dimension greater than or equal to 1.

The paper is organized as follows. In Section 2, we prove the first generalization of the Khimshiashvili formula based on Szafraniec's methods. In Section 3, we give the results on the local topology of locally closed definable sets. Section 4 contains the second generalization of the Khimshiashvili formula, based on the study of generic relative polar curves. In Section 5, we establish the real version of the Lê-Iomdine formula.

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2. SOME GENERAL RESULTS ON THE REAL MILNOR FIBRE

Let $f : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}, 0)$ be a definable function-germ of class C^r , $r \geq 2$. By Lemma 10 in [2] or by the main theorem of [21], we can equip $f^{-1}(0)$ with a finite Whitney stratification that satisfies the Thom (a_f) -condition.

Lemma 2.1. *There exists $\epsilon_0 > 0$ such that for $0 < \epsilon \leq \epsilon_0$, there exists δ_ϵ such that for $0 < \delta \leq \delta_\epsilon$, the topological type of $f^{-1}(\delta) \cap B_\epsilon$ does not depend on the choice of the couple (ϵ, δ) .*

Proof. Let $\epsilon_0 > 0$ be such that for $0 < \epsilon \leq \epsilon_0$, the sphere S_ϵ intersects $f^{-1}(0)$ transversally. Then there exists a neighborhood U_ϵ of 0 in \mathbb{R} such that for each $\delta \in U_\epsilon$, the fibre $f^{-1}(\delta)$ intersects the sphere S_ϵ transversally. If it is not the case, then we can find a sequence of points $(p_m)_{m \in \mathbb{N}}$ in S_ϵ such that the vectors $\frac{p_m}{|p_m|}$ and $\frac{\nabla f(p_m)}{|\nabla f(p_m)|}$ are collinear, and such that the sequence converges to a point p in $S_\epsilon \cap f^{-1}(0)$. If S denotes the stratum of $f^{-1}(0)$ that contains p then, applying the Thom (a_f) -condition, there exists a unit vector V normal to $T_p S$ such that $\frac{p}{|p|}$ and V are collinear. This contradicts the fact that S_ϵ intersects $f^{-1}(0)$ transversally.

Now let us fix $\epsilon > 0$ with $\epsilon \leq \epsilon_0$. Let us choose $\delta_\epsilon > 0$ such that $[0, \delta_\epsilon]$ is included in U_ϵ and δ is a regular value of f for $0 < \delta \leq \delta_\epsilon$. Let (ϵ_1, δ_1) and (ϵ_2, δ_2) be two couples with $0 < \epsilon_i \leq \epsilon$ and $0 < \delta_i \leq \delta_{\epsilon_i}$ for $i = 1, 2$. If $\epsilon_1 = \epsilon_2$ then the Thom-Mather first isotopy lemma implies that

the fibres $f^{-1}(\delta_1) \cap B_{\epsilon_1}$ and $f^{-1}(\delta_2) \cap B_{\epsilon_2}$ are homeomorphic. Now assume that $\epsilon_1 < \epsilon_2$. By the same arguments as above, there exists a neighborhood U of 0 in \mathbb{R} such that for each $\delta \neq 0$ in U , the distance function to the origin has no critical point on $f^{-1}(\delta) \cap (B_{\epsilon_2} \setminus \overset{\circ}{B}_{\epsilon_1})$. Let us choose $\delta_3 \neq 0$ in U such that $0 < \delta_3 \leq \min\{\delta_1, \delta_2\}$. By the first case, $f^{-1}(\delta_3) \cap B_{\epsilon_1}$ is homeomorphic to $f^{-1}(\delta_1) \cap B_{\epsilon_1}$ and $f^{-1}(\delta_3) \cap B_{\epsilon_2}$ is homeomorphic to $f^{-1}(\delta_2) \cap B_{\epsilon_2}$. But, since the distance function to the origin has no critical points on $f^{-1}(\delta_3) \cap (B_{\epsilon_2} \setminus \overset{\circ}{B}_{\epsilon_1})$, the fibres $f^{-1}(\delta_3) \cap B_{\epsilon_2}$ and $f^{-1}(\delta_3) \cap B_{\epsilon_1}$ are homeomorphic. \square

Of course a similar result is true for negative values of f .

Definition 2.2. The (real) Milnor fibres of f are the sets $f^{-1}(\delta) \cap B_\epsilon$ and $f^{-1}(-\delta) \cap B_\epsilon$, where $0 < \delta \ll \epsilon \ll 1$.

Sometimes we call $f^{-1}(\delta) \cap B_\epsilon$ (resp. $f^{-1}(-\delta) \cap B_\epsilon$) the positive (resp. negative) Milnor fibre of f . The Khimshiashvili formula [16] relates the Euler characteristic of the Milnor fibres to the topological degree of ∇f at the origin, when f has an isolated singularity.

Theorem 2.3 (The Khimshiashvili formula). *If f has an isolated critical point at the origin then*

$$\chi(f^{-1}(\delta) \cap B_\epsilon) = 1 - \text{sign}(-\delta)^n \text{deg}_0 \nabla f,$$

where $0 < |\delta| \ll \epsilon \ll 1$.

Proof. We give a proof for we will need a similar argument later. Let U be a small open subset of \mathbb{R}^n such that $0 \in U$ and f is defined in U . We perturb f in a Morse function $\tilde{f} : U \rightarrow \mathbb{R}$. Let p_1, \dots, p_k be the critical points of \tilde{f} , with respective indices $\lambda_1, \dots, \lambda_k$. Let $\delta > 0$, by Morse theory we have:

$$\chi(f^{-1}([-\delta, \delta]) \cap B_\epsilon) - \chi(f^{-1}(-\delta) \cap B_\epsilon) = \sum_{i=1}^k (-1)^{\lambda_i}.$$

Actually we can choose \tilde{f} sufficiently close to f so that the p_i 's lie in $f^{-1}([\frac{-\delta}{4}, \frac{\delta}{4}])$. Now the inclusion $f^{-1}(0) \cap B_\epsilon \subset f^{-1}([-\delta, \delta]) \cap B_\epsilon$ is a homotopy equivalence (Durfee [8] proved this result in the semi-algebraic case, but his argument holds in the C^r definable case, see also [6, 17]) and $f^{-1}(0) \cap B_\epsilon$ is the cone over $f^{-1}(0) \cap S_\epsilon$, so $\chi(f^{-1}([-\delta, \delta]) \cap B_\epsilon) = 1$. This gives the result for the negative Milnor fibre. To get the result for the positive one, it is enough to replace f with $-f$. \square

The following formulas are due to Arnol'd [1] and Wall [37].

Corollary 2.4. *With the same hypothesis on f , we have:*

$$\chi(\{f \leq 0\} \cap S_\epsilon) = 1 - \text{deg}_0 \nabla f,$$

$$\chi(\{f \geq 0\} \cap S_\epsilon) = 1 + (-1)^{n-1} \text{deg}_0 \nabla f.$$

If n is even, we have:

$$\chi(\{f = 0\} \cap S_\epsilon) = 2 - 2 \text{deg}_0 \nabla f.$$

Proof. By a deformation argument due to Milnor [23], $f(-\delta) \cap B_\epsilon$, $\delta > 0$, is homeomorphic to $\{f \leq -\delta\} \cap S_\epsilon$, which is homeomorphic to $\{f \leq 0\} \cap S_\epsilon$ if δ is very small. \square

We start our study of the general case with an easy lemma.

Lemma 2.5. *Let $f : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}, 0)$ be a definable function germ of class C^r , $r \geq 2$, and let $0 < \delta \ll \epsilon$. If n is even then*

$$\chi(f^{-1}(-\delta) \cap B_\epsilon) = \chi(\{f \geq 0\} \cap S_\epsilon),$$

and

$$\chi(f^{-1}(\delta) \cap B_\epsilon) = \chi(\{f \leq 0\} \cap S_\epsilon).$$

If n is odd then

$$\chi(f^{-1}(-\delta) \cap B_\epsilon) = 2 - \chi(\{f \geq 0\} \cap S_\epsilon),$$

and

$$\chi(f^{-1}(\delta) \cap B_\epsilon) = 2 - \chi(\{f \leq 0\} \cap S_\epsilon).$$

Proof. If the Milnor fibre is empty or a compact manifold without boundary then the result is obvious.

Otherwise, if n is even then $f^{-1}(-\delta) \cap B_\epsilon$ is an odd-dimensional manifold with boundary and so

$$\chi(f^{-1}(-\delta) \cap B_\epsilon) = \frac{1}{2} \chi(f^{-1}(-\delta) \cap S_\epsilon) = \chi(\{f \geq -\delta\} \cap S_\epsilon).$$

But for δ small, the inclusion $\{f \geq 0\} \cap S_\epsilon \subset \{f \geq -\delta\} \cap S_\epsilon$ is a homotopy equivalence (see [8]).

If n is odd then $\{f \geq -\delta\} \cap B_\epsilon$ is an odd-dimensional manifold with corners. Rounding the corners, we get

$$\begin{aligned} \chi(\{f \geq -\delta\} \cap B_\epsilon) &= \frac{1}{2} \left(\chi(f^{-1}(-\delta) \cap B_\epsilon) + \chi(\{f \geq -\delta\} \cap S_\epsilon) \right. \\ &\quad \left. - \chi(f^{-1}(-\delta) \cap S_\epsilon) \right) = \frac{1}{2} \left(\chi(f^{-1}(-\delta) \cap B_\epsilon) + \chi(\{f \geq -\delta\} \cap S_\epsilon) \right). \end{aligned}$$

But the inclusion $\{f \geq 0\} \cap B_\epsilon \subset \{f \geq -\delta\} \cap B_\epsilon$ is a homotopy equivalence and so

$$\chi(\{f \geq -\delta\} \cap B_\epsilon) = 1.$$

□

For the rest of this section, we assume that the structure is polynomially bounded. The techniques developed and the results proved by Szafraniec [31] (see also [4]) are valid in this context. Let $\omega(x) = x_1^2 + \cdots + x_n^2$. Then there exists an integer $d > 0$ sufficiently big such that $g_+ = f - \omega^d$ and $g_- = -f - \omega^d$ have an isolated critical point at the origin. Moreover Szafraniec showed that

$$\chi(\{f \leq 0\} \cap S_\epsilon) = 1 - \deg_0 \nabla g_+ \text{ and } \chi(\{f \geq 0\} \cap S_\epsilon) = 1 - \deg_0 \nabla g_-.$$

Applying the previous lemma, we can state our first generalization of the Khimshiashvili formula.

Corollary 2.6. *If $0 < \delta \ll \epsilon$, we have:*

$$\chi(f^{-1}(-\delta) \cap B_\epsilon) = 1 - (-1)^n \deg_0 \nabla g_-,$$

and

$$\chi(f^{-1}(\delta) \cap B_\epsilon) = 1 - (-1)^n \deg_0 \nabla g_+.$$

In general, the exponent d is difficult to estimate. However, in the case of a weighted-homogeneous polynomial, Szafraniec [32] provided another method which is completely effective.

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a real weighted homogeneous polynomial function of type $(d_1, \dots, d_n; d)$ with $\nabla f(0) = 0$. Let p be the smallest positive integer such that $2p > d$ and each d_i divides p . Also let $a_i = \frac{p}{d_i}$ and

$$\omega = \frac{x_1^{2a_1}}{2a_1} + \cdots + \frac{x_n^{2a_n}}{2a_n}.$$

Now consider $g_1 = f - \omega$ and $g_2 = -f - \omega$. Szafraniec proved that g_1 and g_2 have an algebraically isolated critical point at the origin and that

$$\chi(\{f \leq 0\} \cap S^{n-1}) = 1 - \deg_0 \nabla g_1 \text{ and } \chi(\{f \geq 0\} \cap S^{n-1}) = 1 - \deg_0 \nabla g_2.$$

Applying Lemma 2.5, we obtain the following Khimshiashvili type formula for the fibres of a real weighted homogeneous polynomial.

Corollary 2.7. *We have*

$$\chi(f^{-1}(-1)) = 1 - (-1)^n \deg_0 \nabla g_2,$$

and

$$\chi(f^{-1}(1)) = 1 - (-1)^n \deg_0 \nabla g_1.$$

Note that $\deg_0 \nabla g_1$ and $\deg_0 \nabla g_2$ can be computed algebraically thanks to the Eisenbud-Levine-Khimshiashvili formula [14, 16] because they have an algebraically isolated zero at the origin.

Let us apply this corollary to the examples presented in [32].

- (1) Let $f(x, y, z) = x^2y - y^4 - yz^3$. By [32], we have that $\deg_0 \nabla g_1 = \deg_0 \nabla g_2 = 1$. So $\chi(f^{-1}(-1)) = \chi(f^{-1}(1)) = 2$.
- (2) Let $f(x, y, z) = x^3 + x^2z - y^2$. By [32], we have that $\deg_0 \nabla g_1 = 1$ and $\deg_0 \nabla g_2 = -1$. So $\chi(f^{-1}(-1)) = 0$ and $\chi(f^{-1}(1)) = 2$.
- (3) Let $f(x, y, z) = x^3 - xy^2 + xyz + 2x^2y - 2y^3 - y^2z - xz^2 + yz^2$. Then by [32], $\deg_0 \nabla g_1 = 3$, so $\chi(f^{-1}(1)) = 4$.

3. SOME RESULTS ON THE TOPOLOGY OF LOCALLY CLOSED DEFINABLE SETS

Let X be a locally closed definable set. We assume that 0 belongs to X . We equip X with a finite definable C^r , $r \geq 2$, Whitney stratification. The fact that such a stratification exists is due to Loi [22]. Recently Nguyen, Trivedi and Trotman [29] gave another proof of this result. We denote by S_0 the stratum that contains 0 .

Let $g : (X, 0) \rightarrow (\mathbb{R}, 0)$ be a definable function that is the restriction to X of a definable function G of class C^r , $r \geq 2$, defined in a neighborhood of the origin. We assume that g has at worst an isolated critical point (in the stratified sense) at the origin. As in the previous section, the positive and the negative real Milnor fibres of g are the sets $g^{-1}(\delta) \cap X \cap B_\epsilon$ and $g^{-1}(-\delta) \cap X \cap B_\epsilon$, where $0 < \delta \ll \epsilon \ll 1$.

Lemma 3.1. *For $0 < \delta \ll \epsilon \ll 1$, we have*

$$\chi(X \cap g^{-1}(-\delta) \cap B_\epsilon) = \chi(X \cap \{g \leq 0\} \cap S_\epsilon),$$

and

$$\chi(X \cap g^{-1}(\delta) \cap B_\epsilon) = \chi(X \cap \{g \geq 0\} \cap S_\epsilon).$$

Proof. Using the methods developed in [11], we can assume that the critical points of g on $X \cap S_\epsilon$ are isolated, that they lie in $\{g \neq 0\}$ and that they are outwards-pointing (resp. inwards-pointing) in $\{g > 0\}$ (resp. $\{g < 0\}$). Let us denote them by $\{p_1, \dots, p_s\}$.

We recall that if $Z \subset \mathbb{R}^n$ is a locally closed definable set, equipped with a Whitney stratification and p is an isolated critical point of a definable function $\phi : Z \rightarrow \mathbb{R}$, restriction to Z of a C^2 -definable function Φ , then the index of ϕ at p is defined as follows:

$$\text{ind}(\phi, Z, p) = 1 - \chi(Z \cap \{\phi = \phi(p) - \eta\} \cap B_r(p)),$$

where $0 < \eta \ll r \ll 1$ and $B_r(p)$ is the closed ball of radius r centered at p .

As in [11], Section 3, we can apply the results proved in [9]. Namely, by Theorem 3.1 in [9], we can write

$$\chi(\{g \leq 0\} \cap X \cap S_\epsilon) = \sum_{i \mid g(p_i) < 0} \text{ind}(g, X \cap S_\epsilon, p_i),$$

and for $0 < \delta \ll \epsilon$,

$$\chi(\{g \leq \delta\} \cap X \cap B_\epsilon) = \sum_{i \mid g(p_i) < 0} \text{ind}(g, X \cap B_\epsilon, p_i) + \text{ind}(g, X, 0).$$

By Lemma 2.1 in [9], $\text{ind}(g, X \cap S_\epsilon, p_i) = \text{ind}(g, X \cap B_\epsilon, p_i)$ if $g(p_i) < 0$. Moreover,

$$\text{ind}(g, X, 0) = 1 - \chi(g^{-1}(-\delta) \cap X \cap B_\epsilon)$$

and, as explained in the proof of Theorem 2.3, $\chi(\{g \leq \delta\} \cap X \cap B_\epsilon) = 1$ if δ is small enough. Combining these observations, we find that

$$\chi(X \cap g^{-1}(-\delta) \cap B_\epsilon) = \chi(X \cap \{g \leq 0\} \cap S_\epsilon).$$

□

Remark 3.2. We believe that it is possible to establish these equalities applying a stratified version of the Milnor deformation argument mentioned in the proof of Corollary 2.4. This is done by Comte and Merle in [5] when X is conic and g is a generic linear form.

For the rest of this section, we will denote by $\text{Lk}(Y)$ the link at the origin of a definable set Y .

Corollary 3.3. *Assume that $\dim S_0 > 0$ and that $g|_{S_0}$ has no critical point at 0, i.e., $g^{-1}(0)$ intersects S_0 transversally at 0. Then the following equalities hold:*

$$\chi(\text{Lk}(X \cap \{g \leq 0\})) = \chi(\text{Lk}(X \cap \{g \geq 0\})) = 1,$$

and

$$\chi(\text{Lk}(X)) + \chi(\text{Lk}(X \cap \{g = 0\})) = 2.$$

Proof. If $g|_{S_0}$ has no critical point at 0, then $g : X \rightarrow \mathbb{R}$ is a stratified submersion in a neighborhood of 0. Furthermore for $0 < \epsilon \ll 1$, the sphere S_ϵ intersects $X \cap \{g = 0\}$ transversally, so 0 is a regular value of $g|_{X \cap B_\epsilon}$. Therefore if δ is small enough,

$$\chi(X \cap \{g = -\delta\} \cap B_\epsilon) = \chi(X \cap \{g = \delta\} \cap B_\epsilon) = \chi(X \cap \{g = 0\} \cap B_\epsilon) = 1.$$

It is enough to apply the previous lemma and then the Mayer-Vietoris sequence. □

For $v \in S^{n-1}$, we denote by v^* the function $v^*(x) = \langle v, x \rangle$, where $\langle \cdot, \cdot \rangle$ is the standard scalar product. The previous corollary applies to a generic linear form v^* .

Corollary 3.4. *Assume that $\dim S_0 > 0$. If $v \notin S^{n-1} \cap (T_0 S_0)^\perp$, then*

$$\chi(\text{Lk}(X \cap \{v^* \leq 0\})) = \chi(\text{Lk}(X \cap \{v^* \geq 0\})) = 1,$$

and

$$\chi(\text{Lk}(X)) + \chi(\text{Lk}(X \cap \{v^* = 0\})) = 2.$$

Proof. If $v \notin (T_0 S_0)^\perp$, then $v^*|_{S_0}$ has no critical point at 0. □

Let us relate this corollary to results that we proved in earlier papers. Combining Theorem 5.1 in [11] and the comments after Theorem 2.6 in [12], we can write that if $\dim S_0 > 0$,

$$\chi(\text{Lk}(X)) + \frac{1}{g_n^{n-1}} \int_{G_n^{n-1}} \chi(\text{Lk}(X \cap H)) dH = 2,$$

where G_n^{n-1} is the Grassmann manifold of linear hyperplanes in \mathbb{R}^n and g_n^{n-1} is its volume. This last equality is based on the study of the local behaviour of the generalized Lipschitz-Killing curvatures made in [5] and [11]. We see that it is actually a direct consequence of Corollary 3.4, which gives a more precise result on the local topology of locally closed definable sets. Similarly for $0 < k < \dim S_0$, we know that

$$-\frac{1}{g_n^{n-k-1}} \int_{G_n^{n-k-1}} \chi(\text{Lk}(X \cap H)) dH + \frac{1}{g_n^{n-k+1}} \int_{G_n^{n-k+1}} \chi(\text{Lk}(X \cap L)) dL = 0,$$

where G_n^{n-k} is the Grassmann manifold of k -dimensional vector spaces in \mathbb{R}^n and g_n^{n-k} is its volume. In fact a recursive application of Corollary 3.4 shows that $\chi(\text{Lk}(X \cap H)) = \chi(\text{Lk}(X \cap L))$ for H generic in G_n^{n-k-1} and L generic in G_n^{n-k+1} .

Let us give another application of Corollary 3.4 to the topology of real Milnor fibres. As in the previous section, $f : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}, 0)$ is the germ at the origin of a definable function of class C^r , $r \geq 2$. We assume that $f^{-1}(0)$ is equipped with a finite Whitney stratification that satisfies the Thom (a_f) -condition. Let S_0 be the stratum that contains 0.

Corollary 3.5. *If $\dim S_0 > 0$ and if $v \notin S^{n-1} \cap (T_0 S_0)^\perp$, then for $0 < \delta \ll \epsilon \ll 1$, we have*

$$\chi(f^{-1}(\delta) \cap B_\epsilon) = \chi(f^{-1}(\delta) \cap \{v^* = 0\} \cap B_\epsilon),$$

and

$$\chi(f^{-1}(-\delta) \cap B_\epsilon) = \chi(f^{-1}(-\delta) \cap \{v^* = 0\} \cap B_\epsilon).$$

Proof. Applying Corollary 3.4 to the sets $\{f \geq 0\}$ and $\{f \leq 0\}$, we get that

$$\chi(\text{Lk}(\{f \geq 0\})) + \chi(\text{Lk}(\{f \leq 0\} \cap \{v^* = 0\})) = 2,$$

where \geq, \leq . Lemma 2.5 applied to f and $f|_{\{v^*=0\}}$ gives the result. \square

In the next section, we will give a generalization of this result based on generic relative polar curves.

4. MILNOR FIBRES AND RELATIVE POLAR CURVES

Let $f : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}, 0)$ be a definable function-germ of class C^r , $r \geq 2$. We will give a second generalization of the Khimshiashvili formula in this setting. For this we need first to study the behaviour of a generic linear function on the fibres of f and the behaviour of f on the fibres of a generic linear function.

We start with a study of the critical points of $v^*_{|f^{-1}(\delta)}$ for δ small and v generic in S^{n-1} . Let

$$\Gamma_v = \{x \in \mathbb{R}^n \setminus \Sigma_f \mid \text{rank}(\nabla f(x), v) < 2\}.$$

We will need a first genericity condition. We can equip $f^{-1}(0)$ with a finite Whitney stratification that satisfies the Thom (a_f) -condition.

Lemma 4.1. *There exists a definable set $\Sigma_1 \subset S^{n-1}$ of positive codimension such that if $v \notin \Sigma_1$, then $\{v^* = 0\}$ intersects $f^{-1}(0) \setminus \{0\}$ transversally in a neighborhood of the origin.*

Proof. It is a particular case of Lemma 3.8 in [10]. \square

Lemma 4.2. *If $v \notin \Sigma_1$ then $\Gamma_v \cap f^{-1}(0) = \emptyset$.*

Proof. If it is not the case then we can find an arc $\alpha : [0, \nu[\rightarrow f^{-1}(0)$ such that $\alpha(0) = 0$ and for $0 < s < \nu$, $\nabla f(\alpha(s)) \neq 0$ and $\text{rank}(\nabla f(\alpha(s)), v) < 2$. Let S be the stratum that contains $\alpha(]0, \nu[)$. Since $\nabla f(\alpha(s))$ is normal to $T_{\alpha(s)}S$, the points in $\alpha(]0, \nu[)$ are critical points of $v|_S$ and hence lie in $\{v^* = 0\}$. This contradicts Lemma 4.1. \square

Corollary 4.3. *If $v \notin \Sigma_1$ then $\Gamma_v \cap \{v^* = 0\} = \emptyset$.*

Proof. As in the proof of the previous lemma, we see that

$$\text{if } \Gamma_v \cap \{v^* = 0\} \neq \emptyset, \text{ then } \Gamma_v \cap f^{-1}(0) \neq \emptyset.$$

\square

Lemma 4.4. *There exists a definable set $\Sigma_2 \subset S^{n-1}$ of positive codimension such that if $v \notin \Sigma_2$, Γ_v is a curve (possibly empty) in the neighbourhood of the origin.*

Proof. Let

$$M = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \mid \text{rank}(\nabla f(x), y) < 2\}.$$

Let $p = (x_0, y_0)$ be a point in $M \setminus (\Sigma_f \times \mathbb{R}^n)$. We can assume that $f_{x_1}(x_0) \neq 0$. Therefore locally $M \setminus (\Sigma_f \times \mathbb{R}^n)$ is given by the equations $m_{12}(x, y) = \dots = m_{1n}(x, y) = 0$, where

$$m_{ij}(x, y) = \begin{vmatrix} f_{x_i}(x) & f_{x_j}(x) \\ y_i & y_j \end{vmatrix}.$$

The Jacobian matrix of the mapping (m_{12}, \dots, m_{1n}) has the following form

$$\begin{pmatrix} * & \cdots & * & -f_{x_2} & f_{x_1} & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ * & \cdots & * & -f_{x_n} & 0 & \cdots & f_{x_1} \end{pmatrix}.$$

This implies that $M \setminus (\Sigma_f \times \mathbb{R}^n)$ is a C^{r-1} manifold of dimension $n+1$. The Bertini-Sard theorem ([3], 9.5.2) implies that the discriminant D of the projection

$$\begin{array}{ccc} \pi_y : M \setminus (\Sigma_f \times \mathbb{R}^n) & \rightarrow & \mathbb{R}^n \\ & (x, y) & \mapsto y \end{array}$$

is a definable set of dimension less than or equal to $n-1$. Hence for all $v \in S^{n-1} \setminus D$, the dimension of $\pi_y^{-1}(v)$ is less than or equal to 1. But $\pi_y^{-1}(v)$ is exactly Γ_v and we set $\Sigma_2 = D \cap S^{n-1}$. \square

Corollary 4.5. *Let $v \in S^{n-1}$ be such that $v \notin \Sigma_2$. There exists δ'_v such that for $0 < |\delta| \leq \delta'_v$, the critical points of $v|_{f^{-1}(\delta)}$ are Morse critical points in a neighborhood of the origin.*

Proof. After a change of coordinates, we can assume that $v = e_1 = (1, 0, \dots, 0) \in \mathbb{R}^n$ and so that $v^*(x) = x_1$.

Let p be a point in $\Gamma_v = \Gamma_{e_1}$. If $f_{x_1}(p) = 0$ then, since the minors $m_{1i} = \frac{\partial(f, x_1)}{\partial(x_1, x_i)}$, $i = 2, \dots, n$, vanish at p , $f_{x_i}(p) = 0$ for $i = 2, \dots, n$ and so p belongs to Σ_f , which is impossible. Therefore $f_{x_1}(p) \neq 0$ and by the proof of Lemma 4.4, we conclude that Γ_{e_1} is defined by the vanishing of the minors m_{1i} , $i = 2, \dots, n$, and that

$$\text{rank}(\nabla m_{12}, \dots, m_{1n}) = n - 1$$

along Γ_{e_1} . Let \mathbf{a} be an arc (i.e., a connected component) of Γ_{e_1} , and let $\alpha : [0, \nu[\rightarrow \mathbf{a}$ be a C^r definable parametrization such that $\alpha(0) = 0$ and $\alpha(]0, \nu[) \subset \mathbf{a}$. Since f does not vanish on \mathbf{a} , the function $f \circ \alpha$ is strictly monotone which implies that for $s \in]0, \nu[$,

$$(f \circ \alpha)'(s) = \langle \nabla f(\alpha(s)), \alpha'(s) \rangle \neq 0.$$

Hence the vectors

$$\nabla f(\alpha(s)), \nabla m_{12}(\alpha(s)), \dots, \nabla m_{1n}(\alpha(s))$$

are linearly independent since the $\nabla m_{1i}(\alpha(s))$'s are orthogonal to $\alpha'(s)$. By Lemma 3.2 in [33], this is equivalent to the fact that the function $x_1 : f^{-1}(f(\alpha(s))) \rightarrow \mathbb{R}$ has a non-degenerate critical point at $\alpha(s)$. It is easy to conclude because Γ_{e_1} has a finite numbers of arcs. \square

From now on, we will work with $v \in S^{n-1}$ such that $v \notin \Sigma_1 \cup \Sigma_2$. After a change of coordinates, we can assume that $v = e_1 = (1, 0, \dots, 0)$ and so the conclusions of Lemma 4.1, Lemma 4.2, Corollary 4.3, Lemma 4.4 and Corollary 4.5 are valid for Γ_{x_1} and $\{x_1 = 0\}$. Let us study the points of Γ_{x_1} more accurately. By the previous results, we know that if p is a point of Γ_{x_1} close to the origin then p is a Morse critical point of $x_1|_{f^{-1}(f(p))}$, $f_{x_1}(p) \neq 0$, $x_1(p) \neq 0$ and $f(p) \neq 0$.

Lemma 4.6. *Let p be a point in Γ_{x_1} close to the origin. Let $\mu(p)$ be the Morse index of $x_1|_{f^{-1}(f(p))}$ at p . Then p is a Morse critical point of $f|_{x_1^{-1}(x_1(p))}$ and if $\theta(p)$ is the Morse index of $f|_{x_1^{-1}(x_1(p))}$ at p then*

$$(-1)^{\mu(p)} = (-1)^{n-1} \text{sign}(f_{x_1}(p))^{n-1} (-1)^{\theta(p)}.$$

Proof. By Lemma 3.2 in [33], we know that

$$\det[\nabla f(p), \nabla f_{x_2}(p), \dots, \nabla f_{x_n}(p)] \neq 0,$$

and that

$$(-1)^{\mu(p)} = (-1)^{n-1} \text{sign}(f_{x_1}(p))^n \text{sign} \det[\nabla f(p), \nabla f_{x_2}(p), \dots, \nabla f_{x_n}(p)].$$

But $\nabla f(p) = f_{x_1}(p)e_1$ and so $\det[e_1, \nabla f_{x_2}(p), \dots, \nabla f_{x_n}(p)] \neq 0$ and

$$(-1)^{\mu(p)} = (-1)^{n-1} \text{sign}(f_{x_1}(p))^{n-1} \text{sign} \det[e_1, \nabla f_{x_2}(p), \dots, \nabla f_{x_n}(p)].$$

Still using Lemma 3.2 in [33], we see that p is a Morse critical point of $f|_{x_1^{-1}(x_1(p))}$ and that

$$(-1)^{\mu(p)} = (-1)^{n-1} \text{sign}(f_{x_1}(p))^{n-1} (-1)^{\theta(p)}.$$

\square

Lemma 4.7. *Let p be a point in Γ_{x_1} close to the origin. Then*

$$\det[\nabla f_{x_1}(p), \nabla f_{x_2}(p), \dots, \nabla f_{x_n}(p)] \neq 0$$

and

$$(-1)^{\theta(p)} = \text{sign}(x_1(p)f_{x_1}(p)) \text{sign} \det[\nabla f_{x_1}(p), \nabla f_{x_2}(p), \dots, \nabla f_{x_n}(p)].$$

Proof. Since $\det[e_1, \nabla f_{x_2}(p), \dots, \nabla f_{x_n}(p)] \neq 0$, we can write

$$\nabla f_{x_1}(p) = \beta(p)e_1 + \sum_{i=2}^n \beta_i(p)\nabla f_{x_i}(p),$$

and so,

$$\det[\nabla f_{x_1}(p), \nabla f_{x_2}(p), \dots, \nabla f_{x_n}(p)] = \beta(p)\det[e_1, \nabla f_{x_2}(p), \dots, \nabla f_{x_n}(p)].$$

Let $\alpha : [0, \nu[\rightarrow \overline{\Gamma_{x_1}}$ be a parametrization of the arc that contains p . We have

$$(f_{x_1} \circ \alpha)'(s) = \langle \nabla f_{x_1}(\alpha(s)), \alpha'(s) \rangle = \beta(\alpha(s)) \langle e_1, \alpha'(s) \rangle = \beta(\alpha(s))(x_1 \circ \alpha)'(s).$$

But since f_{x_1} and x_1 do not vanish on Γ_{x_1} , $(f_{x_1} \circ \alpha)'(s)$ and $(x_1 \circ \alpha)'(s)$ do not vanish for s small. Therefore for p close to the origin, $\beta(p) \neq 0$ and

$$\text{sign} \beta(p) = \text{sign}(x_1(p)f_{x_1}(p)).$$

\square

Let \mathcal{B} be the set of connected components of Γ_{x_1} . If $\mathbf{b} \in \mathcal{B}$ then \mathbf{b} is a half-branch on which the functions f_{x_1} and $\det[\nabla f_{x_1}, \dots, \nabla f_{x_n}]$ have constant sign. So we can decompose \mathcal{B} into the disjoint union $\mathcal{B}^+ \sqcup \mathcal{B}^-$ where \mathcal{B}^+ (resp. \mathcal{B}^-) is the set of half-branches on which $f_{x_1} > 0$ (resp. $f_{x_1} < 0$). If $\mathbf{b} \in \mathcal{B}$, we denote by $\sigma(\mathbf{b})$ the sign of $\det[\nabla f_{x_1}, \dots, \nabla f_{x_n}]$ on \mathbf{b} .

Definition 4.8. We set $\lambda^+ = \sum_{\mathbf{b} \in \mathcal{B}^+} \sigma(\mathbf{b})$ and $\lambda^- = \sum_{\mathbf{b} \in \mathcal{B}^-} \sigma(\mathbf{b})$.

Remark 4.9. If f has an isolated critical point at the origin then for $\eta \neq 0$ small enough, $(\nabla f)^{-1}(\eta, 0, \dots, 0)$ is exactly $\Gamma_{x_1} \cap f_{x_1}^{-1}(\eta)$. Moreover if $p \in \Gamma_{x_1} \cap f_{x_1}^{-1}(\eta)$, then

$$\text{sign det}[\nabla f_{x_1}(p), \nabla f_{x_2}(p), \dots, \nabla f_{x_n}(p)] \neq 0.$$

Hence $(\eta, 0, \dots, 0)$ is a regular value of ∇f and so

$$\text{deg}_0 \nabla f = \sum_{p \in \Gamma_{x_1} \cap f_{x_1}^{-1}(\eta)} \text{sign det}[\nabla f_{x_1}(p), \nabla f_{x_2}(p), \dots, \nabla f_{x_n}(p)].$$

If $\eta > 0$ (resp. $\eta < 0$), this implies that $\text{deg}_0 \nabla f = \lambda^+$ (resp. λ^-).

The following lemma will enable us to define other indices associated with f and x_1 .

Lemma 4.10. *There exists $\epsilon_0 > 0$ such that for $0 < \epsilon \leq \epsilon_0$, there exists $a_\epsilon > 0$ such that for $0 < a \leq a_\epsilon$, there exists $\alpha_{a,\epsilon} > 0$ such that for $0 < \alpha \leq \alpha_{a,\epsilon}$, the topological type of $f^{-1}(\alpha) \cap \{x_1 = a\} \cap B_\epsilon$ does not depend on the choice of the triple (ϵ, a, α) .*

Proof. For $a > 0$ small enough, we define $\beta(a)$ by

$$\beta(a) = \inf\{|f(p)| \mid p \in \Gamma_{x_1} \cap \{x_1 = a\}\}.$$

The function β is well defined because $\Gamma_{x_1} \cap \{x_1 = a\}$ is finite and $\beta(a) > 0$. Moreover it is definable and so it is continuous on a small interval of the form $]0, u[$. This implies that the set

$$\mathcal{O} = \{(a, \alpha) \in \mathbb{R} \times \mathbb{R}^* \mid a \in]0, u[\text{ and } 0 < \alpha < \beta(a)\}$$

is open and connected.

Since $\{x_1 = 0\}$ intersects $f^{-1}(0) \setminus \{0\}$ transversally (in the stratified sense),

$$\{x_1 = 0\} \cap f^{-1}(0) \setminus \{0\}$$

is Whitney stratified, the strata being the intersections of $\{x_1 = 0\}$ with the strata of $f^{-1}(0) \setminus \{0\}$.

Let $\epsilon_0 > 0$ be such that for $0 < \epsilon \leq \epsilon_0$, the sphere S_ϵ intersects $\{x_1 = 0\} \cap f^{-1}(0)$ transversally. Then there exists a neighborhood \mathcal{U}_ϵ of $(0, 0)$ in \mathbb{R}^2 such that for each (a, α) in $(\mathbb{R} \times \mathbb{R}^*) \cap \mathcal{U}_\epsilon$, the fibre $f^{-1}(\alpha) \cap \{x_1 = a\}$ intersects S_ϵ transversally. If it is not the case, then we can find a sequence of points $(p_m)_{m \in \mathbb{N}}$ in S_ϵ such that the vectors $e_1, \frac{p_m}{|p_m|}$ and $\frac{\nabla f(p_m)}{|\nabla f(p_m)|}$ are linearly dependent, and such that the sequence converges to a point p in $S_\epsilon \cap f^{-1}(0) \cap \{x_1 = 0\}$. If S denotes the stratum of $f^{-1}(0)$ that contains p then, applying the Thom (a_f) -condition and the method of Lemma 3.7 in [11], there exists a unit vector v normal to $T_p S$ such that the vectors $e_1, \frac{p}{|p|}$ and v are linearly dependent. But e_1 and v are linearly independent for $\{x_1 = 0\}$ intersects S transversally at p . Therefore S_ϵ does not intersect $S \cap \{x_1 = 0\}$ transversally at p , which is a contradiction. Moreover we can assume that $\mathcal{U}_\epsilon \cap \mathcal{O}$ is connected.

Now let us fix $\epsilon > 0$ with $\epsilon \leq \epsilon_0$. Let us choose $a_\epsilon > 0$ such that $a_\epsilon < u$ and the interval $]0, a_\epsilon]$ is included in \mathcal{U}_ϵ . For each $a \in]0, a_\epsilon]$, there exists $\alpha'_{a,\epsilon}$ such that $\{a\} \times]0, \alpha'_{a,\epsilon}]$ is included in \mathcal{U}_ϵ . We choose $\alpha_{a,\epsilon}$ such that $\alpha_{a,\epsilon} \leq \alpha'_{a,\epsilon}$ and $\alpha_{a,\epsilon} < \beta(a)$, which implies that (a, α) is a regular value of (x_1, f) for $0 < \alpha \leq \alpha_{a,\epsilon}$.

Let $(\epsilon_1, a_1, \alpha_1)$ and $(\epsilon_2, a_2, \alpha_2)$ be two triples with $0 < \epsilon_i \leq \epsilon$, $0 < a_i \leq a_{\epsilon_i}$ and $0 < \alpha_i \leq \alpha_{a_i, \epsilon_i}$ for $i = 1, 2$. If $\epsilon_1 = \epsilon_2$ then the Thom-Mather first isotopy lemma implies that the fibres

$f^{-1}(\alpha_1) \cap \{x_1 = a_1\} \cap B_{\epsilon_1}$ and $f^{-1}(\alpha_2) \cap \{x_1 = a_2\} \cap B_{\epsilon_2}$ are homeomorphic, because (a_1, α_1) and (a_2, α_2) belong to the connected set $\mathcal{U}_{\epsilon_1} \cap \mathcal{O}$.

Now assume that $\epsilon_1 < \epsilon_2$. By the same arguments as above, there exists a neighborhood \mathcal{U} of $(0, 0)$ in \mathbb{R}^2 such that for each $(a, \alpha) \in (\mathbb{R} \times \mathbb{R}^*) \cap \mathcal{U}$, the distance function to the origin has no critical point on

$$f^{-1}(\alpha) \cap \{x_1 = a\} \cap (B_{\epsilon_2} \setminus \overset{\circ}{B}_{\epsilon_1}).$$

Let us choose $(a_3, \alpha_3) \in (\mathbb{R} \times \mathbb{R}^*) \cap \mathcal{U}$ such that

$$0 < a_3 \leq \min\{a_{\epsilon_1}, a_{\epsilon_2}\} \quad \text{and} \quad \alpha_3 \leq \min\{\alpha_{a_3, \epsilon_1}, \alpha_{a_3, \epsilon_2}\}.$$

Then, by the first case, $f^{-1}(\alpha_3) \cap \{x_1 = a_3\} \cap B_{\epsilon_1}$ is homomorphic to $f^{-1}(\alpha_1) \cap \{x_1 = a_1\} \cap B_{\epsilon_1}$ and $f^{-1}(\alpha_3) \cap \{x_1 = a_3\} \cap B_{\epsilon_2}$ is homomorphic to $f^{-1}(\alpha_2) \cap \{x_1 = a_2\} \cap B_{\epsilon_2}$. But, since the distance function to the origin has no critical point on $f^{-1}(\alpha_3) \cap \{x_1 = a_3\} \cap B_{\epsilon_2} \setminus \overset{\circ}{B}_{\epsilon_1}$, the fibre $f^{-1}(\alpha_3) \cap \{x_1 = a_3\} \cap B_{\epsilon_1}$ is homeomorphic to $f^{-1}(\alpha_3) \cap \{x_1 = a_3\} \cap B_{\epsilon_2}$. \square

Similarly, there exists $\epsilon'_0 > 0$ such that for $0 < \epsilon \leq \epsilon'_0$, there exists $b_\epsilon > 0$ such that for $0 < a \leq b_\epsilon$, the topological types of $f^{-1}(0) \cap \{x_1 = a\} \cap B_\epsilon$ and $f^{-1}(0) \cap \{x_1 = -a\} \cap B_\epsilon$ do not depend on the choice of (ϵ, a) . Therefore we can make the following definition.

Definition 4.11. We set

$$\begin{aligned} \gamma^{+,+} &= \chi(f^{-1}(0) \cap \{x_1 = a\} \cap B_\epsilon) - \chi(f^{-1}(\alpha) \cap \{x_1 = a\} \cap B_\epsilon), \\ \gamma^{+,-} &= \chi(f^{-1}(0) \cap \{x_1 = -a\} \cap B_\epsilon) - \chi(f^{-1}(\alpha) \cap \{x_1 = -a\} \cap B_\epsilon), \\ \gamma^{-,+} &= \chi(f^{-1}(0) \cap \{x_1 = a\} \cap B_\epsilon) - \chi(f^{-1}(-\alpha) \cap \{x_1 = a\} \cap B_\epsilon), \\ \gamma^{-,-} &= \chi(f^{-1}(0) \cap \{x_1 = -a\} \cap B_\epsilon) - \chi(f^{-1}(-\alpha) \cap \{x_1 = -a\} \cap B_\epsilon), \end{aligned}$$

where $0 < \alpha \ll a \ll \epsilon$.

Now we are in position to state the generalization of the Khimshiashvili formula. Remember that e_1 satisfies the genericity conditions of Lemmas 4.1 and 4.4.

Theorem 4.12. Assume that $e_1 \notin \Sigma_1 \cup \Sigma_2$. For $0 < \delta \ll \epsilon \ll 1$, we have

$$\begin{aligned} \chi(f^{-1}(-\delta) \cap B_\epsilon) &= 1 - \lambda^- - \gamma^{-,-} = 1 - \lambda^+ - \gamma^{-,+}, \\ \chi(f^{-1}(\delta) \cap B_\epsilon) &= 1 - (-1)^n \lambda^- - \gamma^{+,+} = 1 - (-1)^n \lambda^+ - \gamma^{+,-}. \end{aligned}$$

Proof. The set of critical points of x_1 on $f^{-1}(-\delta) \cap \overset{\circ}{B}_\epsilon$ is exactly $\Gamma_{x_1} \cap f^{-1}(-\delta)$. Moreover we know that if $p \in \Gamma_{x_1} \cap f^{-1}(-\delta)$ then $x_1(p) \neq 0$. By Morse theory, we have

$$\chi(f^{-1}(-\delta) \cap B_\epsilon \cap \{x_1 \geq 0\}) - \chi(f^{-1}(-\delta) \cap B_\epsilon \cap \{x_1 = 0\}) = \sum_{\substack{p \in \Gamma_{x_1} \cap f^{-1}(-\delta) \\ x_1(p) > 0}} (-1)^{\mu(p)},$$

$$\chi(f^{-1}(-\delta) \cap B_\epsilon \cap \{x_1 \leq 0\}) - \chi(f^{-1}(-\delta) \cap B_\epsilon \cap \{x_1 = 0\}) = (-1)^{n-1} \cdot \sum_{\substack{p \in \Gamma_{x_1} \cap f^{-1}(-\delta) \\ x_1(p) < 0}} (-1)^{\mu(p)}.$$

Here we remark that $f^{-1}(\delta) \cap B_\epsilon$ is a manifold with boundary and x_1 may have critical points on the boundary. But by Lemma 3.7 in [11], these critical points lie in $\{x_1 \neq 0\}$ and are outwards-pointing (resp. inwards-pointing) in $\{x_1 > 0\}$ (resp. $\{x_1 < 0\}$). That is why they do not appear in the above two formulas. Adding the two equalities and applying the Mayer-Vietoris sequence, we obtain

$$\begin{aligned} \chi(f^{-1}(-\delta) \cap B_\epsilon) - \chi(f^{-1}(-\delta) \cap B_\epsilon \cap \{x_1 = 0\}) &= \\ \sum_{\substack{p \in \Gamma_{x_1} \cap f^{-1}(-\delta) \\ x_1(p) > 0}} (-1)^{\mu(p)} + (-1)^{n-1} \sum_{\substack{p \in \Gamma_{x_1} \cap f^{-1}(-\delta) \\ x_1(p) < 0}} (-1)^{\mu(p)}. \end{aligned}$$

Since $\nabla f = f_{x_1} e_1$ on Γ_{x_1} , it is easy to check that p belongs to $\Gamma_{x_1} \cap \{f < 0\} \cap \{x_1 > 0\}$ if and only if p belongs to $\Gamma_{x_1} \cap \{f_{x_1} < 0\} \cap \{x_1 > 0\}$ and p belongs to $\Gamma_{x_1} \cap \{f < 0\} \cap \{x_1 < 0\}$ if and only if p belongs to $\Gamma_{x_1} \cap \{f_{x_1} > 0\} \cap \{x_1 < 0\}$. Let us decompose \mathcal{B}^+ into the disjoint union $\mathcal{B}^+ = \mathcal{B}^{+,+} \sqcup \mathcal{B}^{+,-}$ where $\mathcal{B}^{+,+}$ (resp. $\mathcal{B}^{+,-}$) is the set of half-branches of \mathcal{B}^+ on which $x_1 > 0$ (resp. $x_1 < 0$). Similarly we can write $\mathcal{B}^- = \mathcal{B}^{-,+} \sqcup \mathcal{B}^{-,-}$. Combining Lemma 4.6 and Lemma 4.7, we can rewrite the above equality in the following form:

$$\chi(f^{-1}(-\delta) \cap B_\epsilon) - \chi(f^{-1}(-\delta) \cap B_\epsilon \cap \{x_1 = 0\}) = - \sum_{\mathbf{b} \in \mathcal{B}^{-,+}} \sigma(\mathbf{b}) - \sum_{\mathbf{b} \in \mathcal{B}^{+,-}} \sigma(\mathbf{b}). \quad (1)$$

Since $(-\delta, 0)$ is a regular value of (f, x_1) then there exists $a_\delta > 0$ such that for $0 < a \leq a_\delta$, $(-\delta, \pm a)$ are regular value of (f, x_1) and

$$\chi(f^{-1}(-\delta) \cap \{x_1 = \pm a\} \cap B_\epsilon) = \chi(f^{-1}(-\delta) \cap \{x_1 = 0\} \cap B_\epsilon).$$

Let us fix a such that $0 < a \leq a_\delta$ and let us relate $\chi(f^{-1}(-\delta) \cap \{x_1 = -a\} \cap B_\epsilon)$ to $\chi(f^{-1}(\alpha) \cap \{x_1 = -a\} \cap B_\epsilon)$ where $0 < \alpha \ll a$. Note that the set of critical points of f on $\{x_1 = -a\} \cap B_\epsilon$ is exactly $\Gamma_{x_1} \cap \{x_1 = -a\}$. Moreover this set of critical points is included in $\{f > -\delta\}$. Indeed, if it is not the case, then there is a half-branch of Γ_{x_1} that intersects $\{x_1 = -a\}$ on $\{f \leq -\delta\}$. But since x_1 and f are negative on this branch, this would imply that Γ_{x_1} intersects $\{f = -\delta\}$ on $\{-a \leq x_1 < 0\}$, which is not possible for $a \leq a_\delta$.

Now let us look at the critical points of f on $\{x_1 = -a\} \cap S_\epsilon$. In the proof of Lemma 4.10, we established the existence of a neighborhood \mathcal{U}_ϵ of $(0, 0)$ in \mathbb{R}^2 such that for each $(a, \alpha) \in (\mathbb{R} \times \mathbb{R}^*) \cap \mathcal{U}_\epsilon$, the fibre $f^{-1}(\alpha) \cap \{x_1 = a\}$ intersects S_ϵ transversally. Therefore we can choose δ such that the critical points of f on $\{x_1 = 0\} \cap S_\epsilon \cap \{f \neq 0\}$ lie in $\{|f| > \delta\}$. Moreover by a Curve Selection Lemma argument, they are outwards-pointing in $\{f > \delta\}$ and inwards-pointing in $\{f < -\delta\}$. So, if a is small enough, then the critical points of f on $\{x_1 = -a\} \cap S_\epsilon \cap \{f \neq 0\}$, lying in $\{|f| > \delta\}$, are outwards-pointing (resp. inwards-pointing) in $\{f > \delta\}$ (resp. $\{f < -\delta\}$). By Morse theory, we find that

$$\chi(\{f \leq -\delta\} \cap \{x_1 = -a\} \cap B_\epsilon) - \chi(\{f = -\delta\} \cap \{x_1 = -a\} \cap B_\epsilon) = 0$$

$$\chi(\{-\delta \leq f \leq -\alpha\} \cap \{x_1 = -a\} \cap B_\epsilon) - \chi(\{f = -\delta\} \cap \{x_1 = -a\} \cap B_\epsilon) = \sum_{\substack{p \in \Gamma_{x_1} \cap \{x_1 = -a\} \\ f(p) < 0}} (-1)^{\theta(p)},$$

$$\chi(\{f \geq \alpha\} \cap \{x_1 = -a\} \cap B_\epsilon) - \chi(\{f = \alpha\} \cap \{x_1 = -a\} \cap B_\epsilon) = \sum_{\substack{p \in \Gamma_{x_1} \cap \{x_1 = -a\} \\ f(p) > 0}} (-1)^{\theta(p)}.$$

By the Mayer-Vietoris sequence, we have that

$$\begin{aligned} 1 &= \chi(\{x_1 = -a\} \cap B_\epsilon) = \chi(\{f \leq -\delta\} \cap \{x_1 = -a\} \cap B_\epsilon) \\ &+ \chi(\{-\delta \leq f \leq -\alpha\} \cap \{x_1 = -a\} \cap B_\epsilon) - \chi(\{f = -\delta\} \cap \{x_1 = -a\} \cap B_\epsilon) \\ &- \chi(\{f = -\alpha\} \cap \{x_1 = -a\} \cap B_\epsilon) + \chi(\{-\alpha \leq f \leq \alpha\} \cap \{x_1 = -a\} \cap B_\epsilon) \\ &+ \chi(\{f \geq \alpha\} \cap \{x_1 = -a\} \cap B_\epsilon) - \chi(\{f = \alpha\} \cap \{x_1 = -a\} \cap B_\epsilon). \end{aligned}$$

Using the fact that the inclusion

$$\{f = 0\} \cap \{x_1 = -a\} \cap B_\epsilon \subset \{-\alpha \leq f \leq \alpha\} \cap \{x_1 = -a\} \cap B_\epsilon$$

is a homotopy equivalence and applying the above equalities, we get

$$1 = \sum_{\substack{p \in \Gamma_{x_1} \cap \{x_1 = -a\} \\ f(p) < 0}} (-1)^{\theta(p)} + \sum_{\substack{p \in \Gamma_{x_1} \cap \{x_1 = -a\} \\ f(p) > 0}} (-1)^{\theta(p)} + \chi(f^{-1}(-\delta) \cap \{x_1 = -a\} \cap B_\epsilon) \\ + \chi(f^{-1}(0) \cap \{x_1 = -a\} \cap B_\epsilon) - \chi(f^{-1}(-\alpha) \cap \{x_1 = -a\} \cap B_\epsilon).$$

By Lemma 4.7, we can rewrite this equality in the following form:

$$1 = - \sum_{\mathbf{b} \in \mathcal{B}^{+,-}} \sigma(\mathbf{b}) + \sum_{\mathbf{b} \in \mathcal{B}^{-,-}} \sigma(\mathbf{b}) + \chi(f^{-1}(-\delta) \cap \{x_1 = -a\} \cap B_\epsilon) + \gamma^{-,-}. \quad (2)$$

Combining (1) and (2), we obtain the first equality of the statement. The second one is obtained replacing $-a$ with a in the above discussion. The third and fourth ones are obtained replacing f with $-f$. \square

Remark 4.13. (1) If f has an isolated critical point at the origin then we recover the Khimshiashvili formula because

$$\gamma^{-,-} = \gamma^{+,+} = \gamma^{+,-} = \gamma^{-,+} = 0$$

and $\lambda^+ = \lambda^- = \deg_0 \nabla f$.

- (2) If we denote by S_0 the stratum that contains 0 and if we assume that $\dim S_0 > 0$, then by the Thom (a_f) -condition, the polar curve Γ_v is empty in a neighborhood of 0 if $v \notin S^{n-1} \cap (T_x S_0)^\perp$. Then applying Equality (1) of the previous proof, we recover Corollary 3.5. Actually, we can say more about the relation between the topologies of $f^{-1}(\pm\delta) \cap B_\epsilon$ and $f^{-1}(\pm\delta) \cap B_\epsilon \cap \{v^* = 0\}$. As mentioned in the proof of Theorem 4.12, the critical points of v^* restricted to $f^{-1}(\pm\delta) \cap S_\epsilon$ lie in $\{v^* \neq 0\}$ and are outwards-pointing (resp. inwards-pointing) in $\{v^* > 0\}$ (resp. $\{v^* < 0\}$). So we can apply the arguments of the proof of Theorem 6.3 in [13] to get that $f^{-1}(\pm\delta) \cap B_\epsilon$ is homeomorphic to $f^{-1}(\pm\delta) \cap B_\epsilon \cap \{v^* = 0\} \times [-1, 1]$.

5. ONE DIMENSIONAL CRITICAL LOCUS AND A REAL LÊ-IOMDINE FORMULA

In this section, we apply the results of Section 4 to the case of a one-dimensional singular set, in order to establish a real version of the Lê-Iomdine formula.

Let $f : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}, 0)$ be a definable function-germ of class C^r , $2 \leq r$. We assume that $\dim \Sigma_f = 1$. In a neighborhood of the origin, the partition

$$(f^{-1}(0) \setminus \Sigma_f, \Sigma_f \setminus \{0\}, \{0\})$$

gives a Whitney stratification of $f^{-1}(0)$ which satisfies the Thom (a_f) -condition, because the points where the Whitney conditions and the Thom (a_f) -condition may fail form a 0-dimensional definable set of $\Sigma_f \setminus \{0\}$. Let \mathcal{C} be the set of half-branches of Σ_f , i.e., the set of connected components of $\Sigma_f \setminus \{0\}$.

Lemma 5.1. *There exists a definable set $\Sigma_3 \subset S^{n-1}$ of positive codimension such that if $v \notin \Sigma_3$, v^* does not vanish on $\Sigma_f \setminus \{0\}$ in a neighborhood of the origin.*

Proof. Let $\mathbf{c} \in \mathcal{C}$. If v^* vanishes on \mathbf{c} in a neighborhood of the origin then, if $u \neq 0$ is on $C_0 \mathbf{c}$ (the tangent cone at \mathbf{c} at the origin) then $v^*(u) = 0$ and so $v \in u^\perp$. So if $v \notin \cup_{\mathbf{c} \in \mathcal{C}} (C_0 \mathbf{c})^\perp$ then v^* does not vanish on $\Sigma_f \setminus \{0\}$. But $(\cup_{\mathbf{c} \in \mathcal{C}} (C_0 \mathbf{c})^\perp) \cap S^{n-1}$ has dimension less than or equal to $n - 2$. \square

From now on, we assume that $e_1 \in S^{n-1}$ is generic, i.e., $e_1 \notin \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$. Since $e_1 \notin \Sigma_3$, there exists $a_1 > 0$ such that for $0 < a \leq a_1$, $x_1^{-1}(\pm a)$ intersects $\Sigma_f \setminus \{0\}$ transversally and so, the points in $x_1^{-1}(\pm a) \cap (\Sigma_f \setminus \{0\})$ are isolated critical points of $f|_{\{x_1=\pm a\}}$. For $q \in x_1^{-1}(\pm a) \cap (\Sigma_f \setminus \{0\})$, we denote by $\deg_q \nabla f|_{x_1^{-1}(\pm a)}$ the topological degree of the mapping

$$\frac{\nabla f|_{x_1^{-1}(\pm a)}}{|\nabla f|_{x_1^{-1}(\pm a)}} : x_1^{-1}(\pm a) \cap S_{\epsilon'}(q) \rightarrow S^{n-2},$$

where $S_{\epsilon'}(q)$ is the sphere centered at q of radius ϵ' with $0 < \epsilon' \ll 1$.

Let us write $\mathcal{C} = \mathcal{C}^+ \sqcup \mathcal{C}^-$ where \mathcal{C}^+ (resp. \mathcal{C}^-) is the set of half-branches of \mathcal{C} on which $x_1 > 0$ (resp. $x_1 < 0$).

Lemma 5.2. *Let $\mathbf{c} \in \mathcal{C}^+$. There exists $a_{\mathbf{c}} > 0$ such that the function $a \mapsto \deg_q \nabla f|_{x_1^{-1}(a)}$, where $\{q\} = \mathbf{c} \cap \{x_1 = a\}$, is constant on $]0, a_{\mathbf{c}}[$.*

Proof. It is enough to prove that there exists an interval $]0, a_{\mathbf{c}}[$ on which the function $a \mapsto \deg_q \nabla f|_{x_1^{-1}(a)}$ is locally constant. Let $d : \mathbb{R}^n \rightarrow \mathbb{R}$ be the distance function to \mathbf{c} . It is a continuous definable function and there exists an open definable neighbourhood \mathcal{U} of \mathbf{c} such that d is smooth on $\mathcal{U} \setminus \mathbf{c}$. Moreover we can assume that d is a (stratified) submersion on $\{f \leq 0\} \cap (\mathcal{U} \setminus \mathbf{c})$.

Let $\pi : \{f \leq 0\} \cap (\mathcal{U} \setminus \mathbf{c}) \rightarrow \mathbb{R}^2$ be the mapping defined by $\pi(p) = (x_1(p), d(p))$ and let $\Delta \subset \mathbb{R}^2$ be its (stratified) discriminant. It is a definable curve included in $\mathbb{R} \times \mathbb{R}^*$ and so $\bar{\Delta} \cap (\mathbb{R} \times \{0\})$ is a finite number of points. Let us choose $a_{\mathbf{c}} > 0$ such that

$$a_{\mathbf{c}} < \min \{x_1(u) \mid u \in \bar{\Delta} \cap (\mathbb{R}^* \times \{0\})\}.$$

If $0 < a \leq a_{\mathbf{c}}$, then there exists $t > 0$ and $\epsilon > 0$ such that $]a - t, a + t[\times]0, \epsilon[$ does not meet Δ . Hence the function

$$\begin{aligned}]a - t, a + t[\times]0, \epsilon[&\rightarrow \mathbb{R} \\ (a', \epsilon') &\mapsto \chi(\{f \leq 0\} \cap \{x_1 = a'\} \cap \{d = \epsilon'\}) \end{aligned}$$

is constant. Therefore by Corollary 2.4, the function $a' \mapsto \deg_{q'} \nabla f|_{x_1^{-1}(a')}$ is constant on $]a - t, a + t[$. \square

Of course, a similar result is valid for $\mathbf{c} \in \mathcal{C}^-$. If $\mathbf{c} \in \mathcal{C}$, let us denote by $\tau(\mathbf{c})$ the value that the function $a \mapsto \deg_q \nabla f|_{x_1^{-1}(a)}$, $\{q\} = \mathbf{c} \cap \{x_1 = a\}$, takes close to the origin.

Definition 5.3. We set $\gamma^+ = \sum_{\mathbf{c} \in \mathcal{C}^+} \tau(\mathbf{c})$ and $\gamma^- = \sum_{\mathbf{c} \in \mathcal{C}^-} \tau(\mathbf{c})$.

In this setting, Theorem 4.12 admits the following formulation.

Theorem 5.4. *Assume that $e_1 \notin \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$. For $0 < \delta \ll \epsilon \ll 1$, we have*

$$\begin{aligned} \chi(f^{-1}(-\delta) \cap B_{\epsilon}) &= 1 - \lambda^- - \gamma^- = 1 - \lambda^+ - \gamma^+, \\ \chi(f^{-1}(\delta) \cap B_{\epsilon}) &= 1 - (-1)^n(\lambda^+ - \gamma^-) = 1 - (-1)^n(\lambda^- - \gamma^+). \end{aligned}$$

Proof. Since $\Gamma_{x_1} \cap f^{-1}(0) = \emptyset$, the critical points of $f|_{\{x_1=\pm a\}}$ in $f^{-1}([-\alpha, \alpha])$, $0 < \alpha \ll \delta \ll \epsilon$, are exactly the points in $\Sigma_f \cap \{x_1 = \pm a\}$. An easy adaptation of the proof of the Khimshiashvili formula (Theorem 2.3) gives that

$$\gamma^{-,-} = \gamma^-, \quad \gamma^{-,+} = \gamma^+, \quad \gamma^{+,-} = (-1)^{n-1} \gamma^+ \quad \text{and} \quad \gamma^{+,+} = (-1)^{n-1} \gamma^-.$$

\square

We remark that $\lambda^- + \gamma^- = \lambda^+ + \gamma^+$. Moreover, if n is even, we have

$$\begin{aligned}\chi(f^{-1}(\delta) \cap B_\epsilon) - \chi(f^{-1}(-\delta) \cap B_\epsilon) &= \gamma^+ + \gamma^-, \\ \chi(f^{-1}(\delta) \cap B_\epsilon) + \chi(f^{-1}(-\delta) \cap B_\epsilon) &= 2 - (\lambda^+ + \lambda^-),\end{aligned}$$

and if n is odd, we have

$$\begin{aligned}\chi(f^{-1}(\delta) \cap B_\epsilon) - \chi(f^{-1}(-\delta) \cap B_\epsilon) &= \lambda^+ + \lambda^-, \\ \chi(f^{-1}(\delta) \cap B_\epsilon) + \chi(f^{-1}(-\delta) \cap B_\epsilon) &= 2 - (\gamma^+ + \gamma^-).\end{aligned}$$

Therefore the two sums $\sum_{\mathbf{b} \in \mathcal{B}} \sigma(\mathbf{b})$ and $\sum_{\mathbf{c} \in \mathcal{C}} \tau(\mathbf{c})$ do not depend on the generic choice of linear function that we used to define them. Moreover, applying Lemma 2.5, we get that if n is even, $\chi(\text{Lk}(\{f=0\})) = 2 - (\lambda^+ + \lambda^-)$ and if n is odd, $\chi(\text{Lk}(\{f=0\})) = \gamma^+ + \gamma^-$.

Let us give an example. Let $f(x, y, z) = y^2 - zx^b$, $b > 1$ (see [26], Example 2.2). This polynomial is weighted-homogeneous but we cannot apply Corollary 2.7, for b may be arbitrary large. Then $\Sigma_f = \{(0, 0, z) \mid z \in \mathbb{R}\}$.

Let $v = (1, 1, 1)$ so that $v^*(x, y, z) = x + y + z$. We have to check that v satisfies the conclusions of Lemma 4.1, Lemma 4.4 and Corollary 4.5, and Lemma 5.1. A straightforward computation shows that

$$\Gamma_v = \left\{ \left(x, -\frac{x^b}{2}, \frac{x}{b} \right) \mid x \neq 0 \right\}.$$

Since $\Gamma_v \cap \{v^* = 0\} = \emptyset$, we see that $\{v^* = 0\}$ intersects the stratum $f^{-1}(0) \setminus \Sigma_f$ transversally. Moreover, since v^* does not vanish on $\Sigma_f \setminus \{0\}$, $\{v^* = 0\}$ intersects the stratum $\Sigma_f \setminus \{0\}$ transversally and so v satisfies the conclusion of Lemma 4.1 (and of Lemma 5.1 as well). It is clear that Γ_v is a curve in the neighborhood of the origin. In order to check that the conclusion of Corollary 4.5 holds, thanks to the computations of Lemmas 4.6 and 4.7, it is enough to check that $\det[\nabla f_x, \nabla f_y, \nabla f_z]$ does not vanish on Γ_v . But

$$\det[\nabla f_x, \nabla f_y, \nabla f_z](x, y, z) = -2b^2 x^{2b-2},$$

and so the conclusion of Corollary 4.5 holds. Moreover, since

$$\frac{\partial f}{\partial v}(x, y, z) = -bx^{b-1}z + 2y - x^b,$$

we easily compute that $\lambda^+ = \lambda^- = -1$ if b is odd and that $\lambda^+ = 0$ and $\lambda^- = -2$ if b is even.

It remains to compute γ^+ and γ^- . But γ^+ is the local topological degree at $(0, 0)$ of the function $f(x, y, a - x - y)$, $a > 0$, that is the local topological degree at $(0, 0)$ of the function

$$(x, y) \mapsto y^2 - ax^b + x^{b+1} + yx^b.$$

Then it is not difficult to see that $\gamma^+ = -1$ if b is even and $\gamma^+ = 0$ if b is odd. Similarly $\gamma^- = 1$ if b is even and $\gamma^- = 0$ if b is odd. Therefore, applying Theorem 5.4 and Lemma 2.5, we obtain that

$$\chi(f^{-1}(-1)) = 2, \quad \chi(f^{-1}(1)) = 0 \quad \text{and} \quad \chi(\text{Lk}(\{f=0\})) = 0.$$

In the rest of the section, we will apply Theorem 5.4 to establish a real version of the Lê-Iomdine formula. From now on, we assume that the structure is polynomially bounded.

Lemma 5.5. *There exists $n_0 \in \mathbb{N}$ such that*

$$|f_{x_1}(p)| > |x_1(p)|^{n_0} \quad \text{and} \quad |f(p)| > |x_1(p)|^{n_0},$$

for $p \in \Gamma_{x_1}$ close to the origin.

Proof. For $u > 0$ small, we define $\beta(u)$ by

$$\beta(u) = \inf\{|f_{x_1}(p)| \mid p \in \Gamma_{x_1} \cap \{|x_1| = u\}\}.$$

It is well defined because $\Gamma_{x_1} \cap \{|x_1| = u\}$ is finite and $\beta(u) > 0$. The function β is definable and so is the function $\alpha(R) = \beta(\frac{1}{R})$, defined for $R > 0$ sufficiently big. Then there exists $n_0 \in \mathbb{N}$ such that $\frac{1}{\alpha(R)} < R^{n_0}$ for $R > 0$ sufficiently big. This implies that $\beta(\frac{1}{R}) > \frac{1}{R^{n_0}}$, i.e., $\beta(u) > u^{n_0}$ for $u > 0$ sufficiently small. Hence for $p \in \Gamma_{x_1}$ sufficiently close to the origin, we have

$$|f_{x_1}(p)| > |x_1(p)|^{n_0}.$$

A similar proof works for the second equality because f and x_1 do not vanish on Γ_{x_1} . □

Let us fix $k \in \mathbb{N}$ with $k > n_0 + 1$ and let us set $g(x) = f(x) + x_1^k$.

Lemma 5.6. *The function g has an isolated critical point at the origin.*

Proof. A point p belongs to $(\nabla g)^{-1}(0)$ if and only if

$$\frac{\partial f}{\partial x_1}(p) + kx_1^{k-1}(p) = 0 \text{ and } \frac{\partial f}{\partial x_i}(p) = 0 \text{ for } i \geq 2.$$

Let us suppose first that $p \in \Sigma_f \setminus \{0\}$. This implies that $x_1(p) = 0$. Since x_1 does not vanish on $\Sigma_f \setminus \{0\}$ close to the origin, this case is not possible. Let us suppose now that $p \notin \Sigma_f$. Then p belongs to Γ_{x_1} and so $x_1(p) \neq 0$ and $f_{x_1}(p) \neq 0$. By the previous lemma, $|f_{x_1}(p)| > |x_1(p)|^{n_0}$ which implies that $k|x_1(p)|^{k-1} > |x_1(p)|^{n_0}$, and so $|x_1(p)|^{k-n_0-1} > \frac{1}{k}$ in the neighborhood of the origin. This is impossible by the choice of k . The only possible case is when p is the origin. □

The previous lemma enables us to use the Khimshiashvili formula to compute the Euler characteristic of the Milnor fibre of g . We will relate $\text{deg}_0 \nabla g$ to the indices λ^+ , λ^- , γ^+ and γ^- . Before that we need some auxiliary results. Let

$$\Gamma_{x_1}(g) = \{x \in \mathbb{R}^n \setminus \Sigma_g \mid \text{rank}(\nabla g(x), e_1) < 2\}.$$

Lemma 5.7. *We have $\Gamma_{x_1}(g) \cap \{g = 0\} = \emptyset$.*

Proof. If it is not the case this implies that the following set

$$\{g = 0\} \cap \{g_{x_2} = \dots = g_{x_n} = 0\} \cap \{x_1 = 0\} \setminus \{0\}$$

is not empty in the neighbourhood of the origin. Therefore the set

$$\{f = 0\} \cap \{f_{x_2} = \dots = f_{x_n} = 0\} \cap \{x_1 = 0\} \setminus \{0\}$$

is not empty in the neighbourhood of the origin. But this is not possible because

$$\{f = 0\} \cap \{f_{x_2} = \dots = f_{x_n}\} = \Sigma_f$$

and $\{x_1 = 0\} \setminus \{0\} \cap \Sigma_f = \emptyset$. □

Lemma 5.8. *The set $\Gamma_{x_1}(g)$ admits the following decomposition:*

$$\Gamma_{x_1}(g) = \Gamma_{x_1} \sqcup (\Sigma_f \setminus \{0\}).$$

Proof. We see that $p \in \Gamma_{x_1}(g)$ if and only if $g_{x_2}(p) = \dots = g_{x_n}(p) = 0$ and $g_{x_1}(p) \neq 0$. Since $g_{x_i}(p) = f_{x_i}(p)$, $i = 2, \dots, n$, it is clear that $\Gamma_{x_1} \sqcup (\Sigma_f \setminus \{0\}) \subset \Gamma_{x_1}(g)$. If $p \in \Gamma_{x_1}(g)$ then $f_{x_2}(p) = \dots = f_{x_n}(p) = 0$ and $f_{x_1}(p) + kx_1(p)^{k-1} \neq 0$. If $f_{x_1}(p) \neq 0$ then $p \in \Gamma_{x_1}$. If $f_{x_1}(p) = 0$ then $p \in \Sigma_f \setminus \{0\}$. □

Lemma 5.9. *If $p \in \Gamma_{x_1}$, then $\det[\nabla g_{x_1}(p), \dots, \nabla g_{x_n}(p)] \neq 0$ and*

$$\text{sign det}[\nabla g_{x_1}(p), \dots, \nabla g_{x_n}(p)] = \text{sign det}[\nabla f_{x_1}(p), \dots, \nabla f_{x_n}(p)].$$

Proof. Let $p \in \Gamma_{x_1}$. We have $g_{x_1}(p) = f_{x_1}(p) + kx_1^{k-1}$. By the choice of k , this implies that $\text{sign } g_{x_1}(p) = \text{sign } f_{x_1}(p)$. Using the computations of Lemmas 4.6 and 4.7, we see that

$$\text{sign } \det[\nabla f_{x_1}(p), \dots, \nabla f_{x_n}(p)] = \text{sign} \left(x_1(p) f_{x_1}(p) \frac{\partial(x_1, f_{x_2}, \dots, f_{x_n})}{\partial(x_1, \dots, x_n)}(p) \right).$$

But

$$\frac{\partial(x_1, f_{x_2}, \dots, f_{x_n})}{\partial(x_1, \dots, x_n)}(p) = \frac{\partial(x_1, g_{x_2}, \dots, g_{x_n})}{\partial(x_1, \dots, x_n)}(p)$$

so $\frac{\partial(x_1, g_{x_2}, \dots, g_{x_n})}{\partial(x_1, \dots, x_n)}(p) \neq 0$. Since $x_1(p)g_{x_1}(p) \neq 0$, we obtain that

$$\det[\nabla g_{x_1}(p), \dots, \nabla g_{x_n}(p)] \neq 0$$

and since $\text{sign } g_{x_1}(p) = \text{sign } f_{x_1}(p)$, we conclude that

$$\text{sign } \det[\nabla g_{x_1}(p), \dots, \nabla g_{x_n}(p)] = \text{sign } \det[\nabla f_{x_1}(p), \dots, \nabla f_{x_n}(p)].$$

□

Lemma 5.10. *Assume that k is even. If $q \in \Sigma_f \setminus \{0\}$ is close enough to the origin and $x_1(q) = a$, then $\deg_q \nabla f|_{\{x_1=a\}}$ is equal to $\deg_q(\nabla g - \nabla g(q))$, where $\deg_q(\nabla g - \nabla g(q))$ is the topological degree of the mapping*

$$\frac{\nabla g - \nabla g(q)}{|\nabla g - \nabla g(q)|} : S_{\epsilon'}(q) \rightarrow S^{n-1}$$

with $0 < \epsilon' \ll 1$.

Proof. We have that $\deg_q \nabla f|_{\{x_1=a\}}$ is equal to the topological degree of the mapping $\frac{W}{|W|} : S_{\epsilon'}(q) \rightarrow S^{n-1}$ where $W = (x_1 - a, f_{x_2}, \dots, f_{x_n})$. But

$$\nabla g - \nabla g(q) = (f_{x_1} + kx_1^{k-1} - ka^{k-1}, f_{x_2}, \dots, f_{x_n})$$

and so, since $f_{x_1}(q) = 0$ and $k-1$ is odd, there exists a small neighborhood of q on which $f_{x_1} + kx_1^{k-1} - ka^{k-1}$ and $x_1 - a$ have the same sign. If ϵ' is small enough, then the mappings $\frac{\nabla g - \nabla g(q)}{|\nabla g - \nabla g(q)|}$ and $\frac{W}{|W|}$ are homotopic on $S_{\epsilon'}(q)$. Hence the two topological degrees are equal. □

Proposition 5.11. *If k is odd, then*

$$\deg_0 \nabla g = \lambda^- = \lambda^+ + \gamma^+ - \gamma^-.$$

If k is even, then

$$\deg_0 \nabla g = \lambda^- + \gamma^- = \lambda^+ + \gamma^+.$$

Proof. Let $\eta > 0$ be a small real number. The set $(\nabla g)^{-1}(-\eta, 0, \dots, 0)$ is finite because $\Gamma_{x_1}(g)$ is one-dimensional. Let us write

$$(\nabla g)^{-1}(-\eta, 0, \dots, 0) = \{p_1, \dots, p_s\} \cup \{q_1, \dots, q_r\},$$

where

$$\{p_1, \dots, p_s\} = (\nabla g)^{-1}(-\eta, 0, \dots, 0) \cap \Gamma_{x_1}$$

and

$$\{q_1, \dots, q_r\} = (\nabla g)^{-1}(-\eta, 0, \dots, 0) \cap (\Sigma_f \setminus \{0\}).$$

Therefore we have

$$\deg_0 \nabla g = \sum_{i=1}^s \deg_{p_i}(\nabla g - \nabla g(p_i)) + \sum_{j=1}^r \deg_{q_j}(\nabla g - \nabla g(q_j)).$$

If k is odd, $(\nabla g)^{-1}(-\eta, 0, \dots, 0) \cap (\Sigma_f \setminus \{0\})$ is empty and, by the choice of k ,

$$(\nabla g)^{-1}(-\eta, 0, \dots, 0) \cap \Gamma_{x_1} = (\nabla g)^{-1}(-\eta, 0, \dots, 0) \cap [\cup_{b \in \mathcal{B}} b].$$

Using Lemma 5.9, we conclude that

$$\deg_0 \nabla g = \lambda^- = \lambda^+ + \gamma^+ - \gamma^-.$$

If k is even then

$$(\nabla g)^{-1}(-\eta, 0, \dots, 0) \cap (\Sigma_f \setminus \{0\}) = (\nabla g)^{-1}(-\eta, 0, \dots, 0) \cap [\cup_{c \in \mathcal{C}} c].$$

Using Lemma 5.10, we conclude that $\deg_0 \nabla g = \lambda^- + \gamma^- = \lambda^+ + \gamma^+$. □

Now we are in position to formulate the real version of the Lê-Iomdine formula.

Theorem 5.12. *Assume that $e_1 \notin \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$ and that $k > n_0 + 1$. For $0 < \delta \ll \epsilon \ll 1$, we have*

- if k is odd,

$$\begin{aligned} \chi(g^{-1}(-\delta) \cap B_\epsilon) &= \chi(f^{-1}(-\delta) \cap B_\epsilon) + \gamma^-, \\ \chi(g^{-1}(\delta) \cap B_\epsilon) &= \chi(f^{-1}(\delta) \cap B_\epsilon) + (-1)^{n-1} \gamma^+, \end{aligned}$$

- if k is even,

$$\begin{aligned} \chi(g^{-1}(-\delta) \cap B_\epsilon) &= \chi(f^{-1}(-\delta) \cap B_\epsilon), \\ \chi(g^{-1}(\delta) \cap B_\epsilon) &= \chi(f^{-1}(\delta) \cap B_\epsilon) + (-1)^{n-1} (\gamma^+ + \gamma^-). \end{aligned}$$

Proof. We know that $\chi(g^{-1}(-\delta) \cap B_\epsilon) = 1 - \deg_0 \nabla g$. If k is odd, this gives

$$\chi(g^{-1}(-\delta) \cap B_\epsilon) = 1 - \lambda^- = \chi(f^{-1}(-\delta) \cap B_\epsilon) + \gamma^-.$$

If k is even, this gives

$$\chi(g^{-1}(-\delta) \cap B_\epsilon) = 1 - \lambda^- - \gamma^- = \chi(f^{-1}(-\delta) \cap B_\epsilon).$$

We know that $\chi(g^{-1}(\delta) \cap B_\epsilon) = 1 - (-1)^n \deg_0 \nabla g$. So if k is odd, then

$$\chi(g^{-1}(\delta) \cap B_\epsilon) = 1 - (-1)^n \lambda^- = \chi(f^{-1}(\delta) \cap B_\epsilon) - (-1)^n \gamma^+.$$

If k is even, we get that

$$\begin{aligned} \chi(g^{-1}(\delta) \cap B_\epsilon) &= 1 - (-1)^n (\lambda^- + \gamma^-) \\ &= 1 - (-1)^n (\lambda^- - \gamma^+) - (-1)^n (\gamma^+ + \gamma^-) \\ &= \chi(f^{-1}(\delta) \cap B_\epsilon) - (-1)^n (\gamma^+ + \gamma^-). \end{aligned}$$

□

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NICOLAS DUTERTRE, LABORATOIRE ANGEVIN DE RECHERCHE EN MATHÉMATIQUES, LAREMA, UMR6093, CNRS, UNIV. ANGERS, SFR MATHSTIC, 2 Bd LAVOISIER 49045 ANGERS CEDEX 01, FRANCE.
Email address: nicolas.dutertre@univ-angers.fr