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São Carlos, Brazil, 22–28 July, 2018

Editors:
Grazielle Feliciani Barbosa
José Luis Cisneros Molina
Nivaldo De Góes Grulha Júnior
András Némethi

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Introduction

The International Workshop on Real and Complex Singularities is organized by the Singularities Group of the Institute of Mathematical and Computer Sciences (ICMC) of the University of São Paulo (USP), campus São Carlos. It is one of the key events for people working in singularity theory, algebraic geometry, bifurcation theory and related areas. It brings together world experts and young researchers to report on their recent achievements, to exchange ideas and to address trends of research in a highly stimulating environment. The Workshop is a bi-annual event which started in 1990.

The 15th edition of the International Workshop on Real and Complex Singularities took place from July 22 to 28, 2018 at the ICMC-USP, São Carlos. It was dedicated in honor of Maria Ruas and Terence Gaffney's 70th birthdays and Marcelo Saia's 60th birthday. As in previous editions, the Workshop aimed at the interaction between undergraduate and graduate students, young researchers and world experts of Singularities and other related areas, and the dissemination of scientific works. The activities of the Workshop consisted of lectures, short courses, communications and posters on the most diverse fronts, within the Theory of Singularities. The program consisted of 20 plenary sessions, 6 among them were special plenary sessions, where the speakers presented papers recently developed in collaboration with professors Terence Gaffney, Maria Ruas and Marcelo Saia, honored in this edition, 48 ordinary sessions, and 34 posters.

We consider that the Workshop successfully achieved its goal. The quality of the lectures presented was strongly appreciated by the participants; the themes of the offered short courses and the effective participation of graduate students and young researchers were again highlights of this event.

It had a significant number of participants. The 145 participants came from several Brazilian and foreign institutions, for example, from the USA, Japan, Spain, France, England, Canada, Peru, Mexico, Poland, Germany. We thank the speakers and the participants whose presence was the real success of the Workshop.

The organization of the Workshop was possible thanks to the help of many people and institutions. The members of the Scientific Committee: Enrique Artal Bartolo, Jean-Paul Brasselet, Alexandre César Gurgel Fernandes, Marcelo Escudeiro Hernandes, Zbigniew Jelonek, Ursula Ludwig, Takashi Nishimura, Regilene Delazari dos Santos Oliveira, Maria Aparecida Soares Ruas, José Seade, Mihai Tibăr. The members of the Organizing Committee: Grazielle Feliciani Barbosa, Nivaldo de Góes Grulha Junior, Raimundo Nonato Araújo dos Santos, Thaís Maria Dalbelo, Aurélio Menegon Neto and Miriam da Silva Pereira. The Workshop was funded by Institutions from Brasil: FAPESP, CNPq and INCTMat.

The articles in the present volume were submitted by participants of the 15th International Workshop on Real and Complex Singularities. They show a wide spectrum of topics in Singularity Theory. We thank all the contributors for their high quality research articles.

Grazielle Feliciani Barbosa
José Luis Cisneros Molina
Nivaldo De Góes Grulha Júnior
András Némethi

Figure 1: Maria Ruas and Terence Gaffney 70th birthdays and Marcelo Saia 60th birthday.

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ON THE INDEX OF PRINCIPAL FOLIATIONS OF SURFACES IN $\mathbb{R}^3$ WITH CORANK 1 SINGULARITIES

J. C. F. COSTA, L. F. MARTINS, AND J. J. NUÑO-BALLESTEROS

Abstract. It is well known that the index associated to the principal foliations at a cross-cap point is $\frac{1}{2}$. In this work we study the index for other corank 1 singularities from $(\mathbb{R}^2, 0)$ to $(\mathbb{R}^3, 0)$ which either are simple or are non-simple but included in strata of $\mathcal{A}_e$-codimension ≤ 3. We show that the index, under certain conditions, is always 0 or 1, bearing out that the Loewner conjecture could be true for all corank 1 singularities.

1. Introduction

The classical Loewner conjecture states that the index of the binary differential equation (BDE) which represents the equation of the principal directions of a smooth immersed surface in $\mathbb{R}^3$ at an isolated umbilic point is always less than or equal to 1. The Loewner conjecture is a stronger version of the famous Carathéodory conjecture, which claims that every smooth convex embedding of a 2-sphere in $\mathbb{R}^3$ must have at least two umbilics. In fact, since the sum of the indices of the umbilics of a compact immersed surface is equal to its Euler-Poincaré characteristic (according to the Poincaré-Hopf formula) it follows that the Loewner conjecture implies the Carathéodory conjecture, not only for a convex embedding of a 2-sphere, but for any immersion. The Loewner conjecture is true in the analytic case (cf. [19, 30]) but the smooth case is still open, as far as we know.

A natural question is whether or not the Loewner conjecture is still true when we consider a singular surface parametrised as the image of a smooth non-immersive map germ $f : (\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^3, \mathbf{0})$. In fact, if the non-immersive point is isolated then we have a well defined BDE for the principal directions outside the origin and it makes sense to consider the index of the singular point of the BDE. By definition, the corank of f is the dimension of the kernel of its differential at the origin. When f has corank 2, then it is known that this conjecture is false, since it is not difficult to construct a surface with an isolated singular point of index two (see [11], Remark 4.7). However, we believe that if f has corank 1, then the index is always less than or equal to one and hence, the Loewner conjecture is also true in this case. The main purpose of this paper is to analyse many examples which support this conjecture.

The family of examples we consider here is taken from Mond's classification in [23], where he gives a classification under $\mathcal{A}$-equivalence (that is, changes of coordinates in the source and target) of all smooth germs $f : (\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^3, \mathbf{0})$ which either are simple or are non-simple but included in strata of $\mathcal{A}_e$-codimension ≤ 3. All map germs in this list have corank 1, so we can use them to test our conjecture. Of course this list is far from being a complete classification,

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but they are the most natural examples to begin with the analysis. Our main result is that the index, under certain conditions, is always 0 or 1 in all these examples (Theorem 3.4).

In final of the paper we also consider generic deformations of the singular surface. Let

$$f_\lambda : (\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^3, \mathbf{0}), \quad \lambda \in (-\varepsilon, \varepsilon),$$

be a generic deformation of a corank 1 map germ f, i.e. $f_0 = f$, f_λ is generic for $\lambda \neq 0$ and the map $(\lambda, t) \mapsto f_\lambda(t)$ is smooth. D. Mond showed in [24] how to count the number of cross-caps in f_λ. Using his result, we estimate the number of umbilic points that appear on the image of f_λ in a neighbourhood of its singular point (Proposition 5.3).

Some references for index of BDE's are [4, 6, 7, 8, 20].

2. Surfaces with corank 1 singularities

We shall consider surfaces in $\mathbb{R}^3$ defined as the image of a corank 1 smooth map $f : U \to \mathbb{R}^3$, where U is an open subset of $\mathbb{R}^2$. The differential geometry of singular surfaces has been an object of interest in the past decades and it can be considered with different approaches (cross-caps or Whitney umbrellas, cuspidal edges, swallowtails or more general types of fronts, etc.) For example, see [3, 10, 14, 16, 17, 22, 25, 26, 27, 28]. See also [21], where the authors studied in depth the geometry of surfaces in $\mathbb{R}^3$ with corank 1 singularities.

From the Singularity Theory point of view, if we are concerned in corank 1 map germs $(\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^3, \mathbf{0})$ up to $\mathcal{A}$-equivalence then we have a classification list given by D. Mond in [23]. The Mond's classification is summarized in Table 1 for either simple map germs or non-simple but included in strata of $\mathcal{A}_e$-codimension ≤ 3. When k is even, S_k^+ is equivalent to S_k^-, and C_k^+ to C_k^-.

We recall that two map germs $f, g : (\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^3, \mathbf{0})$ are said to be $\mathcal{A}$-*equivalent*, denoted by $f \sim g$, if there exist germs of diffeomorphisms $h : (\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^2, \mathbf{0})$ and $k : (\mathbb{R}^3, \mathbf{0}) \to (\mathbb{R}^3, \mathbf{0})$ such that $g = k \circ f \circ h^{-1}$. For more details about definitions and notations from Singularity theory used in this work (such as, $\mathcal{A}_e$-codimension, simple germs, etc.), see [31].

Table 1: $\mathcal{A}$-classes of corank 1 map germs $(\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^3, \mathbf{0})$ either simple or non-simple but included in strata of $\mathcal{A}_e$-codimension ≤ 3 (cf. [23]).

Germ	$\mathcal{A}_e$-codimension	Name
(x, y^2, xy)	0	Cross-cap (S_0)
$(x, y^2, y^3 \pm x^{k+1}y)$, $k \geq 1$	k	$S_k^\pm$
$(x, y^2, x^2y \pm y^{2k+1})$, $k \geq 2$	k	$B_k^\pm$
$(x, y^2, xy^3 \pm x^k y)$, $k \geq 3$	k	$C_k^\pm$
$(x, y^2, x^3y + y^5)$	4	F_4
$(x, xy + y^{3k-1}, y^3)$, $k \geq 2$	k	H_k
$(x, xy + y^3, xy^2 + ay^4)$, $a \neq 0, \frac{1}{2}, 1, \frac{3}{2}$	3	P_3

Surfaces in the same $\mathcal{A}$-orbit clearly have diffeomorphic image but not necessarily they have the same local differential geometry. So, we cannot take f as one of the normal forms in the above table. We need parametrisations for corank 1 surfaces in $\mathbb{R}^3$ obtained with changes of coordinates at source and target which preserve the geometry of the image.

The geometry of singular surfaces parametrised locally by a germ of a smooth function $\mathcal{A}$-equivalent to one of those in Table 1 is considered, for instance, in [10, 15, 26].

We summarize in the next result the partition of the set of all corank 1 map germs $f : (\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^3, \mathbf{0})$ according to their 2-jets under the action of the group $\mathcal{A}^2$ (i.e., the group of 2-jets of diffeomorphisms in the source and target). We denote by $J^2(2,3)$ the space of

2-jets $j^2f(\mathbf{0})$ of map germs $f:(\mathbb{R}^2,\mathbf{0})\to(\mathbb{R}^3,\mathbf{0})$ and by $\Sigma^1J^2(2,3)$ the subset of 2-jets of corank 1.

Proposition 2.1. (Classification of 2-jets [23]) *There exist four orbits in* $\Sigma^1J^2(2,3)$ *under the action of* $\mathcal{A}^2$*, which are*

$$(x,y^2,xy),\ (x,y^2,0),\ (x,xy,0),\ (x,0,0).$$

The following result gives relevant parametrisations for corank 1 surfaces in $\mathbb{R}^3$ according to the classification given in Proposition 2.1. The cross-cap case, that is, when $j^2f(\mathbf{0})\sim(x,y^2,xy)$ is done in [10, 32], and the case $j^2f(\mathbf{0})\sim(x,0,0)$ is not of our interest here because f is a non-simple wich is included in a stratum of $\mathcal{A}_e$-codimension >3.

Proposition 2.2. ([15]) *Let* $f:(\mathbb{R}^2,\mathbf{0})\to(\mathbb{R}^3,\mathbf{0})$ *be a corank 1 map germ. Then, after using smooth changes of coordinates in the source and isometries in the target, we can reduce* $j^kf(\mathbf{0})$ *to the form*

$$(x,y)\ \mapsto\ \left(x\,,\ \frac{1}{2}\,y^2+\sum_{i=2}^{k}\frac{b_i}{i!}\,x^i\,,\ \frac{1}{2}\,a_{20}x^2+\sum_{i+j=3}^{k}\frac{a_{ij}}{i!j!}\,x^iy^j\right), \tag{1}$$

if $j^2f(\mathbf{0})$ *is* $\mathcal{A}$*-equivalent to* $(x,y^2,0)$*, or*

$$(x,y)\ \mapsto\ \left(x\,,\ xy+\sum_{i=3}^{k}\frac{b_i}{i!}\,y^i\,,\ \frac{1}{2}\,a_{20}x^2+\sum_{i+j=3}^{k}\frac{a_{ij}}{i!j!}\,x^iy^j\right), \tag{2}$$

if $j^2f(\mathbf{0})$ *is* $\mathcal{A}$*-equivalent to* $(x,xy,0)$*, where* b_i,a_{ij} *are constants.*

Let $f:(\mathbb{R}^2,\mathbf{0})\to(\mathbb{R}^3,\mathbf{0})$ be a map germ of corank 1 and let $j^kf(\mathbf{0})$ be given by (1) in Proposition 2.2. Then, the conditions for f to be $\mathcal{A}$-equivalent to S_k,B_k,C_k or F_4 are as follows (see [15, 26]):

$$\begin{array}{ll} S_1: & a_{03}\neq0,a_{21}\neq0,\\ S_{k\geq2}: & a_{03}\neq0,a_{21}=\cdots=a_{k1}=0,a_{(k+1)1}\neq0,\\ B_2: & a_{03}=0,a_{21}\neq0,3a_{05}a_{21}-5a_{13}^2\neq0,\\ B_{k\geq3}: & a_{03}=0,a_{21}\neq0,3a_{05}a_{21}-5a_{13}^2=0,\\ & \xi_3=\cdots=\xi_{k-1}=0,\xi_k\neq0,\\ C_{k\geq3}: & a_{03}=0,a_{21}=\cdots=a_{(k-1)1}=0,a_{k1}\neq0,a_{13}\neq0,\\ F_4: & a_{03}=0,a_{21}=0,a_{31}\neq0,a_{13}=0,a_{05}\neq0, \end{array} \tag{3}$$

where ξ_m depends on the $(2m+1)$-jet of the third component of (1) in Proposition 2.2 (see [15]).

If f is such that the $j^kf(\mathbf{0})$ is given by (2) in Proposition 2.2, then the conditions for f to be $\mathcal{A}$-equivalent to H_k or P_3 can be deduced in a similar way (see for instance [26]). In particular, we distinguish between the H_k and P_3 singularities by looking at the coefficient a_{03}. We have:

$$\begin{array}{ll} H_{k\geq2}: & a_{03}\neq0,\\ P_3: & a_{03}=0. \end{array} \tag{4}$$

In order to characterize completely the H_k and P_3 singularities some more conditions are necessary (see [26]). Since these other conditions are not used here in our calculations, we will omit them except for the condition $a_{04}-3a_{12}b_3\neq0$ for P_3-singularity which we show now. In fact, let f be $\mathcal{A}$-equivalent to P_3. We compute the double point curve of $f(x,y)=(x,p(x,y),q(x,y))$, which is defined by equations:

$$\frac{p(x,y)-p(x,u)}{y-u}=\frac{q(x,y)-q(x,u)}{y-u}=0.$$

This gives us the two following equations:

$$\begin{aligned}
&24x + 4b_3(u^2+uy+y^2) + b_4(u^3+u^2y+uy^2+y^3) + \text{ h.o.t.} = 0,\\
&a_{04}(u^3+u^2y+uy^2+y^3) + 4xa_{13}(u^2+uy+y^2)\\
&\quad + 6x(a_{22}x+2a_{12})(u+y) + 12a_{21}x^2 + 4a_{31}x^3 + \text{ h.o.t.} = 0,
\end{aligned}$$

where h.o.t. means "higher-order terms".

Now, using the first equation to eliminate the variable x, one obtains a curve in the plane (y,u) which is isomorphic to the double point curve:

$$\mathcal{W} = 1/24(u+y)(a_{04}(u^2+y^2) - 2a_{12}b_3(u^2+uy+y^2) + \text{ h.o.t.} = 0.$$

We know from [24] that if f is $\mathcal{A}$-equivalent to P_3, then the Milnor number of $\mathcal{W}$ at the origin must be equal to 4. This implies that $\mathcal{W}$ is 3-determined and thus, its 3-jet has to be nondegenerate. In other words, the discriminant of $j^3\mathcal{W}(\mathbf{0})$ must be different of 0, that is,

$$(a_{04} - 3a_{12}b_3)(a_{04} - a_{12}b_3) \neq 0$$

holds. In particular, $a_{04} - 3a_{12}b_3 \neq 0$.

3. Index of lines of curvature

Let $f : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a smooth map given by $f(x,y) = (f_1(x,y), f_2(x,y), f_3(x,y))$. The first and the second fundamental forms for f are given, respectively, by

$$I = E\,dx^2 + 2F\,dxdy + G\,dy^2 \quad \text{and} \quad II = L\,dx^2 + 2M\,dxdy + N\,dy^2$$

where

$$E = \langle f_x, f_x\rangle \ , \quad F = \langle f_x, f_y\rangle \ , \quad G = \langle f_y, f_y\rangle \ ,$$

$$L = \frac{\det(f_x,f_y,f_{xx})}{\sqrt{EG-F^2}} \ , \quad M = \frac{\det(f_x,f_y,f_{xy})}{\sqrt{EG-F^2}} \ , \quad N = \frac{\det(f_x,f_y,f_{yy})}{\sqrt{EG-F^2}} \ ,$$

and the subscripts denote partial derivatives. It follows that L, M, N are only defined if the denominator does not vanish; that is, at the regular points of f because $EG - F^2 = \|f_x \times f_y\| \neq 0$ only in these points. For situations which include the case where f may have singularities, we can define

$$L' = \det(f_x,f_y,f_{xx}) \ , \quad M' = \det(f_x,f_y,f_{xy}) \ , \quad N' = \det(f_x,f_y,f_{yy}) \ , \tag{5}$$

and work with this functions instead of L, M, N.

We recall that *umbilics points* are regular points of f in which the second fundamental form is proportional to the first. Then, the rank of the matrix

$$\begin{pmatrix} E & F & G \\ L' & M' & N' \end{pmatrix} \tag{6}$$

is not maximal either at an umbilic or at a singular point of f.

Suppose that (x,y) is a regular point of f which is not umbilic. Then the principal directions of f at (x,y) are defined as the directions determined by the eigenvectors of the second fundamental form at (x,y). The equation of the principal directions of f is given by the binary differential equation (BDE)

$$(FN' - GM')\,dy^2 + (EN' - GL')\,dxdy + (EM' - FL')\,dx^2 = 0 \ . \tag{7}$$

Thus, the principal directions define a pair of orthogonal line fields on the surface, which are singular either at an umbilic or at a singular point of f.

The equation (7) can be seen as a particular case of a positive quadratic differential form (PQD) on $M = f(U)$ in the sense of [13], that is, as a quadratic differential form ω such that for every point p in M the subset $\omega(p)^{-1}(0)$ of the tangent plane T_pM of M at p is either: (i) the union of two transversal lines (in this case p is called a regular point of ω), or (ii) all T_pM (in this case p is called a singular point of ω). In local coordinates (x, y), a PQD form is given by

$$\omega = A(x, y)dy^2 + B(x, y)dxdy + C(x, y)dx^2, \tag{8}$$

where A, B, C are smooth functions, called the *coefficients* of the PQD, such that $B^2 - 4AC \geq 0$. Because (8) is a PQD, $B^2 - 4AC = 0$ if and only if $A = B = C = 0$ ([13]). The points where $A = B = C = 0$ are the singular points of ω and the set

$$\Delta = \{(x, y) \in U; B^2 - 4AC(x, y) = 0\}$$

which is called the discriminant of the PQD coincides with its singular set. (For a general quadratic differential equation which is not necessarily a PQD, the discriminant Δ is different from the set of singular points of the equation; see for example the survey paper [29].)

Therefore, if ω is the PQD (7) associated to f then $(x, y) \in \triangle$ if and only if (x, y) is an umbilic or singular point of f (and hence a singular point of ω), which can be easily seen from the matrix (6). Then, all important features of the equation (8) occur on the discriminant. Taking an isolated singular point p of ω, we can consider the index at p associated with any of the lines of principal curvature determined by ω, which is denoted in the literature by $\operatorname{ind}(\omega, p)$ but we shall denote here by $\operatorname{ind}_{\mathcal{P}}(f, p)$ in order to specify f and with $\mathcal{P}$ indicating principal, as a reference for the equation (7). This means the number of turns of the line field when we run through a small circle centered at p. For instance, we can easily to compute the index of the three types of generic umbilics classified by Darboux (see, for instance, [2, 9, 12, 14]): the lemon (or D_1), the monster (or D_2) and the star (or D_3), which are $1/2$, $1/2$ and $-1/2$, respectively. Moreover, from the description for the principal lines at a cross-cap point p of f, whose configuration can be found in [12], for example, we deduce that the index $\operatorname{ind}_{\mathcal{P}}(f, p)$ is $1/2$ (see Figure 1).

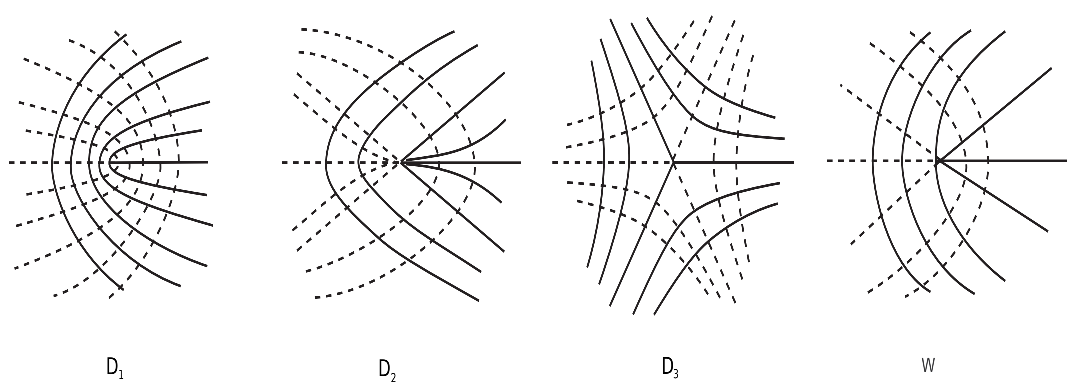

FIGURE 1. From left to right: configuration of integral curves of the principal directions at generic umbilics D_1, D_2 and D_3, and of the principal lines at a cross-cap point of f, W.

In order to consider the index $\operatorname{ind}_{\mathcal{P}}(f, p)$ it is necessary to have p as an isolated singular point of ω (for example, we should eliminate the possibility of the existence of a sequence of umbilic points on the smooth part of the surface that converges to p). We shall consider this question.

For this, we use the following lemma which shows that the index of an isolated singular point of a PQD is related to the mapping degree, given in terms of the coefficients of ω.

Lemma 3.1. ([18], Part 2, VIII, 2.3) *Let p be an isolated singular point of the positive quadratic differential form $\omega = A(x,y)\,dy^2 + B(x,y)\,dxdy + C(x,y)\,dx^2$. Then,*

$$\operatorname{ind}(\omega, p) = -\frac{1}{2}\deg((A,B),p) = -\frac{1}{2}\deg((B,C),p)$$

where $\deg((A,B),p)$ and $\deg((B,C),p)$ denote the mapping degrees of the maps (A,B) and (B,C), respectively, at p.

Let $h : (\mathbb{R}^n, \mathbf{0}) \to (\mathbb{R}^n, \mathbf{0})$ be a continuous map such that $\mathbf{0}$ is isolated in $h^{-1}(\mathbf{0})$. The degree $\deg(h, \mathbf{0})$ of h at $\mathbf{0}$ is defined as follows: choose a ε-ball B^n_ε centered at $\mathbf{0}$ in $\mathbb{R}^n$ so small that $h^{-1}(\mathbf{0}) \cap B^n_\varepsilon = \{\mathbf{0}\}$ and let S^{n-1}_ε be the $(n-1)$-sphere centered at the origin of radius ε. Choose an orientation of each copy of $\mathbb{R}^n$. Then the degree of h at $\mathbf{0}$ is the degree of the mapping $\frac{h}{\|h\|} : S^{n-1}_\varepsilon \to S^{n-1}$ ($S^{n-1} \subset \mathbb{R}^n$ is the unit standard sphere), where the spheres are oriented as $(n-1)$-spheres in $\mathbb{R}^n$. If h is differentiable, this degree can be computed as the sum of the signs of the Jacobian determinant of h (i.e., of its derivative) at all the h-preimages near $\mathbf{0}$ of a regular value of h near $\mathbf{0}$.

We also recall that $h : (\mathbb{R}^n, \mathbf{0}) \to (\mathbb{R}^n, \mathbf{0})$ is a *quasi-homogeneous map germ with weight* $\boldsymbol{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n$ and *quasi-degree* $\boldsymbol{d} = (d_1, \ldots, d_n) \in \mathbb{N}^n$ if

$$h_i(\lambda^{a_1}x_1, \lambda^{a_2}x_2, \ldots, \lambda^{a_n}x_n) = \lambda^{d_i} h_i(x_1, x_2, \ldots, x_n)$$

for each $i = 1, 2, \ldots, n$ and all $\lambda > 0$. We say that a smooth function has *quasi-order* m if all monomials in its Taylor expression have quasi-degree greater than or equal to m. We say that h is a *semi-quasi-homogeneous map with weight* $\boldsymbol{a}$ and *quasi-degree* $\boldsymbol{d}$ if $h = g + G$ with g a quasi-homogeneous map germ with weight $\boldsymbol{a}$ and quasi-degree $\boldsymbol{d}$ such that $\mathbf{0}$ is isolated in $g^{-1}(\mathbf{0})$, and each component G_i of G has quasi-order greater than d_i, $i = 1, 2, \ldots, n$.

The following theorem shows that for semi-quasi-homogeneous map germs, the degree at a zero coincides with the degree at this zero of its quasi-semi-homogeneous part.

Theorem 3.2. ([5]) *With the above notations, let $h = g + G$ be a semi-quasi-homogeneous map germ. Then $\mathbf{0}$ is isolated in $h^{-1}(0)$ and*

$$\deg(h, \mathbf{0}) = \deg(g, \mathbf{0}).$$

Before giving the results about the index of the lines of curvature for a corank 1 surface, we present an illustrative example explaining all our calculations.

Example 3.3. *Let S_1^+-standard be the map germ given by $(x, y^2, y^3 + x^2y)$ as in Table 1. The coefficients of its first and second fundamental forms are, respectively:*

$$E = 1 + 4x^2y^2, \quad F = 2xy(x^2 + 3y^2), \qquad G = 4y^2 + (x^2 + 3y^2)^2$$

and

$$L' = 4y^2, \quad M' = 4xy, \quad N' = -2x^2 + 6y^2.$$

Let $Ady^2 + Bdxdy + Cdx^2 = 0$ the BDE of the principal directions of S_1^+-standard. Then, from (7) we have that

$$A = -8x^5y - 16xy^3 - 24x^3y^3,\ B = -2x^2 + 6y^2 - 12x^4y^2 - 16y^4 - 36y^6,\ C = 4xy + 8x^3y^3 - 24xy^5.$$

Consider the map germ $h = (B, C) : (\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^2, \mathbf{0})$ given by $h = g + G$ taking

$$g(x,y) = (-2x^2 + 6y^2, 4xy) \text{ and } G(x,y) = (-12x^4y^2 - 16y^4 - 36y^6, 8x^3y^3 - 24xy^5).$$

In this case, g is a homogeneous map germ, $\mathbf{0}$ is isolated in $g^{-1}(\mathbf{0})$ and each component G_i of G has quasi-order greater than 2. Then, by Theorem 3.2, the degree of h in $\mathbf{0}$ coincides with the degree of g in $\mathbf{0}$. It is easy to calculate the degree of g in $\mathbf{0}$, which is -2. Hence, by Lemma 3.1, the index of the BDE associated to S_1^+-standard is 1.

The case S_1^--standard is analogous. Repeating this same sketch of calculations, we can conclude that the index of the BDE associated to S_1^--standard is 0. Figure 2 shows S_1^- and S_1^+-standards surfaces with their lines of curvatures.

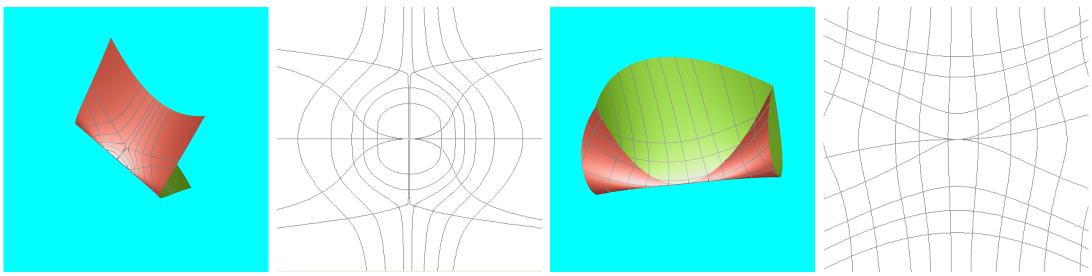

FIGURE 2. Standards S_1^+ and S_1^- surfaces and their lines of curvature.

In Proposition 2.1 are listed four orbits in $\Sigma^1 J^2(2,3)$ under the action of group $\mathcal{A}^2$, with the first one corresponding the known case of the cross-cap (cf. [10, 32]) and the fourth orbit listed corresponding to a non-simple germ wich is included in a stratum of $\mathcal{A}_e$-codimension > 3. Then it is just remaining to consider two cases in the 2-jet classification: $(x, y^2, 0)$ and $(x, xy, 0)$. The next theorem complete the study of the index of an isolated singular point of a BDE which represents the equation of the principal directions of a corank 1 simple map germ $f : (\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^3, \mathbf{0})$ or non simple but in strata of $\mathcal{A}_e$-codimension ≤ 3.

Theorem 3.4. *Let $f : (\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^3, \mathbf{0})$ be a corank 1 simple map germ or non-simple strata of $\mathcal{A}_e$-codimension ≤ 3. Consider $j^k f(\mathbf{0})$ as in Proposition 2.2. If $a_{12}^2 - a_{21}a_{03} \neq 0$ then $\mathbf{0} \in \mathbb{R}^2$ is an isolated singular point of the BDE associated to f given in (7) and*

$$\operatorname{ind}_{\mathcal{P}}(f, \mathbf{0}) = \begin{cases} 0 & if \quad a_{21}a_{03} < 0 \\ 0 & if \quad a_{12}^2 > a_{21}a_{03} \\ 1 & if \quad a_{12}^2 < a_{21}a_{03} \end{cases}$$

if $j^2 f(\mathbf{0})$ has type $(x, y^2, 0)$ and

$$\operatorname{ind}_{\mathcal{P}}(f, \mathbf{0}) = \begin{cases} 0 & if \quad a_{21}a_{03} \leq 0 \\ 1 & if \quad a_{21}a_{03} > 0 \end{cases}$$

if $j^2 f(\mathbf{0})$ has type $(x, xy, 0)$.

Proof. Under hypothesis, we just need to consider map germs which are $\mathcal{A}$-equivalent to one of those given in Table 1 and such that the 2-jet has the type $(x, y^2, 0)$ or $(x, xy, 0)$. We divide the proof in three parts. In all of them, we start with the following procedure:

Given $f : (\mathbb{R}^3, \mathbf{0}) \to (\mathbb{R}^2, \mathbf{0})$, we first calculate the coefficients E, F, G, L', M', N' associated to f; second we get the BDE expression of the principal directions of f given by (7), denoted here by $Ady^2 + Bdxdy + Cdx^2 = 0$.

These calculations can be done quickly using for instance the Mathematica software. Thus, they will be omitted here.

Part 1. $j^2 f(\mathbf{0}) \sim (x, y^2, 0)$**.**

• Consider f $\mathcal{A}$-equivalent to S_1 given in Table 1 (this means f is $\mathcal{A}$-equivalent to S_1^+ or S_1^-). The conditions on the coefficients of a S_1-singularity are $a_{03} \neq 0$ and $a_{21} \neq 0$. We use the same procedure given in Example 3.3. After calculating the coefficients of the first and second fundamental forms associated to f and getting the BDE expression of the principal directions of f, let us take the semi-quasi-homogeneous map $h = (B, C) : (\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^2, \mathbf{0})$. So, in this case we can consider $h = g + G$, where

$$g(x, y) = \left(-\frac{a_{21}}{2}x^2 + \frac{a_{03}}{2}y^2, a_{21}xy + a_{12}y^2\right)$$

is quasi-homogeneous (in fact homogeneous) and G has higher order terms. We call resultant of g to the resultant of the two components of g (with respect to one of the variables). The resultant of g is given by the expression $a_{12}^2 - a_{21}a_{03}$ (which is not zero by hypothesis) then we can conclude that $\mathbf{0}$ is isolated in $g^{-1}(\mathbf{0})$. Therefore, by Theorem 3.2, $\mathbf{0}$ is isolated in $h^{-1}(\mathbf{0})$ and the degree of h in $\mathbf{0}$ coincides with the degree of g in $\mathbf{0}$. Now we apply Lemma 3.1 to calculate the index $\mathrm{ind}_{\mathcal{P}}(f, \mathbf{0})$. To do this, let us calculate the degree of g at $\mathbf{0}$. Since $a_{03}, a_{21} \neq 0$, it may occur:

$$\text{(i)} \quad a_{21}a_{03} < 0 \quad \text{or} \quad \text{(ii)} \quad a_{21}a_{03} > 0.$$

Taking the following change of coordinates in the source of g

$$\begin{cases} X = a_{21}x + a_{12}y \\ Y = y \end{cases}$$

it holds that

$$g \sim \left(-\frac{1}{2a_{21}}(X^2 - 2a_{12}XY + (a_{12}^2 - a_{21}a_{03})Y^2), XY\right).$$

Taking now the change of coordinates in the target $k_1(u, v) = (-2a_{21}u, v)$, we have

$$g \sim (X^2 - 2a_{12}XY + (a_{12}^2 - a_{21}a_{03})Y^2), XY).$$

After one more change of coordinates in the target given by $k_2(u, v) = (u + 2a_{12}v, v)$, it holds that

$$g \sim (X^2 + (a_{12}^2 - a_{21}a_{03})Y^2, XY) = \tilde{g}(X, Y).$$

Due the previous change of coordinates, it follows that

$$\deg\,(g, \mathbf{0}) = -\mathrm{sgn}(2X^2 - 2(a_{12}^2 - a_{21}a_{03})Y^2)\deg\,(\tilde{g}, \mathbf{0}),$$

where sgn denotes the sign of a function.

If $a_{21}a_{03} < 0$ then $a_{12}^2 - a_{21}a_{03} > 0$. So, $\tilde{g}$ is not surjective and thus $\deg\,(\tilde{g}, \mathbf{0}) = 0$. Hence $\deg\,(g, \mathbf{0}) = 0$ and thus $\mathrm{ind}_{\mathcal{P}}(f, \mathbf{0}) = 0$.

If $a_{21}a_{03} > 0$, we have two possibilities: $a_{12}^2 > a_{21}a_{03}$ or $a_{12}^2 < a_{21}a_{03}$. If $a_{12}^2 > a_{21}a_{03}$ then $a_{12}^2 - a_{21}a_{03} > 0$ and as already done, $\mathrm{ind}_{\mathcal{P}}(f, \mathbf{0}) = 0$. If $a_{12}^2 < a_{21}a_{03}$ then $a_{12}^2 - a_{21}a_{03} < 0$. In this case, the Jacobian determinant of $\tilde{g}$ is equal to

$$2X^2 - 2(a_{12}^2 - a_{21}a_{03})Y^2 > 0$$

for any (X, Y).

Taking any regular value of $\tilde{g}$, there always exist two $\tilde{g}$-preimages for which the signs of the Jacobian determinants are 1. Hence $\deg\,(\tilde{g}, \mathbf{0}) = 2$, which implies that $\deg\,(g, \mathbf{0}) = -2$ and thus $\mathrm{ind}_{\mathcal{P}}(f, \mathbf{0}) = 1$.

• Consider f $\mathcal{A}$-equivalent to S_k given in Table 1, for any $k \geq 2$. By conditions on the coefficients of a S_k-singularity given in (3) and by hypothesis $a_{12}^2 - a_{21}a_{03} \neq 0$, one has that $a_{12} \neq 0$.

We reproduce the same steps as in the previous case. From the coefficients B and C of (7) for f, we can take, for all $k \geq 2$, the semi-quasi-homogeneous map $h = (B, C) : (\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^2, \mathbf{0})$ given by $h = g + G$, where

$$g(x, y) = \left(-\frac{a_{(k+1)1}}{(k+1)!} x^{k+1} + \frac{a_{03}}{2} y^2, a_{12} y^2 \right) \tag{9}$$

is quasi-homogeneous and G has higher-order terms. In the expression of the resultant of g appears just a_{12}, which is not zero in this case. Therefore, for all $k \geq 2$, the map germ g in (9) is clearly not surjective and hence its degree is 0. By Theorem 3.2, $\deg(h, \mathbf{0}) = \deg(g, \mathbf{0}) = 0$. As consequence, by Lemma 3.1, the $\mathrm{ind}_{\mathcal{P}}(f, \mathbf{0}) = 0$.

• Consider f $\mathcal{A}$-equivalent to B_k given in Table 1, for any $k \geq 2$. A B_k-singularity is characterized by conditions which appear in (3). Since $a_{03} = 0$, the general hypothesis reduces to $a_{12} \neq 0$. We proceed in the same way as in the previous cases, following the same steps.

In this case, for all $k \geq 2$, we can take the semi-quasi-homogeneous map

$$h = (B, C) : (\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^2, \mathbf{0})$$

given by $h = g + G$, where

$$g(x, y) = \left(-\frac{1}{2} a_{21} x^2, a_{12} y^2 + a_{21} xy \right) \tag{10}$$

is homogeneous and G has higher-order terms. The resultant of g is given by $a_{12}^2 a_{21}$ which is not zero. Therefore, for all $k \geq 2$, the map germ g in (10) clearly is not surjective and hence its degree is 0. Then, again we have that $\mathrm{ind}_{\mathcal{P}}(f, \mathbf{0}) = 0$.

• Consider f $\mathcal{A}$-equivalent to C_k given in Table 1, for any $k \geq 3$. A C_k-singularity is characterized by conditions

$$a_{03} = 0, a_{21} = a_{31} = \cdots = a_{(k-1)1} = 0, a_{k1} \neq 0 \text{ and } a_{13} \neq 0.$$

Then, the general hypothesis again reduces to $a_{12} \neq 0$. Proceeding in the same way as in the previous cases, for all $k \geq 3$, we can take the semi-quasi-homogeneous map

$$h = (B, C) : (\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^2, \mathbf{0})$$

such that $h = g + G$, where

$$g(x, y) = \left(-\frac{1}{k!} a_{k1} x^k, a_{12} y^2 \right), \tag{11}$$

is quasi-homogeneous and G has higher-order terms.

In the expression of the resultant of g just appears a_{12}, which is not zero. Therefore, for all $k \geq 3$, the map germ g in (11) clearly is not surjective and hence its degree is 0 from which one concludes that $\mathrm{ind}_{\mathcal{P}}(f, \mathbf{0}) = 0$.

• Consider f $\mathcal{A}$-equivalent to F_4 given in Table 1. The F_4-singularity is characterized by conditions

$$a_{03} = a_{21} = a_{13} = 0, a_{31} \neq 0 \text{ and } a_{05} \neq 0.$$

Then, the general hypothesis again reduces for $a_{12} \neq 0$. In this case, we can take the semi-quasi-homogeneous map $h = (B, C) : (\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^2, \mathbf{0})$ given by $h = g + G$, where

$$g(x, y) = \left(-\frac{1}{6} a_{31} x^3, a_{12} y^2 \right) \tag{12}$$

is quasi-homogeneous and G has higher-order terms.

The resultant of g is given by a_{12} which is not zero. Therefore, the map germ g in (12) is also non surjective and hence its degree is 0. Thus $\mathrm{ind}_{\mathcal{P}}(f, \mathbf{0}) = 0$.

Part 2. $j^2f(\mathbf{0}) \sim (x, xy, 0)$ **and** f **is a simple map germ.**

In this case f is $\mathcal{A}$-equivalent to H_k given in Table 1, with $k \geq 2$. We have already seen in Section 2 that a necessary condition to H_k-singularity occurs is $a_{03} \neq 0$.

In this case, we can take the semi-quasi-homogeneous map $h = (B, C) : (\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^2, \mathbf{0})$, $h = g + G$, where

$$g(x, y) = \left(a_{12}x^2 + a_{03}xy, \frac{1}{2}a_{21}x^2 - \frac{1}{2}a_{03}y^2\right) \tag{13}$$

is homogeneous and G has higher-order terms.

The resultant of g is given by the expression $-a_{03}^2(a_{12}^2 - a_{21}a_{03})$, which is not zero. Then $\mathbf{0}$ is isolated in $g^{-1}(\mathbf{0})$.

Consider the following change of coordinates in the source of g:

$$\begin{cases} X = x \\ Y = a_{12}x + a_{03}y\,. \end{cases}$$

Then

$$g \sim \left(XY, \frac{1}{2a_{03}}\left(-(a_{12}^2 - a_{21}a_{03})X^2 + 2a_{12}XY - Y^2\right)\right).$$

Taking another change of coordinates $k_1(u, v) = (u, 2a_{03}v)$, now in the target, it holds that

$$g \sim (XY, -(a_{12}^2 - a_{21}a_{03})X^2 + 2a_{12}XY - Y^2).$$

After one more change of coordinates in the target given by $k_2(u, v) = (u, v - 2a_{12}u)$, we have

$$g \sim (XY, -(a_{12}^2 - a_{21}a_{03})X^2 + 2a_{12}XY - Y^2) = \tilde{g}(X, Y).$$

Due the previous changes of coordinates applied in g, it follows that $\deg(g, \mathbf{0}) = \deg(\tilde{g}, \mathbf{0})$, which does not depend on the sign of a_{03}.

If $a_{12}^2 - a_{21}a_{03} > 0$ then $\tilde{g}$ is not surjective. In fact, take for instance $(0, \epsilon) \in \mathbb{R}^2, \epsilon > 0$ small enough. Then there is not (X, Y) such that $\tilde{g}(X, Y) = (0, \epsilon)$. Suppose by absurd that

$$XY = 0 \ \text{ and } \ -(a_{12}^2 - a_{21}a_{03})X^2 + 2a_{12}XY - Y^2 = \epsilon.$$

From the first expression, $X = 0$ or $Y = 0$. If $X = 0$, then the second equation reduces to $-Y^2 = \epsilon > 0$. Otherwise, if $Y = 0$, then we obtain $-(a_{12}^2 - a_{21}a_{03})X^2 = \epsilon > 0$ while $(a_{12}^2 - a_{21}a_{03}) > 0$.

Thus, $\tilde{g}$ is not surjective and $\deg(\tilde{g}, \mathbf{0}) = 0 = \deg(g, \mathbf{0})$. Hence, $\mathrm{ind}_{\mathcal{P}}(f, \mathbf{0}) = 0$.

If $a_{12}^2 - a_{21}a_{03} < 0$, the Jacobian determinant of $\tilde{g}$ is equal to

$$-2Y^2 + 2(a_{12}^2 - a_{21}a_{03})X^2 < 0.$$

For any regular value of $\tilde{g}$, there always exist two $\tilde{g}$-preimages for which the sign of the Jacobian determinants of $\tilde{g}$ are -1. Hence $\deg(\tilde{g}, \mathbf{0}) = -2$, which implies that $\deg(g, \mathbf{0}) = -2$ and thus $\mathrm{ind}_{\mathcal{P}}(f, \mathbf{0}) = 1$.

Part 3. $j^2f(\mathbf{0}) \sim (x, xy, 0)$ **and** f **is a non-simple strata of** $\mathcal{A}_e$**-codimension** ≤ 3.

In this case f is $\mathcal{A}$-equivalent to P_3 given in Table 1. We have already seen in Section 2 that necessary conditions to P_3-singularity occurs are $a_{03} = 0$ and $a_{04} - 3a_{12}b_3 \neq 0$.

In this case, we can take the semi-quasi-homogeneous map $h = (A, B) : (\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^2, \mathbf{0})$ given by $h = g + G$, where

$$g(x, y) = \left(\frac{1}{2}a_{21}x^2 + \left(-\frac{1}{6}a_{04} + \frac{1}{2}a_{12}b_3\right)y^3, a_{12}x^2\right),$$

is quasi-homogeneous with weight $(3, 2)$ and quasi-degree $(6, 6)$ and G has only higher-order terms. Moreover, since $a_{12} \neq 0$ and $a_{04} - 3a_{12}b_3 \neq 0$, the resultant of g given by $a_{12}^2(a_{04} - 3a_{12}b_3)^2$

is not zero. Therefore $g^{-1}(\mathbf{0}) = \mathbf{0}$. In particular, h is semi-quasi-homogeneous and $\deg(h) = \deg(g) = 0$ because g is not surjective. So, $\mathrm{ind}_{\mathcal{P}}(f, \mathbf{0}) = 0$. □

From Theorem 3.4 it holds that:

Corollary 3.5. *Let f be a map germ in the $\mathcal{A}$-class of one of the map germs given in Table 1, with $j^k f(\mathbf{0})$ as in Proposition 2.2. Suppose that $a_{12}^2 - a_{21}a_{03} \neq 0$.*
(i) *If $f \sim S_1^{\pm}$ or H_k then $\mathrm{ind}_{\mathcal{P}}(f, \mathbf{0}) = 0$ or 1.*
(ii) *If $f \sim S_{k\geq 2}^{\pm}, B_k^{\pm}, C_k^{\pm}, F_4$ or P_3 then $\mathrm{ind}_{\mathcal{P}}(f, \mathbf{0}) = 0$.*

Remark 3.6. *It follows from Theorem 3.4 that for any corank 1 map germ f satisfying its hypothesis, the singularity of the BDE of the principal directions of f is an isolated point, i.e. there is not sequence of umbilic points on the smooth part of the surface that converges to the singular point of the surface.*

4. Geometric interpretation of the condition $a_{12}^2 - a_{21}a_{03} \neq 0$

Let $f : (\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^3, \mathbf{0})$ be a corank 1 map germ whose 2-jet has $\mathcal{A}^2$-type either $(x, y^2, 0)$ or $(x, xy, 0)$. We want to analyze the circles which have a special contact with f at the origin. To do this, we need to look at the singularity type of the contact map germ $C_{\mathbf{v},\mathbf{u}} : (\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^2, \mathbf{0})$ given by

$$C_{\mathbf{v},\mathbf{u}}(x, y) = (\langle f(x, y), \mathbf{v}\rangle, \|f(x, y) - \mathbf{u}\|^2 - \|\mathbf{u}\|^2),$$

where $\mathbf{v}, \mathbf{u} \in \mathbb{R}^3$, $\|\mathbf{v}\| = 1$ is the unit normal vector of the circle and $\mathbf{u}$ is its centre. Note that the first component is nothing but the height function which measures the contact of f with the normal plane to $\mathbf{v}$ and the second component the squared distance function which measures the contact of f with the sphere of centre $\mathbf{u}$.

In order to consider the desired contact we use the umbilic curvature, the binormal and asymptotic directions defined in [21], which are related to contact properties of the surface given by f with planes and spheres. The umbilic curvature κ_u is an important second-order invariant of the f: when it is non-zero, then $1/\kappa_u$ is the radius of the unique sphere with umbilical contact (that is, contact of type $\Sigma^{2,2}$ in Thom-Boardman terminology) with the surface at the singular point. See [21] for details.

We recall that a map germ $g : (\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^2, \mathbf{0})$ has type $\Sigma^{2,1}$ if and only if its 2-jet is equivalent to $(x^2, 0)$.

Lemma 4.1. *Let $f : (\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^3, \mathbf{0})$ be a corank 1 map germ with $j^k f(\mathbf{0})$ as in Proposition 2.2 and with non-zero umbilic curvature κ_u at the origin.*

(i) *If $j^2 f(\mathbf{0}) \sim (x, y^2, 0)$, there are exactly two circles with contact of type $\Sigma^{2,1}$ with f at the origin, given by $\mathbf{u} = (0, 0, 1/a_{20})$ and $\mathbf{v} = (0, 0, 1)$ or $\mathbf{v} = (0, -a_{20}, b_2)/\sqrt{a_{20}^2 + b_2^2}$.*
(ii) *If $j^2 f(\mathbf{0}) \sim (x, xy, 0)$, there is exactly one circle with contact of type $\Sigma^{2,1}$ with f at the origin, given by $\mathbf{u} = (0, 0, 1/a_{20})$ and $\mathbf{v} = (0, 0, 1)$.*

Proof. Notice that the circle determined by $\mathbf{u}, \mathbf{v}$ has contact of type $\Sigma^{2,1}$ if and only if the sphere of centre $\mathbf{u}$ has umbilical contact and the plane normal to $\mathbf{v}$ is binormal (i.e., it has a degenerate contact $\Sigma^{2,1}$). Then, our results follow from the analysis of contacts with spheres and planes in [21], where the umbilic curvature at the origin is $\kappa_u(\mathbf{0}) = |a_{20}|$. □

We observe that if $j^2 f(\mathbf{0}) \sim (x, y^2, xy)$ then there is not circle with contact of type $\Sigma^{2,1}$ with f at the origin (because there is not sphere with contact of type $\Sigma^{2,2}$ with f, see [21] for details). The circles with contact of type $\Sigma^{2,1}$ with f given in the above lemma will be called $\Sigma^{2,1}$-*circles* for simplicity.

Definition 4.2. Let $g : (\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^2, \mathbf{0})$ be a map germ of type $\Sigma^{2,1}$. We say that g is $\Sigma^{2,1}$-*generic* if it is $\mathcal{A}$-equivalent to a finitely determined map germ of the form

$$(x^2, c_0x^3 + 3c_1x^2y + 3c_2xy^2 + c_3y^3),$$

for some $c_0, c_1, c_2, c_3 \in \mathbb{R}$.

Remark 4.3. *It follows from the definition that if $j^3g(\mathbf{0}) = (x^2, c_0x^3 + 3c_1x^2y + 3c_2xy^2 + c_3y^3)$, then a necessary condition for g being $\Sigma^{2,1}$-generic is that $c_2^2 - c_1c_3 \neq 0$. In fact, a necessary condition for finite determinacy for map germs $(\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^2, \mathbf{0})$ is that its Jacobian determinant has to be non-degenerate. A simple computation shows that the Jacobian determinant of $j^3g(\mathbf{0})$ is $6x(c_1x^2 + 2c_2xy + c_3y^2)$, so we must have $c_3 \neq 0$ and $c_2^2 - c_1c_3 \neq 0$.*

Corollary 4.4. *Let $f : (\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^3, \mathbf{0})$ be a corank 1 map germ with $j^kf(\mathbf{0})$ as in Proposition 2.2 and with non-zero umbilic curvature κ_u at the origin. Assume that the $\Sigma^{2,1}$-circles of f have $\Sigma^{2,1}$-generic contact. Then, $a_{12}^2 - a_{21}a_{03} \neq 0$.*

Proof. It is easy to show that for $\mathbf{u} = (0, 0, 1/a_{20})$ and $\mathbf{v} = (0, 0, 1)$, we have:

$$j^3C_{\mathbf{v},\mathbf{u}}(\mathbf{0}) = \left(\frac{1}{2}a_{20}x^2, -\frac{1}{3a_{20}}(a_{30}x^3 + 3a_{21}x^2y + 3a_{12}xy^2 + a_{03}y^3)\right).$$

When $j^2f(\mathbf{0}) \sim (x, y^2, 0)$ and we consider $\mathbf{u} = (0, 0, 1/a_{20})$ and $\mathbf{v} = (0, -a_{20}, b_2)/\sqrt{a_{20}^2 + b_2^2}$, we get

$$j^3C_{\mathbf{v},\mathbf{u}}(\mathbf{0}) = \left(-\frac{1}{2\sqrt{a_{20}^2 + b_2^2}}a_{20}y^2, -\frac{1}{3a_{20}}(a_{30}x^3 + 3a_{21}x^2y + 3a_{12}xy^2 + a_{03}y^3)\right).$$

So the result follows from Remark 4.3. □

5. Umbilics and cross-caps of generic deformations

Let $f : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a smooth map. It was shown in [12] that f is principally structurally stable at an umbilic point if and only if it is one of the Darbouxian umbilics D_i, $i = 1, 2, 3$ (see also [2]). Furthermore, the unique stable singularity for f is a cross-cap point.

The map f is said to be *generic* if the ulfoldings

$$D : \mathbb{R}^3 \times U \to \mathbb{R}^3 \times \mathbb{R}, \quad (\mathbf{u}, (x, y)) \mapsto (\mathbf{u}, d_{\mathbf{u}}(x, y)), \quad d_{\mathbf{u}}(x, y) = \frac{1}{2}\|f(x, y) - \mathbf{u}\|^2$$

and

$$H : S^2 \times U \to S^2 \times \mathbb{R}, \quad (\mathbf{v}, (x, y)) \mapsto (\mathbf{v}, h_{\mathbf{v}}(x, y)), \quad h_{\mathbf{v}}(x, y) = \langle f(x, y), \mathbf{v}\rangle$$

are generic in the Thom-Boardman sense (see [11] for details). So, if the map f is not generic, we can take a generic deformation $f_\lambda : U_0 \subset U \to \mathbb{R}^3$, $\lambda \in (-\varepsilon, \varepsilon)$, of f, i.e. $f_0 = f$, f_λ is generic for $\lambda \neq 0$ and the map $(\lambda, t) \mapsto f_\lambda(t)$ is smooth, and the index $\mathrm{ind}_{\mathcal{P}}(f, p)$ is equal to $(D_1 + D_2 - D_3 + W)/2$, where D_1, D_2, D_3 also denote the number of umbilics of each type and W the number of cross-caps points that appear in f_λ near p, for $\lambda \neq 0$ small enough.

When $f : (\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^3, \mathbf{0})$ is a corank 1 map germ and f_λ, $\lambda \in (-\varepsilon, \varepsilon)$, is a generic deformation of f, D. Mond showed in [24] how to count the number of cross-caps in f_λ. More precisely, it is showed the following possibilities for W in f_λ according the $\mathcal{A}$-types of f given in

Table 1:

$$\begin{array}{ll}
S_k^{\pm}, k \geq 1: & W = \begin{cases} 2n\,;\, n = 0, 1, \ldots, \frac{k+1}{2} & \text{if } k \text{ is odd} \\ 2n+1\,;\, n = 0, 1, \ldots, \frac{k}{2} & \text{if } k \text{ is even} \end{cases} \\
B_k^{\pm}, H_k, k \geq 2: & W = 0, 2 \\
C_k^{\pm}, k \geq 3: & W = \begin{cases} 2n+1\,;\, n = 0, 1, \ldots, \frac{k-1}{2} & \text{if } k \text{ is odd} \\ 2n\,;\, n = 0, 1, \ldots, \frac{k}{2} & \text{if } k \text{ is even} \end{cases} \\
F_4, P_3: & W = 1, 3.
\end{array}$$

As an immediate consequence of this, we obtain some information about the number of umbilic points in f_λ. In fact, this number is equal to $D_1 + D_2 + D_3 = 2(\mathrm{ind}_{\mathcal{P}}(f, \mathbf{0}) + D_3) - W$. So, if W is even (respec. odd), the number of umbilic points that appear in f_λ is even (resp. odd). Consequently, we have:

Lemma 5.1. *Let $f : (\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^3, \mathbf{0})$ be a corank 1 map germ simple or non-simple but including in strata of $\mathcal{A}_e$-codimension ≤ 3. If $f_\lambda : (\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^3, \mathbf{0})$ is a generic deformation of f then the number of umbilic points that appear in f_λ near $\mathbf{0}$, for λ small enough, is:*
(i) *even if $f \sim S_k^{\pm}$ (with k odd), $B_k^{\pm}$, $C_k^{\pm}$ (with k even) or H_k;*
(ii) *odd if $f \sim S_k^{\pm}$ (with k even), $C_k^{\pm}$ (with k odd), F_4 or P_3.*

We shall give more precise information about the number of umbilic points in f_λ. Before stating the result and proving it, we need recall some facts about multiplicity for special types of singular points of a map.

Given a smooth map germ $f : (\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^3, \mathbf{0})$, we say that $\mathbf{0}$ is a 2-*rounding* of f if $\mathbf{0}$ is either a 2-*flattening* (that is, there is a unit vector $\mathbf{v} \in \mathbb{R}^3$ such that $\mathbf{0}$ is a singularity of type $\Sigma^{2,2}$ of $h_{\mathbf{v}}$) or a *non-flat* 2-*rounding* (that is, it is not a 2-flattening and there is $\mathbf{u} \in \mathbb{R}^3$ such that $\mathbf{0}$ is a singularity of type $\Sigma^{2,2}$ of $d_{\mathbf{u}}$). It is known that a regular (resp. singular) point of f is a 2-rounding if and only if it is an umbilic point (resp. it is not a cross-cap point). See [11] for details. So, since a generic deformation of f only has umbilics of type D_i, $i = 1, 2, 3$, and cross-caps, and since cross-caps are not 2-roundings, then in order to estimate the number of umbilic points in f_λ it is enough to estimate the number of its 2-roundings, which is denoted by $n_{\mathcal{R}}(f_\lambda, \mathbf{0})$.

The number $n_{\mathcal{R}}(f_\lambda, \mathbf{0})$ is related with the multiplicity of $\mathbf{0}$ as a rounding of f, $\mu_{\mathcal{R}}(f, \mathbf{0})$, as follows:

$$n_{\mathcal{R}}(f_\lambda, \mathbf{0}) \leq \mu_{\mathcal{R}}(f, \mathbf{0}) \quad \text{and} \quad n_{\mathcal{R}}(f_\lambda, \mathbf{0}) \equiv \mu_{\mathcal{R}}(f, \mathbf{0}) \,(\mathrm{mod}\, 2),$$

for λ small enough, if $\mu_{\mathcal{R}}(f, \mathbf{0})$ is finite (see Theorem 2.9 of [11]), where

$$\mu_{\mathcal{R}}(f, \mathbf{0}) = \dim_{\mathbb{R}} \frac{C^\infty(\mathbb{R}^2, \mathbf{0})}{\mathcal{R}(f, \mathbf{0})} \ ,$$

with $C^\infty(\mathbb{R}^2, \mathbf{0})$ being the ring of germs at $\mathbf{0}$ of smooth real-valued functions on $\mathbb{R}^2$ and $\mathcal{R}(f, \mathbf{0})$ the ideal generated by the germs at $\mathbf{0}$ of the 4-minors of the matrix given by

$$\begin{pmatrix} f_{1_x} & f_{2_x} & f_{3_x} & 0 \\ f_{1_y} & f_{2_y} & f_{3_y} & 0 \\ f_{1_{xx}} & f_{2_{xx}} & f_{3_{xx}} & E \\ f_{1_{xy}} & f_{2_{xy}} & f_{3_{xy}} & F \\ f_{1_{yy}} & f_{2_{yy}} & f_{3_{yy}} & G \end{pmatrix},$$

where $f = (f_1, f_2, f_3)$. See [11] for details.

We also recall that if $h : (\mathbb{R}^n, \mathbf{0}) \to (\mathbb{R}^n, \mathbf{0})$ is a smooth map germ with $\mathbf{0}$ isolated in $h^{-1}(\mathbf{0})$, then the multiplicity $\mu(h, \mathbf{0})$ of h at $\mathbf{0}$ is defined by

$$\mu(h, \mathbf{0}) = \dim_{\mathbb{R}} \frac{C^\infty(\mathbb{R}^n, \mathbf{0})}{\langle h \rangle},$$

where $\langle h\rangle$ is the ideal generated by the components of h. It is known that $\mu(h,\mathbf{0})$ is the number of complex h-preimages near $\mathbf{0}$ of a regular value of h near $\mathbf{0}$. If $h=(h_1,\dots,h_n)$, with each h_i being a homogeneous polynomial such that $\mathbf{0}$ is isolated in $h^{-1}(\mathbf{0})$, it is well known that $\mu(h,\mathbf{0})=d_1\cdots d_n$, where d_i is the degree of each h_i. On the other hand, writing $h=g+G$, where $g=(g_1,\dots,g_n)$ with g_i being the first non-zero jet of h_i, then $\mu(h,\mathbf{0})=\mu(g,\mathbf{0})$, if $\mathbf{0}$ is isolated in $g^{-1}(\mathbf{0})$. When $\mathbf{0}$ is not isolated in $g^{-1}(\mathbf{0})$ in the above construction, we can take a suitable selection of weights associated with any variable in order to make possible a different decomposition $h=g'+G'$ satisfying $\mu(h,\mathbf{0})=\mu(g',\mathbf{0})$. In fact, it is valid the same statement of Theorem 3.2, with multiplicity instead of index (see Remark 3.1 of [5]). Furthermore, one shall use the following result:

Proposition 5.2. ([1, 5]) *Using the above notations, let $h=g+G$ be a semi-quasi-homogeneous map germ with weight $\mathbf{a}=(a_1,\dots,a_n)$ and quasi-degree $\mathbf{d}=(d_1,\dots,d_n)$. Suppose that $\mu(h,\mathbf{0})<\infty$. Then*

$$\mu(h,\mathbf{0})=\mu(g,\mathbf{0})=\frac{d_1\cdots d_n}{a_1\cdots a_n}.$$

Let us denote by ΣD_i the number of umbilic points of f_λ, that is, $\Sigma D_i=D_1+D_2+D_3$. So, one gets the following result:

Proposition 5.3. *Under the same assumptions in Theorem 3.4, if the umbilic curvature of f is non-zero at the origin and f_λ is a generic deformation of f, then the number of umbilic points of f_λ, for λ small enough, if finite, satisfies:*

(i) $f\sim S_k^\pm, k\geq 1$: $\Sigma D_i\leq k+1$ *with* $\Sigma D_i\equiv k+1\,(\mathrm{mod}\,2)$.
(ii) $f\sim C_k^\pm, k\geq 3$: $\Sigma D_i\leq k$ *with* $\Sigma D_i\equiv k\,(\mathrm{mod}\,2)$.
(iii) $f\sim B_k^\pm$ *or* $H_k, k\geq 2$: $\Sigma D_i = 0$ *or* 2.
(iv) $f\sim F_4$ *or* P_3: $\Sigma D_i=1$ *or* 3.

Furthermore, $D_3\geq W$ when $\mathrm{ind}_{\mathcal{P}}(f,\mathbf{0})=0$, and $D_3\geq \frac{W}{2}$ when $\mathrm{ind}_{\mathcal{P}}(f,\mathbf{0})=1$.

Proof. We shall count the number $n_{\mathcal{R}}(f_\lambda,\mathbf{0})$ of 2-roundings of f_λ. Let us take $f=(x,f_2,f_3)$ as in Proposition 2.2.

Since f is not a cross-cap and $\kappa_u(\mathbf{0})=|a_{20}|\neq 0$, it follows from Corollary 2.17 of [21] that $\mathbf{0}$ is a non-flat 2-rounding of f. From [11] we conclude that $\mathcal{R}(f,\mathbf{0})=\langle P_y,P_{xy}\rangle$ if $j^2f(\mathbf{0})\sim(x,y^2,0)$ and $\mathcal{R}(f,\mathbf{0})=\langle P_y,P_{yy}\rangle$ if $j^2f(\mathbf{0})\sim(x,xy,0)$, where

$$P_y=\begin{vmatrix} f_x & 0\\ f_{xx} & E\\ f_{xy} & F\\ f_{yy} & G\end{vmatrix},\quad P_{xy}=\begin{vmatrix} f_x & 0\\ f_y & 0\\ f_{xx} & E\\ f_{yy} & G\end{vmatrix}\quad\text{and}\quad P_{yy}=\begin{vmatrix} f_x & 0\\ f_y & 0\\ f_{xx} & E\\ f_{xy} & F\end{vmatrix}.$$

Let $h:(\mathbb{R}^2,\mathbf{0})\to(\mathbb{R}^2,\mathbf{0})$ given by $h=(P_y,P_{xy})$ or (P_y,P_{yy}). Then

$$\mu_{\mathcal{R}}(f,\mathbf{0}) \;=\; \dim_{\mathbb{R}}\frac{C^\infty(\mathbb{R}^2,\mathbf{0})}{\langle h\rangle} \;=\; \mu(h,\mathbf{0}).$$

• Let us suppose that $j^2f(\mathbf{0})\sim(x,y^2,0)$.
If $f\sim S_1^\pm$ or $B_k^\pm$ then $a_{21}\neq 0$. After some calculations we take $h=g+G$, where

$$g(x,y)=(-a_{21}x-a_{12}y,\frac{1}{2}a_{21}x^2-\frac{1}{2}a_{03}y^2)$$

and G has higher-order terms. Since the resultant of g is given by the expression $\frac{1}{2}a_{21}(a_{12}^2-a_{21}a_{03})$ and $a_{12}^2-a_{21}a_{03}\neq 0$ by hypothesis, we have that $\mathbf{0}$ is isolated in $g^{-1}(\mathbf{0})$ and it holds that

$$n_{\mathcal{R}}(f_\lambda, \mathbf{0}) \leq \mu_{\mathcal{R}}(f, \mathbf{0}) \;=\; \mu(g, \mathbf{0}) \;=\; 2\,.$$

By Lemma 5.1, $\varSigma D_i$ is even for $S_1^{\pm}$ and $B_k^{\pm}$ and, therefore, $\varSigma D_i = 0, 2$.

If $f \sim S_{k\geq 2}$, $C_k^{\pm}$ or F_4 then we reproduce the same steps as in previous case, taking an apropriated g such that $h = g + G$ satisfies the Corollary 5.2, getting after calculations the desired results.

• Let us suppose now that $j^2 f(\mathbf{0}) \sim (x, xy, 0)$. We take $f \sim H_k$ or P_3, depending on a_{03} is non-zero or zero, respectively. Since $h = (P_y, P_{yy})$, we take $h = g + G$, where

$$g(x,y) = \left(a_{12}x + a_{03}y, -\frac{1}{2}a_{21}x^2 + \frac{1}{2}a_{03}y^2\right)$$

when f is of H_k type, or $g(x,y) = (a_{12}x + (\frac{1}{2}a_{04} - a_{12}b_3)y^2, -\frac{1}{2}a_{21}x^2)$, when f is of P_3 type with $a_{21} \neq 0$, or $g(x,y) = (a_{12}x + (\frac{1}{2}a_{04} - a_{12}b_3)y^2, (\frac{1}{6}a_{04} - \frac{1}{2}a_{12}b_3)y^3)$, when f is of P_3 type with $a_{21} = 0$, with G having higher-order terms. Since $a_{12} \neq 0$ from hypothesis, and $a_{04} - 3a_{12}b_3 \neq 0$ when f is of P_3 type, which appear in the expression of the resultant of g, then we conclude that h is semi-quasi-homogeneous and so, it follows that $\mu(h, \mathbf{0}) = \mu(g, \mathbf{0}) = 2$ if $f \sim H_k$, and $\mu(h, \mathbf{0}) = \mu(g, \mathbf{0}) = 4$ if $f \sim P_3$ type. So, the result on $\varSigma D_i$ follows from Lemma 5.1.

For the second part of the proposition, it is enough to use the relation

$$D_1 + D_2 - D_3 = 2\,\mathrm{ind}_{\mathcal{P}}(f, \mathbf{0}) + W.$$

□

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References

[1] V. Arnold, A. Varchenko and S. Gusein-Zade, *Singularités des applications différentiables*, Mir, Moscow, 1982.

[2] J. W. Bruce and D. L. Fidal, *On binary differential equations and umbilics*, Proc. Royal Soc. Edinburgh Sect A, 111 (1989), 147-168.

[3] J. W. Bruce and J. M. West, *Functions on a crosscap*, Math. Proc. Cambridge Philos. Soc., 123 (1998), 19-39.

[4] J. W. Bruce and F. Tari, *On the multiplicity of implicit differential equations*, J. Diff. Eq. 148 (1998), 122-147. DOI: 10.1006/jdeq.1998.3454

[5] A. Cima, A. Gasull, J. Torregrosa, *On the relation between index and multiplicity*, J. London. Math. Soc. (2) 57 (1998), 757-768. DOI: 10.1112/s0024610798006188

[6] L S. Challapa, *Index of quadratic differentail forms*, Real and complex singularities, Contemp. Math. 459, Amer. Math. Soc., Providence, RI (2008), 179-193. DOI: 10.1090/conm/459/08970

[7] L. S. Challapa, *Invariants of binary differential equations*, J. Dyn. Control Syst. 15 (2009), no. 2, 157-176. DOI: 10.1007/s10883-009-9066-z

[8] L. S. Challapa and M. A. S. Ruas, *Index of an implicit differential equation*, Publ. Mat. 54 (2010), no. 1, 173-186.

[9] G. Darboux, *Leçons sur la théorie générales des surfaces*, vol. 4, Gauthier-Villars, Paris, 1896.

[10] T. Fukui and M. Hasegawa, *Fronts of Whitney umbrella - a differential geometric approach via blowing up*, J. Singularities 4 (2012), 35-67. DOI: 10.5427/jsing.2012.4c

[11] T. Fukui and J.J. Nuño-Ballesteros, *Isolated roundings and flattenings of submanifolds in Euclidean spaces*, Tohoku Math. J. 57 (2005), 469-503. DOI: 10.2748/tmj/1140727069

[12] R. Garcia, J. Sotomayor and C. Gutierrez, *Lines of principal curvature around umbilics and Whitney umbrellas*, Tohoku Math. J. (2) 52 (2000), 163-172. DOI: 10.2748/tmj/1178224605

[13] V. Guiñez, *Positive quadratic differential forms and foliations with singularities on surfaces*, Trans. Amer. Math. Soc. 309 (1988), 477-502. DOI: 10.1090/s0002-9947-1988-0961601-4

[14] C. Gutierrez and J. Sotomayor, *Lines of principal curvature for mappings with Whitney umbrella singularities*, Tohoku Math. J. (2) 38 (1986), 551-559. DOI: 10.2748/tmj/1178228407

[15] M. Hasegawa, *Functions on singular surfaces*, Math. Anal. Lab. Records, Kyoto University Research Information Repository (2013), 1868: 1–8.
[16] M. Hasegawa, A. Honda, K. Naokawa, M. Umehara and K. Yamada, *Intrinsic invariants of cross caps*, Selecta Math. (N.S.) 20 (2014), no. 3, 769-785. DOI: 10.1007/s00029-013-0134-6
[17] M. Hasegawa, A. Honda, K. Naokawa, K. Saji, M. Umehara and K. Yamada, *Intrinsic properties of surfaces with singularities*, Internat. J. Math. 26 (2015), no. 4, 1540008, 34 pp. DOI: 10.1142/s0129167x1540008x
[18] H. Hopf, *Differential geometry in the large*, Lecture Notes in Math. 1000, Springer, New York, 1983.
[19] V. V. Ivanov, *An analytic conjecture of Carathéodory*, Siberian Math. J. 43 (2002), 251-322.
[20] S. Izumiya, M. C. R. Fuster, M. A. S. Ruas and F. Tari, *Differential geometry from a singularity theory viewpoint*, World Scientific Pub. Co. Pte. Ltd., Hackesanck, NJ, 2016. DOI: /doi.org/10.1142/9108
[21] L. F. Martins and J. J. Nuño-Ballesteros, *Contact properties of surfaces in $\mathbb{R}^3$ with corank 1 singularities*, Tohoku Math. J. (2) 67 (2015), no. 1, 105-124. DOI: 10.2748/tmj/1429549581
[22] L. F. Martins and K. Saji, *Geometric invariants of cuspidal edges*, Canadian J. Math. 68 (2016), no. 2, 445-462. DOI: /10.4153/cjm-2015-011-5
[23] D. Mond, *On the classification of germs of maps from $\mathbb{R}^2$ to $\mathbb{R}^3$*, Proc. London Math. Soc. (3) 50 (1985), 333-369. DOI: 10.1112/plms/s3-50.2.333
[24] D. Mond, *Some remarks on the geometry and classification of germs of maps from surfaces to 3-space*, Topology (26) 3 (1987), 361-383. DOI: 10.1016/0040-9383(87)90007-3
[25] J. J. Nuño-Ballesteros and F. Tari, *Surfaces in $\mathbb{R}^4$ and their projections to 3-spaces*, Proc. Roy. Soc. Edinburgh Sect. A, 137 (2007), 1313-1328.
[26] R. Oset Sinha and F. Tari, *Projections of surfaces in $\mathbb{R}^4$ to $\mathbb{R}^3$ and the geometry of their singular images*, Rev. Mat. Iberoam. 31 (2015), no. 1, 33-50. DOI: 10.4171/rmi/825
[27] R. Oset Sinha and F. Tari, *On the flat geometry of cuspidal edges*, Osaka J. Math. 55 (2018), no. 3, 393-421.
[28] K. Saji, M. Umehara and K. Yamada, *The geometry of fronts*, Ann. of Math. (2) 169 (2009), 491-529. DOI: 10.4007/annals.2009.169.491
[29] F. Tari, *Pairs of foliations on surfaces*, Real and complex singularities, 305-337, London Math. Soc. Lecture Note Ser., 380, Cambridge Univ. Press, Cambridge, 2010.
[30] C. J. Titus, *A proof of a conjecture of Loewner and of the conjecture of Caratheodory on umbilic points*, Acta Math. 131 (1973), 43-77. DOI: 10.1007/bf02392036
[31] C. T. C. Wall, *Finite determinacy of smooth map-germs*, Bull. London Math. Soc. 13 (1981), no. 6, 481–539.
[32] J. West, *The differential geometry of the cross-cap*, Ph.D. Thesis, Liverpool University, 1995.

J. C. F. Costa, Departamento de Matemática, UNESP - Universidade Estadual Paulista, R. Cristóvão Colombo, 2265, CEP 15054-000, São José do Rio Preto-SP, Brazil
Email address: `joao.costa@unesp.br`

L. F. Martins, Departamento de Matemática, UNESP - Universidade Estadual Paulista, R. Cristóvão Colombo, 2265, CEP 15054-000, São José do Rio Preto-SP, Brazil
Email address: `luciana.martins@unesp.br`

J. J. Nuño-Ballesteros, Departament de Geometria i Topologia, Universitat de València, Campus de Burjassot 46100, Spain
Email address: `Juan.Nuno@uv.es`

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FINITE TYPE ξ-ASYMPTOTIC LINES OF PLANE FIELDS IN $\mathbb{R}^3$

DOUGLAS H. DA CRUZ AND RONALDO A. GARCIA

Abstract. We prove that a finite type curve is a ξ-asymptotic line (without parabolic points) of a suitable plane field. It is also given an explicit example of a hyperbolic closed finite type ξ-asymptotic line. These results obtained here are generalizations, for plane fields, of the results of V. Arnold.

1. Introduction

A regular plane field in $\mathbb{R}^3$ is usually defined by the kernel of a differential one form or a unit vector field $\xi\colon \mathbb{R}^3 \to \mathbb{R}^3$. In this last case $\xi(p)$ is the normal vector to the plane at point p. The classical and germinal work about plane fields in $\mathbb{R}^3$ is [14].

The normal curvature of a plane field is defined by (see [2] and [5])

$$k_n(p, dr) = -\frac{\langle d\xi(p), dr\rangle}{\langle dr, dr\rangle}.$$

For integrable plane fields the normal curvature is the usual concept of curves on surfaces.

The regular curves $\gamma\colon I \to \mathbb{R}^3$ such that $k_n(\gamma(t), \gamma'(t)) = 0$ are called *ξ-asymptotic lines* and the directions dr such $k_n(p, dr) = 0$ are called *ξ-asymptotic directions*.

Recall that asymptotic lines on surfaces are regular curves γ such that $k_n(\gamma(t), \gamma'(t)) = 0$. Also, asymptotic lines are the curves γ such that the osculating plane of γ coincides with the tangent plane of the surface along it, so asymptotic lines are of extrinsic nature.

The local study, and singular aspects of asymptotic lines on surfaces in $\mathbb{R}^3$, near parabolic points, is a very classical subject, see [3, 6, 7, 8], [9] and references therein.

The study of closed asymptotic lines of surfaces in $\mathbb{R}^3$ under the viewpoint of qualitative theory of differential equations is more recent, see [6, 7, 8]. It is worth to mention that existence of closed asymptotic lines on the tubes of "T-surfaces" is still an open problem. See [1, page 107] and [11].

Also, it is not known if there is a surface in $\mathbb{R}^3$ having a cylindrical region foliated by closed asymptotic lines (see [13, page 110]). In $\mathbb{S}^3$, all asymptotic lines of the Clifford torus are globally defined, and they are the Villarceau circles.

V. Arnold in [4] studied the topology of asymptotic lines being curves of type (t, t^m, t^n) near $t = 0$, which are called of finite type. Also, it was shown in [4] that the projection of a closed asymptotic line of a hyperbolic surface of graph type $(x, y, h(x, y))$ in the horizontal plane (x, y) cannot be a starlike curve.

The main results of this work are the following.

The Theorem 3.1 states that any finite type curve is a ξ-asymptotic line (without parabolic points) of a suitable plane field in $\mathbb{R}^3$.

The Theorem 4.3 gives an example of a hyperbolic closed finite type ξ-asymptotic line of a plane field in $\mathbb{R}^3$.

2. Preliminaries and Previous Results

In this paper, the space $\mathbb{R}^3$ is endowed with the Euclidean norm $|\cdot| = \langle \cdot, \cdot \rangle^{\frac{1}{2}}$.

Definition 2.1 ([10, Definition 5.15])**.** A subset $\Omega \subset \mathbb{R}^2$ is called a *starlike convex set* if there is a point $p \in \Omega$, called the *star point*, such that, for every $q \in \Omega$, the segment $\overline{pq}$ lies in Ω. The boundary of a starlike convex set is called a *starlike curve*.

Theorem 2.2 (D. Panov, see [4])**.** *The projection of a closed asymptotic line of a surface $z = \varphi(x, y)$ to the plane $\{z = 0\}$ cannot be a starlike curve* (*in particular, this projection cannot be a convex curve*).

Definition 2.3 ([4])**.** A smoothly immersed curve $\gamma : I \to \mathbb{R}^3$ is said to be of *finite type* at a point x, if $\{\gamma'(x), \gamma''(x), \ldots, \gamma^{(k)}(x)\}$ generate all the tangent space $T_{\gamma(x)}\mathbb{R}^3$ for some $k \in \mathbb{N}$. Here $\gamma^{(k)}(x)$ denotes the derivative of order k of γ. In a neighborhood of this point, the curve is parametrized locally by $\gamma(x) = (x, a_m x^m + \mathcal{O}^{m+1}(x), b_n x^n + \mathcal{O}^{n+1}(x))$, where $m, n \in \mathbb{N}$, $a_m b_n \neq 0$ and $1 < m < n$.

The set $\{1, m, n\}$, $(1 < m < n)$, of the degrees of γ is called the *symbol* of the point. If $n = m + 1$, then γ is said to be of *rotating type* at the point.

If a curve is of finite type (resp. rotating type) at every point, then it is called of *finite type curve* (resp. *rotating type curve*).

A finite type curve γ can have inflection points, i.e., points where the curvature of γ vanishes.

Arnold's Theorem (See [4])**.** *An asymptotic curve of finite type on a hyperbolic surface is a rotating curve.*

Every rotating space curve of finite type is an asymptotic line on a suitable hyperbolic surface.

A new proof of Arnold's Theorem will be given in the appendix.

2.1. Plane fields in $\mathbb{R}^3$.

Let $\xi : \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field of class C^k, where $k \geq 3$.

Definition 2.4. A *plane field* ξ in $\mathbb{R}^3$, orthogonal to the vector field ξ, is defined by the 1-form $\langle \xi, dr \rangle = 0$, where dr is a direction in $\mathbb{R}^3$. See Fig. 1.

Theorem 2.5 ([2, Jacobi Theorem, p.2])**.** *There exists a family of surfaces orthogonal to ξ if, and only if, $\langle \xi, curl(\xi) \rangle \equiv 0$.*

A plane field ξ is said to be *completely integrable* if $\langle \xi, curl(\xi) \rangle \equiv 0$. A surface of the family of surfaces orthogonal to ξ is called an integral surface.

2.2. Normal curvature of a plane field.

Definition 2.6 ([2, p. 8])**.** The *normal curvature* k_n of a plane field in the direction dr orthogonal to ξ is defined by

$$k_n = \frac{\langle \xi, d^2 r \rangle}{\langle dr, dr \rangle} = -\frac{\langle d\xi, dr \rangle}{\langle dr, dr \rangle}.$$

This definition agrees with the classical one given by L. Euler, see [5].

The geometric interpretation of k_n is given by means of the curvature of a plane curve, which we shall now describe.

In the plane $\pi(p_0, dr)$ generated by $\xi(p_0)$ and dr (direction orthogonal to $\xi(p_0)$) we have a line field $\ell(p)$ orthogonal to vector $\bar{\xi}(p) \in \pi(p_0, dr)$ obtained projecting $\xi(p)$ in the plane $\pi(p_0, dr)$, with $p \in \pi(p_0, dr)$. The integral curves $\varphi_p(t)$ of the line field ℓ are regular curves and $k_n(p_0, dr)$ is the plane curvature of $\varphi_{p_0}(t)$ at $t = 0$. See Fig. 2.

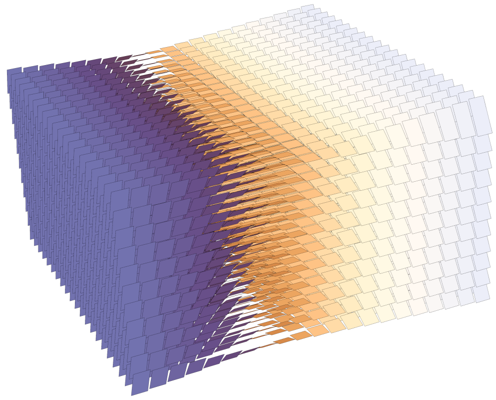


FIGURE 1. Plane field ξ in $\mathbb{R}^3$ defined by the 1-form $\langle \xi, dr\rangle = dz - ydx = 0$, where $\xi(x, y, z) = (-y, 0, 1)$ and $dr = (dx, dy, dz)$.

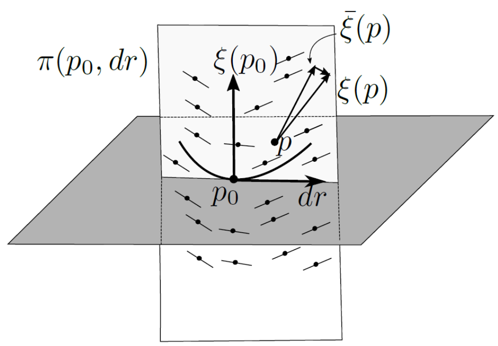


FIGURE 2. Line field and normal curvature $k_n(p_0, dr)$.

2.3. **ξ-asymptotic lines and parabolic points of a plane field.** The ξ-asymptotic directions of a plane field ξ are defined by the following implicit differential equation

$$\langle \xi, dr\rangle = 0, \quad \langle d\xi, dr\rangle = 0. \tag{2.1}$$

and will referred as the implicit differential equation of the ξ-asymptotic lines.

A solution dr of equation (2.1) is called a ξ-*asymptotic direction*. A curve γ in $\mathbb{R}^3$ is a ξ-*asymptotic line* if γ is an integral curve of equation (2.1). Analogously to the case of asymptotic lines on surfaces, for plane fields the osculating plane of a $\xi-$asymptotic line coincides with the plane of the distribution of planes passing through the point of the curve. See also [2, page 29].

Definition 2.7. If at a point r there exists two real distinct ξ-asymptotic directions (resp. two complex ξ-asymptotic directions), then r is called a *hyperbolic point* (resp. *elliptic point*).

Definition 2.8. If at r the two ξ-asymptotic directions coincide or all the directions are ξ-asymptotic directions then r is called a *parabolic point*.

Example 2.9. The circle in $\mathbb{R}^3$ given by $x^2 + y^2 = 1$, $z = 0$, is a ξ-asymptotic line without parabolic points of the plane field ξ defined by the orthogonal vector field $\xi = (\rho, \varrho, \sigma)$, where $\rho = x^2yz + y^3z - x^2y - y^3 + xz - 2yz + y$, $\varrho = x^3 - x^3z - xy^2z + xy^2 + 2xz + yz - x$ and $\sigma = -x^2 - y^2$. See Fig. 3. The plane field ξ is not completely integrable. By the Theorem 2.2, this circle cannot be an asymptotic line of a regular surface $z = \varphi(x, y)$.

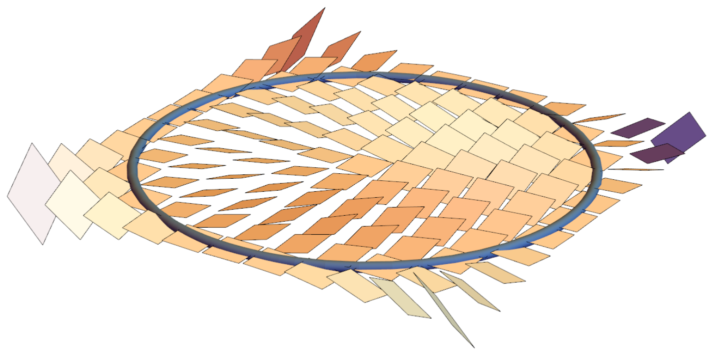


FIGURE 3. The circle is a ξ-asymptotic line without parabolic points of the plane field defined by the orthogonal vector field $\xi = (\rho, \varrho, \sigma)$, where $\rho = x^2yz + y^3z - x^2y - y^3 + xz - 2yz + y$, $\varrho = x^3 - x^3z - xy^2z + xy^2 + 2xz + yz - x$ and $\sigma = -x^2 - y^2$.

Proposition 2.10. *Given a plane field ξ, let $\varphi : \mathbb{R}^3 \to \mathbb{R}$ be a differentiable nonvanishing function. Then a curve γ is a ξ-asymptotic line if, and only if, γ is a ξ-asymptotic line of the plane field $\widetilde{\xi}$ orthogonal to the vector field $\widetilde{\xi} = \varphi\xi$.*

Proof. The implicit differential equation of ξ-asymptotic lines of $\widetilde{\xi}$ is given by

$$\langle \widetilde{\xi}, dr \rangle = \varphi \langle \xi, dr \rangle = 0, \quad \langle d\widetilde{\xi}(dr), dr \rangle = d\varphi(dr)\langle \xi, dr \rangle + \varphi \langle d\xi(dr), dr \rangle = 0.$$

Then γ is a ξ-asymptotic line of ξ if, and only if, γ is a ξ-asymptotic line of the plane field $\widetilde{\xi}$. □

2.4. **Tubular neighborhood of an integral curve of a plane field.** Let ξ be a plane field orthogonal to a vector field $\xi(x, y, z)$. Then $d\xi = \xi_x dx + \xi_y dy + \xi_z dz$. Let

$$\gamma(x) = (\gamma_1(x), \gamma_2(x), \gamma_3(x))$$

be a curve such that $(\gamma_1'(x), \gamma_2'(x)) \neq (0, 0)$ for all x. Set $X(x) = \gamma'(x)$, $Y(x) = (\gamma_2'(x), -\gamma'(x), 0)$, $Z(x) = (X \wedge Y)(x)$ and $\alpha : \mathbb{R}^3 \to \mathbb{R}^3$,

$$\alpha(x, y, z) = \gamma(x) + yY(x) + zZ(x). \tag{2.2}$$

The map (2.2) is a parametrization of a tubular neighborhood of γ. At this neighborhood, the position point is given by $r = \alpha(x, y, z)$ and then $dr = d\alpha = \alpha_x dx + \alpha_y dy + \alpha_z dz$. It follows that the implicit differential equation (2.1) of the ξ-asymptotic lines is given by

$$\begin{aligned} &\langle \xi, d\alpha \rangle = adx + bdy + cdz = 0, \\ &\langle d\xi, d\alpha \rangle = L_1 dx^2 + L_2 dxdy + L_3 dy^2 + L_4 dxdz + L_5 dydz + L_6 dz^2 = 0, \end{aligned} \tag{2.3}$$

where,

$$a = \langle \xi, \alpha_x \rangle, \quad b = \langle \xi, \alpha_y \rangle, \quad c = \langle \xi, \alpha_z \rangle,$$

and

$$L_1 = \langle \xi_x, \alpha_x \rangle, \quad L_2 = \langle \xi_x, \alpha_y \rangle + \langle \xi_y, \alpha_x \rangle, \quad L_3 = \langle \xi_y, \alpha_y \rangle,$$
$$L_4 = \langle \xi_x, \alpha_z \rangle + \langle \xi_z, \alpha_x \rangle, \quad L_5 = \langle \xi_y, \alpha_z \rangle + \langle \xi_z, \alpha_y \rangle, \quad L_6 = \langle \xi_z, \alpha_z \rangle.$$

Proposition 2.11. *Let $\gamma(x) = (\gamma_1(x), \gamma_2(x), \gamma_3(x))$ be a curve such that, for all x,*

$$(\gamma_1'(x), \gamma_2'(x)) \neq (0, 0).$$

Consider a tubular neighborhood of γ parametrized by equation (2.2). If ξ is a plane field such that $\frac{a}{c}$ and $\frac{b}{c}$ are well defined in a neighborhood of γ, where a, b, c are given by (2.3), then the implicit differential equation of the ξ-asymptotic lines, in this neighborhood, is given by

$$dz = -\left(\frac{a}{c}\right) dx - \left(\frac{b}{c}\right) dy, \quad e dx^2 + 2f dx dy + g dy^2 = 0, \tag{2.4}$$

where,

$$e = L_1 - \frac{aL_4}{c} + \frac{a^2 L_6}{c^2}, \quad g = L_3 - \frac{bL_5}{c} + \frac{b^2 L_6}{c^2}, \quad f = \frac{L_2}{2} - \frac{(aL_5 + bL_4)}{2c} + \frac{abL_6}{2c^2}.$$

Furthermore, in this neighborhood, the parabolic set of ξ is given by $eg - f^2 = 0$.

Proof. In a neighborhood of γ, solve the first equation of (2.3) in the variable dz to get the first equation of (2.4). Replace this dz in the second equation of (2.3) to get the second equation of (2.4).

If $eg - f^2 < 0$ at a point (resp. $eg - f^2 > 0$), then the equations (2.4) define two distinct ξ-asymptotic directions at this point (resp. two complex ξ-asymptotic directions).

If $eg - f^2 = 0$ at a point, then at it the ξ-asymptotic directions coincide or, if $e = g = f = 0$, all directions are ξ-asymptotic directions. $\square$

Definition 2.12 ([2, p. 11])**.** Let ξ be a plane field satisfying the assumptions of Lemma 2.11. The function defined by $\mathcal{K} = eg - f^2$ is called the *Gaussian curvature* of ξ.

Lemma 2.13. *Let $\gamma(x) = (\gamma_1(x), \gamma_2(x), \gamma_3(x))$ be a ξ-asymptotic line of a plane field ξ, such that $(\gamma_1'(x), \gamma_2'(x)) \neq (0, 0)$ for all x. Consider a tubular neighborhood of γ parametrized by equation (2.2). Then, in a neighborhood of γ, the vector field ξ is given by*

$$\begin{aligned}
\xi(x, y, z) &= l_0(x)Y(x) + k_0(x)Z(x) \\
&+ \left(yk_1(x) + zl_1(x) + \left(\frac{y^2}{2}\right)\widetilde{k}_1(x) + yz\widetilde{j}_1(x) + \left(\frac{z^2}{2}\right)\widetilde{l}_1(x) + \widetilde{A}(x, y, z)\right) X(x) \\
&+ \left(yk_2(x) + zl_2(x) + \left(\frac{y^2}{2}\right)\widetilde{k}_2(x) + yz\widetilde{j}_2(x) + \left(\frac{z^2}{2}\right)\widetilde{l}_2(x) + \widetilde{B}(x, y, z)\right) Y(x) \\
&+ \left(yk_3(x) + zl_3(x) + \left(\frac{y^2}{2}\right)\widetilde{k}_3(x) + yz\widetilde{j}_3(x) + \left(\frac{z^2}{3}\right)\widetilde{l}_3(x) + \widetilde{C}(x, y, z)\right) Z(x),
\end{aligned} \tag{2.5}$$

where

$$X(x) = \gamma'(x), \ Y(x) = (\gamma_2'(x), -\gamma_1'(x), 0), \ Z(x) = (X \wedge Y)(x),$$
$$\widetilde{A}(x, 0, 0) = \widetilde{B}(x, 0, 0) = \widetilde{C}(x, 0, 0) = 0$$

and

$$[(\gamma_3'\gamma_1'' - \gamma_1'\gamma_3'')\gamma_1' + (\gamma_3'\gamma_2'' - \gamma_2'\gamma_3'')\gamma_2']k_0 - (\gamma_1'\gamma_2'' - \gamma_2'\gamma_1'')l_0 = 0. \tag{2.6}$$

Furthermore, if

$$k_0 = \gamma_1'\gamma_2'' - \gamma_2'\gamma_1'', \quad l_0 = (\gamma_3'\gamma_1'' - \gamma_1'\gamma_3'')\gamma_1' + (\gamma_3'\gamma_2'' - \gamma_2'\gamma_3'')\gamma_2' \tag{2.7}$$

and $\gamma_1'(x)\gamma_2''(x) - \gamma_2'(x)\gamma_1''(x) \neq 0$ for all x, then the implicit differential equation of the ξ-asymptotic lines is given by (2.4).

Proof. The expression (2.5) holds, since γ is an integral curve of the plane field defined by ξ. Also, as γ is a ξ-asymptotic line, $\langle \xi(x), \gamma''(x)\rangle = 0$ for all x, which gives the equation (2.6).

If $\gamma_1'(x)\gamma_2''(x) - \gamma_2'(x)\gamma_1''(x) \neq 0$, then $c(x,0,0)\neq 0$. The conclusion then follows from Proposition 2.11. $\square$

3. Finite type ξ-asymptotic lines of plane fields

In this section the following result is established.

Theorem 3.1. *Any finite type curve is a ξ-asymptotic line (without parabolic points) of a suitable plane field.*

Proof. Let $\gamma(x) = (\gamma_1(x), \gamma_2(x), \gamma_3(x)) = (x, a_m x^m + \mathcal{O}^{m+1}(x), a_n x^n + \mathcal{O}^{n+1}(x))$ be a finite type curve. Consider a tubular neighborhood of γ parametrized by equation (2.2) and the vector field ξ given by (2.5). Set $k_0(x)\equiv 1$ and solve (2.6) for $l_0(x)$. Then γ is a ξ-asymptotic line of the plane field orthogonal to ξ.

We have that

$$a(x,0,0)=0,\quad b(0,0,0)=0,\quad \text{and}\quad c(0,0,0)=a_m m(m-1)\neq 0.$$

By Proposition 2.11, in a neighborhood of $(0,0,0)$, the equation of ξ-asymptotic lines are given by (2.4).

Set $l_1(x)\equiv 0$ and define $k_1(x)$ by

$$\begin{aligned} k_1 = {} & \frac{((\gamma_1')^2+(\gamma_2')^2)^2[(\gamma_2''\gamma_3'''-\gamma_3''\gamma_2''')\gamma_1' + (\gamma_3''\gamma_1'''-\gamma_1''\gamma_3''')\gamma_2' + (\gamma_1''\gamma_2'''-\gamma_2''\gamma_1''')\gamma_3']}{((\gamma_1')^2+(\gamma_2')^2+(\gamma_3')^2)(\gamma_1'\gamma_2''-\gamma_2'\gamma_1'')} \\ & + \frac{2(\gamma_1'\gamma_2''-\gamma_2'\gamma_1'')}{((\gamma_1')^2+(\gamma_2')^2+(\gamma_3')^2)}. \end{aligned}$$

Then $\mathcal{K}(x,0,0)=-1$. $\square$

4. Hyperbolic closed finite type ξ-asymptotic line

Examples of hyperbolic asymptotic lines on surfaces are given in [6, 7, 8].

In this section it will be given an example of a hyperbolic closed ξ-asymptotic line of finite type for a suitable plane field.

Proposition 4.1. *Let γ, $\gamma(x) = (\gamma_1(x), \gamma_2(x), \gamma_3(x))$, be a curve such that*

$$(\gamma_1'(x),\gamma_2'(x))\neq(0,0),\quad \gamma_1'(x)\gamma_2''(x)-\gamma_2'(x)\gamma_1''(x)\neq 0$$

for all x. Consider the tubular neighborhood α given by (2.2) and the vector field ξ given by (2.5), with $k_0(x)$, $l_0(x)$ given by (2.7). Let $H(x)$ be a nonvanishing function and define $k_1(x)$ by

$$\begin{aligned} k_1 = {} & \frac{((\gamma_1')^2+(\gamma_2')^2)^2[(\gamma_2''\gamma_3'''-\gamma_3''\gamma_2''')\gamma_1' + (\gamma_3''\gamma_1'''-\gamma_1''\gamma_3''')\gamma_2' + (\gamma_1''\gamma_2'''-\gamma_2''\gamma_1''')\gamma_3']}{((\gamma_1')^2+(\gamma_2')^2+(\gamma_3')^2)(\gamma_1'\gamma_2''-\gamma_2'\gamma_1'')} \\ & + \frac{[(\gamma_1'\gamma_1''+\gamma_2'\gamma_2'')\gamma_3' - ((\gamma_1)^2+(\gamma_2)^2)\gamma_3'']l_1 + 2(\gamma_1'\gamma_2''-\gamma_2'\gamma_1'')H}{((\gamma_1')^2+(\gamma_2')^2+(\gamma_3')^2)(\gamma_1'\gamma_2''-\gamma_2'\gamma_1'')}. \end{aligned} \tag{4.1}$$

Then, γ is a ξ-asymptotic line, without parabolic points, of the plane field orthogonal to the vector field ξ.

Furthermore, $\mathcal{K}(x,0,0) = -(H(x))^2$.

Proof. By direct calculations, we can see that γ is a ξ-asymptotic line. The implicit differential equation of the ξ-asymptotic lines are given by (2.4) and $e(x,0,0)=0$, $f(x,0,0)=H(x)$. Since $e(x,0,0)=0$, then $\mathcal{K}(x,0,0) = -(H(x))^2$ for all x.

$\square$

4.1. **Poincaré map associated to a closed ξ-asymptotic line.** Let $\gamma : [0,l] \to \mathbb{R}^3$, $\gamma(x) = (\gamma_1(x), \gamma_2(x), \gamma_3(x))$, be a closed ξ-asymptotic line, without parabolic points, of a plane field ξ, such that $\gamma(0) = \gamma(l)$, $(\gamma_1'(x), \gamma_2'(x)) \neq (0,0)$, $\gamma_1'(x)\gamma_2''(x) - \gamma_2'(x)\gamma_1''(x) \neq 0$ for all x, and consider the tubular neighborhood α given by (2.2).

This means that γ is a regular curve having a projection in a plane which is a strictly locally convex curve.

By the Proposition 2.13, ξ is given by (2.5) and the implicit differential equations of the ξ-asymptotic lines is given by (2.4).

Let $\Sigma_{x_0} = \{(x_0, y, z)\}$ be a transversal section. Then $\alpha(\Sigma_{x_0})$ is the plane spanned by $Y(x_0)$ and $Z(x_0)$. By Lemma 2.13, in a neighborhood of γ, the ξ-asymptotic line passing through $\alpha(x_0, y_0, z_0)$ intersects $\alpha(\Sigma_{x_0})$ again at the point

$$\alpha(x_0 + l, y(x_0 + l, y_0, z_0), z(x_0 + l, y_0, z_0)),$$

where $(y(x, y_0, z_0), z(x, y_0, z_0))$ is solution of the following Cauchy problem

$$\begin{aligned} &\frac{dz}{dx} = -\frac{a}{c} - \left(\frac{b}{c}\right)\frac{dy}{dx} = A + B\frac{dy}{dx}, \\ &e + 2f\frac{dy}{dx} + g\left(\frac{dy}{dx}\right)^2 = 0, \\ &(y(x_0, y_0, z_0), z(x_0, y_0, z_0)) = (y_0, z_0). \end{aligned} \tag{4.2}$$

The *Poincaré map* $\mathcal{P}$, also called *first return map*, associated to γ is defined by $\mathcal{P} : \mathcal{U} \subset \Sigma \to \Sigma$, $\mathcal{P}(y_0, z_0) = (y(l, y_0, z_0), z(l, y_0, z_0))$. See Fig. 4.

A closed ξ-asymptotic line γ is said to be *hyperbolic* if the eigenvalues of $d\mathcal{P}_{(0,0)}$ does not belong to $\mathbb{S}^1$. See [12] for the generic properties of the Poincaré map associated to closed orbits of vector fields.

We will denote by $d\mathcal{P}_{(0,0)}$ the matrix of the first derivative of the Poincaré map evaluated at $(y_0, z_0) = (0,0)$.

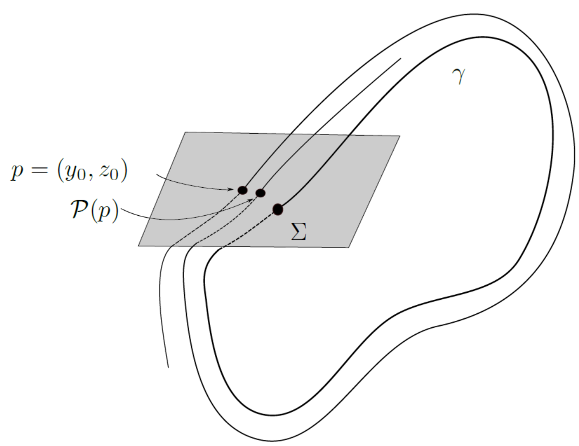


FIGURE 4. Poincaré return map.

Proposition 4.2. *Let $\gamma : [0,l] \to \mathbb{R}^3$, $\gamma(x) = (\gamma_1(x), \gamma_2(x), \gamma_3(x))$, be a closed ξ-asymptotic line, having a projection in a plane which is a locally strictly convex curve.*

Let $\mathcal{P}$ be the Poincaré map associated to γ. Then $d\mathcal{P}_{(0,0)} = \mathcal{Q}(l)$, where $\mathcal{Q}(x)$ is solution of the following Cauchy problem:

$$\frac{d}{dx}\left(\mathcal{Q}(x)\right) = \mathcal{M}(x)\mathcal{Q}(x), \quad \mathcal{Q}(0) = \mathcal{I}, \tag{4.3}$$

where $\mathcal{I}$ is the identity matrix, and $\mathcal{M}(x)$, $\mathcal{Q}(x)$ are the matrices given by

$$\mathcal{M}(x) = \begin{pmatrix} -\frac{e_y(x,0,0)}{2f(x,0,0)} & -\frac{e_z(x,0,0)}{2f(x,0,0)} \\ (A)_y(x,0,0) & (A)_z(x,0,0) \end{pmatrix}, \mathcal{Q}(x) = \begin{pmatrix} \frac{dy}{dy_0}(x,0,0) & \frac{dy}{dz_0}(x,0,0) \\ \frac{dz}{dy_0}(x,0,0) & \frac{dz}{dz_0}(x,0,0) \end{pmatrix},$$

where $A = -\frac{a}{c}$.

Proof. To fix the notation suppose that

$$\gamma(0) = \gamma(l), \quad (\gamma_1'(x), \gamma_2'(x)) \neq (0,0), \quad \text{and} \quad \gamma_1'(x)\gamma_2''(x) - \gamma_2'(x)\gamma_1''(x) \neq 0 \quad \text{for all} \quad x$$

.

Let $(y(x,y_0,z_0), z(x,y_0,z_0))$ be solution of the Cauchy problem given by equation (4.2). Then, at $(y,z) = (0,0)$, $\frac{dy}{dx}(x,0,0) = \frac{dz}{dx}(x,0,0) = 0$.

Differentiating the first equation of (4.2) with respect to y_0 (resp. z_0), it results that:

$$\frac{d}{dx}\left(\frac{dz}{dy_0}\right) = A_y\frac{dy}{dy_0} + A_z\frac{dz}{dy_0} + B\frac{d}{dx}\left(\frac{dy}{dy_0}\right) + \left(B_y\frac{dy}{dy_0} + B_z\frac{dz}{dy_0}\right)\frac{dy}{dx}, \tag{4.4}$$

respectively,

$$\frac{d}{dx}\left(\frac{dz}{dz_0}\right) = A_y\frac{dy}{dz_0} + A_z\frac{dz}{dz_0} + B\frac{d}{dx}\left(\frac{dy}{dz_0}\right) + \left(B_y\frac{dy}{dz_0} + B_z\frac{dz}{dz_0}\right)\frac{dy}{dx}. \tag{4.5}$$

Differentiating the second equation of (4.2) with respect to y_0 (resp. z_0), it results that:

$$\begin{aligned} &e_y\frac{dy}{dy_0} + e_z\frac{dz}{dy_0} + 2f\frac{d}{dx}\left(\frac{dy}{dy_0}\right) + 2\left(f_y\frac{dy}{dy_0} + f_z\frac{dz}{dy_0} + g\frac{d}{dx}\left(\frac{dy}{dy_0}\right)\right)\frac{dy}{dx} \\ &+ \left(g_y\frac{dy}{dy_0} + g_z\frac{dz}{dy_0}\right)\left(\frac{dy}{dx}\right)^2 = 0, \end{aligned} \tag{4.6}$$

respectively,

$$\begin{aligned} &e_y\frac{dy}{dz_0} + e_z\frac{dz}{dz_0} + 2f\frac{d}{dx}\left(\frac{dy}{dz_0}\right) + 2\left(f_y\frac{dy}{dz_0} + f_z\frac{dz}{dz_0} + g\frac{d}{dx}\left(\frac{dy}{dz_0}\right)\right)\frac{dy}{dx} \\ &+ \left(g_y\frac{dy}{dz_0} + g_z\frac{dz}{dz_0}\right)\left(\frac{dy}{dx}\right)^2 = 0. \end{aligned} \tag{4.7}$$

Evaluating (4.4), (4.5), (4.6), (4.7) at $(y,z) = (0,0)$, it follows that:

$$\begin{aligned} A_y\frac{dy}{dy_0} + A_z\frac{dz}{dy_0} &= \frac{d}{dx}\left(\frac{dz}{dy_0}\right), \quad e_y\frac{dy}{dy_0} + e_z\frac{dz}{dy_0} + 2f\frac{d}{dx}\left(\frac{dy}{dy_0}\right) = 0, \\ A_y\frac{dy}{dz_0} + A_z\frac{dz}{dz_0} &= \frac{d}{dx}\left(\frac{dz}{dz_0}\right), \quad e_y\frac{dy}{dz_0} + e_z\frac{dz}{dz_0} + 2f\frac{d}{dx}\left(\frac{dy}{dz_0}\right) = 0. \end{aligned}$$

Then $\frac{d}{dx}\left(\mathcal{Q}(x)\right) = \mathcal{M}(x)\mathcal{Q}(x)$. Since $(y(0,y_0,z_0), z(0,y_0,z_0)) = (y_0,z_0)$, it follows that $\mathcal{Q}(0) = \mathcal{I}$.

Since $\mathcal{P}(y_0,z_0) = (y(l,y_0,z_0), z(l,y_0,z_0))$, the first derivative $d\mathcal{P}_{(0,0)}$ is given by $\mathcal{Q}(l)$. $\square$

4.2. **Example of a hyperbolic closed finite type ξ-asymptotic line.** An explicit example of a hyperbolic closed ξ-asymptotic line is given in the next result.

Theorem 4.3. *Let $\gamma : [0, 2\pi] \to \mathbb{R}^3$, $\gamma(x) = (sin(x), cos(x), sin^3(x))$, see Fig. 5. Then it is a hyperbolic finite type ξ-asymptotic line of a suitable plane field.*

Proof. Let ξ be a plane field orthogonal to the vector field ξ given by (2.5), where $k_0(x)$ and $l_0(x)$ are given by (2.7). Let $k_1(x)$ given by (4.1), with $H(x) \equiv 1$. Then

$$k_1(x) = \frac{3(3cos^2(x) - 1)sin(x)l_1(x) + 24cos^3(x) - 18cos(x) - 2}{9cos^6(x) - 18cos^4(x) + 9cos^2(x) + 1}.$$

By Proposition 4.1, γ is a ξ-asymptotic line without parabolic points and $\mathcal{K}(x, 0, 0) = -1$. Performing the calculations, $e_z(x, 0, 0) = \mathcal{E}(x) + l_2(x)$. Solve $e_z(x, 0, 0) = 0$ for $l_2(x)$. This vanishes the entry $\left(-\frac{e_z(x,0,0)}{2f(x,0,0)}\right)$ of $\mathcal{M}(x)$ given by Theorem 4.2. From (4.3), it follows that the eigenvalues of $d\mathcal{P}_{(0,0)}$ are given by

$$\exp\left(\int_0^{2\pi} -\frac{e_y(x, 0, 0)}{2f(x, 0, 0)}dx\right) \quad \text{and} \quad \exp\left(\int_0^{2\pi} A_z(x, 0, 0)dx\right).$$

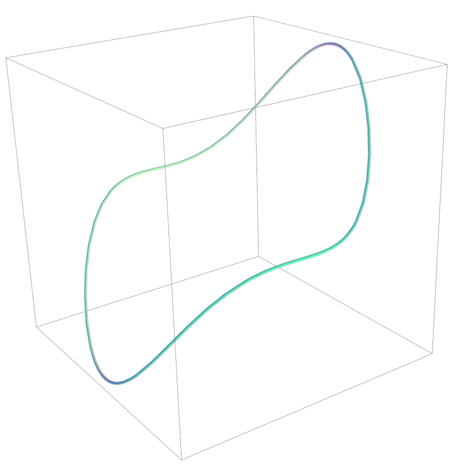
Set $l_1(x) = cos(x)$. Then

$$\begin{aligned} A_z(x, 0, 0) = {} & 9sin(x)cos^8(x) + 54sin(x)cos^6(x) - 9cos^6(x) - 117sin(x)cos^4(x) \\ & + 18cos^4(x) + 55cos^2(x)sin(x) - 9cos^2(x) - 1. \end{aligned}$$

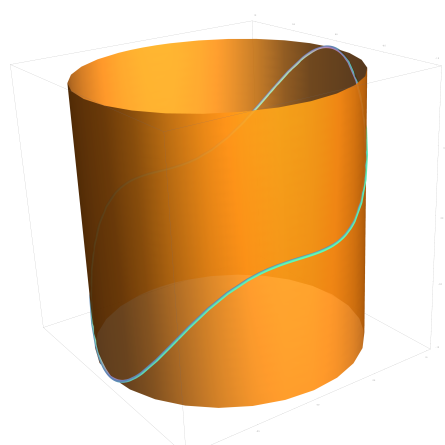
It follows that $\int_0^{2\pi} A_z(x, 0, 0)dx = -\frac{25\pi}{8}$. Let $k_3(x) = 0$ and $k_2(x)$ a solution of the equation $e_y(x, 0, 0) + 2f(x, 0, 0) = 0$. It follows that

$$\int_0^{2\pi} \left(-\frac{e_y(x, 0, 0)}{2f(x, 0, 0)}\right) dx = 2\pi.$$

□



(A) Curve $\gamma(x) = (sin(x), cos(x), sin^3(x))$.



(B) Curve $\gamma(x)$ on the cylinder $\beta(x, y) = (sin(x), cos(x), y)$.

FIGURE 5. Finite type curve $\gamma(x) = (sin(x), cos(x), sin^3(x))$.

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References

[1] *Geometry. II*, volume 29 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 1993. Spaces of constant curvature, A translation of Geometriya. II, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1988, Translation by V. Minachin [V. V. Minakhin], Translation edited by È. B. Vinberg.

[2] Y. Aminov. *The geometry of vector fields*. Gordon and Breach Publishers, Amsterdam, 2000.

[3] Y. Aminov. *The geometry of submanifolds*. Gordon and Breach Science Publishers, Amsterdam, 2001.

[4] V. I. Arnold. Topological problems in the theory of asymptotic curves. *Tr. Mat. Inst. Steklova*, 225(Solitony Geom. Topol. na Perekrest.):11–20, 1999.

[5] L. Euler. Recherches sur la courbure des surfaces. *Mémoires de l'Académie des Sciences de Berlin*, 16(119–143):9, 1760.

[6] R. Garcia, C. Gutierrez, and J. Sotomayor. Structural stability of asymptotic lines on surfaces immersed in $\mathbb{R}^3$. *Bull. Sci. Math.*, 123(8):599–622, 1999. DOI: 10.1016/S0007-4497(99)00116-5

[7] R. Garcia and J. Sotomayor. Structural stability of parabolic points and periodic asymptotic lines. *Mat. Contemp.*, 12:83–102, 1997.

[8] R. Garcia and J. Sotomayor. *Differential equations of classical geometry, a qualitative theory*. Publicações Matemáticas do IMPA. [IMPA Mathematical Publications]. Instituto Nacional de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, 2009.

[9] S. Izumiya, M. d. C. Romero Fuster, M. A. S. Ruas, and F. Tari. *Differential geometry from a singularity theory viewpoint*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2016. DOI: 10.1142/9108

[10] S. G. Krantz. *Convex analysis*. CRC Press, 2015.

[11] L. Nirenberg. Rigidity of a class of closed surfaces. In *Nonlinear Problems (Proc. Sympos., Madison, Wis., 1962)*, pages 177–193. Univ. of Wisconsin Press, Madison, Wis., 1963.

[12] J. Palis, Jr. and W. de Melo. *Geometric theory of dynamical systems*. Springer-Verlag, New York-Berlin, 1982. An introduction, Translated from the Portuguese by A. K. Manning.

[13] E. R. Rozendorn. Surfaces of negative curvature. In *Current problems in mathematics. Fundamental directions, Vol. 48 (Russian)*, Itogi Nauki i Tekhniki, pages 98–195. Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1989.

[14] A. Voss. Geometrische Interpretation der Differentialgleichung $Pdx+Qdy+Rdz=0$. *Math. Ann.*, 16(4):556–559, 1880. DOI: 10.1007/bf01446224

Appendix

A new proof of Arnold's Theorem

The proof of Arnold's Theorem [4] is given on graph surfaces $z = z(x, y)$. Using affine coordinates, the surface takes the form $z = xy + \dots$, where the dots denote the terms of higher order. Arnold showed that an asymptotic line $x = x(t)$, $y = y(t)$, $z = z(t)$ of finite type is a rotating curve.

After that, he proves that given a rotating curve $x = x(t)$, $y = y(t)$, $z = z(t)$ then there exists an appropriated function $H(x, y)$ such that the rotating curve is an asymptotic line of the surface $z = H(x, y)$.

Below, will be given a geometric proof of Arnold's Theorem, with an explicit parametrization of the surface.

Proof. Let γ be a curve of finite type (u, u^m, u^n), $n \geq m$. Set $N(u) = (\gamma_2'(u), -\gamma_1'(u), 0)$. Consider the local surface parametrized by

$$\alpha(u, v) = \gamma(u) + vN(u) + (k_1(u)v + k_2(u)v^2 + k_3(u)v^3 + \mathcal{O}^4(v))(\gamma' \wedge N)(u).$$

Let N_α be the unit normal vector

$$N_\alpha = \frac{\alpha_u \wedge \alpha_v}{|\alpha_u \wedge \alpha_v|}.$$

The implicit differential equation of the asymptotic lines of α is given by

$$edu^2 + 2fdudv + gdv^2 = 0,$$

where $e = \langle \alpha_{uu}, N_\alpha \rangle$, $f = \langle \alpha_{uv}, N_\alpha \rangle$ and $g = \langle \alpha_{vv}, N_\alpha \rangle$.

Supposing that γ is an asymptotic line of α, and parametrized by $v = 0$, we have that $e(u, 0) = 0$. Then by equation (4.1) it follows that

$$k_1(u) = \frac{[(n-m)m^2u^{2(m-1)} + n - 1]nu^{n-m}}{[1 + m^2u^{2(m-1)} + n^2u^{2(n-1)}](m-1)m}. \tag{A.8}$$

Direct calculations show that

$$f(u, 0) = \frac{(n-m)(n-1)n(1 + m^2u^{2(m-1)})^2u^{n-m-1}}{(m-1)m}.$$

It follows that $f(0, 0) \neq 0$ if, and only if, $n = m + 1$, i.e., γ is a rotating curve.

If γ is a rotating space curve of finite type (u, u^m, u^{m+1}), $m \geq 2$, set $N(u) = (\gamma_2'(u), -\gamma_1'(u), 0)$ and let

$$\beta(u, v) = \gamma(u) + vN(u) + k_1(u)v(\gamma' \wedge N)(u),$$

where $k_1(u)$ is given by (A.8) with $n = m + 1$. Therefore, $e(u, 0) = [\beta_u, \beta_v, \beta_{uu}](0, 0) = 0$ and $f(0, 0) = [\beta_u, \beta_v, \beta_{uv}](0, 0) = \frac{m+1}{m-1} \neq 0$. Then γ is an asymptotic line, without parabolic points, of the surface parametrized by β in a neighborhood of $(u, v) = (0, 0)$. $\square$

Douglas H. da Cruz, Instituto de Matemática e Estatística, Universidade Federal de Goiás, Campus Samambaia, 74690-900 - Goiânia -G0 - Brazil
Email address: douglashilariodacruz@gmail.com

Ronaldo A. Garcia, Instituto de Matemática e Estatística, Universidade Federal de Goiás, Campus Samambaia, 74690-900 - Goiânia -G0 - Brazil
Email address: ragarcia@ufg.br

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ON THE CHARACTERISTIC CURVES ON A SURFACE IN $\mathbb{R}^4$

JORGE LUIZ DEOLINDO-SILVA

Abstract. We study some robust features of characteristic curves on smooth surfaces in $\mathbb{R}^4$. These curves are analogous to the asymptotic curves in the elliptic region. A $P_3(c)$-point is an isolated special point at which the unique characteristic (or asymptotic) direction is tangent to the parabolic curve. At this point, by considering the cross-ratio invariant, we show that the 2-jet of the curve formed by the inflections of the characteristic curves is projectively invariant. In addition, we exhibit the possible configurations of the characteristic curves at a $P_3(c)$-point.

1. Introduction

For surfaces in $\mathbb{R}^3$, an asymptotic direction is a self-conjugate tangent direction, and a characteristic direction is a tangent direction such that the angle it forms with its conjugate direction is extremal. At a hyperbolic (resp. parabolic or elliptic) point there are two (resp. one or 0) asymptotic directions and at an elliptic (resp. parabolic or hyperbolic) point there are two (resp. one or 0) characteristic directions. The asymptotic and characteristic curves are the integral curves of asymptotic and characteristic directions, respectively. It is well known that the characteristic curves are, in many ways, analogous to the asymptotic curves in the elliptic region (see [4, 5, 20]) and both curves are given, in a local chart, by a binary differential equation (BDE)

$$A(x,y)dx^2 + 2B(x,y)dxdy + C(x,y)dy^2 = 0, \tag{1}$$

where the coefficients A, B, and C are smooth functions defined in an open subset U of $\mathbb{R}^2$. The discriminant curve of equation (1) of the asymptotic and characteristic curves coincides with the parabolic curve. At cusps of Gauss the unique asymptotic and characteristic direction is tangent to the parabolic curve (see for example [1]). Although asymptotic curves can be also defined using the contact of the surface with lines, the characteristic curves do not satisfy this property.

In [20], Oliver used Uribe-Vargas's cr-invariant defined in [24], to show that the topological type of the singularity of the characteristic curves at a cusp of Gauss is invariant under projective transformations. Furthermore, the locus of inflection points of the characteristic curves (characteristic inflection curve) has some geometrical meaning. In particular, he classified a cusp of Gauss in terms of the relative position of the parabolic curve, the characteristic inflection curve and conodal curve. In this paper, we extend the results in [20] on characteristic curves for surfaces in $\mathbb{R}^4$.

The study of the differential geometry of immersed surfaces in 4-space was carried out by several authors, for example [2, 3, 10, 11, 16, 17, 19, 21, 23]. The study of characteristic curves did not receive the same treatment in the current literature. The definition of characteristic

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curves for surfaces in $\mathbb{R}^4$ is inspired from (and is analogous to) that for surfaces in $\mathbb{R}^3$ in the following way: for surfaces in $\mathbb{R}^3$, there is a relation between the BDEs of the asymptotic curves, of the characteristic curves and of the lines of principal curvature. By considering the BDE (1) as a point in the projective plane, the BDEs of the asymptotic curves and of the lines of principal curvature determine the BDE of the characteristic curves, such that the three BDEs define (at each point on the surface) a self polar triangle in the projective plane. In fact, the BDE of the asymptotic curves determines the other two BDEs ([4, 23]). Asymptotic directions are also defined on surfaces in $\mathbb{R}^4$ and are given by a BDE (see §3). Its equation is used to define in a unique way, two other BDEs such that the three equations form a self-polar triangle in the projective plane. One of them is what is called the BDE of the characteristic curves (called a *characteristic* BDE, for short) (see [23]). In this sense, the asymptotic and characteristic directions on surface in $\mathbb{R}^4$ behave as solutions of BDEs in the same way as its analogue on surfaces in $\mathbb{R}^3$.

For a surface in $\mathbb{R}^4$, the asymptotic directions are also captured by the contact of the surface with lines. This contact reveals aspects of the differential geometry of the surface in the closure of its hyperbolic region and is described by the $\mathcal{A}$-singularities of the family of orthogonal projections to 3-spaces. The projection along an asymptotic directions at a point on the parabolic set may have a $P_3(c)$-point. Away from inflection points, the characteristic and asymptotic curves are generically a family of cusps at ordinary parabolic points and have a folded singularity at a $P_3(c)$-point.

This point has similar behavior to the cusps of Gauss on surfaces in $\mathbb{R}^3$ (see [3, 10, 19, 24]). In [9, 10], we defined the cr-invariant at $P_3(c)$-points and showed that the S_2-curve, flecnodal curve and multi-local singularities curves are robust features of the surface in 4-space (Euclidean, affine or projective). Although the characteristic curves are not projective invariant of the surface, our goal is to produce results on the characteristic curves at $P_3(c)$-points similar to those results of Oliver [20]. At a $P_3(c)$-point, we show that the 2-jet of the curve formed by the inflection points of the characteristic curves (*characteristic inflection curve*) and the topological type of singularity of the characteristic curves in the elliptic domain are invariants under projective transformations. In addition, we list the possible configurations of the parabolic, S_2 and characteristic inflection curves using the cross-ratio invariant of this set of curves.

2. Binary differential equation

To study the configurations of characteristic curves, we need some results on BDEs which are studied extensively (see for example [22] for a survey article). We recall some results concerning the configurations of the solution curves of a BDE. A BDE defines two directions in the region where $\delta = B^2 - AC > 0$, a double (repeated) direction on the set $\Delta = \{\delta = 0\}$ and no direction where $\delta < 0$. The set Δ is the discriminant of the BDE. For generic BDEs and at generic points on Δ, the integral curves of (1) is a family of cusps, and the discriminant curve is a smooth curve traced by these cusps, except at isolated points called folded singularity (see below).

Consider the manifold of contact elements to the plane, that is, $PT^*\mathbb{R}^2 = \mathbb{R}^2 \times \mathbb{R}P^1$, and take the affine chart $q = dx/dy$, then $PT^*\mathbb{R}^2$ is endowed with the canonical contact structure determined by the 1-form $dx - qdy$. The projection associated to the contact structure is $\pi : PT^*\mathbb{R}^2 \to \mathbb{R}^2$ and given by $\pi(x, y, q) = (x, y)$. When the coefficients of a BDE do not vanish simultaneously, we may assume that $A \neq 0$ and take

$$\Omega(x, y, q) = A(x, y)q^2 + 2B(x, y)q + C(x, y). \tag{2}$$

The set $\Omega = 0$ is a surface $\mathcal{M}$. The directions defined by (1) lift to a single valued field

$$\xi = \Omega_q \partial y + q\Omega_q \partial x - (\Omega_y + q\Omega_x)\partial q \tag{3}$$

on $\mathcal{M}$ obtained by intersecting the contact planes with the tangent planes to $\mathcal{M}$. (See, for example, [8] for a suitable lifted field). The regions where $\delta > 0$, the image of $\pi|_{\mathcal{M}}$ is a two-fold covering. The critical set of $\pi|_{\mathcal{M}}$ given by $\Omega = \Omega_q = 0$ is called the criminant (its projection is the discriminant curve).

Stable topological models of (1) arise when the discriminant is a regular curve (or is empty). At almost all points of the discriminant, the field ξ is regular i.e., the unique direction at a point of the discriminant is transverse to it, then the BDE is smoothly equivalent to $dx^2 + ydy^2 = 0$ ([6], [7]). When ξ has an elementary singularity, the unique direction is tangent to the discriminant at that point, then equation (1) is smoothly equivalent to $dx^2 + (-x + \lambda y^2)dy^2 = 0$ with $\lambda \neq 0, \frac{1}{16}$ ([8]); the corresponding point in the plane is called a *folded singularity* of the BDE. There are three topological models: a *folded saddle* if $\lambda < 0$, a *folded node* if $0 < \lambda < \frac{1}{16}$ and a *folded focus* if $\frac{1}{16} < \lambda$. These occur when the lifted field ξ has a saddle, node or focus, respectively (see Figure 1 and [8]).

A solution curve of (1) has an *inflection point* at the projection of a point on $\mathcal{M}$ where

$$\Omega = \Omega_y + q\Omega_x = 0. \tag{4}$$

There is a smooth curve of such points which is tangent to the discriminant curve at folded singularities of equation (1) ([5]).

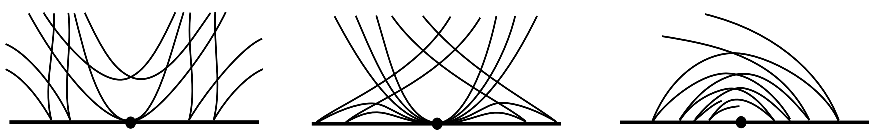

FIGURE 1. A folded saddle (left), node (center) and focus (right).

3. CHARACTERISTIC CURVES ON SURFACES IN $\mathbb{R}^4$

Let M be a regular surface in $\mathbb{R}^4$. For a given point $p \in M$, consider the unit circle in T_pM parametrized by $\theta \in [0, 2\pi]$. The curvature vectors $\eta(\theta)$ of the normal sections of M by the hyperplane $\langle\theta\rangle \oplus N_pM$ form an ellipse in the normal plane N_pM called the *curvature ellipse* and is the image this unit circle by a pair of quadratic forms

$$(Q_1, Q_2) = (ax^2 + 2bxy + cy^2, lx^2 + 2mxy + ny^2),$$

where a, b, c, l, m, n are the coefficients of the second fundamental form of M at p ([16]). Points on the surface are classified according to the position of the point p with respect to the ellipse (N_pM is viewed as an affine plane through p). The point p is called *elliptic/parabolic/hyperbolic* if it is inside/on/outside the ellipse at p, respectively.

Following the approach in [2], a binary form $Ax^2+2Bxy+Cy^2$ is represented by its coefficients $(A, B, C) \in \mathbb{R}^3$, there is a cone Γ given by $B^2 - AC = 0$ representing the perfect squares. If the forms Q_1 and Q_2 are independent, they determine a line in the projective plane $\mathbb{R}P^2$ and the cone a conic that we still denoted by Γ. This line meets the conic in 0/1/2 points according as $\delta(p) < 0/ = 0/ > 0$, where

$$\delta(p) = (an - cl)^2 - 4(am - bl)(bn - cm).$$

A point p is elliptic/parabolic/hyperbolic if $\delta < 0/ = 0/ > 0$. The parabolic set is denoted by Δ-set. If Q_1 and Q_2 are dependent, the rank of the matrix $\begin{pmatrix} a & b & c \\ l & m & n \end{pmatrix}$ is 1 (provided either of the forms is non-zero); the corresponding points on the surface are referred to as inflection points. There is an action of $\mathbf{GL}(2,\mathbb{R}) \times \mathbf{GL}(2,\mathbb{R})$ on pairs of binary forms. The orbits of this action are as follows (see for example [13]):

(x^2, y^2)	hyperbolic point
$(xy, x^2 - y^2)$	elliptic point
(x^2, xy)	parabolic point
$(x^2 \pm y^2, 0)$	inflection point
$(x^2, 0)$	degenerate inflection point
$(0, 0)$	degenerate inflection point.

The *asymptotic directions* (labelled by conjugate directions in [16]) are defined as the directions along θ such that the curvature vector $\eta(\theta)$ is tangent to the curvature ellipse (see also [17]). A curve on M whose tangent direction at each point is an asymptotic direction is called an *asymptotic curve*. The asymptotic curves of M are solution curves of the BDE

$$\Psi(x, y, q) = (am - bl)q^2 + (an - cl)q + (bn - cm) = 0, \tag{5}$$

([17, 16]). We call this equation the asymptotic BDE. The discriminant of the BDE (5) is the Δ-set and is a generic smooth curve on surface. Away from inflection points, at a hyperbolic (resp. parabolic or elliptic) point there are 2 (resp. 1 or 0) asymptotic directions at that point.

Since we do not distinguish between a BDE and its non-zero multiples, at each point (x, y), we can view a BDE (1) as a quadratic form in dx, dy and represent it by the point $(A : 2B : C)$ in $\mathbb{R}P^2$. To a point $(A : 2B : C)$ is associated a polar line with respect to the conic Γ. Three points in $\mathbb{R}P^2$ form a self-polar triangle if the polar of any of the three points is the line through the remaining two points. In our case the point $(A : 2B : C)$ is parametrized by $(x, y) \in U$ (for more details, see [15] chapter 7). The metric on M is given by $ds^2 = X_1 dx^2 + 2X_2 dxdy + X_3 dy^2$ and determines a point $(X_1 : 2X_2 : X_3)$ in the projective plane. It turns out that the polar line of $(X_1 : 2X_2 : X_3)$ consists of BDEs whose solutions are orthogonal curves on M ([4, 23]). This polar line intersects the polar line of the asymptotic BDE (5) at a unique point (P) which represent a BDE, called the BDE of the lines of principal curvature ([23]). The BDEs (A) of the asymptotic curves and the BDE (P) determine a unique BDE (C), the characteristic BDE, such the three of them form a self-polar triangle in the projective plane. In fact, (C) is the Jacobian of (A) and (P) ([23]), and if the surface M is parametrized by $\phi(x, y)$, the characteristic BDE is given by

$$\begin{aligned}\Phi(x, y, q) =& (L(GL - EN) - 2M(FL - EM))q^2 + 2((M(EN + GL) - 2LNF))q \\ &+ 2M(GM - FN) - N(GL - EN) = 0,\end{aligned} \tag{6}$$

where $E = \langle \phi_x, \phi_x \rangle$, $F = \langle \phi_x, \phi_y \rangle$, $G = \langle \phi_y, \phi_y \rangle$, $L = (am - bl)$, $2M = (an - cl)$ and $N = (bn - cm)$. A *characteristic curve* is the a curve on M whose tangent direction at each point is a characteristic direction. The discriminant curve of the BDE (6) coincides with the parabolic set. At elliptic point there are two characteristic directions and at each parabolic point there is one.

The asymptotic directions can be described via the singularities of the projections of M to 3-spaces (see [2]). Consider the family of orthogonal projections given by

$$\begin{array}{rcl} P : M \times S^3 & \to & TS^3 \\ (p, \mathbf{u}) & \mapsto & (\mathbf{u}, p - \langle p, \mathbf{u} \rangle \mathbf{u}). \end{array}$$

For $\mathbf{u}$ fixed, the projection can be viewed, locally at a point p, as a map germ

$$P_{\mathbf{u}} : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0).$$

(Two germs f and g are said to be $\mathcal{A}$-equivalent and write $f \sim_{\mathcal{A}} g$, if $g = k \circ f \circ h^{-1}$ for some germs of diffeomorphisms h and k of, respectively, the source and target.) The generic $\mathcal{A}$-singularities of $P_{\mathbf{u}}$ are those that have $\mathcal{A}_e$-codimension ≤ 3 (which is the dimension of S^3), see Table 1 and Table 2.

Table 1. The generic local singularities of orthogonal projections of M to 3-spaces ([18]).

Name	Normal form	$\mathcal{A}_e$-codimension
Immersion	$(x, y, 0)$	0
Cross-cap	(x, y^2, xy)	0
$B_k^{\pm}$	$(x, y^2, x^2y \pm y^{2k+1}), k = 2, 3$	k
$S_k^{\pm}$	$(x, y^2, y^3 \pm x^{k+1}y), k = 1, 2, 3$	k
$C_k^{\pm}$	$(x, y^2, xy^3 \pm x^k y), k = 3$	k
H_k	$(x, xy + y^{2k+2}, y^3), k = 2, 3$	k
$P_3(c)$	$(x, xy + y^3, xy^2 + cy^4),\ c \neq 0, \frac{1}{2}, 1, \frac{3}{2}$	3^*

* The codimension of $P_3(c)$ is that of its stratum.

Table 2. Bi-germs of $\mathcal{A}_e$-codimension 2 of orthogonal projections of M to 3-spaces ([14]).

Name	Normal Form	$\mathcal{A}_e$-codimension
$[A_2]$	$(x, y, 0; X, Y, X^2 + Y^3)$	2
$(A_0S_0)_2$	$(x, y, 0; Y^2, XY + Y^5, X)$	2
$A_0S_1^{\pm}$	$(x, y, 0; Y^3 \pm X^2Y, Y^2, X)$	2
$A_0S_0\|A_1^{\pm}$	$(x, y, 0, X, XY, Y^2 \pm X^2)$	2

For a complete table see [14].

The projection $P_{\mathbf{u}}$ is singular at p if and only if $\mathbf{u} \in T_pM$. The singularity is a cross-cap unless $\mathbf{u}$ is an asymptotic direction at p. The $\mathcal{A}_e$-codimension 2 singularities occur on curves on a generic surface and the $\mathcal{A}_e$-codimension 3 ones occur at special points on these curves. When projecting the surface along an asymptotic direction at a parabolic point, the projection may have a $P_3(c)$-singularity ([3, 10]). If we call S_2-curve (resp. B_2, $(A_0S_0)_2$, $A_0S_1^{\pm}$, $A_0S_0|A_1^{\pm}$-curve) the closure of the set of points p on M for which there exists a projection $P_{\mathbf{u}}$ having an S_2 (resp. B_2, $(A_0S_0)_2$, $A_0S_1^{\pm}$, $A_0S_0|A_1^{\pm}$)-singularity at p, then these curves meet the parabolic set tangentially at a $P_3(c)$-singularity (see Proposition 3.1 and for a complete proof [9, 10]). At a $P_3(c)$-singularity the unique asymptotic (or characteristic) direction is tangent to the parabolic set. This point is called a $P_3(c)$-point and is also a point where the asymptotic (or characteristic) curves have a folded singularity (see §2).

Throughout this paper, we consider the family of orthogonal projections P where the map $P_{\mathbf{u}}$ has $P_3(c)$-point. We can take $\mathbf{u} = (0, 1, 0, 0)$ as an asymptotic direction. We choose local coordinates at p such that the surface is given in Monge form

$$\phi(x, y) = (x, y, f^1(x, y), f^2(x, y))$$

where $(j^1f^1(0,0), j^1f^2(0,0)) = (0,0)$ and with 2-jet of $(f^1, f^2) = (Q_1, Q_2)$. We denote by (X, Y, Z, W) the coordinates in $\mathbb{R}^4$ and we parametrize the directions near $\mathbf{u}$ by $(u, 1, v, w)$. Instead of the orthogonal projection to the plane $(u, 1, v, w)^\perp$, we project to the fixed plane (X, Z, W). The modified family of projections is given by

$$\begin{array}{rccl} P: & (\mathbb{R}^2 \times \mathbb{R}^3, 0) & \to & (\mathbb{R}^3, 0) \\ & ((x,y),(u,v,w)) & \mapsto & P_{\mathbf{u}} = (x - uy, f^1(x,y) - vy, f^2(x,y) - wy), \end{array}$$

with $P_0(x,y) = (x, f^1(x,y), f^2(x,y))$. As the $P_3(c)$-point belongs to Δ-set and if we denote by $o(k)$ the terms of order greater than k in $x_1, \ldots, x_r$, then we can take $(Q_1, Q_2) = (x^2, xy)$ and write

$$\begin{aligned} f^1(x,y) &= x^2 + \textstyle\sum_{i=0}^3 a_{3i}x^{3-i}y^i + \sum_{i=0}^4 a_{4i}x^{4-i}y^i + \sum_{i=0}^5 a_{5i}x^{5-i}y^i + o(5), \\ f^2(x,y) &= xy + \textstyle\sum_{i=0}^3 b_{3i}x^{3-i}y^i + \sum_{i=0}^4 b_{4i}x^{4-i}y^i + \sum_{i=0}^5 b_{5i}x^{5-i}y^i + o(5). \end{aligned} \tag{7}$$

The 2-jet of the coefficients of a, b, c, l, m, and n of (Q_1, Q_2) are given as follows

$$\begin{array}{rcl} a &=& \frac{1}{2}f^1_{xx} = 1 + 3a_{30}x + a_{31}y + 6a_{40}x^2 + 3a_{41}xy + a_{42}y^2, \\ b &=& \frac{1}{2}f^1_{xy} = a_{31}x + a_{32}y + \frac{3}{2}a_{41}x^2 + 2a_{42}xy + \frac{3}{2}a_{43}y^2, \\ c &=& \frac{1}{2}f^1_{yy} = a_{32}x + 3a_{33}y + a_{42}x^2 + 3a_{43}xy + 6a_{44}y^2, \\ l &=& \frac{1}{2}f^2_{xx} = 3b_{30}x + b_{31}y + 6b_{40}x^2 + 3b_{41}xy + b_{42}y^2, \\ m &=& \frac{1}{2}f^2_{xy} = \frac{1}{2} + b_{31}x + b_{32}y + \frac{3}{2}b_{41}x^2 + 2b_{42}xy + \frac{3}{2}b_{43}y^2, \\ n &=& \frac{1}{2}f^2_{yy} = b_{32}x + 3b_{33}y + b_{42}x^2 + 3b_{43}xy + 6b_{44}y^2. \end{array}$$

The curve formed by the locus of geodesic inflection points of the characteristic (resp. asymptotic) curves we call *characteristic inflection curve* (resp. *flecnodal curve* (see [9, 10])) and denoted by C_h-curve (resp. F_l-curve). We have the following result.

Proposition 3.1. *Let M be a surface in $\mathbb{R}^4$ given in Monge form as in (7), and suppose that the origin is a $P_3(c)$-point. Then we have the following initial terms of the following curves:*

a) *the parabolic curve (Δ-curve):*

$$x = \frac{6a_{32}b_{33} - 9b_{33}^2 - 6a_{44}}{a_{32}}y^2 + o(2).$$

b) *the B_2-curve:*

$$x = \frac{2(3a_{32}^3b_{33} - 4a_{32}^2b_{33}^2 - 3a_{44}a_{32}^2 - 8a_{44}a_{32}b_{33} + 12a_{44}b_{33}^2 + 8a_{44}^2)}{a_{32}(a_{32} - 2b_{33})^2}y^2 + o(2).$$

c) *the S_2-curve:*

$$x = \frac{6(a_{32}^3b_{33} + 48a_{32}^2b_{33}^2 - 72a_{32}b_{33}^3 - a_{44}a_{32}^2 - 72a_{44}a_{32}b_{33} + 36a_{44}b_{33}^2 + 24a_{44}^2)}{a_{32}(a_{32} + 6b_{33})^2}y^2 + o(2).$$

d) *the $A_0S_1^{\pm}$-curve:*

$$x = \frac{3a_{32}^2b_{33}^2 - 4a_{32}a_{44}b_{33} + 3a_{44}b_{33}^2 + 2a_{44}^2}{a_{32}(4a_{32}b_{33} - 4b_{33}^2 - 3a_{44})}y^2 + o(2).$$

e) *the $(A_0S_0)_2$-curve:*

$$x = \frac{12a_{32}b_{33} - 9b_{33}^2 - 6a_{44}}{a_{32}}y^2 + o(2).$$

f) *the $A_0S_0|A_1^{\pm}$-curve:*

$$x = \frac{3a_{32}^2b_{33}^2 - 16a_{32}a_{44}b_{33} + 12a_{44}b_{33}^2 + 8a_{44}^2}{4(a_{32}b_{33} - b_{33}^2 - a_{44})a_{32}}y^2 + o(2).$$

g) *the F_l-curve:*

$$x = \frac{6(a_{32}b_{33} - a_{44})(24a_{32}b_{33} - 36b_{33}^2 + a_{32}^2 - 24a_{44})}{a_{32}(6b_{33} - a_{32})^2}y^2 + o(2).$$

h) *the C_h-curve:*

$$x = \frac{6(a_{32}b_{33} - a_{44} - 3b_{33}^2)(36b_{33}^2 - 24a_{32}b_{33} + a_{32}^2 + 24a_{44})}{a_{32}(a_{32} + 6b_{33})^2}y^2 + o(2).$$

All the above curves are tangent to the parabolic curve at the $P_3(c)$-point and any two have contact of order 2 at the origin.

Proof. The singularity of the projection P_0 is $\mathcal{A}$-equivalent to a $P_3(c)$-singularity when $a_{33} = 0$, $a_{32}, a_{44}, b_{33} \neq 0$, $a_{44}/(a_{32}b_{33}) \neq 0, 1/2, 1, 3/2$, and $5a_{32}b_{33} - 6b_{33}^2 - 4a_{44} \neq 0$ ([9, 21]). All the curves Δ, B_2, S_2, $A_0S_1^{\pm}$, $(A_0S_0)_2$, $A_0S_0|A_1^{\pm}$ are determined in [9, 10] using adjacencies of the $P_3(c)$-singularity.

The curves in $g)$ and $h)$ are obtained using the asymptotic and characteristic BDEs. In fact, the 2-jet of the characteristic BDE (6) is written as

$$\begin{aligned} j^2\Phi =& q^2 + (2b_{32}x + 6b_{33}y)q + (2a_{32}b_{32} - 6a_{31}b_{33} + a_{31}a_{32} + 12b_{32}b_{33} + 3a_{43})xy + a_{32}x \\ & (a_{42} + a_{32}^2 + 2b_{32}^2 - 2a_{31}b_{32} + 3a_{30}a_{32} + 4b_{31}a_{32})x^2 + (18b_{33}^2 - 6a_{32}b_{33} + 6a_{44})y^2. \end{aligned} \tag{8}$$

Thus, we can write by the implicit function theorem

$$x = \frac{6(-a_{44} + a_{32}b_{33} - 3b_{33}^2)}{a_{32}}y^2 - (6b_{33}a_{32})yq - \frac{1}{a_{32}}q^2 + o(2).$$

Substituting the expression of x into $\Phi_y + q\Phi_x = 0$ we obtain

$$\Big(18b_{33}^2 - 6a_{44} + 6a_{32}b_{33}\Big)y + \Big(3b_{33} + \frac{1}{2}a_{32}\Big)q + o(1) = 0.$$

Again, solving implicitly the last equality, we get

$$q = \frac{12(a_{32}b_{33} - a_{44} - 3b_{33}^2)}{(6b_{33} + a_{32})}y + o(1).$$

Substituting q in the expression of x gives the 2-jet of the characteristic inflection curve. The 2-jet of the flecnodal curve is also determined in [9, 10] using the same approach above for the asymptotic BDE. $\square$

We denote the tangent lines to the Legendrian lifts of the parabolic, B_2, S_2, flecnodal, characteristic inflection, $(A_0S_0)_2$, A_0S_1, and $A_0S_0|A_1^{\pm}$ curves in PT^*M at a $P_3(c)$-point by l_P, l_B, l_S, l_F, l_C, $l_{s_{02}}$, l_{s_1}, and $l_{s_{01}}$, respectively. We denote by l_g the contact element at the $P_3(c)$-point (i.e., the vertical line in the contact plane at that point).

Remark 3.1. By Proposition 3.1, l_P, l_S , l_B , l_F, and l_C are distinct unless

$$(5a_{32}b_{33} - 6b_{33}^2 - 4a_{44}) = 0.$$

This condition is precisely that for the family of the orthogonal projections P to fail to be a versal unfolding of a $P_3(c)$-singularity ([10]). In a generic one-parameter family of surfaces case (see [3, Proposition 4.3]) there are double $P_3(c)$-points.

Theorem 1. *At a generic $P_3(c)$-point, the 2-jet of the C_h-curve is projectively invariant.*

Proof. The cross-ratio of lines l_P, l_g, l_S, l_C is given by

$$(l_P, l_g : l_S, l_C) = \frac{c_S - c_P}{c_C - c_P} = \frac{\frac{9(5a_{32}b_{33}-6b_{33}^2-4a_{44})^2}{a_{32}(a_{32}+6b_{33})^2}}{-\frac{9(5a_{32}b_{33}-6b_{33}^2-4a_{44})^2}{a_{32}(a_{32}+6b_{33})^2}} = -1,$$

where c_P, c_S, and c_C are the coefficients of order 2 of the parabolic curve, S_2-curve and C_h-curve, respectively. The result follows from the fact that the 2-jet of the C_h-curve depends on the S_2-curve and parabolic curve which are projective invariants ([9, 10]). □

Proposition 3.2. *The topological type of the singularity of the characteristic BDE at a $P_3(c)$-point is invariant under projective transformations.*

Proof. The singularity type is determined by equation (8). It is given by the type of the singularity of the lifted field ξ: a saddle, node or focus. Since a $P_3(c)$-point is a folded singularity, the characteristic BDE is locally smoothy equivalent to $dx^2 + (-x + \lambda y^2)dy^2 = 0$, where

$$\lambda = -\frac{3}{2}\frac{(5a_{32}b_{33} - 6b_{33}^2 - 4a_{44})}{a_{32}^2}$$

determines the topological type of singularity if and only if $\lambda \neq 0, \frac{1}{16}$ (see [5]). Observe that the coefficients a_{44} and b_{33} of λ depend on a combination of the cross-ratios $\rho_1 = (l_P, l_B : l_S, l_F)$, $\rho_2 = (l_P, l_g : l_{s_{01}}, l_{s_{02}})$, $\rho_3 = (l_P, l_g : l_{s_1}, l_{s_{02}})$, and a_{32}. In fact,

$$\begin{aligned}
\rho_1 &= \frac{a_{32} - 3b_{33}}{a_{32} - 6b_{33}},\\
\rho_2 &= -\frac{21a_{32}^2b_{33}^2 - 60a_{32}b_{33}^3 + 36b_{33}^4 - 32a_{32}a_{44}b_{33} + 48a_{44}b_{33}^2 + 16a_{44}^2}{24a_{32}(a_{32}b_{33} - b_{33}^2 - a_{44})b_{33}},\\
\rho_3 &= -\frac{21a_{32}^2b_{33}^2 - 60a_{32}b_{33}^3 + 36b_{33}^4 - 32a_{32}a_{44}b_{33} + 48a_{44}b_{33}^2 + 16a_{44}^2}{6(4a_{32}b_{33} - 4b_{33}^2 - 3a_{44})a_{32}b_{33}}.
\end{aligned}$$

Using ρ_1 we get $b_{33} = \dfrac{1}{3}\dfrac{(\rho_1 - 1)a_{32}}{2\rho_1 - 1}$. From ρ_2 and ρ_3 it follows that

$$6a_{32}b_{33}((3\rho_3 - 4\rho_2 + 1)a_{44} + 4b_{33}(6\rho_2 - \rho_3)(a_{32} - b_{33})) = 0.$$

Replacing b_{33} in the above equation, we obtain

$$a_{44} = \frac{4}{9}\frac{a_{32}^2(\rho_1 - 1)(\rho_2 - \rho_3)(5\rho_1 - 2)}{(2\rho_1 - 1)^2(4\rho_2 - 3\rho_3 - 1)}.$$

Since $a_{32} \neq 0$, substituting b_{33} and a_{44} into λ, shows that the type of singularity of the characteristic BDE depends only on the values of the cross-ratios ρ_1, ρ_2 and ρ_3, all of which are projective invariants. □

At a $P_3(c)$-point, the 4-jet of the parametrization $\phi(x, y) = (x, y, f^1(x, y), f^2(x, y))$ of the surface M is equivalent, by projective transformations, to the normal form

$$(x, y, x^2 + xy^2 + \alpha y^4, xy + \beta y^3 + \psi), \tag{9}$$

where $6\beta^2 + 4\alpha - 15\beta + 5 \neq 0$, $\alpha \neq 0, 1/2, 1, 3/2$, and ψ is a polynomial of degree 4 (see [11]).

According to Proposition 3.2, we can use the normal form (9) to present the topological type of the singularity of the characteristic BDE at a $P_3(c)$-point. In [9, 10] we showed that α and β in (9) are also projective invariants described as functions of ρ_1, ρ_2 and ρ_3. This allows us to recalculate the expressions of the curves in Proposition 3.1 in terms of α and β. In fact, consider representing M locally as a surface $\bar{M}$ in $\mathbb{P}^4$, given in the affine chart $\{[x : y : z : w : 1]\}$ in Monge form $[x : y : f^1(x, y) : f^2(x, y) : 1]$. We can take (f^1, f^2) with 4-jet as in (9) and use the equations of the curves in Proposition 3.1 with $a_{32} = 1$, $a_{44} = \alpha$, and $b_{33} = \beta$.

Theorem 2. *At a $P_3(c)$-point, the characteristic BDE has a folded singularity if and only if $\gamma = -(5\beta - 6\beta^2 - 4\alpha) \neq 0, \frac{1}{24}$. The singularity is a folded saddle if $\gamma < 0$, a folded node if $0 < \gamma < \frac{1}{24}$, and folded focus if $\gamma > \frac{1}{24}$.*

Proof. The proof follows from Proposition 3.2. Note that $\lambda = -\frac{3}{2}(5\beta - 6\beta^2 - 4\alpha) \neq 0, \frac{1}{16}$. Thus the singularity of the characteristic BDE is determined by values of γ. □

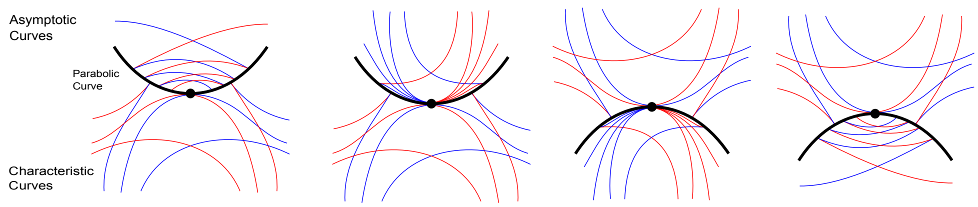

FIGURE 2. The asymptotic and characteristic curves at a $P_3(c)$-point. $\gamma < -1/24$ (first); $-1/24 < \gamma < 0$ (second); $0 < \gamma < 1/24$ (third) and $\gamma > 1/24$ (fourth).

Remark 3.2. The types of the singularities of the asymptotic and characteristic BDE are not related ([23]). However, for surfaces in $\mathbb{R}^4$, thanks to Theorem 2, the types of these singularities have opposite indices at a $P_3(c)$-point, that is, on one side of the parabolic curve we have a folded saddle and on the other a folded node or focus or vice-versa. This also happens for surfaces in $\mathbb{R}^3$ at cusps of Gauss [4]. Figure 2 shows the generic configurations of asymptotic and characteristic curves at a folded singularity.

Following the approach in [20], we denoted by ρ_c the cross-ratio $(l_P, l_g : l_C, l_B)$ and call it the *characteristic cross-ratio*. It can be written in terms of the coefficients of normal form (9) as follows

$$\rho_c = -\frac{9(2\beta - 1)^2}{(1 + 6\beta)^2}.$$

As the generic relative positions of the relevant curves at a $P_3(c)$-point are determined by their 2-jets, we can give the their relative positions in terms of the values of ρ_c. In what follows, we present the relative positions of the curves Δ, B_2, S_2, F_l, and C_h.

Theorem 3. *Let c_P, c_B, c_S, c_F, and c_C be the coefficients of order 2 associated to curves Δ, B_2, S_2, F_l, and C_h, respectively, at a $P_3(c)$-point of a smooth surface in $\mathbb{R}^4$. Then there are 4 possible relative positions of these curves depending on the values of ρ_c:*

(i) *If $\rho_c < -9$, then $c_C < c_P < c_B < c_F < c_S$*
(ii) *If $-9 < \rho_c < -1$, then $c_C < c_P < c_B < c_S < c_F$*
(iii) *If $-1 < \rho_c < -1/9$, then $c_C < c_P < c_S < c_B < c_F$*
(iv) *If $-1/9 < \rho_c < 0$, then $c_C < c_P < c_S < c_F < c_B$.*

Proof. The proof follows from Proposition 3.1 with $a_{32} = 1$, $a_{44} = \alpha$ and $b_{33} = \beta$. It is easy to check that the coefficients c_P, c_B, c_S, c_F, and c_C satisfy $c_C < c_P < c_B, c_S, c_F$ for all value of

α, β. Furthermore,

$$c_B - c_S = \frac{8(6\beta-1)(4\alpha+6\beta^2-5\beta)^2}{(2\beta-1)^2(1+6\beta)^2},$$
$$c_B - c_F = \frac{8(3\beta-1)(4\alpha+6\beta^2-5\beta)^2}{(2\beta-1)^2(1+6\beta)^2},$$
$$c_S - c_F = -\frac{216\beta(4\alpha+6\beta^2-5\beta)^2}{(6\beta-1)^2(1+6\beta)^2}.$$

Since $4\alpha+6\beta^2-5\beta \neq 0$ (see Remark 3.1), we have $c_B > c_S$ if and only if $\beta > 1/6$; $c_B > c_F$ if and only if $\beta > 1/3$; and $c_S > c_F$ if and only if $\beta < 0$. This and the fact that $\rho_c = -\frac{9(2\beta-1)^2}{(1+6\beta)^2}$, for each value of β we obtain the desired result. □

Theorem 4. *With notation in Theorem 3, consider the 2-jets of curves Δ, S_2, and C_h represented by the parabolas $x = c_P \cdot y^2$, $x = c_S \cdot y^2$, and $x = c_C \cdot y^2$, respectively. There are four possible configurations for Δ, S_2, and C_h and these are determined by α and β. They are described by Figure 3.*

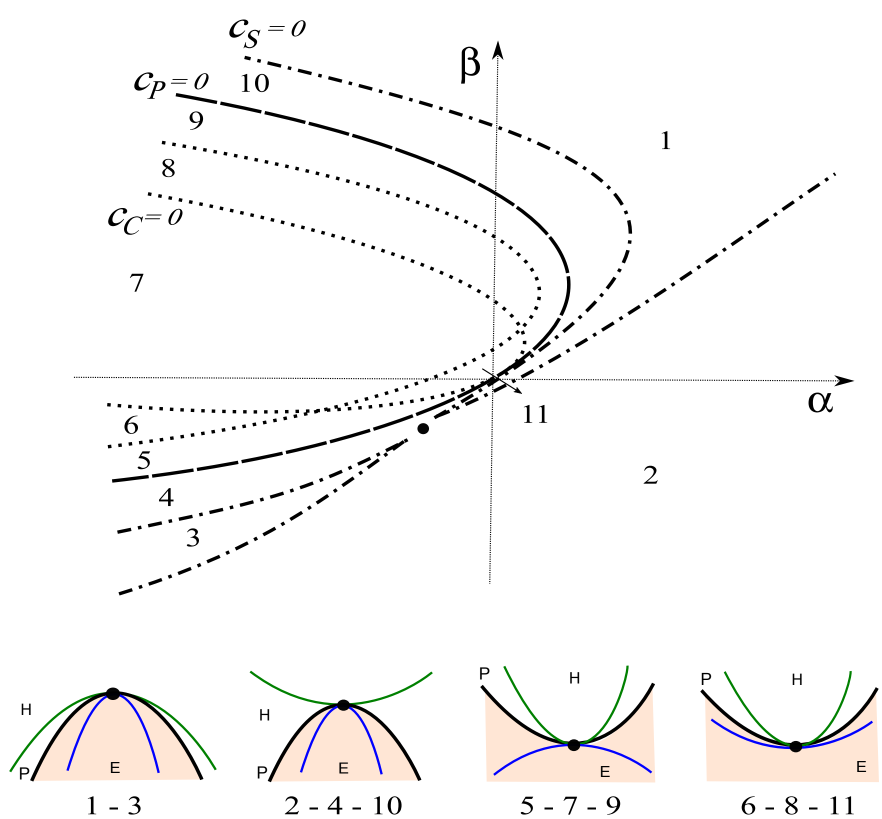

FIGURE 3. Partition of (α, β)-plane. The bottom pictures are the configurations of Δ-curve (black), S_2-curve (green), and C_h-curve (blue) at a $P_3(c)$-point. H, P, and E mean hyperbolic, parabolic, and elliptic region, respectively.

Proof. Consider the 2-jets of the parametrisation of the Δ-curve, S_2-curve, C_h-curve with the second order coefficients given by

$$\begin{aligned} c_P &= 3(2\beta - 3\beta^2 - 2\alpha),\\ c_S &= \frac{6(36\alpha\beta^2 - 72\beta^3 + 24\alpha^2 - 72\alpha\beta + 48\beta^2 - \alpha + \beta)}{(1+6\beta)^2},\\ c_C &= \frac{6(-3\beta^2 - \alpha + \beta)(36\beta^2 + 24\alpha - 24\beta + 1)}{(1+6\beta)^2}. \end{aligned}$$

The generic configurations of these curves occur when α and β avoid the set

$$\{c_P = 0\} \cup \{c_S = 0\} \cup \{c_C = 0\}.$$

The conditions $c_P = 0$, $c_S = 0$, and $c_C = 0$ determine curves in (α, β)-plane represented by dashed curve, dot-dashed curve, and doted curve in Figure 3, respectively. Then the (α, β)-plane is partitioned into 11 open regions. There are four different configurations of the Δ-curve, S_2-curve, and C_h-curve that are given at the bottom of Figure 3. For instance, in regions 1 and 3, the configurations of the Δ-curve, S_2-curve, C_h-curve are described in the first bottom picture; in regions 2, 4 and 10, the configurations are described in the second bottom picture and so on. $\square$

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References

[1] T. Banchoff, T. Gaffney and C. McCrory, Cusps of Gauss mappings. *Research Notes in Mathematics* 55 (1982), Pitman (Advanced Publishing Program).

[2] J. W. Bruce and A. C. Nogueira, Surfaces in $\mathbb{R}^4$ and duality. *Quart. J. Math. Oxford Ser.* Ser. (2) 49 (1998), 433–443.

[3] J. W. Bruce and F. Tari, Families of surfaces in $\mathbb{R}^4$. *Proc. Edinb. Math. Soc.* 45 (2002), 181–203.

[4] J. W. Bruce and F. Tari, Dupin indicatrices and families of curve congruence. *Trans. Amer. Math. Soc.* 357 (2004), 267–285. DOI: 10.1090/s0002-9947-04-03497-x

[5] J. W. Bruce and F. Tari, Duality and implicit differential equations. *Nonlinearity.* 13 (2000), 791–811. DOI: 10.1088/0951-7715/13/3/315

[6] M. Cibrario, Sulla reduzione a forma delle equatione linearialle derviate parziale di secondo ordine di tipo misto, *Accamdemia di Scienze e Lettrere, Instituto Lomardo Redicconti.* 65 (1932), 889–906. DOI: 10.1007/bf03017728

[7] L. Dara, Singularites generiques des equations differentielles multi-forms, *Bol. Soc. Bras. Mat.* No. 6 (1975) 95–129. DOI: 10.1007/bf02584779

[8] A. A. Davydov, *Qualitative control theory.* Translations of Mathematical Monographs 142, AMS, Providence, RI, 1994.

[9] J. L. Deolindo-Silva, Cr-invariantes para surperfícies em $\mathbb{R}^4$. Doctoral Thesis, Instituto de Ciências Matemáticas e de Computação, University of São Paulo, São Carlos, 2016. DOI: 10.11606/T.55.2016.tde-03102016-104837

[10] J. L. Deolindo-Silva, Cross-ratio invariants for surfaces in 4-space (2018). arχiv: 1807.11133

[11] J. L. Deolindo-Silva and Y. Kabata, Projective classification of jets of surfaces in 4-space. *Hiroshima Mathematical Journal*, 49 (2019) 35–46. DOI: 10.32917/hmj/1554516037

[12] R. A. Garcia and J. Sotomayor, Harmonic mean curvature lines on surfaces immersed in $\mathbb{R}^3$. *Bull. Braz. Math. Soc.*, 34 (2) (2003), 303–331.

[13] C. G. Gibson, *Singular points of smooth mappings.* Pitman Research Notes in Mathematics, vol. 25 (1979).

[14] C. A. Hobbs and N. P. Kirk, On the classification and bifurcation of multi-germs of maps from surfaces to 3-space, Math. Scand. 89 (2001), 57–96

[15] S. Izumyia, M. C. Romero-Fuster, M. A. S. Ruas and F. Tari, *Differential geometry from a singularity theory viewpoint.* World Scientific, 2016. DOI: 10.1142/9108
[16] J. A. Little, On the singularities of submanifolds of heigher dimensional Euclidean space. *Annli Mat. Pura et Appl.* (4A) 83 (1969), 261–336.
[17] D. K. H. Mochida, M. C. Romero-Fuster and M. A. S. Ruas, The geometry of surfaces in 4-space from a contact viewpoint. *Geometriae Dedicata* 54 (1995), 323–332. DOI: 10.1007/bf01265348
[18] D. M. Q. Mond, On the classification of germs of maps from $\mathbb{R}^2$ to $\mathbb{R}^3$. *Proc. London Math. Soc.* 50 (1985), 333–369.
[19] J. J. Nuño-Ballesteros and F. Tari, Surfaces in $\mathbb{R}^4$ and their projections to 3-spaces. *Roy. Proc. Edinburgh Math. Soc.* 137A (2007), 1313–1328.
[20] J. Oliver, On the characteristic curves on a smooth surface. *J. London Math. Soc.* (2) 83 (2011), 755–767.
[21] R. Oset-Sinha and F. Tari, Projections of surfaces in $\mathbb{R}^4$ to $\mathbb{R}^3$ and the geometry of their singular images. *Rev. Mat. Iberoam.* 31, 1 (2015), 33–50. DOI: 10.4171/rmi/825
[22] F. Tari, Pairs of foliations on surfaces. *Real and Complex Singularities*, Edited by M. Manoel, M. C. Romero Fuster, C. T. C. Wall. London Mathematical Society Lecture Notes Series 380 (2010), 305–337. DOI: 10.1017/cbo9780511731983.023
[23] F. Tari, Self-adjoint operators on surfaces in $\mathbb{R}^n$. *Differential Geom. Appl.* 27 (2009), 296–306.
[24] R. Uribe-Vargas, A projective invariant for swallowtails and godrons, and global theorems on the flecnodal curve. *Mosc. Math. J.* 6 (2006), 731–768. DOI: 10.17323/1609-4514-2006-6-4-731-768

Jorge Luiz Deolindo-Silva, Departamento de Matemática, Universidade Federal de Santa Catarina, Blumenau-SC, Brazil.
Email address: `jorge.deolindo@ufsc.br`

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HORO-FLAT SURFACES ALONG CUSPIDAL EDGES IN THE HYPERBOLIC SPACE

SHYUICHI IZUMIYA, MARIA CARMEN ROMERO-FUSTER, KENTARO SAJI, AND MASATOMO TAKAHASHI

Abstract. There are two important classes of surfaces in the hyperbolic space. One of class consists of extrinsic flat surfaces, which is an analogous notion to developable surfaces in the Euclidean space. Another class consists of horo-flat surfaces, which are given by one-parameter families of horocycles. We use the Legendrian dualities between hyperbolic space, de Sitter space and the lightcone in the Lorentz-Minkowski 4-space in order to study the geometry of flat surfaces defined along the singular set of a cuspidal edge in the hyperbolic space. Such flat surfaces can be considered as flat approximations of the cuspidal edge. We investigate the geometrical properties of a cuspidal edge in terms of the special properties of its flat approximations.

1. Introduction

The tangent plane at a point of a regular surface is a flat approximation of the surface at a point, which is the basic idea to define the curvatures of the surface at the point. In this sense, the curvature at a point measures how far or near is the shape of the surface from a plane at the given point. On the other hand, the normal plane of a surface at a point also provides important information of the surface, for instance, the notion of normal section plays an important role in surface theory. One of the possible generalizations of this viewpoint consists in considering flat surfaces which are tangent or normal to the surface along a given curve. In [12, 18], osculating (and normal) flat surfaces along a curve on a surface in the Euclidean space are investigated, and with the help of these notions, the geometrical behaviour of a curve lying on a given surface was studied in [11, 16].

On the other hand, several articles on the differential geometry of surfaces with singularities have appeared during the two last decades [4, 7–14, 21, 25, 26, 29–34, 36]. An important class of singular surfaces is provided by the wave fronts, on which a smooth unit normal vector field of the surface even at a singular point exists. This means that a tangent and thus normal planes can be defined at any point of a wave front. One of the simplest and generic wave fronts is a *cuspidal edge*, whose set of singular points is a regular space curve. In [23], osculating and normal flat surfaces along the singular points of a cuspidal edge in the Euclidean space are defined and investigated.

In the present paper we analyze the geometry of cuspidal edges in the hyperbolic space. We point out that in the hyperbolic 3-space there exist two notions of flatness of surfaces [19, 22] other than that of flat Gaussian curvature surfaces. We shall consider extrinsic flat surfaces and

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horospherical flat surfaces. The notion of *extrinsic flat surfaces* is a direct analogy to that of flat surfaces in the Euclidean space. However, the notion of *horospherical flat surfaces* has completely different properties [22]. It is a one-parameter family of horocycles, namely, a surface swept by a horocycle. We call them *horocyclic surfaces*. We call each horocycle a *generating horocycle*. It is known that a horospherical flat surface is (at least locally) parametrized as a horocyclic surface [22, Theorem 4.4]. We introduce osculating and normal horospherical flat surfaces along a cuspidal edge and we call them *flat approximations*. The main purpose of this paper is to investigate the geometrical properties of a cuspidal edge in terms of the special properties of its flat approximations. We use in §2 the Legendrian duality theorem obtained in [15] in order to define the flat wave fronts as well as some invariants of cuspidal edges in the hyperbolic space. Moreover, certain families of functions of the cuspidal edge are introduced in §2 as the main tool in this paper. In §3, we quickly review the general theory of horocyclic surfaces given in [22]. The basic properties of the above families of functions are investigated in §4 and §5. In §6.3 we analyze special cuspidal edges depending on special properties of flat approximations. Finally, in §7 we make a remark on the global properties of a curve in the hyperbolic space from the view point of the Legendrian duality.

We shall assume throughout the whole paper that all the maps and manifolds are of class C^∞ unless the contrary is explicitly stated.

2. Flat fronts in the hyperbolic space

The hyperbolic space is realized as a spacelike pseudo-hypersphere with an imaginary radius in the Lorentz-Minkowski 4-space. The first author obtained in [15] a general theory on Legendrian dualities for pseudo-spheres in the Lorentz-Minkowski space leading to a commutative diagram between certain contact manifolds defined by the dual relations. Such dualities have proven to be useful in the study of the differential geometry of submanifolds of the pseudo-spheres and the results obtained have been described in several papers [2,5,17,22,24]. See also [6,27,28].

We observe that the flatness of a surface contained in a three dimensional pseudo-sphere is determined by the degeneration of the dual surface. By taking this fact into account, we investigate in the present paper the flat approximations of cuspidal edges contained in the hyperbolic 3-space.

Consider the Lorentz-Minkowski 4-space $\mathbb{R}^4_1 = (\mathbb{R}^4, \langle\ ,\ \rangle)$ with the pseudo-inner product $\langle\ ,\ \rangle = (-{+}{+}{+})$ and the following subspaces

$$H^3 = \{\boldsymbol{v} \in \mathbb{R}^4_1 \mid \langle \boldsymbol{v}, \boldsymbol{v}\rangle = -1\}, \quad S^3_1 = \{\boldsymbol{v} \in \mathbb{R}^4_1 \mid \langle \boldsymbol{v}, \boldsymbol{v}\rangle = 1\}, \quad LC^* = \{\boldsymbol{v} \in \mathbb{R}^4_1 \mid \langle \boldsymbol{v}, \boldsymbol{v}\rangle = 0\}$$

that we call respectively, the *hyperbolic* 3*-space*, the *de Sitter* 3*-space* and the *lightcone*. We take now the submanifolds,

$$\Delta_1 = \{(\boldsymbol{v}, \boldsymbol{w}) \in H^3 \times S^3_1 \mid \langle \boldsymbol{v}, \boldsymbol{w}\rangle = 0\},$$

$$\Delta_2 = \{(\boldsymbol{v}, \boldsymbol{w}) \in H^3 \times LC^* \mid \langle \boldsymbol{v}, \boldsymbol{w}\rangle = -1\},$$

together with their corresponding canonical projections

$$\pi_{11} : \Delta_1 \to H^3, \quad \pi_{12} : \Delta_1 \to S^3_1, \quad \pi_{21} : \Delta_2 \to H^3, \quad \pi_{22} : \Delta_2 \to LC^*.$$

We can consider the 1-forms $\langle d\boldsymbol{v}, \boldsymbol{w}\rangle$ and $\langle \boldsymbol{v}, d\boldsymbol{w}\rangle$ on $\mathbb{R}^4_1 \times \mathbb{R}^4_1$, given by

$$\langle d\boldsymbol{v}, \boldsymbol{w}\rangle = -w_0dv_0 + w_1dv_1 + w_2dv_2 + w_3dv_3, \quad \langle \boldsymbol{v}, d\boldsymbol{w}\rangle = -v_0dw_0 + v_1dw_1 + v_2dw_2 + v_3dw_3,$$

for $\boldsymbol{v} = (v_0, v_1, v_2, v_3)$, $\boldsymbol{w} = (w_0, w_1, w_2, w_3) \in \mathbb{R}^4_1$. Clearly, the restrictions

$$\theta_{i1} = \langle d\boldsymbol{v}, \boldsymbol{w}\rangle|_{\Delta_i}, \quad \theta_{i2} = \langle \boldsymbol{v}, d\boldsymbol{w}\rangle|_{\Delta_i} \quad (i = 1, 2)$$

determine the same hyperplane field over Δ_i. Moreover, Δ_i is a contact manifold with the contact form $\theta_{i1}(=\theta_{i2})$, and π_{i1}, π_{i2} are Legendrian fibrations [15, Theorem 2.2]. There is a contact diffeomorphism $\Phi_{12}:\Delta_1\to\Delta_2$, given by $\Phi_{12}(\boldsymbol{v},\boldsymbol{w})=(\boldsymbol{v},\boldsymbol{v}\pm\boldsymbol{w})$ [15, page 330].

For a non-zero vector $\boldsymbol{v}\in\mathbb{R}^4_1$ and a real number c, we define a *hyperplane with pseudo normal* $\boldsymbol{v}$ by

$$HP(\boldsymbol{v},c)=\{\boldsymbol{x}\in\mathbb{R}^4_1 \mid \langle\boldsymbol{x},\boldsymbol{v}\rangle=c\ \}.$$

We say that $HP(\boldsymbol{v},c)$ is a *spacelike*, a *timelike* or a *lightlike hyperplane* according $\boldsymbol{v}$ satisfies that $\langle\boldsymbol{v},\boldsymbol{v}\rangle<0$, $\langle\boldsymbol{v},\boldsymbol{v}\rangle>0$ or $\langle\boldsymbol{v},\boldsymbol{v}\rangle=0$ respectively. We then have three kinds of totally umbilical surfaces in H^3, given by the intersection of H^3 with the different hyperplanes of $\mathbb{R}^4_1$: A surface $H^3\cap HP(\boldsymbol{v},c)$ is said to be a *sphere*, an *equidistant surface* or a *horosphere* provided $HP(\boldsymbol{v},c)$ is a spacelike, a timelike or a lightlike hyperplane respectively. Moreover, an equidistant surface $H^3\cap HP(\boldsymbol{v},0)$ is called a *hyperbolic plane*.

Let $U\subset\mathbb{R}^2$ be an open subset. We say that two maps $f:U\to H^3$ and $g:U\to S^3_1$ are Δ_1-*dual* (one to each other) if the map $(f,g):U\to\Delta_1$ is isotropic [15]. Then a map $f:U\to H^3$ is said to be a *frontal* if it has a Δ_1-dual $g:U\to S^3_1$. Moreover, we say that $f:U\to H^3$ is a *front* provided it has a Δ_1-dual $g:U\to S^3_1$, such that $(f,g):U\to\Delta_1$ is an immersion. Analogous concepts for the Δ_2-duality can be introduced too.

A map $f:U\to H^3$ is said to be *flat* (or more precisely, *extrinsically flat*) if its Δ_1-dual $g:U\to S^3_1$ satisfies that $\operatorname{rank} dg_p\le 1$ for any $p\in U$. On the other hand, $f:U\to H^3$ is said to be *horospherically flat* (or *horo-flat*) provided its Δ_2-dual, $g:U\to LC^*$, satisfies that $\operatorname{rank} dg_p\le 1$ for any $p\in U$.

Let M^3 be a 3-dimensional manifold. A singular point p of the map-germ $f:(U,p)\to M^3$ is a *cuspidal edge* if f is $\mathcal{A}$-equivalent to the germ $(u_1,u_2)\mapsto(u_1,u_2^2,u_2^3)$ at 0. If a singular point p of $f:(U,p)\to M^3$ is a cuspidal edge, then we also say that the germ f is a *cuspidal edge*. Here we recall that two map-germs $f,g:(\mathbb{R}^2,0)\to(\mathbb{R}^3,0)$ are $\mathcal{A}$-equivalent provided there exist diffeomorphism germs $\phi:(\mathbb{R}^2,0)\to(\mathbb{R}^2,0)$ and $\Phi:(\mathbb{R}^3,0)\to(\mathbb{R}^3,0)$ such that $\Phi\circ f\circ\phi^{-1}=g$.

It is well-known that a cuspidal edge $f:(U,p)\to H^3$ is a front, namely, there exists a Δ_1 dual $g:(U,p)\to S^3_1$ of f such that (f,g) is an immersion (see [1, 25], for example). Since both, the singular set $S(f)$ of f and its image $f(S(f))$, are regular curves, we can take a local coordinate system (u_1,u_2) centered at p on U such that

$$S(f)=\{(u_1,u_2)|u_2=0\},\quad |\langle f_{u_1}(u_1,0),f_{u_1}(u_1,0)\rangle|=1,\quad\text{and}\quad \det(f_{u_1},f_{u_2u_2},g,f)>0.$$

We set $u_1=u$ and $\boldsymbol{\gamma}(u)=f(u,0)$ and define vector fields along $\boldsymbol{\gamma}$ as follows:

$$\begin{array}{rcl}\boldsymbol{t}(u)&=&f_u(u,0),\\ \boldsymbol{\nu}(u)&=&g(u,0),\\ \boldsymbol{b}(u)&=&\boldsymbol{\gamma}(u)\wedge\boldsymbol{t}(u)\wedge\boldsymbol{\nu}(u),\\ \boldsymbol{l}^{\varepsilon}_{\nu}(u)&=&\boldsymbol{\gamma}(u)+\varepsilon\boldsymbol{\nu}(u),\\ \boldsymbol{l}^{\varepsilon}_{b}(u)&=&\boldsymbol{\gamma}(u)+\varepsilon\boldsymbol{b}(u),\end{array}\tag{2.1}$$

where $\varepsilon=\pm1$. Here, for any $\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3\in\mathbb{R}^4_1$, we define a vector $\boldsymbol{x}_1\wedge\boldsymbol{x}_2\wedge\boldsymbol{x}_3$ by

$$\boldsymbol{x}_1\wedge\boldsymbol{x}_2\wedge\boldsymbol{x}_3=\begin{vmatrix}-\boldsymbol{e}_0&\boldsymbol{e}_1&\boldsymbol{e}_2&\boldsymbol{e}_3\\ x^1_0&x^1_1&x^1_2&x^1_3\\ x^2_0&x^2_1&x^2_2&x^2_3\\ x^3_0&x^3_1&x^3_2&x^3_3\end{vmatrix},$$

where $\{\boldsymbol{e}_0,\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3\}$ is the canonical basis of $\mathbb{R}^4_1$ and $\boldsymbol{x}_i=(x^i_0,x^i_1,x^i_2,x^i_3)$. Then $\{\boldsymbol{\gamma},\boldsymbol{t},\boldsymbol{\nu},\boldsymbol{b}\}$ is a pseudo-orthonormal frame satisfying $\det(\boldsymbol{\gamma},\boldsymbol{t},\boldsymbol{\nu},\boldsymbol{b})=1$, and $\{\boldsymbol{\gamma},\boldsymbol{t},\boldsymbol{l}^{\varepsilon}_{\nu},\boldsymbol{b}\}$, $\{\boldsymbol{\gamma},\boldsymbol{t},\boldsymbol{l}^{\varepsilon}_{b},\boldsymbol{\nu}\}$ are moving frames along $\boldsymbol{\gamma}$. We then have the following Frenet-Serret type formulae:

$$
\begin{pmatrix} \boldsymbol{\gamma}' \\ \boldsymbol{t}' \\ \boldsymbol{\nu}' \\ \boldsymbol{b}' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & \kappa_\nu^h & \kappa_b^h \\ 0 & -\kappa_\nu^h & 0 & \kappa_t^h \\ 0 & -\kappa_b^h & -\kappa_t^h & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\gamma} \\ \boldsymbol{t} \\ \boldsymbol{\nu} \\ \boldsymbol{b} \end{pmatrix}, \tag{2.2}
$$

$$
\begin{pmatrix} \boldsymbol{\gamma}' \\ \boldsymbol{t}' \\ (\boldsymbol{l}_\nu^\varepsilon)' \\ \boldsymbol{b}' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \mathfrak{h}_\nu^\varepsilon & 0 & \varepsilon\kappa_\nu^h & \kappa_b^h \\ 0 & \mathfrak{h}_\nu^\varepsilon & 0 & \varepsilon\kappa_t^h \\ \varepsilon\kappa_t^h & -\kappa_b^h & -\varepsilon\kappa_t^h & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\gamma} \\ \boldsymbol{t} \\ \boldsymbol{l}_\nu^\varepsilon \\ \boldsymbol{b} \end{pmatrix}, \tag{2.3}
$$

and

$$
\begin{pmatrix} \boldsymbol{\gamma}' \\ \boldsymbol{t}' \\ (\boldsymbol{l}_b^\varepsilon)' \\ \boldsymbol{\nu}' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \mathfrak{h}_b^\varepsilon & 0 & \varepsilon\kappa_b^h & \kappa_\nu^h \\ 0 & \mathfrak{h}_b^\varepsilon & 0 & -\varepsilon\kappa_t^h \\ -\varepsilon\kappa_t^h & -\kappa_\nu^h & \varepsilon\kappa_t^h & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\gamma} \\ \boldsymbol{t} \\ \boldsymbol{l}_b^\varepsilon \\ \boldsymbol{\nu} \end{pmatrix}, \tag{2.4}
$$

where

$$
\begin{array}{rcl} \kappa_\nu^h & = & \langle \boldsymbol{\gamma}'', \boldsymbol{\nu} \rangle, \\ \kappa_b^h & = & -\det(\boldsymbol{\gamma}, \boldsymbol{\gamma}', \boldsymbol{\gamma}'', \boldsymbol{\nu}), \\ \kappa_t^h & = & \det(\boldsymbol{\gamma}, \boldsymbol{\gamma}', \boldsymbol{\nu}, \boldsymbol{\nu}'), \\ \mathfrak{h}_\nu^\varepsilon & = & 1 - \varepsilon\kappa_\nu^h, \\ \mathfrak{h}_b^\varepsilon & = & 1 - \varepsilon\kappa_b^h. \end{array} \tag{2.5}
$$

Here, we call κ_ν^h the *normal curvature*, κ_b^h the *geodesic curvature*, κ_t^h the *cuspidal torsion*, $\mathfrak{h}_\nu^\varepsilon$ the *horospherical normal curvature*, $\mathfrak{h}_b^\varepsilon$ the *horospherical geodesic curvature* of the cuspidal edge respectively. Since $\boldsymbol{b} = \boldsymbol{\gamma} \wedge \boldsymbol{t} \wedge \boldsymbol{\nu}$, the horospherical geodesic curvature corresponds to the singular curvature [34].

We denote $I = U \cap S(f)$ and introduce the following functions on $H^3 \times I$:

$$
\begin{array}{rcl} H_{l\nu}^\varepsilon(\boldsymbol{x}, u) & = & \langle \boldsymbol{x}, \boldsymbol{l}_\nu^\varepsilon(u) \rangle + 1, \\ H_{lb}^\varepsilon(\boldsymbol{x}, u) & = & \langle \boldsymbol{x}, \boldsymbol{l}_b^\varepsilon(u) \rangle + 1. \end{array} \tag{2.6}
$$

One can also consider $H_\nu(\boldsymbol{x}, u) = \langle \boldsymbol{x}, \boldsymbol{\nu}(u) \rangle$ and $H_b(\boldsymbol{x}, u) = \langle \boldsymbol{x}, \boldsymbol{b}(u) \rangle$. Considering these functions is analogous notion in the Euclidean space [23]. See Appendix A for these cases.

We can take $\boldsymbol{x}$ as a parameter and regard these functions as parameter families of functions of u, then we can look at their corresponding discriminant set.

Let $g : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ be a function. For a manifold N and $p \in N$, a function

$$
G : (N \times \mathbb{R}, (p, 0)) \to (\mathbb{R}, 0)
$$

is called an *unfolding* of g if $G(p, u) = g(u)$ holds. In this setting, we regard G as a parameter family of a function g. We assume that $g'(0) = 0$ and define the set Σ_G and the *discriminant set* $\mathcal{D}_G$ of G as

$$
\Sigma_G = \{(q, u) \in N \times \mathbb{R} \mid G(q, u) = G_u(q, u) = 0\},
$$

$$
\mathcal{D}_G = \{q \in N \mid \text{ there exists } u \in \mathbb{R} \text{ such that } G(q, u) = G_u(q, u) = 0\}.
$$

If the map (G, G_u) is submersion at $(p, 0)$, then Σ_G is a manifold. By definition, the discriminant set is the envelope of the family $\{q \in N | G(q, u) = 0\}_{u \in \mathbb{R}}$ (see [3, Section 7] or [21, Section 5] for the general theory of unfoldings and their discriminant sets).

Now apply $(N, p) = (H^3, p)$ for $p \in H^3$ and $G = H_{l\nu}^\varepsilon, H_{lb}^\varepsilon$. Since $\boldsymbol{l}_\nu^\varepsilon$ and $\boldsymbol{l}_b^\varepsilon$ are lightlike, the discriminant sets $\mathcal{D}_{H_{l\nu}^\varepsilon}$ and $\mathcal{D}_{H_{lb}^\varepsilon}$ are the envelopes of families of horospheres. For a fixed

u, $\{\boldsymbol{x} \in H^3(-1) \mid H^{\varepsilon}_{l\nu}(\boldsymbol{x}, u) = 0\}$ are two horospheres tangent to the cuspidal edge at $\boldsymbol{\gamma}(u)$ and $\{\boldsymbol{x} \in H^3(-1) \mid H^{\varepsilon}_{lb}(\boldsymbol{x}, u) = 0\}$ are two horospheres normal to the cuspidal edge at $\boldsymbol{\gamma}(u)$, respectively. We investigate these functions and discriminant sets in Sections 4 and 5.

In what follows, we shall use the following abbreviation:

$$\kappa_\nu = \kappa^h_\nu, \quad \kappa_b = \kappa^h_b, \quad \kappa_t = \kappa^h_t.$$

3. Horocyclic surfaces

In this section, we give a quick review of general treatment of horocyclic surfaces. See [22] for detail. Let $\boldsymbol{g} : I \to H^3(-1)$ be a regular curve. Since $H^3(-1)$ is a Riemannian manifold, we can reparametrize $\boldsymbol{g}$ by the arc-length. Hence, we may assume that $\boldsymbol{g}(s)$ is a unit speed curve. Then the hyperbolic curvature κ_h and the hyperbolic torsion τ_h is defined by $\kappa_h(s) = |\boldsymbol{g}''(s) - \boldsymbol{g}(s)|$ and

$$\tau_h(s) = -\frac{\det(\boldsymbol{g}(s), \boldsymbol{g}'(s), \boldsymbol{g}''(s), \boldsymbol{g}'''(s))}{(\kappa_h(s))^2},$$

where $|\boldsymbol{v}| = \sqrt{|\langle \boldsymbol{v}, \boldsymbol{v}\rangle|}$ for $\boldsymbol{v} \in \mathbb{R}^4_1$. It can be shown that the curve $\boldsymbol{g}(s)$ satisfies the condition $\kappa_h(s) \equiv 0$ if and only if there exists a lightlike vector $\boldsymbol{c}$ such that $\boldsymbol{g}(s) - \boldsymbol{c}$ is a geodesic, where $\equiv$ stands for the equality holds identically. Such a curve is called an *equidistant curve*. Moreover $\boldsymbol{g}$ is called a *horocycle* if $\kappa_h(s) \equiv 1$ and $\tau_h(s) \equiv 0$. Let $\{\boldsymbol{\gamma}, \boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3\}$ be a pseudo-orthonormal basis of $\mathbb{R}^4_1$ which satisfies $\langle \boldsymbol{\gamma}, \boldsymbol{\gamma}\rangle = -1$ and $\langle \boldsymbol{a}_i, \boldsymbol{a}_i\rangle = 1$ $(i = 1, 2, 3)$. Setting

$$\boldsymbol{g}(s) = \boldsymbol{\gamma} + s\boldsymbol{a}_1 + \frac{s^2}{2}(\boldsymbol{\gamma} + \boldsymbol{a}_2),$$

we see that $\kappa_h(s) \equiv 1$ and $\tau_h(s) \equiv 0$. Thus $s \mapsto \boldsymbol{g}(s)$ is a horocycle. Furthermore, let $\{\boldsymbol{\gamma}(u), \boldsymbol{a}_1(u), \boldsymbol{a}_2(u), \boldsymbol{a}_3(u)\}$ be a pseudo-orthonormal frame on an open interval I which satisfies $\langle \boldsymbol{\gamma}(u), \boldsymbol{\gamma}(u)\rangle = -1$ and $\langle \boldsymbol{a}_i(u), \boldsymbol{a}_i(u)\rangle = 1$ $(i = 1, 2, 3)$. Then the surface

$$F : (u, s) \mapsto \boldsymbol{\gamma}(u) + s\boldsymbol{a}_1(u) + \frac{s^2}{2}(\boldsymbol{\gamma}(u) + \boldsymbol{a}_2(u)) \tag{3.1}$$

is a one-parameter family of horocycles, namely, a horocyclic surface. We define fundamental invariants of horocyclic surfaces. Since a horocyclic surface (3.1) is determined by the frame $\{\boldsymbol{\gamma}(u), \boldsymbol{a}_1(u), \boldsymbol{a}_2(u), \boldsymbol{a}_3(u)\}$, the six functions $c_1(u), \dots, c_6(u)$ is defined by the following Frenet-Serre type equations:

$$\begin{pmatrix} \boldsymbol{\gamma}'(u) \\ \boldsymbol{a}_1'(u) \\ \boldsymbol{a}_2'(u) \\ \boldsymbol{a}_3'(u) \end{pmatrix} = \begin{pmatrix} 0 & c_1(u) & c_2(u) & c_3(u) \\ c_1(u) & 0 & c_4(u) & c_5(u) \\ c_2(u) & -c_4(u) & 0 & c_6(u) \\ c_3(u) & -c_5(u) & -c_6(u) & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\gamma}(u) \\ \boldsymbol{a}_1(u) \\ \boldsymbol{a}_2(u) \\ \boldsymbol{a}_3(u) \end{pmatrix}. \tag{3.2}$$

Let α be a function of u, and set $\bar{F}(u, s) = F(u, s - \alpha(u))$. Then the images $\bar{F}(\mathbb{R} \times I)$ and $F(\mathbb{R} \times I)$ coincide. We set $\bar{c}_1, \dots, \bar{c}_6$ be the invariants defined by (3.2) of $\bar{F}(u, s)$. Then we have the equation

$$
\left\{\begin{array}{lcl}
\bar{c}_1(u) & = & c_1(u)+\dfrac{\alpha(u)^2}{2}(c_4(u)-c_1(u))+\alpha(u)c_2(u)+\alpha'(u),\\
\bar{c}_2(u) & = & c_2(u)+\alpha(u)(c_4(u)-c_1(u)),\\
\bar{c}_3(u) & = & \left(1+\dfrac{\alpha(u)^2}{2}\right)c_3(u)+\alpha(u)c_5(u)+\dfrac{\alpha(u)^2}{2}c_6(u),\\
\bar{c}_4(u) & = & c_4(u)+\dfrac{\alpha(u)^2}{2}(c_4(u)-c_1(u))+\alpha(u)c_2(u)+\alpha'(u),\\
\bar{c}_5(u) & = & c_5(u)+\alpha(u)(c_3(u)+c_6(u)),\\
\bar{c}_6(u) & = & \left(1-\dfrac{\alpha(u)^2}{2}\right)c_6(u)-\alpha(u)c_5(u)-\dfrac{\alpha(u)^2}{2}c_3(u).
\end{array}\right. \tag{3.3}
$$

Then we see that $\bar{c}_1(u)-\bar{c}_4(u)=c_1(u)-c_4(u)$ and

$$
\bar{c}_1(u)-\bar{c}_4(u)=\bar{c}_2(u)=0 \text{ if and only if } c_1(u)-c_4(u)=c_2(u)=0. \tag{3.4}
$$

Furthermore, the following proposition holds (see [22, Proposition 5.3]).

Proposition 3.1. *The horocyclic surface F is horo-flat if and only if $c_1(u)-c_4(u)=c_2(u)=0$ for any $u\in I$.*

4. Osculating horo-flat surfaces

In this section, we construct a parametrization of the discriminant set of $H^\varepsilon_{l\nu}$.

Let $f:(U,p)\to H^3$ be a cuspidal edge. As in Section 2, we assume $I=S(f)\cap U=\{(u,0)\}\cap U$ and set $\boldsymbol{\gamma}(u)=f(u,0)$. Then we have vector fields along $\boldsymbol{\gamma}$ as in (2.1). We consider invariants defined in (2.5). We assume $(\kappa_t,\mathfrak{h}^\varepsilon_\nu)(u)\neq(0,0)$ for any $u\in I$ unless otherwise stated.

4.1. **The discriminant set of $H^\varepsilon_{l\nu}$.** By differentiating (2.3), we have

$$
\begin{array}{rcl}
(\boldsymbol{l}^\varepsilon_\nu)' & = & \mathfrak{h}^\varepsilon_\nu\boldsymbol{t}+\varepsilon\kappa_t\boldsymbol{b},\\
(\boldsymbol{l}^\varepsilon_\nu)'' & = & \mathfrak{h}^\varepsilon_\nu\boldsymbol{\gamma}+\varepsilon(-\kappa_b\kappa_t-\kappa'_\nu)\boldsymbol{t}+(\kappa_\nu-\varepsilon\kappa_\nu^2+\varepsilon\kappa_t^2)\boldsymbol{\nu}+(\kappa_b-\varepsilon\kappa_\nu\kappa_b+\varepsilon\kappa'_t)\boldsymbol{b},\\
(\boldsymbol{l}^\varepsilon_\nu)''' & = & \varepsilon(-\kappa_b\kappa_t-2\kappa'_\nu)\boldsymbol{\gamma}\\
& & +\big(1-\kappa_b^2-\varepsilon\kappa_t\kappa'_b-2\varepsilon\kappa_b\kappa'_t-\varepsilon\kappa''_\nu+\varepsilon(-1+\kappa_b^2+\kappa_t^2)\kappa_\nu-\kappa_\nu^2+\varepsilon\kappa_\nu^3\big)\boldsymbol{t}\\
& & +(-\kappa_b\kappa_t+\kappa'_\nu-3\varepsilon\kappa_t\kappa'_t-3\varepsilon\kappa'_\nu\kappa_\nu)\boldsymbol{\nu}\\
& & +\big(-\varepsilon\kappa_b^2\kappa_t-\varepsilon\kappa_t^3-2\varepsilon\kappa_b\kappa'_\nu+\kappa'_b+\varepsilon\kappa''_t+(\kappa_t-\varepsilon\kappa'_b)\kappa_\nu-\varepsilon\kappa_t\kappa_\nu^2\big)\boldsymbol{b}.
\end{array} \tag{4.1}
$$

Since $\{\boldsymbol{\gamma},\boldsymbol{t},\boldsymbol{\nu},\boldsymbol{b}\}$ is a basis of $\mathbb{R}^4_1$, we can set $\boldsymbol{x}=x_\gamma\boldsymbol{\gamma}+x_t\boldsymbol{t}+x_\nu\boldsymbol{\nu}+x_b\boldsymbol{b}$. Then $H^\varepsilon_{l\nu}(\boldsymbol{x},u)=0$ if and only if $x_\gamma=\varepsilon x_\nu+1$. Moreover, $H^\varepsilon_{l\nu}(\boldsymbol{x},u)=(H^\varepsilon_{l\nu})_u(\boldsymbol{x},u)=0$ if and only if the equalities

$$
x_\gamma=\varepsilon x_\nu+1,\quad x_t=-\varepsilon\kappa_t s,\quad x_b=\mathfrak{h}^\varepsilon_\nu s
$$

hold for some $s\in\mathbb{R}$, under the assumption $(\kappa_t,\mathfrak{h}^\varepsilon_\nu)\neq(0,0)$. Since $\boldsymbol{x}\in H^3$, we have that $x_\nu=\varepsilon s^2(\kappa_t^2+(\mathfrak{h}^\varepsilon_\nu)^2)/2$. Thus $H^\varepsilon_{l\nu}(\boldsymbol{x},u)=(H^\varepsilon_{l\nu})_u(\boldsymbol{x},u)=0$ if and only if

$$
\boldsymbol{x}=\left(\frac{s^2}{2}(\kappa_t^2+(\mathfrak{h}^\varepsilon_\nu)^2)+1\right)\boldsymbol{\gamma}-\varepsilon\kappa_t s\boldsymbol{t}+\frac{\varepsilon s^2}{2}(\kappa_t^2+(\mathfrak{h}^\varepsilon_\nu)^2)\boldsymbol{\nu}+\mathfrak{h}^\varepsilon_\nu s\boldsymbol{b}
$$

for some $s\in\mathbb{R}$. Thus $\mathcal{D}_{H^\varepsilon_{l\nu}}$ is parameterized by

$$
(u,s)\mapsto\boldsymbol{x}=\boldsymbol{\gamma}+\Big(-\varepsilon\kappa_t\boldsymbol{t}+\mathfrak{h}^\varepsilon_\nu\boldsymbol{b}\Big)s+\frac{s^2}{2}\Big(\kappa_t^2+(\mathfrak{h}^\varepsilon_\nu)^2\Big)\boldsymbol{l}^\varepsilon_\nu.
$$

We set

$$
\overline{D^\varepsilon_l}=\frac{-\varepsilon\kappa_t\boldsymbol{t}+\mathfrak{h}^\varepsilon_\nu\boldsymbol{b}}{\sqrt{\kappa_t^2+(\mathfrak{h}^\varepsilon_\nu)^2}},
$$

and call it the *normalized* $\boldsymbol{b}$*-Darboux vector field*. By applying a parameter change

$$\tilde{s} = s\sqrt{\kappa_t^2 + (\mathfrak{h}_\nu^\varepsilon)^2},$$

we obtain the following parameterization of $\mathcal{D}_{H^\varepsilon_{l\nu}}$

$$(u,s) \mapsto \boldsymbol{x} = \boldsymbol{\gamma} + \overline{D^\varepsilon_l} s + \frac{s^2}{2}\boldsymbol{l}^\varepsilon_\nu.$$

Since $|\overline{D^\varepsilon_l}| = 1$, for a fixed u, $s \mapsto \boldsymbol{\gamma} + \overline{D^\varepsilon_l}s + s^2\boldsymbol{l}^\varepsilon_\nu/2$ is a horocycle, see §3. We also see that $\{\boldsymbol{\gamma}, \overline{D^\varepsilon_l}, \varepsilon\boldsymbol{\nu}, (\boldsymbol{l}^\varepsilon_\nu)'/|(\boldsymbol{l}^\varepsilon_\nu)'|\}$ is a pseudo-orthonormal frame of $\mathbb{R}^4_1$. Following §3, we set

$$\{\boldsymbol{\gamma}, \boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3\} = \{\boldsymbol{\gamma}, \overline{D^\varepsilon_l}, \varepsilon\boldsymbol{\nu}, (\boldsymbol{l}^\varepsilon_\nu)'/|(\boldsymbol{l}^\varepsilon_\nu)'|\},$$

and

$$F_{l\nu}(u,s) = F^\varepsilon_{l\nu}(u,s) = \boldsymbol{\gamma} + \boldsymbol{a}_1 s + \frac{s^2}{2}(\boldsymbol{\gamma} + \boldsymbol{a}_2). \tag{4.2}$$

By definition, $F_{l\nu}$ is a Δ_2-dual of $\boldsymbol{l}^\varepsilon_\nu$. An example of the osculating horo-flat surface $F_{l\nu}$ of

$$f(u,v) = \Big(f_1(u,v), f_2(u,v), f_3(u,v), \sqrt{f_1(u,v)^2 + f_2(u,v)^2 + f_3(u,v)^2 - 1}\Big), \tag{4.3}$$

where $f_1(u,v) = 3 + u,\ f_2(u,v) = u^2/2 + v^2/2,\ f_3(u,v) = u^2/2 + uv^2/2 + v^3/2$ near $(0,0)$ is provided by Figure 1. We can now define invariants $c_{\nu,1}, \dots, c_{\nu,6}$ as in (3.2), namely,

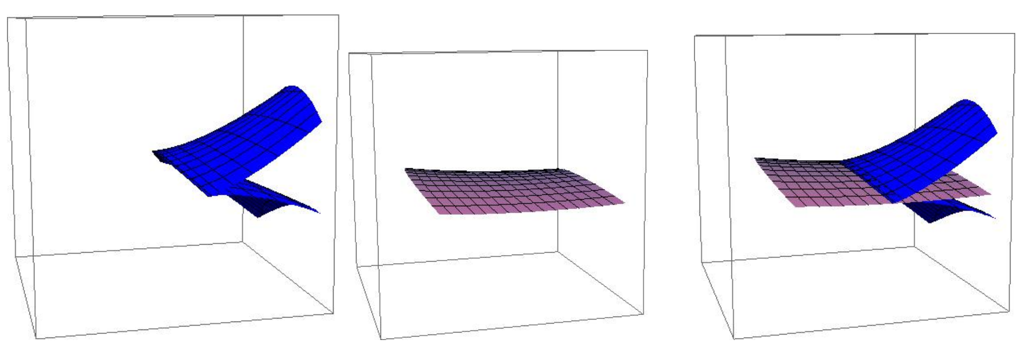

FIGURE 1. Cuspidal edge, $F_{l\nu}$ and the both surfaces together

$$\begin{pmatrix} \boldsymbol{\gamma}' \\ \boldsymbol{a}_1' \\ \boldsymbol{a}_2' \\ \boldsymbol{a}_3' \end{pmatrix} = \begin{pmatrix} 0 & c_{\nu,1} & c_{\nu,2} & c_{\nu,3} \\ c_{\nu,1} & 0 & c_{\nu,4} & c_{\nu,5} \\ c_{\nu,2} & -c_{\nu,4} & 0 & c_{\nu,6} \\ c_{\nu,3} & -c_{\nu,5} & -c_{\nu,6} & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\gamma} \\ \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \\ \boldsymbol{a}_3 \end{pmatrix}. \tag{4.4}$$

It is not difficult to see that

$$
\begin{aligned}
c_{\nu,1} &= \frac{-\varepsilon\kappa_t}{\sqrt{\kappa_t^2+(\mathfrak{h}_\nu^\varepsilon)^2}}, \\
c_{\nu,2} &= 0, \\
c_{\nu,3} &= \frac{\mathfrak{h}_\nu^\varepsilon}{\sqrt{\kappa_t^2+(\mathfrak{h}_\nu^\varepsilon)^2}}, \\
c_{\nu,4} &= c_{\nu,1} = \frac{-\varepsilon\kappa_t}{\sqrt{\kappa_t^2+(\mathfrak{h}_\nu^\varepsilon)^2}}, \\
c_{\nu,5} &= \frac{\delta_o^h}{\kappa_t^2+(\mathfrak{h}_\nu^\varepsilon)^2}, \\
c_{\nu,6} &= \frac{-\varepsilon\kappa_\nu+\kappa_\nu^2+\kappa_t^2}{\sqrt{\kappa_t^2+(\mathfrak{h}_\nu^\varepsilon)^2}}, \\
c_{\nu,3}+c_{\nu,6} &= \sqrt{\kappa_t^2+(\mathfrak{h}_\nu^\varepsilon)^2},
\end{aligned}
\tag{4.5}
$$

where we set

$$\delta_o^h = -\kappa_b\big((\mathfrak{h}_\nu^\varepsilon)^2+\kappa_t^2\big)+\varepsilon\kappa_t(\mathfrak{h}_\nu^\varepsilon)'-\varepsilon\mathfrak{h}_\nu^\varepsilon\kappa_t'.$$

By the condition $(\kappa_t,\mathfrak{h}_\nu^\varepsilon)\neq(0,0)$, we have $c_{\nu,3}+c_{\nu,6}\neq 0$. The invariant $c_{\nu,5}$ corresponds to the invariant δ of the Euclidean case (see [18, 23]). Note that if $(\kappa_t,\mathfrak{h}_\nu^\varepsilon)\equiv(0,0)$, then by (2.3), it holds that $(\boldsymbol{l}_\nu^\varepsilon)'\equiv 0$. This implies that $F_{l\nu}$ is a horosphere.

By (4.4) and (4.5), we see

$$
\begin{aligned}
F_{l\nu}' &= \boldsymbol{\gamma}'+\boldsymbol{a}_1's+\frac{s^2}{2}(\boldsymbol{\gamma}'+\boldsymbol{a}_2') \\
&= c_{\nu,1}s\boldsymbol{\gamma}+c_{\nu,1}\boldsymbol{a}_1+c_{\nu,1}s\boldsymbol{a}_2+\left(c_{\nu,3}+c_{\nu,5}s+\frac{s^2}{2}(c_{\nu,3}+c_{\nu,6})\right)\boldsymbol{a}_3
\end{aligned}
\tag{4.6}
$$

$$(F_{l\nu})_s = s\boldsymbol{\gamma}+\boldsymbol{a}_1+s\boldsymbol{a}_2, \tag{4.7}$$

where $' = \partial/\partial t$. We set

$$\lambda = (c_{\nu,3}+c_{\nu,6})s^2+2c_{\nu,5}s+2c_{\nu,3} \tag{4.8}$$

and

$$\eta = \partial u - c_{\nu,1}\partial s, \tag{4.9}$$

then we see $S(F_{l\nu})=\{(u,s)\in I\times\mathbb{R}\,|\,\lambda(u,s)=0\}$ by (4.6) and (4.7). We also see $\ker dF_{l\nu}=\langle\eta\rangle_{\mathbb{R}}$ on $S(F_{l\nu})$ holds. By (4.6), (4.7) and (4.5),

$$\boldsymbol{\nu}_l = \boldsymbol{a}_2 - s\boldsymbol{a}_1 - \frac{s^2}{2}(\gamma+\boldsymbol{a}_2)\in S_1^3$$

is a Δ_1-dual of $F_{l\nu}$, and $F_{l\nu}+\boldsymbol{\nu}_l=\boldsymbol{\gamma}+\boldsymbol{a}_2$ is a Δ_2-dual of $F_{l\nu}$. Since the Δ_2-dual of $F_{l\nu}$ degenerates to a curve, $F_{l\nu}$ is a horo-flat surface. On the other hand, since $c_{\nu,1}-c_{\nu,4}\equiv c_{\nu,2}\equiv 0$, we also see that $F_{l\nu}$ is a horo-flat surface by Proposition 3.1. It follows that each of $F_{l\nu}$ is a horo-flat surface tangent to the cuspidal edge at any $\boldsymbol{\gamma}(u)$, so that we call it an *osculating horo-flat surface* (*along the cuspidal edge*).

4.2. **Singularities of osculating horo-flat surface.** We consider singularities of osculating horo-flat surface $F_{l\nu}$. A singular point p of the map-germ $f:(U,p)\to(\mathbb{R}^3,0)$ is a *swallowtail* if f is $\mathcal{A}$-equivalent to $(u,v)\mapsto(u,4v^3+2uv,3v^4+uv^2)$ at 0. A singular point p of f is a *cuspidal lip* (respectively, a *cuspidal beak*) if f is $\mathcal{A}$-equivalent to $(u,v)\mapsto(u,2v^3+\sigma u^2v,3v^4+\sigma u^2v^2)$ at 0 with $\sigma=+1$ (respectively, $\sigma=-1$). A singular point p of f is a *cuspidal cross cap* if f is $\mathcal{A}$-equivalent to $(u,v)\mapsto(u,v^2,uv^3)$ at 0.

Since $\boldsymbol{\nu}_l : U \to\in S^3_1$, the map $(F_{l\nu}, \boldsymbol{\nu}_l) : U \to \Delta_1$ is an immersion if and only if

$$\det(F'_{l\nu}, \nabla^F_\eta \boldsymbol{\nu}_l, \boldsymbol{\nu}_l, F_{l\nu})|_{S(F_{l\nu})} \neq 0, \tag{4.10}$$

where η is given by (4.9), and ∇^F_η be the canonical covariant derivative by η along F induced from the Levi-Civita connection on H^3. Since

$$\nabla^F_\eta \boldsymbol{\nu}_l = \alpha_0 \boldsymbol{\gamma} + \alpha_1 \boldsymbol{a}_1 + \alpha_2 \boldsymbol{a}_2 + (-c_{\nu,6} + s c_{\nu,5} + (s^2/2)(c_{\nu,3} + c_{\nu,6}))\boldsymbol{a}_3$$

($\alpha_0, \alpha_1, \alpha_2$ are some functions), the left hand side of (4.10) is $c_{\nu,1}(c_{\nu,3} + c_{\nu,6} - c_{\nu,1}\lambda)$. Thus by the assumption $c_{\nu,3} + c_{\nu,6} \neq 0$, the condition (4.10) is equivalent to $c_{\nu,1}(u) \neq 0$. Let Q be the discriminant of $\lambda = (c_{\nu,3} + c_{\nu,6})s^2 + 2c_{\nu,5}s + 2c_{\nu,3}$ (in (4.8)) regarding a quadratic equation of s:

$$Q(u) = c_{\nu,5}(u)^2 - 2c_{\nu,3}(u)(c_{\nu,3}(u) + c_{\nu,6}(u)) = c_{\nu,5}(u)^2 - 2\mathfrak{h}^\varepsilon_\nu(u).$$

If $Q < 0$, then there is no singular point. If $Q(u_0) = 0$, we set $s_0 = -c_{\nu,5}(u_0)/(c_{\nu,3}(u_0)+c_{\nu,6}(u_0))$. Then (u_0, s_0) is a singular point of $F_{l\nu}$.

Proposition 4.1. *Under the above notation, we have the following.*

(I) *If $Q(u_0) = 0$, the singular point (u_0, s_0) of $F_{l\nu}$ is a cuspidal edge if and only if*

$$c_{\nu,1}((c'_{\nu,3} + c'_{\nu,6})s^2 + 2c'_{\nu,5}s + 2c'_{\nu,3}) \neq 0$$

at u_0. Moreover, there are no swallowtails. The singular point (u_0, s_0) is a cuspidal lip if and only if $c_{\nu,1} \neq 0$, $(c'_{\nu,3} + c'_{\nu,6})s^2 + 2c'_{\nu,5}s + 2c'_{\nu,3} = 0$, and

$$\det \begin{pmatrix} (c''_{\nu,3} + c''_{\nu,6})s^2 + 2c''_{\nu,5}s + 2c''_{\nu,3} & 2(c'_{\nu,3} + c'_{\nu,6})s + 2c'_{\nu,5} \\ 2(c'_{\nu,3} + c'_{\nu,6})s + 2c'_{\nu,5} & 2(c_{\nu,3} + c_{\nu,6}) \end{pmatrix} > 0 \tag{4.11}$$

at (u_0, s_0). The singular point (u_0, s_0) is a cuspidal beak if and only if $c_{\nu,1} \neq 0$,

$$(c'_{\nu,3} + c'_{\nu,6})s^2 + 2c'_{\nu,5}s + 2c'_{\nu,3} = 0,$$

the left hand side of the determinant (4.11) is negative, and

$$\begin{aligned} s^2(c''_{\nu,3} + c''_{\nu,6}) + 2sc''_{\nu,5} + c''_{\nu,3} - 2c'_{\nu,1}(s(c_{\nu,3} + c_{\nu,6}) + c_{\nu,5}) \\ - 4c_{\nu,1}(s(c'_{\nu,3} + c'_{\nu,6}) + c'_{\nu,5}) + 2c^2_{\nu,1}(c_{\nu,3} + c_{\nu,6}) \neq 0 \end{aligned} \tag{4.12}$$

at (u_0, s_0). The singular point (u_0, s_0) is a cuspidal cross cap if and only if $c_{\nu,1} = 0$ and $c'_{\nu,1}((c'_{\nu,3} + c'_{\nu,6})s^2 + 2c'_{\nu,5}s + 2c'_{\nu,3}) \neq 0$ at u_0.

(II) *If $Q(u) > 0$, let s be the solution of $\lambda = 0$, namely,*

$$s = \frac{-c_{\nu,5} \pm \sqrt{c^2_{\nu,5} - 2c_{\nu,3}(c_{\nu,3} + c_{\nu,6})}}{c_{\nu,3} + c_{\nu,6}}. \tag{4.13}$$

Then (u, s) is a singular point. The singular point is a cuspidal edge if and only if $c_{\nu,1} \neq 0$ and

$$(c'_{\nu,3} + c'_{\nu,6})s^2 + 2c'_{\nu,5}s + 2c'_{\nu,3} - 2c_{\nu,1}((c_{\nu,3} + c_{\nu,6})s + c_{\nu,5}) \neq 0 \tag{4.14}$$

at (u, s). The singular point is a swallowtail if and only if $c_{\nu,1} \neq 0$ and the left hand side of (4.14) vanishes, $(c_{\nu,3} + c_{\nu,6})s + c_{\nu,5} \neq 0$, and (4.12) holds at (u, s). Moreover, there are no cuspidal lips and cuspidal beaks. The singular point (u, s) is a cuspidal cross cap if and only if $c_{\nu,1} = 0$ and $c'_{\nu,1}((c'_{\nu,3} + c'_{\nu,6})s^2 + 2c'_{\nu,5}s + 2c'_{\nu,3}) \neq 0$ at (u, s).

There are criteria for these singularities of horo-flat surfaces in [22, Theorem 6.2]. However, since the condition $c_{\nu,3} \equiv 0$ is assumed in [22, Theorem 6.2], we give a proof.

Proof. Since (4.10) is equivalent to $c_{\nu,1}(u) \neq 0$, $F_{l\nu}$ is a front at a singular point if and only if $c_{\nu,1} \neq 0$ when $Q \geq 0$. We show the proposition by using Proposition B.1. By (4.6) and (4.7), λ in (4.8) is an identifier of singularities which is defined just before Proposition B.1. If $Q(u_0) = 0$, then $\lambda_s(u_0, s_0) = 0$. Thus $\eta\lambda(u_0, s_0) \neq 0$ if and only if $\lambda_u(u_0, s_0) \neq 0$. This proves the assertion for a cuspidal edge. Furthermore, since $\eta\lambda(u_0, s_0) = 0$ implies $(\lambda_u, \lambda_s)(u_0, s_0) = (0, 0)$, this proves the assertion for a swallowtail. When $(\lambda_u, \lambda_s)(u_0, s_0) = (0, 0)$, calculating the Hesse matrix of λ and $\eta\eta\lambda$, we have the assertion of the case of $Q(u_0) = 0$ by (3) of Proposition B.1. If $Q(u) > 0$, by Proposition B.1 with the data $\lambda = (c_{\nu,3}+c_{\nu,6})s^2+2c_{\nu,5}s+2c_{\nu,3}$ and $\eta = \partial u - c_{\nu,1}\partial s$, we can show the assertion. □

By (4.8), if $c_{\nu,3} \equiv 0$, then $(u, 0)$ is a singular point of $F_{l\nu}$. This means that all generating horocycles are tangent to $F_{l\nu}|_{S(F_{l\nu})}$ at all the regular points of this curve. Thus $F_{l\nu}$ is said to be *horo-flat tangent* if $c_{\nu,2} \equiv c_{\nu,3} \equiv c_{\nu,1} - c_{\nu,4} \equiv 0$ holds (see [22, Section 5] for detail). See also Section 6.3. If $F_{l\nu}$ is horo-flat tangent, then we have the following corollary. In this case, since $c_{\nu,3} + c_{\nu,6} \neq 0$, it holds that $c_{\nu,6} \neq 0$, and $S(F_{l\nu}) = \{s(c_{\nu,6}s + 2c_{\nu,5}) = 0\}$.

Corollary 4.2. *Under the assumptions $c_{\nu,2} \equiv c_{\nu,3} \equiv c_{\nu,1} - c_{\nu,4} \equiv 0$ and $c_{\nu,6} \neq 0$ on the singularities of $F_{l\nu}$, the map $F_{l\nu}$ is a front, and the following assertions hold:*
(I) *If $c_{\nu,5}(u_0) = 0$, then $Q(u_0) = 0$ and $d\lambda(u_0, 0) = 0$ hold, in particular there are no cuspidal edge and swallowtail. The singular point $(u_0, 0)$ is a cuspidal beak if and only if*

$$c'_{\nu,5}(-2c'_{\nu,5} + c_{\nu,1}c_{\nu,6}) \neq 0$$

at u_0, Moreover, there are no cuspidal lips.
(II) *If $c_{\nu,5}(u) \neq 0$, then $Q(u_0) > 0$ and*

(1) *$d\lambda \neq 0$ at both $(u, 0)$ and $(u, -c_{\nu,5}/c_{\nu,6})$.*
(2) *A singular point $(u, 0)$ is a cuspidal edge. A singular point $(u, -c_{\nu,5}/c_{\nu,6})$ is a cuspidal edge if and only if $c_{\nu,5}c'_{\nu,6} - 2c'_{\nu,5}c_{\nu,6} \neq 0$ at u.*
(3) *A singular point $(u, 0)$ is not a swallowtail. A singular point $(u, -c_{\nu,5}/c_{\nu,6})$ is a swallowtail if and only if $c_{\nu,5}c'_{\nu,6}-2c'_{\nu,5}c_{\nu,6} = 0$ and a formula (4.12) with $c_{\nu,3} \equiv 0$, $s = -c_{\nu,5}/c_{\nu,6}$ holds at u.*

Proof. Since $c_{\nu,3} \equiv 0$, we have $\mathfrak{h}^\varepsilon_\nu \equiv 0$ by (4.5). By the assumption $c_{\nu,6} \neq 0$, it holds that $\kappa_t \neq 0$. Again by (4.5), we get $c_{\nu,1} \neq 0$. By (4.10), this condition is equivalent to that $F_{l\nu}$ is a front, we have the first assertion. One can easily show the other assertions by applying Proposition 4.1. □

5. Normal horo-flat surfaces

In this section, we construct a parametrization of the discriminant set of H^ε_{lb}.

Let $f : (U, p) \to H^3$ be a cuspidal edge. Under the same notation as in Section 4, we assume $(\kappa_t, \mathfrak{h}^\varepsilon_b)(u) \neq (0, 0)$ for any $u \in I$ unless otherwise stated.

By using similar arguments to those of Section 4, we obtain the following. Since

$$(\boldsymbol{l}^\varepsilon_b)' = \mathfrak{h}^\varepsilon_b \boldsymbol{t} - \varepsilon\kappa_t \boldsymbol{\nu},$$

we have that $H_{lb}(\boldsymbol{x}, u) = (H_{lb})_u(\boldsymbol{x}, u) = 0$ if and only if

$$x_\gamma = \varepsilon x_b + 1, \quad x_t = \varepsilon\kappa_t s, \quad x_\nu = \mathfrak{h}^\varepsilon_b s, \quad \text{for some } s \in \mathbb{R},$$

where $\boldsymbol{x} = x_\gamma \boldsymbol{\gamma} + x_t \boldsymbol{t} + x_\nu \boldsymbol{\nu} + x_b \boldsymbol{b}$.

Since $\boldsymbol{x} \in H^3$, it holds that $x_b = \varepsilon \dfrac{s^2}{2}(\kappa_t^2 + (\mathfrak{h}_b^\varepsilon)^2)$ and thus

$$x_\gamma = \frac{s^2}{2}(\kappa_t^2 + (\mathfrak{h}_b^\varepsilon)^2) + 1, \quad x_t = \varepsilon\kappa_t s, \quad x_\nu = \mathfrak{h}_b^\varepsilon s, \quad x_b = \frac{\varepsilon s^2}{2}(\kappa_t^2 + (\mathfrak{h}_b^\varepsilon)^2).$$

We set

$$\overline{D_{lb}^\varepsilon} = \frac{\varepsilon\kappa_t \boldsymbol{t} + \mathfrak{h}_b^\varepsilon \boldsymbol{\nu}}{\sqrt{\kappa_t^2 + (\mathfrak{h}_b^\varepsilon)^2}},$$

and call it the *normalized $\boldsymbol{\nu}$-Darboux vector field.* Now, by a parameter change,

$$\tilde{s} = s\sqrt{\kappa_t^2 + (\mathfrak{h}_b^\varepsilon)^2},$$

and rewriting $\tilde{s}$ as s, we obtain the following parameterization of $\mathcal{D}_{H_{lb}^\varepsilon}$

$$(u, s) \mapsto \boldsymbol{x} = \boldsymbol{\gamma} + \overline{D_{lb}^\varepsilon} s + \frac{s^2}{2}\boldsymbol{l}_b^\varepsilon.$$

As seen in the case of $H_{l\nu}$, since $|\overline{D_{lb}^\varepsilon}| = 1$, for a fixed u, $s \mapsto \boldsymbol{\gamma} + \overline{D_{lb}^\varepsilon} s + \dfrac{s^2}{2}\boldsymbol{l}_b^\varepsilon$ is a parabola and thus a horocycle ([22, Section 4]).

We have that $\{\boldsymbol{\gamma}, \overline{D_{lb}^\varepsilon}, \varepsilon\boldsymbol{b}, (\boldsymbol{l}_b^\varepsilon)'/|(\boldsymbol{l}_b^\varepsilon)'|\}$ is a pseudo-orthonormal frame. Analogously to Section 4, we set

$$\{\boldsymbol{\gamma}_b, \boldsymbol{a}_{b,1}, \boldsymbol{a}_{b,2}, \boldsymbol{a}_{b,3}\} = \{\boldsymbol{\gamma}, \overline{D_{lb}^\varepsilon}, \varepsilon\boldsymbol{b}, (\boldsymbol{l}_b^\varepsilon)'/|(\boldsymbol{l}_b^\varepsilon)'|\}$$

and

$$F_{lb}(u, s) = F_{lb}^\varepsilon(u, s) = \boldsymbol{\gamma}_b + \boldsymbol{a}_{b,1} s + \frac{s^2}{2}(\boldsymbol{\gamma}_b + \boldsymbol{a}_{b,2}). \tag{5.1}$$

By definition, F_{lb} is a Δ_2-dual of $\boldsymbol{l}_b^\varepsilon$. Similarly to the case of $H_{l\nu}^\varepsilon$, the invariants $c_{b,1}, \dots, c_{b,6}$ are defined by the relation

$$\begin{pmatrix} \boldsymbol{\gamma}_b' \\ \boldsymbol{a}_{b,1}' \\ \boldsymbol{a}_{b,2}' \\ \boldsymbol{a}_{b,3}' \end{pmatrix} = \begin{pmatrix} 0 & c_{b,1} & c_{b,2} & c_{b,3} \\ c_{b,1} & 0 & c_{b,4} & c_{b,5} \\ c_{b,2} & -c_{b,4} & 0 & c_{b,6} \\ c_{b,3} & -c_{b,5} & -c_{b,6} & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\gamma}_b \\ \boldsymbol{a}_{b,1} \\ \boldsymbol{a}_{b,2} \\ \boldsymbol{a}_{b,3} \end{pmatrix}. \tag{5.2}$$

Then we have

$$\begin{aligned} c_{b,1} &= \frac{\varepsilon\kappa_t}{\sqrt{\kappa_t^2 + (\mathfrak{h}_b^\varepsilon)^2}}, \\ c_{b,2} &= 0, \\ c_{b,3} &= \frac{\mathfrak{h}_b^\varepsilon}{\sqrt{\kappa_t^2 + (\mathfrak{h}_b^\varepsilon)^2}}, \\ c_{b,4} &= c_{b,1} = \frac{\varepsilon\kappa_t}{\sqrt{\kappa_t^2 + (\mathfrak{h}_b^\varepsilon)^2}}, \\ c_{b,5} &= \frac{\delta_n^h}{\kappa_t^2 + (\mathfrak{h}_b^\varepsilon)^2}, \\ c_{b,6} &= \frac{-\varepsilon\kappa_b + \kappa_b^2 + \kappa_t^2}{\sqrt{\kappa_t^2 + (\mathfrak{h}_b^\varepsilon)^2}} \\ c_{b,3} + c_{b,6} &= \sqrt{\kappa_t^2 + (\mathfrak{h}_b^\varepsilon)^2}, \end{aligned} \tag{5.3}$$

where we set

$$\delta_n^h = -\kappa_\nu((\mathfrak{h}_b^\varepsilon)^2 + \kappa_t^2) - \varepsilon\kappa_t(\mathfrak{h}_b^\varepsilon)' + \varepsilon\mathfrak{h}_b^\varepsilon\kappa_t'.$$

By (5.3), $\boldsymbol{\nu}_{lb} = -(s^2/2)\boldsymbol{\gamma}_b - s\boldsymbol{a}_{b,1} + (1 - s^2/2)\boldsymbol{a}_{b,2}$ is a Δ_1-dual of F_{lb}, and $F_{lb} + \boldsymbol{\nu}_{lb} = \boldsymbol{\gamma}_b + \boldsymbol{a}_{b,2}$ is a Δ_2-dual of F_{lb}. Since the Δ_2-dual of F_{lb} degenerates to a curve, F_{lb} is a horo-flat surface.

It follows that each of F_{lb} is a horo-flat surface normal to the cuspidal edge at any $\boldsymbol{\gamma}(u)$, so that we call it a *normal horo-flat surface* (*along the cuspidal edge*). An example of the normal horo-flat surface F_{lb} of f as in (4.3) near $(0,0)$ is provided by Figure 2. Similar calculations

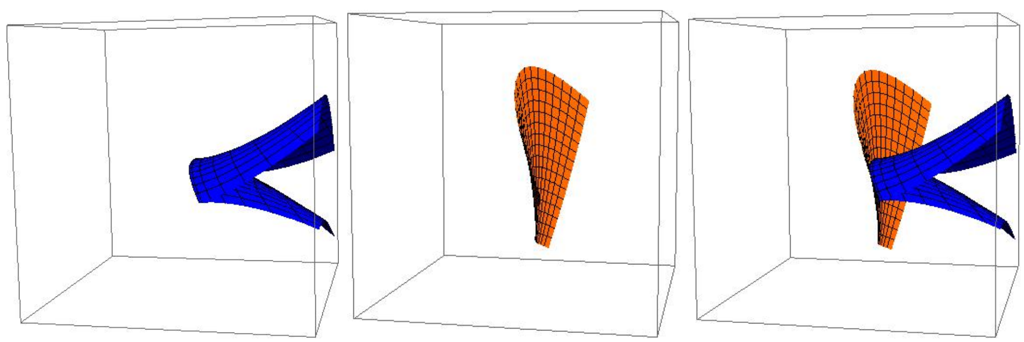

FIGURE 2. Cuspidal edge, F_{lb} and the both surfaces together

to those in Section 4, lead to the characterization of the singularities of F_{lb} (just substitute $c_{b,i}$ into $c_{\nu,i}$ $(i=1,\dots,6)$ in Proposition 4.1 and Corollary 4.2). By comparing (4.5) and (5.3), we see that changing κ_ν to κ_b and κ_t to $-\kappa_t$ in the formulae for $c_{\nu,i}$, leads to the formulae for $c_{b,i}$ $(i=1,\dots,6)$.

6. Special cuspidal edges

We consider a cuspidal edge f, where either $F_{l\nu}$ or F_{lb} has special properties. The special horo-flat surfaces which are one-parameter families of horocycles (horo-flat horocyclic surfaces) are classified in [22, pp815–818]. We consider here the cases of the horo-cylinder and the horocone. We review the special horo-flat surfaces given in [22].

Definition 6.1. A horocyclic surface with the invariants $c_1,\dots,c_6$ is called a *regular horocylindrical surface* if $c_1\equiv c_2\equiv c_4\equiv c_5\equiv 0$, and $c_3(c_3+c_6)>0$. A horocyclic surface is called a *secondary regular horocylindrical surface* if $c_1\equiv c_2\equiv c_4\equiv c_6\equiv 0$, and $c_5^2-2c_3^2<0$.

Definition 6.2. A horocyclic surface with the invariants $c_1,\dots,c_6$ is called a *generalized horocone* if $c_1\equiv c_2\equiv c_3\equiv c_4\equiv 0$. A generalized horocone is called a *horocone with a single vertex* if $c_5\equiv 0$ and there is no subinterval $J\subset I$ such that $c_6|_J=0$. A *horocone with two vertices* is a generalized horocone with the property that there is no subinterval $J\subset I$ such that $c_5|_J=0$, and there exists $\lambda\in\mathbb{R}$ such that $c_6=\lambda c_5$. A generalized horocone is called a *semi-horocone* if the following holds for $(i,j)=(5,6)$ or $(i,j)=(6,5)$: There is no subinterval $J\subset I$ such that $c_i|_J=0$ and c_j/c_i is not constant on $\{t\in I\,|\,c_i(u)\neq 0\}$. If the condition $c_1\equiv c_2\equiv c_3\equiv c_4\equiv c_6\equiv 0$ holds and there is no subinterval $J\subset I$ such that $c_5|_J=0$, then the image of the horocyclic surface is a horosphere. We call this a *conical horosphere.*

Let α be a function. By (4.5) and substituting $c_{\nu,1} - c_{\nu,4} \equiv 0, c_{\nu,2} \equiv 0$ in (3.3), we get

$$(6.1)\qquad \begin{cases} \bar{c}_{\nu,1} &= \bar{c}_{\nu,4} = \dfrac{-\varepsilon\kappa_t}{\sqrt{\kappa_t^2 + (\mathfrak{h}_\nu^\varepsilon)^2}} + \alpha', \\ \bar{c}_{\nu,2} &= 0, \\ \bar{c}_{\nu,3} &= \dfrac{\alpha^2}{2}\sqrt{\kappa_t^2 + (\mathfrak{h}_\nu^\varepsilon)^2} + \alpha\dfrac{\delta_o^h}{\kappa_t^2 + (\mathfrak{h}_\nu^\varepsilon)^2} + \dfrac{\mathfrak{h}_\nu^\varepsilon}{\sqrt{\kappa_t^2 + (\mathfrak{h}_\nu^\varepsilon)^2}}, \\ \bar{c}_{\nu,5} &= \dfrac{\delta_o^h}{\kappa_t^2 + (\mathfrak{h}_\nu^\varepsilon)^2} + \alpha\sqrt{\kappa_t^2 + (\mathfrak{h}_\nu^\varepsilon)^2}, \\ \bar{c}_{\nu,6} &= -\dfrac{\alpha^2}{2}\sqrt{\kappa_t^2 + (\mathfrak{h}_\nu^\varepsilon)^2} - \alpha\dfrac{\delta_o^h}{\kappa_t^2 + (\mathfrak{h}_\nu^\varepsilon)^2} + \dfrac{-\varepsilon\kappa_\nu + \kappa_\nu^2 + \kappa_t^2}{\sqrt{\kappa_t^2 + (\mathfrak{h}_\nu^\varepsilon)^2}}, \\ \bar{c}_{\nu,3} + \bar{c}_{\nu,6} &= \sqrt{\kappa_t^2 + (\mathfrak{h}_\nu^\varepsilon)^2}, \end{cases}$$

and similarly, for a function β, we get

$$(6.2)\qquad \begin{cases} \bar{c}_{b,1} &= \bar{c}_{b,4} = \dfrac{\varepsilon\kappa_t}{\sqrt{\kappa_t^2 + (\mathfrak{h}_b^\varepsilon)^2}} + \beta', \\ \bar{c}_{b,2} &= 0, \\ \bar{c}_{b,3} &= \dfrac{\beta^2}{2}\sqrt{\kappa_t^2 + (\mathfrak{h}_b^\varepsilon)^2} + \beta\dfrac{\delta_n^h}{\kappa_t^2 + (\mathfrak{h}_b^\varepsilon)^2} + \dfrac{\mathfrak{h}_b^\varepsilon}{\sqrt{\kappa_t^2 + (\mathfrak{h}_b^\varepsilon)^2}}, \\ \bar{c}_{b,5} &= \dfrac{\delta_n^h}{\kappa_t^2 + (\mathfrak{h}_b^\varepsilon)^2} + \beta\sqrt{\kappa_t^2 + (\mathfrak{h}_b^\varepsilon)^2}, \\ \bar{c}_{b,6} &= -\dfrac{\beta(u)^2}{2}\sqrt{\kappa_t^2 + (\mathfrak{h}_b^\varepsilon)^2} - \beta\dfrac{\delta_n^h}{\kappa_t^2 + (\mathfrak{h}_b^\varepsilon)^2} + \dfrac{-\varepsilon\kappa_b + \kappa_b^2 + \kappa_t^2}{\sqrt{\kappa_t^2 + (\mathfrak{h}_b^\varepsilon)^2}}, \\ \bar{c}_{b,3} + \bar{c}_{b,6} &= \sqrt{\kappa_t^2 + (\mathfrak{h}_b^\varepsilon)^2}. \end{cases}$$

We remark that one can obtain the formula for F_{lb} by interchanging κ_ν to κ_b and κ_t to $-\kappa_t$ in the formula for $F_{l\nu}$.

6.1. **Horocylinders as osculating and normal horo-flat surfaces.** We consider the condition for $F_{l\nu}$ and F_{lb} to be horocylinders. By (6.1) and (6.2), setting

$$\alpha_c = \frac{-\delta_o^h}{(\kappa_t^2 + (\mathfrak{h}_\nu^\varepsilon)^2)\sqrt{\kappa_t^2 + (\mathfrak{h}_\nu^\varepsilon)^2}}, \tag{6.3}$$

and

$$\beta_c = \frac{-\delta_n^h}{(\kappa_t^2 + (\mathfrak{h}_b^\varepsilon)^2)\sqrt{\kappa_t^2 + (\mathfrak{h}_b^\varepsilon)^2}}, \tag{6.4}$$

we see that $c_{\nu,5} \equiv 0$, $c_{b,5} \equiv 0$. Thus, $\bar{c}_{\nu,1} = \bar{c}_{\nu,4} \equiv 0$ if and only if

$$\frac{-\varepsilon\kappa_t}{\sqrt{\kappa_t^2 + (\mathfrak{h}_\nu^\varepsilon)^2}} + \alpha_c' \equiv 0, \tag{6.5}$$

and $\bar{c}_{b,1} = \bar{c}_{b,4} \equiv 0$ if and only if

$$\frac{\varepsilon\kappa_t}{\sqrt{\kappa_t^2 + (\mathfrak{h}_b^\varepsilon)^2}} + \beta_c' \equiv 0. \tag{6.6}$$

Set

$$C_o^h = -2\varepsilon\kappa_t(\kappa_t^2 + (\mathfrak{h}_\nu^\varepsilon)^2)^2 - 2(\kappa_t^2 + (\mathfrak{h}_\nu^\varepsilon)^2)(\delta_o^h)' + 3\delta_o^h(\kappa_t^2 + (\mathfrak{h}_\nu^\varepsilon)^2)'$$

and

$$C_n^h = 2\varepsilon\kappa_t(\kappa_t^2 + (\mathfrak{h}_b^\varepsilon)^2)^2 - 2(\kappa_t^2 + (\mathfrak{h}_b^\varepsilon)^2)(\delta_n^h)' + 3\delta_n^h(\kappa_t^2 + (\mathfrak{h}_b^\varepsilon)^2)'.$$

Then the condition (6.5) is equivalent to $C^h_o \equiv 0$, and (6.6) is equivalent to $C^h_n \equiv 0$. Moreover, if α_c satisfies (6.3), then $\bar{c}_{\nu,3}$ is equal to a positive functional multiplication of

$$\frac{-(\delta^h_o)^2}{2(\kappa_t^2+(\mathfrak{h}^\varepsilon_\nu)^2)^2}+\mathfrak{h}^\varepsilon_\nu,$$

and if β_c satisfies (6.4), then $\bar{c}_{b,3}$ is equal to a positive functional multiplication of

$$\frac{-(\delta^h_n)^2}{2(\kappa_t^2+(\mathfrak{h}^\varepsilon_b)^2)^2}+\mathfrak{h}^\varepsilon_b.$$

Thus we obtain the following proposition.

Proposition 6.3. *The horocyclic surface $F_{l\nu}$ is a regular horocylindrical surface if and only if $C^h_o \equiv 0$ and*

$$\frac{-(\delta^h_o)^2}{2(\kappa_t^2+(\mathfrak{h}^\varepsilon_\nu)^2)^2}+\mathfrak{h}^\varepsilon_\nu>0.$$

The horocyclic surface F_{lb} is a regular horocylindrical surface if and only if $C^h_n \equiv 0$ and

$$\frac{-(\delta^h_n)^2}{2(\kappa_t^2+(\mathfrak{h}^\varepsilon_b)^2)^2}+\mathfrak{h}^\varepsilon_b>0.$$

We see that if $\kappa_t \equiv 0$. Then $\delta^h_o=-\kappa_b(\mathfrak{h}^\varepsilon_\nu)^2$ and $\delta^h_n=-\kappa_\nu(\mathfrak{h}^\varepsilon_b)^2$ hold, and also $\bar{c}_{\nu,1}=\bar{c}_{\nu,4}=\alpha'_c$ and $\bar{c}_{b,1}=\bar{c}_{b,4}=\beta'_c$. We give examples of cuspidal edge whose osculating and normal horo-flat surfaces are horocylinders.

Example 6.4. (regular horocylindrical surface) We set $\kappa_t \equiv \kappa_b \equiv 0$ and κ_ν satisfies $\mathfrak{h}^\varepsilon_\nu>0$. Setting $\alpha_c=0$, then we see that $\bar{c}_{\nu,1}=\bar{c}_{\nu,2}=\bar{c}_{\nu,4}=\bar{c}_{\nu,5}=0$, and $\bar{c}_{\nu,3}(\bar{c}_{\nu,3}+\bar{c}_{\nu,6})>0$. Then by definition, $F_{l\nu}$ is a regular horocylindrical surface. Similarly, we set $\kappa_t \equiv \kappa_\nu \equiv 0$ and κ_b satisfies $\mathfrak{h}^\varepsilon_b>0$. Setting $\beta_c=0$, then we see that $\bar{c}_{b,1}=\bar{c}_{b,2}=\bar{c}_{b,4}=\bar{c}_{b,5}=0$, and $\bar{c}_{b,3}(\bar{c}_{b,3}+\bar{c}_{b,6})>0$. Then by definition, F_{lb} is a regular horocylindrical surface.

Example 6.5. (secondary regular horocylindrical surface) We set $\kappa_t \equiv \kappa_\nu \equiv 0$ and $\kappa_b=1$. Setting $\alpha_c=0$, then we see that $\bar{c}_{\nu,1}=\bar{c}_{\nu,2}=\bar{c}_{\nu,4}=\bar{c}_{\nu,6}=0$, and $\bar{c}^2_{\nu,5}-2\bar{c}^2_{\nu,3}<0$. Then by definition, $F_{l\nu}$ is a secondary regular horocylindrical surface. We set $\kappa_t \equiv \kappa_b \equiv 0$ and $\kappa_\nu=1$. Setting $\beta_c=0$, then we see that $\bar{c}_{b,1}=\bar{c}_{b,2}=\bar{c}_{b,4}=\bar{c}_{b,6}=0$, and $\bar{c}^2_{b,5}-2\bar{c}^2_{b,3}<0$. Then by definition, F_{lb} is a secondary regular horocylindrical surface.

6.2. **Horocones as osculating and normal horo-flat surfaces.** If the discriminant $Q_{l\nu}$ (respectively, Q_{lb}) of

$$\bar{c}_{\nu,3}=\frac{\alpha^2}{2}\sqrt{\kappa_t^2+(\mathfrak{h}^\varepsilon_\nu)^2}+\alpha\frac{\delta^h_o}{\kappa_t^2+(\mathfrak{h}^\varepsilon_\nu)^2}+\frac{\mathfrak{h}^\varepsilon_\nu}{\sqrt{\kappa_t^2+(\mathfrak{h}^\varepsilon_\nu)^2}}=0$$

$$\left(\text{respectively}, \bar{c}_{b,3}=\frac{\beta^2}{2}\sqrt{\kappa_t^2+(\mathfrak{h}^\varepsilon_b)^2}+\beta\frac{\delta^h_n}{\kappa_t^2+(\mathfrak{h}^\varepsilon_b)^2}+\frac{\mathfrak{h}^\varepsilon_b}{\sqrt{\kappa_t^2+(\mathfrak{h}^\varepsilon_b)^2}}=0\right)$$

as an equation of α (respectively, β) is non-negative, then we have a solution α (respectively, β). We set

$$\sigma^h_o=\frac{-\varepsilon\kappa_t}{\sqrt{\kappa_t^2+(\mathfrak{h}^\varepsilon_\nu)^2}}+\alpha'$$

and

$$\sigma^h_n=\frac{\varepsilon\kappa_t}{\sqrt{\kappa_t^2+(\mathfrak{h}^\varepsilon_b)^2}}+\beta'.$$

Then if $\sigma^h_o \equiv 0$, (respectively, $\sigma^h_n \equiv 0$,) $F_{l\nu}$ (respectively, F_{lb}) is a generalized horocone. Thus we can state the following:

Proposition 6.6. *The horocyclic surface $F_{l\nu}$ is a generalized horocone if and only if $\sigma_o^h \equiv 0$ and $Q_{l\nu} \geq 0$. The horocyclic surface F_{lb} is a generalized horocone if and only if $\sigma_n^h \equiv 0$ and $Q_{lb} \geq 0$.*

We give examples of cuspidal edge whose osculating and normal horo-flat surfaces are horocones.

Example 6.7. (horocone with single and two vertices) We take $\kappa_t \equiv 0$, a non-zero constant κ_b and a constant κ_ν satisfying $\mathfrak{h}_\nu^\varepsilon > 0$ and $\kappa_b^2 - 2\mathfrak{h}_\nu^\varepsilon \geq 0$. We also take a constant α which is a solution that $\bar{c}_{\nu,3} = 0$. Then $\bar{c}_{\nu,1} \equiv \bar{c}_{\nu,2} \equiv \bar{c}_{\nu,3} \equiv \bar{c}_{\nu,4} \equiv 0$ holds. Moreover, we see

$$\bar{c}_{\nu,5} = -\kappa_b + \alpha\mathfrak{h}_\nu^\varepsilon, \quad \bar{c}_{\nu,6} = \mathfrak{h}_\nu^\varepsilon.$$

Thus setting κ_b and κ_ν satisfying $-\kappa_b + \alpha\mathfrak{h}_\nu^\varepsilon = 0$, then we obtain a horocone with a single vertex. On the other hand, $-\kappa_b + \alpha\mathfrak{h}_\nu^\varepsilon \neq 0$, then we obtain a horocone with two vertices.

Similarly, we take $\kappa_t \equiv 0$, a non-zero constant κ_ν and a constant κ_b satisfying $\mathfrak{h}_b^\varepsilon > 0$ and $\kappa_\nu^2 - 2\mathfrak{h}_b^\varepsilon \geq 0$. We also take a constant β which is a solution that $\bar{c}_{b,3} = 0$. Then

$$\bar{c}_{b,1} \equiv \bar{c}_{b,2} \equiv \bar{c}_{b,3} \equiv \bar{c}_{b,4} \equiv 0$$

holds. Moreover, we see

$$\bar{c}_{b,5} = -\kappa_\nu + \beta\mathfrak{h}_b^\varepsilon, \quad \bar{c}_{b,6} = \mathfrak{h}_b^\varepsilon.$$

Thus setting that κ_ν and κ_b satisfy $-\kappa_\nu + \beta\mathfrak{h}_b^\varepsilon = 0$, then we obtain a horocone with a single vertex. On the other hand, if $-\kappa_\nu + \beta\mathfrak{h}_b^\varepsilon \neq 0$, then we obtain a horocone with two vertices.

Example 6.8. (semi-horocone) We set $\varepsilon\kappa_\nu \equiv 1$ and $\varepsilon\kappa_t < 0$. By (6.1),

$$\bar{c}_{\nu,1} = \bar{c}_{\nu,4} = -\varepsilon\kappa_t / \sqrt{\kappa_t^2} + \alpha'.$$

Let α be a solution of $1 + \alpha' = 0$, i.e., $\alpha = -u + A$, where A is a sufficiently large positive constant such that $-\varepsilon\kappa_t$ is positive around $u = 0$. We take $\kappa_t = -2\varepsilon(u + A)$ and $\kappa_b = -u^2 + A^2$. Then $\bar{c}_{\nu,1} \equiv \bar{c}_{\nu,2} \equiv \bar{c}_{\nu,3} \equiv \bar{c}_{\nu,4} \equiv 0$. Moreover, $\bar{c}_{\nu,5} = -u^2 + A^2$ and $\bar{c}_{\nu,6} = 2(u + A)$. Thus we get a semi-horocone $F_{l\nu}$.

Similarly, set $\varepsilon\kappa_b \equiv 1$ and $\varepsilon\kappa_t > 0$. Let β be a solution of $1 + \beta' = 0$ i.e., $\beta = -u + B$, where B is a sufficiently small negative constant such that $\varepsilon\kappa_t$ is positive around $u = 0$. We take $\kappa_t = -2\varepsilon(u + B)$ and $\kappa_\nu = u^2 - B^2$. Then, we see that $\bar{c}_{b,1} \equiv \bar{c}_{b,2} \equiv \bar{c}_{b,3} \equiv \bar{c}_{b,4} \equiv 0$, $\bar{c}_{b,5} = u^2 - B^2$ and $\bar{c}_{b,6} = -2(u + B)$. Thus we get a semi-horocone F_{lb}.

6.3. **Special cases.** If $\kappa_b \equiv 0$, then $\boldsymbol{\nu}$ is the principal normal direction of $\boldsymbol{\gamma}$, or equivalently, $\boldsymbol{b}$ is the bi-normal direction of $\boldsymbol{\gamma}$. If $\kappa_\nu \equiv 0$, then $\boldsymbol{\nu}$ is the bi-normal direction of $\boldsymbol{\gamma}$, and which to say that $\boldsymbol{b}$ is the principal normal direction of $\boldsymbol{\gamma}$.

Important particular cases are:

(i) $\kappa_\nu \equiv \varepsilon$ (i.e., $\mathfrak{h}_\nu^\varepsilon \equiv 0$) in $H_{l\nu}^\varepsilon$,
(ii) $\kappa_b \equiv \varepsilon$ (i.e., $\mathfrak{h}_b^\varepsilon \equiv 0$) in H_{lb}^ε.

If (i) is satisfied, then $c_{\nu,3} \equiv 0$ holds, and if (ii) is satisfied, then $c_{b,3} \equiv 0$ holds. Namely, the singular set of the original cuspidal edge and the singular set of the osculating and the normal horo-flat surfaces coincide respectively. By Proposition 4.1, we have the conditions of singularities of the osculating and normal horo-flat surfaces in terms of the information of the singular locus of the cuspidal edge.

7. Duals of the singular set of the cuspidal edge

Since the curve $\boldsymbol{\gamma}$ of the parameterization (4.2) takes values in H^3, we can consider the Δ_1 and Δ_2 duals of $\boldsymbol{\gamma}$. We set $H^s_\gamma : S^3_1 \times I \to \mathbb{R}$ (respectively, $H^l_\gamma : LC^* \times I \to \mathbb{R}$) by

$$H^s_\gamma(\boldsymbol{x}, u) = \langle \boldsymbol{x}, \boldsymbol{\gamma}(u)\rangle \Big(\text{respectively, } H^l_\gamma(\boldsymbol{x}, u) = \langle \boldsymbol{x}, \boldsymbol{\gamma}(u)\rangle + 1\Big).$$

Then we have a parameterization of the discriminant set of H^s_γ given by

$$DD_l(\phi, u) = \cos\phi\boldsymbol{\nu}(u) + \sin\phi\boldsymbol{b}(u).$$

The corresponding singular set $S(DD_l)$ is

$$S(DD_l) = \left\{(\phi, u)\middle|\cos\phi = \pm\kappa_b/\sqrt{\kappa_\nu^2+\kappa_b^2},\ \sin\phi = \mp\kappa_\nu/\sqrt{\kappa_\nu^2+\kappa_b^2}\right\},$$

with

$$\begin{pmatrix}\boldsymbol{\gamma}'\\ \boldsymbol{t}'\\ \boldsymbol{n}'\\ \boldsymbol{e}'\end{pmatrix} = \begin{pmatrix}0 & 1 & 0 & 0\\ 1 & 0 & \kappa_h & 0\\ 0 & -\kappa_h & 0 & \tau_h\\ 0 & 0 & -\tau_h & 0\end{pmatrix}\begin{pmatrix}\boldsymbol{\gamma}\\ \boldsymbol{t}\\ \boldsymbol{n}\\ \boldsymbol{e}\end{pmatrix},$$

where $\{\boldsymbol{\gamma}, \boldsymbol{t}, \boldsymbol{n}, \boldsymbol{e}\}$ is the hyperbolic Frenet frame along $\boldsymbol{\gamma}$ and $\kappa_h = |\boldsymbol{t}' - \gamma|$ (see Section 3). Since $DD_l|_{S(DD_l)} = \pm\boldsymbol{e}$, it follows that $\pm\boldsymbol{e} = \pm(\kappa_b\boldsymbol{\nu} - \kappa_\nu\boldsymbol{b})/\sqrt{\kappa_b^2+\kappa_\nu^2}$, and $\tau_h = -\kappa_t + \dfrac{\kappa_\nu'\kappa_b - \kappa_b'\kappa_\nu}{\kappa_b^2+\kappa_\nu^2}$ ([20, p109]).

On the other hand, we have a parameterization of the discriminant set of H^l_γ given by

$$HS_l(\phi, u) = \boldsymbol{\gamma}(u) + \cos\phi\boldsymbol{\nu}(u) + \sin\phi\boldsymbol{b}(u),$$

where $\phi \in [0, 2\pi)$. We also have

$$S(HS_l) = \left\{(\phi, u)\middle|\cos\phi = \frac{\kappa_\nu \pm \sqrt{\kappa_\nu + \kappa_b^2 - 1}}{\kappa_\nu^2+\kappa_b^2}\right\}.$$

Thus DD_l and HS_l are Δ_3-dual each other. Here, $\Delta_3 = \{(\boldsymbol{v}, \boldsymbol{w}) \in LC^* \times S^3_1 \,|\, \langle \boldsymbol{v}, \boldsymbol{w}\rangle = 1\}$ and as in Section 2, the phrase "DD_l and HS_l are Δ_3-dual" amounts to say that the map $(DD_l, HS_l) : U \to \Delta_3$ is isotropic with respect to the contact structure defined by the restrictions of the 1-forms

$$\theta_{31} = \langle d\boldsymbol{v}, \boldsymbol{w}\rangle\,|_{\Delta_3}, \quad \theta_{32} = \langle \boldsymbol{v}, d\boldsymbol{w}\rangle\,|_{\Delta_3}.$$

See [15] for details.

Now, we give a global property of a curve in the hyperbolic space. There is a relation

$$\begin{pmatrix}\boldsymbol{\gamma}'\\ \boldsymbol{t}'\\ \boldsymbol{n}'\\ \boldsymbol{e}'\end{pmatrix} = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & \pm\dfrac{\kappa_\nu}{\sqrt{\kappa_\nu^2+\kappa_b^2}} & \pm\dfrac{\kappa_b}{\sqrt{\kappa_\nu^2+\kappa_b^2}}\\ 0 & 0 & \pm\dfrac{\kappa_b}{\sqrt{\kappa_\nu^2+\kappa_b^2}} & \mp\dfrac{\kappa_\nu}{\sqrt{\kappa_\nu^2+\kappa_b^2}}\end{pmatrix}\begin{pmatrix}\boldsymbol{\gamma}\\ \boldsymbol{t}\\ \boldsymbol{\nu}\\ \boldsymbol{b}\end{pmatrix} \tag{7.1}$$

between $\{\boldsymbol{\gamma}, \boldsymbol{t}, \boldsymbol{n}, \boldsymbol{e}\}$ and $\{\boldsymbol{\gamma}, \boldsymbol{t}, \boldsymbol{\nu}, \boldsymbol{b}\}$. If we define θ by

$$\cos\theta = \pm\frac{\kappa_\nu}{\sqrt{\kappa_\nu^2+\kappa_b^2}}, \quad \sin\theta = \pm\frac{\kappa_b}{\sqrt{\kappa_\nu^2+\kappa_b^2}},$$

then we get that $\kappa_t = \theta' - \tau_h$. And in the case that the singular set forms a circle $C = \mathbb{R}/\mathbb{Z}$, we obtain

$$\int_C (\tau_h + \kappa_t)\, du = \theta(1) - \theta(0) = 2n\pi \qquad (n \in \mathbb{Z}).$$

Observe that the integer n is the *linking number* of $\{\boldsymbol{n}, \boldsymbol{e}\}$ around $\{\boldsymbol{\gamma}, \boldsymbol{t}\}$ along $\boldsymbol{\gamma}(C)$, with respect to $\{\boldsymbol{\nu}, \boldsymbol{b}\}$.

Appendix A. Osculating and normal extrinsic flat surfaces

We consider the following smooth functions on $H^3(-1) \times I$:

$$\begin{aligned} H_\nu(\boldsymbol{x}, u) &= \langle \boldsymbol{x}, \boldsymbol{\nu}(u) \rangle, \\ H_b(\boldsymbol{x}, u) &= \langle \boldsymbol{x}, \boldsymbol{b}(u) \rangle. \end{aligned}$$

Then by using the functions H_ν and H_b, we can obtain analogous results. The discriminant set of these functions are envelopes of the osculating or the rectifying hyperbolic planes. In the Euclidean case, the discriminant set of the functions corresponding to them are envelopes of the osculating or the rectifying planes. The results and the geometric meaning of them for these cases are quite similar to those of the case in the Euclidean space [18, 23]. Thus we only give here the parameterizations for the discriminant sets of H_ν and H_b.

The discriminant set $\mathcal{D}_{H_\nu}$ of the function H_ν can be parameterized by

$$(u, \phi) \mapsto \cosh\phi\boldsymbol{\gamma}(u) + \sinh\phi\overline{D}_\nu(u), \quad \overline{D}_\nu(u) = \frac{\kappa_t \boldsymbol{t} + \kappa_\nu \boldsymbol{b}}{\kappa_t^2 + \kappa_\nu^2}(u),$$

where we assume $(\kappa_t, \kappa_\nu) \neq (0, 0)$. This is a one-parameter family of geodesics tangent to the cuspidal edge. Therefore, $\mathcal{D}_{H_\nu}$ is called an *osculating extrinsic flat surface along the cuspidal edge*.

The discriminant set $\mathcal{D}_{H_b}$ of the function H_b can be parameterized by

$$(u, \phi) \mapsto \cosh\phi\boldsymbol{\gamma}(u) + \sinh\phi\overline{D}_b(u), \quad \overline{D}_b(u) = \frac{-\kappa_t \boldsymbol{t} + \kappa_b \boldsymbol{\nu}}{\kappa_t^2 + \kappa_b^2}(u),$$

where we assume $(\kappa_t, \kappa_b) \neq (0, 0)$. This is a one-parameter family of geodesics normal to the cuspidal edge, so that $\mathcal{D}_{H_b}$ is called a *normal extrinsic flat surface along the cuspidal edge*.

Appendix B. Criteria for singularities

We state the some criteria to characterize the singularities used in Sections 4 and 5. Let $f : U \to H^3$ be a frontal with a Δ_1-dual $g : U \to S^3_1$. A function Λ is called an *identifier of singularities* if it is a non-zero functional multiplication of the function $\det(f_u, f_v, g, f)$ for a coordinate system (u, v) on U. If $p \in U$ satisfies $\operatorname{rank} df_p = 1$, then there exists a vector field η such that $\langle \eta_q \rangle_{\mathbb{R}} = \ker df_q$ for all $q \in S(f)$. We call η a *null vector field*. Let $p \in U$ be a singular point satisfying $d\Lambda(p) \neq 0$. Then there exists a parametrization $c : ((-z, z), 0) \to (U, p)$ of $S(f)$ near p, where $z > 0$. Let ∇^f_η be the canonical covariant derivative by η along a map f induced from the Levi-Civita connection on H^3. We set

$$\psi(u) = \det\left(\frac{df(\gamma(u))}{dt},\ \frac{d(\nabla^f_\eta g)(\gamma(u))}{dt},\ g(\gamma(u)),\ f(\gamma(u)) \right).$$

Then we have the following criteria for singularities:

Proposition B.1. *Let $p \in U$ be a singular point of f satisfying* $\operatorname{rank} df_p = 1$. *Then p is*

(1) *a cuspidal edge if and only if f is a front at p, and $\eta\Lambda(p) \neq 0$.*
(2) *a swallowtail if and only if f is a front at p, $d\Lambda(p) \neq 0$, $\eta\Lambda(p) = 0$ and $\eta\eta\Lambda(p) \neq 0$.*
(3) *a cuspidal beak (respectively, cuspidal lip) if and only if f is a front at p, $d\Lambda(p) = 0$, $\det \operatorname{Hess} \Lambda(p) < 0$ and $\eta\eta\Lambda(p) \neq 0$ (respectively, $\det \operatorname{Hess} \Lambda(p) > 0$).*
(4) *a cuspidal cross cap if and only if $\eta\Lambda(p) \neq 0$, $\psi(0) = 0$ and $\psi'(0) \neq 0$.*

These criteria for singularities in H^3 can be easily shown by well-known criteria in [35, Corollary 2.5] (see also [25, Proposition 1.3]) for (1) and (2), in [22, Theorem A.1] for (3), and in [8, Corollary 1.5] for (4).

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References

[1] V. I. Arnold, S. M. Gusein-Zade, and A. N. Varchenko, *Singularities of differentiable maps*, Vol. **1**, Monographs in Mathematics **82**, Birkhäuser, Boston, 1985. DOI: 10.1007/978-1-4612-5154-5

[2] M. Asayama, S. Izumiya and A. Tamaoki and H. Yıldırım, *Slant geometry of spacelike hypersurfaces in hyperbolic space and de Sitter space*, Rev. Mat. Iberoam. **28** (2012), no. 2, 371–400.

[3] J. W. Bruce and P. J. Giblin, *Curves and singularities* (*second edition*), Cambridge University press, Cambridge, 1992.

[4] J. W. Bruce and J. M. West, *Functions on a crosscap*, Math. Proc. Cambridge Philos. Soc. **123** (1998), no. 1, 19–39.

[5] L. Chen and S. Izumiya, *A mandala of Legendrian dualities for pseudo-spheres in semi-Euclidean space*, Proc. Japan Acad. Ser. A Math. Sci. **85** (2009), no. 4, 49–54. DOI: 10.3792/pjaa.85.49

[6] J. M. Espinar, J. A. Gálvez and P. Mira, *Hypersurfaces in H^{n+1} and conformally invariant equations: the generalized Christoffel and Nirenberg problems*, J. Eur. Math. Soc. **11** (2009), no. 4, 903–939. DOI: 10.4171/jems/170

[7] S. Fujimori, Y. Kawakami, M. Kokubu, W. Rossman, M. Umehara and K. Yamada, *CMC-*1 *trinoids in hyperbolic* 3*-space and metrics of constant curvature one with conical singularities on the* 2*-sphere*, Proc. Japan Acad. Ser. A Math. Sci. **87** (2011), no. 8, 144–149. DOI: 10.3792/pjaa.87.144

[8] S. Fujimori, K. Saji, M. Umehara and K. Yamada, *Singularities of maximal surfaces*, Math. Z. **259** (2008), 827–848. DOI: 10.1007/s00209-007-0250-0

[9] T. Fukui and M. Hasegawa, *Fronts of Whitney umbrella-a differential geometric approach via blowing up*, J. Singul. **4** (2012), 35–67. DOI: 10.5427/jsing.2012.4c

[10] J. A. Gálvez and P. Mira, *Embedded isolated singularities of flat surfaces in hyperbolic* 3*-space*, Calc. Var. Partial Differential Equations **24** (2005), no. 2, 239–260. DOI: 10.1007/s00526-004-0321-6

[11] S. Hananoi, N. Ito and S. Izumiya, *Spherical Darboux images of curves on surfaces*, Beitr. Algebra Geom. **56** (2015), 575–585. DOI: 10.1007/s13366-015-0240-z

[12] S. Hananoi and S. Izumiya, *Normal developable surfaces of surfaces along curves*, Proc. Roy. Soc. Edinburgh Sect. A **147** (2017), no. 1, 177–203. DOI: 10.1017/s030821051600007x

[13] M. Hasegawa, A. Honda, K. Naokawa, K. Saji, M. Umehara, and K. Yamada, *Intrinsic properties of singularities of surfaces*, Internat. J. Math. **26** (2015), no. 4, 1540008, 34 pp. DOI: 10.1142/s0129167x1540008x

[14] G. Ishikawa, *Singularities of flat extensions from generic surfaces with boundaries*, Differential Geom. Appl. **28** (2010), no. 3, 341–354. DOI: 10.1016/j.difgeo.2010.02.001

[15] S. Izumiya, *Legendrian dualities and spacelike hypersurfaces in the lightcone*, Moscow Math. J. **9** (2009), no. 2, 325–357. DOI: 10.17323/1609-4514-2009-9-2-325-357

[16] S. Izumiya, A. C. Nabarro and A. de Jesus Sacramento, *Pseudo-spherical normal Darboux images of curves on a timelike surface in three dimensional Lorentz-Minkowski space*, J. Geom. Phys. **97** (2015), 105–118. DOI: 10.1016/j.geomphys.2015.07.014

[17] S. Izumiya, A. C. Nabarro and A. de Jesus Sacramento, *Horospherical and hyperbolic dual surfaces of spacelike curves in de Sitter space*, J. Singul. **16** (2017), 180–193. DOI: 10.5427/jsing.2017.16h

[18] S. Izumiya and S. Otani, *Flat Approximations of surfaces along curves*, Demonstr. Math. **48** (2015), no. 2, 217–241. DOI: 10.1515/dema-2015-0018

[19] S. Izumiya, D. Pei and T. Sano, *Singularities of hyperbolic Gauss maps*, Proc. London Math. Soc. **86** (2003), 485–512. DOI: 10.1112/s0024611502013850

[20] S. Izumiya, D. Pei and M. Takahashi, *Curves and surfaces in hyperbolic space*. Geometric singularity theory, 107–123, Banach Center Publ., **65**, Polish Acad. Sci. Inst. Math., Warsaw, 2004. DOI: 10.4064/bc65-0-8

[21] S. Izumiya, M. C. Romero-Fuster, M. A. S. Ruas and F. Tari, *Differential geometry from a singularity theory viewpoint*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2016. DOI: 10.1142/9108

[22] S. Izumiya, K. Saji and M. Takahashi, *Horospherical flat surfaces in Hyperbolic* 3*-space*, J. Math. Soc. Japan **62** (2010), no. 3, 789–849. DOI: 10.2969/jmsj/06230789

[23] S. Izumiya, K. Saji and N. Takeuchi, *Flat surfaces along cuspidal edges*, J. Singul. **16** (2017), 73–100. DOI: 10.5427/jsing.2017.16c

[24] S. Izumiya, M. Takahashi and F. Tari, *Folding maps on spacelike and timelike surfaces and duality*, Osaka J. Math. **47** (2010), no. 3, 839–862.

[25] M. Kokubu, W. Rossman, K. Saji, M. Umehara, and K. Yamada, *Singularities of flat fronts in hyperbolic* 3*-space*, Pacific J. Math. **221** (2005), 303–351. DOI: 10.2140/pjm.2005.221.303

[26] M. Kokubu and M. Umehara, *Orientability of linear Weingarten surfaces, spacelike CMC-*1 *surfaces and maximal surfaces*, Math. Nachr. **284** (2011), no. 14–15, 1903–1918. DOI: 10.1002/mana.200910176

[27] H. Liu and S. D. Jung, *Hypersurfaces in lightlike cone*, J. Geom. Phys. **58** (2008), no. 7, 913–922.

[28] H. Liu, M. Umehara and K. Yamada, *The duality of conformally flat manifolds*, Bull. Braz. Math. Soc. **42** (2011), no. 1, 131–152. DOI: 10.1007/s00574-011-0007-6

[29] L. F. Martins and K. Saji, *Geometric invariants of cuspidal edges*, Canad. J. Math. **68** (2016), no. 2, 445–462. DOI: 10.4153/cjm-2015-011-5

[30] L. F. Martins, K. Saji, M. Umehara and K. Yamada, *Behavior of Gaussian curvature near non-degenerate singular points on wave fronts*, Geometry and topology of manifolds, Springer Proc. Math. Stat., **154**, 247–281. DOI: 10.1007/978-4-431-56021-0_14

[31] S. Murata and M. Umehara, *Flat surfaces with singularities in Euclidean* 3*-space*, J. Differential Geom. **82** (2009), 279–316. DOI: 10.4310/jdg/1246888486

[32] K. Naokawa, M. Umehara and K. Yamada, *Isometric deformations of cuspidal edges*, Tohoku Math. J. (2) **68** (2016), 73–90. DOI: 10.2748/tmj/1458248863

[33] R. Oset Sinha and F. Tari, *On the flat geometry of the cuspidal edge*, Osaka J. Math. **55** (2018), no. 3, 393–421.

[34] K. Saji, M. Umehara, and K. Yamada, *The geometry of fronts*, Ann. of Math. **169** (2009), 491–529. DOI: 10.4007/annals.2009.169.491

[35] K. Saji, M. Umehara, and K. Yamada, A_k *singularities of wave fronts*, Math. Proc. Cambridge Philos. Soc. **146** (2009), 731-746.

[36] C. Takizawa and K. Tsukada, *Horocyclic surfaces in hyperbolic* 3*-space*, Kyushu J. Math. **63** (2009), no. 2, 269-284. DOI: 10.2206/kyushujm.63.269

Shyuichi Izumiya, Department of Mathematics, Hokkaido University, Sapporo 060-0810, Japan
Email address: izumiya@math.sci.hokudai.ac.jp

Maria del Carmen Romero Fuster, Department de Geometria i Topologia, Universitat de Valencia, 46100 Valencia, Spain
Email address: Carmen.Romero@uv.es

Kentaro Saji, Department of Mathematics, Kobe University, Rokko 1-1, Nada, Kobe 657-8501, Japan
Email address: saji@math.kobe-u.ac.jp

Masatomo Takahashi, Muroran Institute of Technology, Muroran 050-8585, Japan
Email address: masatomo@mmm.muroran-it.ac.jp

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DUALITY ON GENERALIZED CUSPIDAL EDGES PRESERVING SINGULAR SET IMAGES AND FIRST FUNDAMENTAL FORMS

A. HONDA, K. NAOKAWA, K. SAJI, M. UMEHARA, AND K. YAMADA

Dedicated to Professor Toshizumi Fukui for his sixtieth birthday.

Abstract. In the second, fourth and fifth authors' previous work, a duality on generic real analytic cuspidal edges in the Euclidean 3-space $\boldsymbol{R}^3$ preserving their singular set images and first fundamental forms, was given. Here, we call this an "isometric duality". When the singular set image has no symmetries and does not lie in a plane, the dual cuspidal edge is not congruent to the original one. In this paper, we show that this duality extends to generalized cuspidal edges in $\boldsymbol{R}^3$, including cuspidal cross caps, and 5/2-cuspidal edges. Moreover, we give several new geometric insights on this duality.

Introduction

Consider a generic cuspidal edge germ f whose singular set image is a given space curve C. In the second, fourth and fifth authors' previous work [14], the existence of an isometric dual $\check{f}$ of f was shown, which is a cuspidal edge germ having the same first fundamental form as f. Roughly speaking, a cuspidal edge which has the same first fundamental form and the same singular set image as f but is not right equivalent to f, is called an "isomer" of f (see Definition 0.6 for details). The isometric dual $\check{f}$ is a typical example of isomers of f. Recently, the authors found that if we reverse the orientation of C, two other candidates of isomers of f denoted by f_* and $\check{f}_*$ are obtained by imitating the construction of $\check{f}$. These two map germs f_* and $\check{f}_*$ are cuspidal edge germs which are called the *inverse* and the *inverse dual* of f, respectively ($\check{f}_*$ is just the isometric dual of f_*). In this paper, we will show that all of isomers of f are right equivalent to one of

$$\check{f},\ f_*,\ \check{f}_*.$$

We will also determine the number of congruence classes in the set of isomers of f.

By the terminology "C^r-differentiable" we mean C^∞-differentiability if $r = \infty$ and real analyticity if $r = \omega$. We denote by $\boldsymbol{R}^3$ the Euclidean 3-space. Let U be a neighborhood of the origin $(0,0)$ in the uv-plane $\boldsymbol{R}^2$, and let $f : U \to \boldsymbol{R}^3$ be a C^r-map. Without loss of generality, we may assume $f(o) = \mathbf{0}$, where

$$o := (0,0), \qquad \mathbf{0} := (0,0,0). \tag{0.1}$$

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A point $p \in U$ is called a *singular point* if f is not an immersion at p. A singular point $p \in U$ is called a *cuspidal edge point* (resp. a *generalized cuspidal edge point*) if there exist local C^r-diffeomorphisms φ on $\boldsymbol{R}^2$ and Φ on $\boldsymbol{R}^3$ such that $\varphi(o) = p$, $\Phi(f(p)) = \mathbf{0}$ and

$$(f_{3/2} :=)(u, v^2, v^3) = \Phi \circ f \circ \varphi(u, v) \quad \bigl(\text{resp. } (u, v^2, v^3\alpha(u, v)) = \Phi \circ f \circ \varphi(u, v)\bigr),$$

where $\alpha(u, v)$ is a C^r-function. Similarly, a singular point $p \in U$ is called a 5/2-*cuspidal edge point* (resp. a *fold singular point*) if there exist local C^r-diffeomorphisms φ on $\boldsymbol{R}^2$ and Φ on $\boldsymbol{R}^3$ such that $\varphi(o) = p$, $\Phi(f(p)) = \mathbf{0}$ and

$$(f_{5/2} :=)(u, v^2, v^5) = \Phi \circ f \circ \varphi(u, v) \quad \bigl(\text{resp. } (u, v^2, 0) = \Phi \circ f \circ \varphi(u, v)\bigr).$$

Also, a singular point $p \in U$ is called a *cuspidal cross cap point* if there exist local C^r-diffeomorphisms φ on $\boldsymbol{R}^2$ and Φ on $\boldsymbol{R}^3$ such that $\varphi(o) = p$, $\Phi(f(p)) = \mathbf{0}$ and

$$(f_{\mathrm{ccr}} :=)(u, v^2, uv^3) = \Phi \circ f \circ \varphi(u, v).$$

These singular points are all generalized cuspidal edge points.

Let $\mathcal{G}^r_{3/2}(\boldsymbol{R}^2_o, \boldsymbol{R}^3)$ (resp. $\mathcal{G}^r(\boldsymbol{R}^2_o, \boldsymbol{R}^3)$) be the set of germs of C^r-cuspidal edges (resp. generalized C^r-cuspidal edges) $f(u, v)$ satisfying $f(o) = \mathbf{0}$. We fix $l > 0$ and consider an embedding (i.e. a simple regular space curve)

$$\mathbf{c} : J \to \boldsymbol{R}^3 \qquad (J := [-l, l])$$

such that $\mathbf{c}(0) = \mathbf{0}$. We do not assume here that $u \mapsto \mathbf{c}(u)$ is the arc-length parametrization (if necessary, we assume this in latter sections). We denote by C the image of $\mathbf{c}$. Here, we ignore the orientation of C and think of it as the singular set image (i.e. the image of the singular set) of f. We let $\mathcal{G}^r_{3/2}(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C)$ (resp. $\mathcal{G}^r(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C)$) be the subset of $\mathcal{G}^r_{3/2}(\boldsymbol{R}^2_o, \boldsymbol{R}^3)$ (resp. $\mathcal{G}^r(\boldsymbol{R}^2_o, \boldsymbol{R}^3)$) such that the singular set image of f is contained in C (we call C the *edge* of f). Similarly, a subset of $\mathcal{G}^r(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C)$ denoted by

$$\mathcal{G}^r_{\mathrm{ccr}}(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C), \qquad (\text{resp. } \mathcal{G}^r_{5/2}(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C))$$

consisting of germs of cuspidal cross caps (resp. 5/2-cuspidal edges) is also defined.

Throughout this paper, we assume the curvature function $\kappa(u)$ of $\mathbf{c}(u)$ satisfies

$$\kappa(u) > 0 \qquad (u \in J). \tag{0.2}$$

Let U be a neighborhood of $J \times \{0\}$ of $\boldsymbol{R}^2$ and $f : U \to \boldsymbol{R}^3$ a C^r-map consisting only of generalized cuspidal edge points along $J \times \{0\}$ such that

$$f(u, 0) = \mathbf{c}(u) \qquad (u \in J). \tag{0.3}$$

We denote by $\mathcal{G}^r(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$ the set of such f (f is called a *generalized cuspidal edge along* C). Like as the case of map germs at o, the sets

$$\mathcal{G}^r_{3/2}(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C), \quad \mathcal{G}^r_{\mathrm{ccr}}(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C), \quad \mathcal{G}^r_{5/2}(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$$

are also canonically defined. For each point P on the edge C, the plane $\Pi(P)$ passing through P which is perpendicular to the curve C is called the *normal plane* of f at P. The section of the image of f by the normal plane $\Pi(P)$ of C at P is a planar curve with a singular point at P. We call this the *sectional cusp* of f at P. Moreover, we can find a tangent vector $\mathbf{v} \in T_P\boldsymbol{R}^3$ at P, which points in the tangential direction of the sectional cusp at P. We call $\mathbf{v}$ the *cuspidal direction* (cf. (3.6) and Figure 1). The angle θ_P of the cuspidal direction from the principal normal vector of C at P is called the *cuspidal angle*.

If we normalize the initial value $\theta_{\mathbf{c}(0)} \in (-\pi, \pi]$ at $\mathbf{c}(0)(= \mathbf{0})$, then the cuspidal angle

$$\theta(u) := \theta_{\mathbf{c}(u)} \qquad (u \in J)$$

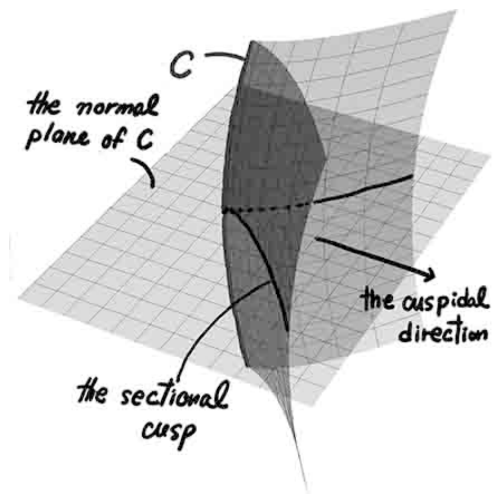

FIGURE 1. A cuspidal edge and its sectional cusp

at $\mathbf{c}(u)$ can be uniquely determined as a C^r-function on J. In [12, 16], the singular curvature $\kappa_s(u)$ and the limiting normal curvature $\kappa_\nu(u)$ along the edge $\mathbf{c}(u)$ are defined. In our present situation, they can be expressed as (cf. [3, Remark 1.9])

$$\kappa_s(u) := \kappa(u)\cos\theta(u), \qquad \kappa_\nu(u) := \kappa(u)\sin\theta(u) \qquad (u \in J). \tag{0.4}$$

By definition, $\kappa(u) = \sqrt{\kappa_s(u)^2 + \kappa_\nu(u)^2}$ holds on J. We say that $f \in \mathcal{G}^r(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C)$ is *generic* at o if

$$|\kappa_s(0)| < \kappa(0). \tag{0.5}$$

We denote by $\mathcal{G}^r_*(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C)$ the set of germs of generic generalized C^r-cuspidal edges in $\mathcal{G}^r(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C)$, and set

$$\begin{aligned}
\mathcal{G}^r_{*,3/2}(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C) &:= \mathcal{G}^r_*(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C) \cap \mathcal{G}^r_{3/2}(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C),\\
\mathcal{G}^r_{*,\mathrm{ccr}}(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C) &:= \mathcal{G}^r_*(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C) \cap \mathcal{G}^r_{\mathrm{ccr}}(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C),\\
\mathcal{G}^r_{*,5/2}(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C) &:= \mathcal{G}^r_*(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C) \cap \mathcal{G}^r_{5/2}(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C).
\end{aligned} \tag{0.6}$$

On the other hand, for $f \in \mathcal{G}^r(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$, we consider the condition

$$|\kappa_s(u)| < \kappa(u) \qquad (u \in J), \tag{0.7}$$

which implies that all singular points of f along the curve C are generic. We denote by

$$\mathcal{G}^r_*(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C) \tag{0.8}$$

the set of $f \in \mathcal{G}^r(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$ satisfying (0.7). Moreover, if

$$\max_{u \in J} |\kappa_s(u)| < \min_{u \in J} \kappa(u) \tag{0.9}$$

holds, then f is said to be *admissible*. We denote by

$$\mathcal{G}^r_{**}(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C) \tag{0.10}$$

the set of admissible $f \in \mathcal{G}^r(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$. Then by imitating (0.6),

$$\mathcal{G}^r_{*,3/2}(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C), \qquad \mathcal{G}^r_{**,3/2}(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C) \tag{0.11}$$

are also defined. The following assertion is obvious:

Lemma 0.1. *Suppose that f belongs to $\mathcal{G}^r_{3/2}(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C)$ (resp. $\mathcal{G}^r_{*,3/2}(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C)$). Then there exists $\varepsilon(>0)$ such that f is an element of $\mathcal{G}^r_{3/2}(\boldsymbol{R}^2_{J(\varepsilon)}, \boldsymbol{R}^3, C)$ (resp. $\mathcal{G}^r_{**,3/2}(\boldsymbol{R}^2_{J(\varepsilon)}, \boldsymbol{R}^3, C)$), where $J(\varepsilon) := [-\varepsilon, \varepsilon]$.*

Let O(3) (resp. SO(3)) be the orthogonal group (resp. the special orthogonal group) as the isometry group (resp. the orientation preserving isometry group) of $\boldsymbol{R}^3$ fixing the origin $\mathbf{0}$.

Definition 0.2. Suppose that f_i $(i = 1, 2)$ are generalized cuspidal edges belonging to $\mathcal{G}^r(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C)$ (resp. $\mathcal{G}^r(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$). Then the image of f_1 is said to have the *same image* as f_2 if there exists a neighborhood $U_i(\subset \boldsymbol{R}^2)$ of o (resp. $J \times \{0\}$) such that $f_1(U_1) = f_2(U_2)$. On the other hand, f_1 is said to be *congruent* to f_2 if there exists an orthogonal matrix $T \in \mathrm{O}(3)$ such that $T \circ f_1$ has the same image as f_2.

We then define the following two equivalence relations:

Definition 0.3. For a given f belonging to $\mathcal{G}^r(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C)$ (resp. $\mathcal{G}^r(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$), we denote by ds^2_f its first fundamental form. A generalized cuspidal edge g belonging to $\mathcal{G}^r(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C)$ (resp. $\mathcal{G}^r(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$) is said to be *right equivalent* to f if there exists a diffeomorphism φ defined on a neighborhood of o (resp. $J \times \{0\}$) in $\boldsymbol{R}^2$ such that $g = f \circ \varphi$.

Definition 0.4. For a given generalized cuspidal edge $f \in \mathcal{G}^r(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C)$ (resp. $\mathcal{G}^r(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$), we denote by ds^2_f its first fundamental form. A generalized cuspidal edge $g \in \mathcal{G}^r(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C)$ (resp. $\mathcal{G}^r(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$) is said to be *isometric* to f if there exists a diffeomorphism φ defined on a neighborhood of o (resp. $J \times \{0\}$) in $\boldsymbol{R}^2$ such that $\varphi^* ds^2_f = ds^2_g$.

In particular, we consider the case $f = g$. If $\varphi^* ds^2_f = ds^2_f$ and φ is not the identity map, then φ is called a *symmetry* of ds^2_f. Moreover, if φ reverses the orientation of the singular curve of f, then φ is said to be *effective*.

Remark 0.5. A cuspidal edge $g \in \mathcal{G}^r_{3/2}(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C)$ (resp. $\mathcal{G}^r_{3/2}(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$) has the same image as a given germ $f \in \mathcal{G}^r_{3/2}(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C)$ (resp. $\mathcal{G}^r_{3/2}(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$) if and only if g is right equivalent to f (cf. [10]).

If two generalized cuspidal edges $f, g \in \mathcal{G}^r(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C)$ (resp. $\mathcal{G}^r(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$) are right equivalent, then they are isometric each other. However, the converse may not be true. So we give the following:

Definition 0.6. For a given $f \in \mathcal{G}^r(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C)$ (resp. $\mathcal{G}^r(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$), a generalized cuspidal edge $g \in \mathcal{G}^r(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C)$ (resp. $\mathcal{G}^r(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$) is called an *isomer* of f (cf. [14]) if it satisfies the following conditions;

(1) g is isometric to f, and
(2) g is not right equivalent to f.

In this situation, we say that g is a *faithful isomer* of f if

- there exists a local diffeomorphism φ such that $\varphi^* ds^2_f = ds^2_g$, and
- the orientations of C induced by $u \mapsto f \circ \varphi(u, 0)$ and $u \mapsto g(u, 0)$ are compatible with respect to the one induced by $u \mapsto f(u, 0)$.

In [14, Corollary D], it was shown the existence of an involution

$$\mathcal{G}^\omega_{*,3/2}(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C) \ni f \mapsto \check{f} \in \mathcal{G}^\omega_{*,3/2}(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C). \tag{0.12}$$

To construct $\check{f}$, we need to apply the so-called Cauchy-Kowalevski theorem on partial differential equations of real analytic category (cf. Theorem 3.8). Here, $\check{f}$ is called the *isometric dual* of f, which satisfies the following properties:

(i) The first fundamental form of $\check{f}$ coincides with that of f.

(ii) The map $\check{f}$ is a faithful isomer of f.

(iii) If $\theta(P)$ is the cuspidal angle of f at $P(\in C)$, then $-\theta(P)$ is the cuspidal angle of $\check{f}$ at P.

In [14], a necessary and sufficient condition for a given positive semi-definite metric to be realized as the first fundamental form of a cuspidal edge along C is given. In this paper, we first prove the following using the method given in [14]:

Theorem I. *There exists an involution* (*called the first involution*)

$$\mathcal{I}_C : \mathcal{G}^{\omega}_{*}(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C) \ni f \mapsto \check{f} \in \mathcal{G}^{\omega}_{*}(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C) \tag{0.13}$$

defined on $\mathcal{G}^{\omega}_{*}(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$ (*cf.* (0.8)) *satisfying the properties* (i), (ii) *and* (iii) *above. Moreover, regarding* f *and* $\check{f}$ *as map germs at* o (*cf. Lemma 0.1*), $\mathcal{I}_C$ *induces a map*

$$\mathcal{I}_o : \mathcal{G}^{\omega}_{*}(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C) \ni f \mapsto \check{f} \in \mathcal{G}^{\omega}_{*}(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C), \tag{0.14}$$

which gives a generalization of the map as in (0.12).

The existence of the map $\mathcal{I}_o$ follows also from [5, Theorem B], since $\check{f}$ is strongly congruent to f in the sense of [5, Definition 3]. However, the existence of the map $\mathcal{I}_C$ itself does not follow from [5], since $\check{f}$ given in Theorem I is not a map germ at o, but a map germ along the curve C. Some variants of this result for germs of swallowtails and cuspidal cross caps were given in [5, Theorem B] using a method different from [14]. (For swallowtails, the duality corresponding to the above properties (i), (ii) and (iii) are not obtained, see item (4) below.) The authors find Theorem I to be suggestive of the following geometric problems:

(1) How many right equivalence classes of isomers of f exist other than $\check{f}$?

(2) When are isomers non-congruent to each other?

(3) The existence of the isometric dual can be proved by applying the Cauchy-Kowalevski theorem. So we need to assume that the given generalized cuspidal edges are real analytic. It is then natural to ask if one can find a new method for constructing the isometric dual in the C^∞-differentiable category.

(4) Can one extend isometric duality to a much wider class, say, for swallowtails?

In this paper, we show the following:

- For a given generalized cuspidal edge $f \in \mathcal{G}^{\omega}_{**}(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$, there exists a unique generalized cuspidal edge $f_* \in \mathcal{G}^{\omega}_{**}(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$ (called the *inverse* of f) having the same first fundamental form as f along the space curve $\mathbf{c}(-u)$ whose cuspidal angle has the same sign as that of f. Moreover, any isomers of f are right equivalent to one of $\{f, \check{f}, f_*, \check{f}_*\}$ (see Theorem II), where $\check{f}_* := \mathcal{I}_C(f_*)$ is called the *inverse dual* of f.
- The four maps $f,\ \check{f},\ f_*,\ \check{f}_*$ are non-congruent in general. Moreover, the right equivalence classes and congruence classes of these four surfaces are determined in terms of the properties of C and ds^2_f (cf. Theorems III and IV).
- Suppose that the image of a C^∞-differentiable cuspidal edge f is invariant under a non-trivial symmetry $T \in \mathrm{SO}(3)$ (cf. Definition 1.2) of $\boldsymbol{R}^3$. Then explicit construction of $\check{f}$ without use of the Cauchy-Kowalevski theorem is given (see Example 5.3).

About the last question (4), the authors do not know whether the isomers of a given swallowtail will exist in general, since the method given in this paper does not apply directly. So it left here as an open problem. (A possible isometric deformations of swallowtails are discussed in authors' previous work [5].)

The paper is organized as follows: In Section 1, we explain our main results. In Section 2, we review the definition and properties of Kossowski metrics. In Section 3, we prove Theorem I as a modification of the proof of [14]. In Section 4, we recall a representation formula for generalized cuspidal edges given in Fukui [3], and prove Theorem II. In Section 5, we investigate the properties of generic cuspidal edges with symmetries. Moreover, we prove Theorems III and IV. Several examples are given in Section 6. Finally, in the appendix, a representation formula for generalized cusps in the Euclidean plane is given.

1. Results

Let ds^2 be a C^r-differentiable positive semi-definite metric on a C^r-differentiable 2-manifold M^2. A point $o \in M^2$ is called a *regular point* of ds^2 if it is positive definite at o, and is called a *singular point* (or a *semi-definite point*) if ds^2 is not positive definite at o. Kossowski [8] defined a certain kind of positive semi-definite metrics called "Kossowski metrics" (cf. Section 2). We let ds^2 be such a metric. Then for each singular point $o \in M^2$, there exists a regular curve $\gamma : (-\varepsilon, \varepsilon) \to M^2$ such that $\gamma(0) = o$ and γ parametrizes the singular set of ds^2 near o. Such a curve is called the *singular curve* of ds^2 near o. In this situation, if $ds^2(\gamma'(0), \gamma'(0))$ does not vanish, then we say that "ds^2 is of type I at o". The first fundamental forms (i.e. the induced metrics) of germs of generalized cuspidal edges are Kossowski metrics of type I (cf. Proposition 3.1).

Setting $M^2 := (\boldsymbol{R}^2; u, v)$, we denote by $\mathcal{K}^r_{\mathrm{I}}(\boldsymbol{R}^2_o)$ the set of germs of C^r-Kossowski metrics of type I at $o := (0,0)$. We fix such a $ds^2 \in \mathcal{K}^r_{\mathrm{I}}(\boldsymbol{R}^2_o)$. Then the metric is expressed as

$$ds^2 = E du^2 + 2F du dv + G dv^2,$$

and there exists a C^r-function λ such that $EG - F^2 = \lambda^2$. Let K be the Gaussian curvature of ds^2 defined at points where ds^2 is positive definite. Then

$$\hat{K} := \lambda K \tag{1.1}$$

can be considered as a C^r-differentiable function defined on a neighborhood $U(\subset \boldsymbol{R}^2)$ of o (cf. [12, 5]). If $\hat{K}$ vanishes (resp. does not vanish) at a singular point $q \in U$ of ds^2, then ds^2 is said to be *parabolic* (resp. *non-parabolic*) at q (see Definition 2.6). We denote by $\mathcal{K}^r_*(\boldsymbol{R}^2_o)$ (resp. $\mathcal{K}^r_p(\boldsymbol{R}^2_o)$) the set of germs of non-parabolic (resp. parabolic) C^r-Kossowski metrics of type I at o. The subset of $\mathcal{K}^r_p(\boldsymbol{R}^2_o)$ defined by

$$\begin{aligned}\mathcal{K}^r_{p,*}(\boldsymbol{R}^2_o) &:= \{ds^2 \in \mathcal{K}^r_p(\boldsymbol{R}^2_o)\,;\, \hat{K}'(o) \neq 0\} \\ &\Bigl(= \{ds^2 \in \mathcal{K}^r_{\mathrm{I}}(\boldsymbol{R}^2_o)\,;\, \hat{K}(o) = 0,\ \hat{K}'(o) \neq 0\}\Bigr)\end{aligned}$$

plays an important role in this paper, where $\hat{K}' = \partial \hat{K}/\partial u$. Metrics belonging to $\mathcal{K}^r_{p,*}(\boldsymbol{R}^2_o)$ are called *p-generic*. On the other hand, if $\hat{K}$ vanishes identically along the singular curve of $ds^2 \in \mathcal{K}^r_{\mathrm{I}}(\boldsymbol{R}^2_o)$, we call ds^2 an *asymptotic* Kossowski metric of type I. We let $\mathcal{K}^r_a(\boldsymbol{R}^2_o)$ be the set of germs of such metrics. This terminology comes from the following two facts:

- for a regular surface, a direction where the normal curvature vanishes is called an asymptotic direction, and
- the induced metric of a cuspidal edge whose limiting normal curvature κ_ν vanishes identically along its singular set belongs to $\mathcal{K}^r_a(\boldsymbol{R}^2_o)$. (Such a cuspidal edge is called an *asymptotic cuspidal edge*, see Proposition 4.12.)

By definition, we have

$$\mathcal{K}^r_*(\boldsymbol{R}^2_o) \cap \mathcal{K}^r_p(\boldsymbol{R}^2_o) = \emptyset, \qquad \mathcal{K}^r_*(\boldsymbol{R}^2_o) \cup \mathcal{K}^r_p(\boldsymbol{R}^2_o) = \mathcal{K}^r_{\mathrm{I}}(\boldsymbol{R}^2_o),$$
$$\mathcal{K}^r_a(\boldsymbol{R}^2_o) \subset \mathcal{K}^r_p(\boldsymbol{R}^2_o) \subset \mathcal{K}^r_{\mathrm{I}}(\boldsymbol{R}^2_o).$$

For $ds^2 \in \mathcal{K}^r_a(\boldsymbol{R}^2_o)$, the Gaussian curvature K can be extended on a neighborhood of o as a C^r-differentiable function. Let $\eta \in T_o\boldsymbol{R}^2$ be the null vector at the singular point o of the asymptotic Kossowski metric ds^2. If

$$dK(\eta)(o) \neq 0, \tag{1.2}$$

then ds^2 is said to be *a-generic*, and we denote by $\mathcal{K}^r_{a,*}(\boldsymbol{R}^2_o)(\subset \mathcal{K}^r_a(\boldsymbol{R}^2_o))$ the set of germs of a-generic asymptotic C^r-Kossowski metrics. Considering the first fundamental form ds^2_f of f, we can define a map

$$\mathcal{J}_o : \mathcal{G}^r_*(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C) \ni f \mapsto ds^2_f \in \mathcal{K}^r_{\mathrm{I}}(\boldsymbol{R}^2_o). \tag{1.3}$$

Theorem II. *There exists an involution (called the second involution)*

$$\mathcal{I}^*_C : \mathcal{G}^\omega_{**}(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C) \ni f \mapsto f_* \in \mathcal{G}^\omega_{**}(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$$

defined on $\mathcal{G}^\omega_{**}(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$ (cf. (0.11)) satisfying the following properties:

(1) *f_* has the same first fundamental form as f, and is a non-faithful isomer of f,*
(2) *$\mathcal{I}^*_C \circ \mathcal{I}_C = \mathcal{I}_C \circ \mathcal{I}^*_C$, where $\mathcal{I}_C$ is the first involution as in Theorem* I.
(3) *Regarding f and f_* as map germs at o (cf. Lemma 0.1), $\mathcal{I}^*_C$ canonically induces a map*

$$\mathcal{I}^*_o : \mathcal{G}^\omega_*(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C) \ni f \mapsto f_* \in \mathcal{G}^\omega_*(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C) \tag{1.4}$$

such that $\mathcal{J}_o \circ \mathcal{I}^*_o = \mathcal{J}_o$ and $\mathcal{I}^*_o \circ \mathcal{I}_o = \mathcal{I}_o \circ \mathcal{I}^*_o$.

(4) *Suppose that g belongs to $\mathcal{G}^\omega_*(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C)$ (resp. $\mathcal{G}^\omega_{**}(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$). If the first fundamental form of g is isometric to that of f, then g is right equivalent to one of $f, \check{f}, f_*$ and $\check{f}_*$.*

Recently, Fukui [3] gave a representation formula for generalized cuspidal edges along their edges in $\boldsymbol{R}^3$. (In [3], a similar formula for swallowtails is also given, although it is not applied in this paper.) We denote by $C^r(\boldsymbol{R}_o)$ (resp. $C^r(\boldsymbol{R}^2_o)$) the set of C^r-function germs at the origin of $\boldsymbol{R}$ (resp. $\boldsymbol{R}^2$). We fix a generalized cuspidal edge $f \in \mathcal{G}^r(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C)$ arbitrarily. The sectional cusp of f at $\mathbf{c}(u)$ induces a function $\mu(u,t) \in C^r(\boldsymbol{R}^2_o)$ which is called the "extended half-cuspidal curvature function" giving the normalized curvature function of the sectional cusp at $\mathbf{c}(u)$ (see the appendix). The value

$$\kappa_c(u) := \frac{\mu(u,0)}{2} \tag{1.5}$$

coincides with the cuspidal curvature at the singular point of the sectional cusp, and so it is called the *cuspidal curvature function* of f (cf. [12]). In Section 4, we give a Björling-type representation formula for cuspidal edges (cf. Proposition 4.3), which is a modification of the formula given in Fukui [3]. (In fact, Fukui [3] expressed the sectional cusp as a pair of functions, but did not use the function μ.) Fukui [3] explained several geometric invariants of cuspidal edges in terms of κ_s, κ_ν and θ. In Section 4, using several properties of modified Fukui's formula together with the proof of Theorem I, we reprove the following assertion which determine the images of the maps $\mathcal{I}_o$ and $\mathcal{J}_o$ (the assertions for the map $\mathcal{I}^*_o$ are not given in [14, 5, 6]):

Fact 1.1. *The maps $\mathcal{I}_o$, $\mathcal{I}^*_o$ and $\mathcal{J}_o$ (cf. (0.14), (1.3) and (1.4)) satisfy the followings:*

(1) *These two maps $\mathcal{I}_o$ and $\mathcal{I}^*_o$ are involutions on $\mathcal{G}^\omega_{*,3/2}(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C)$, and $\mathcal{J}_o$ maps $\mathcal{G}^\omega_{*,3/2}(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C)$ onto $\mathcal{K}^\omega_*(\boldsymbol{R}^2_o)$ (cf.* [14, Theorem 12]).

(2) *The two maps* $\mathcal{I}_o$ *and* $\mathcal{I}_o^*$ *are involutions on* $\mathcal{G}^\omega_{*,\mathrm{ccr}}(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C)$*, and* $\mathcal{J}_o$ *maps* $\mathcal{G}^\omega_{*,\mathrm{ccr}}(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C)$ *onto* $\mathcal{K}^\omega_{p,*}(\boldsymbol{R}^2_o)$ (*cf.* [5, Theorem A]).

(3) *The two maps* $\mathcal{I}_o$ *and* $\mathcal{I}_o^*$ *are involutions on* $\mathcal{G}^\omega_{*,5/2}(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C)$*, and* $\mathcal{J}_o$ *maps* $\mathcal{G}^\omega_{*,5/2}(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C)$ *onto* $\mathcal{K}^\omega_{a,*}(\boldsymbol{R}^2_o)$ (*cf.* [6, Theorem 5.6]).

We may assume that the origin $\mathbf{0}$ is the midpoint of C, and give here the following terminologies:

Definition 1.2. The curve C admits a *symmetry* at $\mathbf{0}$ if there exists $T \in \mathrm{O}(3)$ such that $T(C) = C$ and T is not the identity. Moreover, T is said to be *trivial* if $T(P) = P$ for all $P \in C$. A symmetry of C which is not trivial is called a *non-trivial symmetry*. (Obviously, each non-trivial symmetry reverses the orientation of C.) A non-trivial symmetry is called *positive* (resp. *negative*) if $T \in \mathrm{SO}(3)$ (resp. $T \in \mathrm{O}(3) \setminus \mathrm{SO}(3)$).

If C lies in a plane, then there exists a reflection $S \in \mathrm{O}(3)$ with respect to the plane. Then S is a trivial symmetry of C. We prove the following assertion.

Theorem III. *Let* $f \in \mathcal{G}^\omega_{**,3/2}(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$*, that is,* f *is admissible. Then the number of the right equivalence classes of* f*,* $\check{f}$*,* f_* *and* $\check{f}_*$ *is four if and only if* ds^2_f *has no symmetries* (*cf. Definition 0.4*).

Moreover, we can prove the following:

Theorem IV. *Let* $f \in \mathcal{G}^\omega_{**,3/2}(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$*. Then the number* N_f *of the congruence classes of the images of* $f, \check{f}$*,* f_* *and* $\check{f}_*$ *satisfies the following properties:*

(1) *If* C *has no non-trivial symmetries, and also* ds^2_f *has no symmetries, then* $N_f = 4$*,*
(2) *if not the case in* (1)*, it holds that* $N_f \le 2$*,*
(3) $N_f = 1$ *if and only if*
    (a) C *lies in a plane and has a non-trivial symmetry,*
    (b) C *lies in a plane and* ds^2_f *has a symmetry, or*
    (c) C *has a positive symmetry and* ds^2_f *also has a symmetry.*

2. Kossowski metrics

In this section, we quickly review several fundamental properties of Kossowski metrics.

Definition 2.1. Let p be a singular point of a given positive semi-definite metric ds^2 on M^2. Then a non-zero tangent vector $\boldsymbol{v} \in T_pM^2$ is called a *null vector* if

$$ds^2(\boldsymbol{v}, \boldsymbol{v}) = 0. \tag{2.1}$$

Moreover, a local coordinate neighborhood $(U; u, v)$ is called *adjusted* at $p \in U$ if $\partial_v := \partial/\partial v$ gives a null vector of ds^2 at p.

It can be easily checked that (2.1) implies that $ds^2(\boldsymbol{v}, \boldsymbol{w}) = 0$ for all $\boldsymbol{w} \in T_pM^2$. If $(U; u, v)$ is a local coordinate neighborhood adjusted at $p \in U$, then $F(p) = G(p) = 0$ holds, where

$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2. \tag{2.2}$$

Definition 2.2. A singular point $p \in M^2$ of a C^r-differentiable positive semi-definite metric ds^2 on M^2 is called *K-admissible* if there exists a local coordinate neighborhood $(U; u, v)$ adjusted at p satisfying

$$E_v(p) = 2F_u(p), \qquad G_u(p) = G_v(p) = 0, \tag{2.3}$$

where E, F, G are the C^r-functions on U given in (2.2).

If ds^2_f is the induced metric of a C^r-map $f : U \to \boldsymbol{R}^3$ and $f_v(p) = \mathbf{0}$, then (2.3) is satisfied automatically (cf. Proposition 3.1). The property (2.3) does not depend on the choice of a local coordinate system adjusted at p, as shown in [8] and [4, Proposition 2.7]. In fact, a coordinate-free treatment for the K-admissibility of singular points is given in [8] and [4, Definition 2.3].

Definition 2.3. A positive semi-definite C^r-differentiable metric ds^2 is called a *Kossowski metric* if each singular point $p \in M^2$ of ds^2 is K-admissible and there exists a C^r-function $\lambda(u, v)$ defined on a local coordinate neighborhood $(U; u, v)$ of p such that

$$EG - F^2 = \lambda^2 \qquad (\text{on } U), \tag{2.4}$$

$$(\lambda_u(p), \lambda_v(p)) \neq (0, 0), \tag{2.5}$$

where E, F, G are C^r-functions on U given in (2.2).

The above function λ is determined up to $\pm$-ambiguity (see [5, Proposition 3]). We call such a λ the *signed area density function* of ds^2 with respect to the local coordinate neighborhood $(U; u, v)$. The following fact is known (cf. [8, 16]).

Fact 2.4. *Let ds^2 be a C^r-differentiable Kossowski metric defined on a domain U of the uv-plane. Then the 2-form $d\hat{A} := \lambda du \wedge dv$ on U is defined independently of the choice of adjusted local coordinates (u, v).*

We call $d\hat{A}$ the *signed area form* of ds^2. Let K be the Gaussian curvature defined on the complement of the singular set of ds^2.

Fact 2.5 ([8] and [4, Theorem 2.15])**.** *The 2-form $\Omega := K d\hat{A}$ can be extended as a C^r-differential form on U.*

Definition 2.6. We call Ω the *Euler form* of ds^2. If Ω vanishes (resp. does not vanish) at a singular point $p \in U$ of ds^2, then p is called a *parabolic point* (resp. *non-parabolic point*).

The following fact is also known (cf. [8, 4, 5]).

Fact 2.7. *Let p be a singular point of a Kossowski metric ds^2. Then the null space (i.e. the subspace generated by null vectors at p) of ds^2 is 1-dimensional.*

By applying the implicit function theorem for λ (cf. (2.5)), there exists a regular curve $\gamma(t)$ ($|t| < \varepsilon$) in the uv-plane (called the *singular curve*) parametrizing the singular set of ds^2 such that $\gamma(0) = p$. Then there exists a C^r-differentiable non-zero vector field $\eta(t)$ along $\gamma(t)$ which points in the null direction of the metric ds^2. We call $\eta(t)$ a *null vector field* along the singular curve $\gamma(t)$.

Definition 2.8 ([4])**.** A singular point $p \in M^2$ of a Kossowski metric ds^2 is said to be of *type I* or an A_2 *point* if the derivative $\gamma'(0)$ of the singular curve at p (called the *singular direction* at $\gamma(t)$) is linearly independent of the null vector $\eta(0)$. Moreover, ds^2 is called of *type I* if all of the singular points of ds^2 are of type I.

3. Generalized cuspidal edges

Fix a bounded closed interval $J (\subset \boldsymbol{R})$ and consider a C^r-embedding $\mathbf{c} : J \to \boldsymbol{R}^3$ with arc-length parameter. We assume that the curvature function $\kappa(u)$ of $\mathbf{c}(u)$ is positive everywhere. We fix a C^r-map $\tilde{f} : \tilde{U} \to \boldsymbol{R}^3$ defined on a domain $\tilde{U}$ in the xy-plane $\boldsymbol{R}^2$ containing $J_1 \times \{0\}$ such that each point of $J_1 \times \{0\}$ is a generalized cuspidal edge point and

$$\tilde{f}(J_1 \times \{0\}) = C \qquad (C := \mathbf{c}(J)),$$

where J_1 is a bounded closed interval in $\boldsymbol{R}$. Such an $\tilde{f}$ is called a *generalized cuspidal edge along* C. For such an $\tilde{f}$, there exists a diffeomorphism

$$\varphi : U \ni (u,v) \mapsto (x(u,v), y(u,v)) \in \varphi(U)(\subset \tilde{U})$$

such that

$$f(u,v) := \tilde{f}(x(u,v), y(u,v)) \tag{3.1}$$

satisfies

$$f(u,v) = \mathbf{c}(u) + \frac{v^2}{2}\hat{\xi}(u,v), \tag{3.2}$$

where $\hat{\xi}(u,0)$ gives a vector field along $\mathbf{c}$ which is linearly independent of $\mathbf{c}'(u)$.

Proposition 3.1. *The induced metrics of C^r-differentiable generalized cuspidal edges are C^r-differentiable Kossowski metrics whose singular points are of type I.*

Proof. Let f be a generalized cuspidal edge as in (3.2), and let $ds_f^2 = Edu^2 + 2Fdudv + Gdv^2$ be the first fundamental form of f. Then

$$E = f_u \cdot f_u, \quad F = f_u \cdot f_v, \quad G := f_v \cdot f_v$$

hold, where "$\cdot$" is the inner product of $\boldsymbol{R}^3$. Since $f_v(u,0) = \mathbf{0}$, one can easily check (2.3). By (3.2), we have

$$EG - F^2 = |f_u \times f_v|^2 = v^2 \left| \left(\mathbf{c}' + \frac{v^2}{2}\hat{\xi}_u\right) \times \left(\hat{\xi} + \frac{v}{2}\hat{\xi}_v\right) \right|^2,$$

where $\times$ denotes the cross product in $\boldsymbol{R}^3$. Since two vectors $\mathbf{c}'(u)$, $\hat{\xi}(u,0)$ are linearly independent, the function λ on U given by

$$\lambda := v\lambda_0, \qquad \lambda_0 := \left| \left(\mathbf{c}' + \frac{v^2}{2}\hat{\xi}_u\right) \times \left(\hat{\xi} + \frac{v}{2}\hat{\xi}_v\right) \right| \tag{3.3}$$

is C^r-differentiable and $\lambda_0(u,0) \neq 0$. Moreover, λ^2 coincides with $EG - F^2$. Since $\lambda_v \neq 0$, ds_f^2 is a Kossowski metric. Since $f_v(u,0) = \mathbf{0}$, $\partial_v := \partial/\partial v$ gives the null-direction, which is linearly independent of the singular direction ∂_u. So all singular points of ds_f^2 are of type I. □

Let ds_f^2 be the induced metric of C^r-differentiable generalized cuspidal edge $f \in \mathcal{G}^r(\boldsymbol{R}_J^2, \boldsymbol{R}^3, C)$. We set $\hat{K}(:= \lambda K)$ (cf. (1.1)), where K is the Gaussian curvature of ds_f^2 defined at points where ds_f^2 is positive definite. As mentioned in the introduction, $\hat{K}$ can be extended as a C^r-function on U. Moreover, $\check{K} := vK$ also can be considered as a C^r-function on U (cf. [12, 5]).

Corollary 3.2. *The following assertions hold:*

1. $\hat{K}(u,0) \neq 0$ *if and only if* $\check{K}(u,0) \neq 0$, *and*
2. $\hat{K}_u(u,0) \neq 0$ *if and only if* $\check{K}_u(u,0) \neq 0$, *under the assumption* $\hat{K}(u,0) = 0$.

Proof. By (3.3), we have the expression $\lambda = v\lambda_0$, where $\lambda_0(u,0) \neq 0$. So if we set $\check{K} = vK$, then $\hat{K} = \lambda_0\check{K}$, and $\hat{K}(u,0) = \lambda_0(u,0)\check{K}(u,0)$ hold, and so the first assertion is obvious. Differentiating $\hat{K} = \lambda_0\check{K}$, we have

$$\hat{K}_u = (\lambda_0)_u\check{K} + \lambda_0\check{K}_u.$$

Since $\hat{K}(u,0) = 0$ implies $\check{K}(u,0) = 0$, we have $\hat{K}_u(u,0) = \lambda_0(u,0)\check{K}_u(u,0)$, proving the second assertion. □

Remark 3.3. For a generalized cuspidal edge f,

$$\nu(u,v) := \frac{(2\mathbf{c}'(u) + v^2\hat{\xi}_u(u,v)) \times (2\hat{\xi}(u,v) + v\hat{\xi}_v(u,v))}{|(2\mathbf{c}'(u) + v^2\hat{\xi}_u(u,v)) \times (2\hat{\xi}(u,v) + v\hat{\xi}_v(u,v))|}$$

gives a C^r-differentiable unit normal vector field on U. So f is a frontal map.

Definition 3.4. A parametrization (u, v) of $f \in \mathcal{G}^r(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$ is called an *adapted coordinate system* (cf. [12, Definition 3.7]) if

(1) $f_v(u,0) = \mathbf{0}$ and $|f_u(u,0)| = |f_{vv}(u,0)| = 1$ along the u-axis,
(2) $f_{vv}(u,0)$ is perpendicular to $f_u(u,0)$.

To show the existence of an adapted coordinate system, we prepare the following under the assumption that the curve $\mathbf{c}(u)$ is real analytic:

Lemma 3.5 ([5, Proposition 6])**.** *Let ds^2 be a C^ω-differentiable Kossowski metric defined on an open subset $U(\subset \boldsymbol{R}^2)$. Suppose that $\gamma : J \to U$ is a real analytic singular curve with respect to ds^2 such that*

$$ds^2(\gamma'(t), \gamma'(t)) > 0 \qquad (t \in J). \tag{3.4}$$

Then, for each $t_0 \in J$, there exists a C^ω-differentiable local coordinate system $(V; u, v)$ containing $(t_0, 0)$ such that $V \subset U$ and the coefficients E, F, G of the first fundamental form

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2$$

satisfy the following three conditions:

(1) *$\gamma(u) = (u, 0)$, $E(u, 0) = 1$ and $E_v(u, 0) = 0$ hold along the u-axis,*
(2) *$F(u, v) = 0$ on V, and*
(3) *there exists a C^ω-function G_0 defined on V such that $G(u, v) = v^2G_0(u, v)/2$ and $G_0(u, 0) = 2$.*

Proof. Applying [5, Proposition 6] at the point $(t_0, 0)$ on a singular curve of ds^2, we obtain the desired local coordinate system. □

Corollary 3.6. *For each generalized cuspidal edge $f \in \mathcal{G}^\omega(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$ along C and for each singular point p of f, there exists a local coordinate neighborhood $(V; u, v)$ of p such that the restriction $f|_V$ of f is parametrized by an adapted coordinate system.*

Proof. We let ds^2_f be the first fundamental form of $f(x, y)$. By Lemma 3.5, we obtain a parameter change $(x, y) \mapsto (u(x, y), v(x, y))$ on a neighborhood of p such that the new parameter (u, v) of $f(u, v)$ defined by (3.1) satisfies (1)-(3) of Lemma 3.5 for the first fundamental form ds^2_f of f. Then we can show that this new coordinate system (u, v) is the desired one: Since the u-axis is the singular set of ds^2_f, we have $f_v(u, 0) = \mathbf{0}$. On the other hand, $f_u(u, 0) \cdot f_u(u, 0) = E(u, 0) = 1$ and

$$f_{vv}(u,0) \cdot f_u(u,0) = \left.\frac{\partial F(u,v)}{\partial v}\right|_{v=0} = 0. \tag{3.5}$$

Finally, we have

$$f_{vv}(u,0) \cdot f_{vv}(u,0) = \frac{1}{2} \left.\frac{\partial^2 G(u,v)}{\partial v^2}\right|_{v=0} = \frac{G_0(u,0)}{2} = 1,$$

proving the assertion. □

From now on, we assume that $f(u,v)$ is parametrized by the local coordinate system as in Definition 3.4. Then u is the arc-length parameter of the edge $\mathbf{c}(u) := f(u,0)$. In this section, we assume that the curvature function $\kappa(u)$ of $\mathbf{c}(u)$ is positive for each u. Then the torsion function $\tau(u)$ is well-defined. We can take the unit tangent vector $\mathbf{e}(u) := \mathbf{c}'(u)$ $('=d/du)$, and the unit principal normal vector $\mathbf{n}(u)$ satisfying $\mathbf{c}''(u) = \kappa(u)\mathbf{n}(u)$. We set

$$\mathbf{b}(u) := \mathbf{e}(u) \times \mathbf{n}(u),$$

which is the binormal vector of $\mathbf{c}(u)$. Since $f_{vv}(u,0)$ is perpendicular to $\mathbf{e}(u)$, we can write

$$f_{vv}(u,0) = \cos\theta(u)\mathbf{n}(u) - \sin\theta(u)\mathbf{b}(u), \tag{3.6}$$

which is called the *cuspidal direction*. As defined in the introduction,

- the plane $\Pi(\mathbf{c}(u))$ passing through $\mathbf{c}(u)$ spanned by $\mathbf{n}(u)$ and $\mathbf{b}(u)$ is the normal plane of the space curve $\mathbf{c}(u)$,
- the section of the image of f by $\Pi(\mathbf{c}(u))$ is a plane curve, which is called the *sectional cusp* at $\mathbf{c}(u)$, and
- the vector $f_{vv}(u,0)$ points in the tangential direction of the sectional cusp at $\mathbf{c}(u)$. So we call $\theta(u)$ the *cuspidal angle function*.
- By using $\theta(u)$, the *singular curvature* κ_s and the *limiting normal curvature* κ_ν along the edge of f (cf. [16]) are given in (0.4).

The following fact is important:

Lemma 3.7 ([16]). *The singular curvature is intrinsic. In particular, it is defined along the singular curve with respect to a given Kossowski metric* (cf. [4, (2.17)]). *More precisely,*

$$\kappa_s(u) = \frac{-E_{vv}(u,0)}{2} \tag{3.7}$$

holds, where (u,v) is the coordinate system as in Lemma 3.5.

Proof. As shown in [16, Proposition 1.8], κ_s is expressed as

$$\kappa_s = \frac{-F_vE_u + 2EF_{uv} - EE_{vv}}{2E^{3/2}\lambda_v}, \tag{3.8}$$

where (u,v) is a local coordinate system such that the u-axis is the singular set and ∂_v points in the null direction. If (u,v) is the local coordinate system as in Lemma 3.5, then $F=0$, $\lambda = v\sqrt{EG_0}$ and $E(u,0)=1$ hold. So we can obtain (3.7). $\square$

We now prove the following theorem under the assumption that the curve $\mathbf{c}$ is real analytic:

Theorem 3.8. *We let U be an open subset of the uv-plane $\boldsymbol{R}^2$ containing $J\times\{0\}$ and ds^2 a real analytic Kossowski metric satisfying* (3.4). *Suppose that the curvature function κ of the curve $\mathbf{c}$ is positive everywhere and the absolute value of the singular curvature $\kappa_s(u)$ of ds^2 along the singular curve*

$$J \ni u \mapsto (u,0) \in U$$

is less than $\kappa(u)$ for each $u\in J$. Then there exist two real analytic generalized cuspidal edges g_+, g_- defined on an open subset $V(\subset U)$ containing $J\times\{0\}$ satisfying the following properties:

1. *The maps $u\mapsto g_+(u,0)$ and $u\mapsto g_-(u,0)$ parametrize C, which induce the same orientation as $\mathbf{c}: J\to\boldsymbol{R}^3$.*
2. *ds^2 is the common first fundamental form of g_+ and g_-.*
3. *g_- is a faithful isomer of g_+.*
4. *If $\kappa_\nu^{\pm}: J\to\boldsymbol{R}$ are the limiting normal curvature functions of $g_\pm$, then $\kappa_\nu^- = -\kappa_\nu^+$ holds on J.*

(5) *If ds^2 is non-parabolic at $(u,0)$, then g_+ and g_- have cuspidal edges at $(u,0)$.*

Moreover, suppose that $h : U \to \boldsymbol{R}^3$ is a generalized cuspidal edge whose first fundamental form is ds^2. If $u \mapsto h(u,0)$ parametrizes C giving the same orientation as $\mathbf{c} : J \to \boldsymbol{R}^3$, then h coincides with g_+ or g_-.

We prove this theorem from here on out, as a modification of the proof given in [14].

Remark 3.9. For each $t_0 \in J$, we can take a connected local coordinate neighborhood $(V(t_0); u, v)$ of $(t_0, 0)$ satisfying (1), (2) and (3) of Lemma 3.5. Since J is compact, we can find finite points $t_1, ..., t_k \in J$ such that $\{V(t_j)\}_{j=1}^k$ covers the singular curve $J \times \{0\}$. It is sufficient to prove Theorem 3.8 by replacing U by each $V(t_j)$ $(j = 1, ..., k)$. (In fact, the assertion of Theorem 3.8 contains the uniqueness of $g_\pm$ on each $V(t_j)$, and so $g_\pm$ obtained in $V(t_j)$ can be uniquely extended to $V(t_j) \cup V(t_{j+1})$ for each $j = 1, ..., k-1$.)

The statements of Theorem 3.8 are properties of the maps $g_\pm$ which do not depend on the choice of a local coordinate system containing $J \times \{0\}$. As explained in Remark 3.9, we may assume the existence of a local coordinate system $(U; u, v)$ satisfying (1), (2) and (3) of Lemma 3.5, without loss of generality. Then U contains a bounded closed interval I on the u-axis such that $I \times \{0\}$ gives the singular set of ds^2. We now show the existence of a real analytic generalized cuspidal edge $g(u,v)$ such that $g(u,0) = \mathbf{c}(u)$, $g_v(u,0) = \mathbf{0}$ and

$$g_u \cdot g_u = E, \quad g_u \cdot g_v = 0, \quad g_v \cdot g_v = G,$$

which is defined on a neighborhood of $I \times \{0\}$ in U using the Cauchy-Kowalevski theorem. (We remark that $\mathbf{c}(u)$ is parametrized as an arc-length parameter.) As in Lemma 3.5, we can write $G = v^2 G_0/2$. The following lemma holds:

Lemma 3.10. *If there exists a real analytic generalized cuspidal edge g $(= g_\pm)$ as in Theorem 3.8, then it is a solution of the following system of partial differential equations*

$$\begin{cases} g_v & = v\zeta, \\ \xi_v & (= g_{uv}) = v\zeta_u, \\ \zeta_v & = \dfrac{1}{4}\left((\zeta, g_u, \xi_u)^T\right)^{-1}\left((G_0)_v, -v(G_0)_u, 2r - v(G_0)_{uu} + 4v\zeta_u \cdot \zeta_u\right)^T \end{cases} \tag{3.9}$$

of unknown $\boldsymbol{R}^3$-valued functions g, ξ, ζ with the initial data

$$g(u,0) = \mathbf{c}(u), \quad \xi(u,0) = \mathbf{c}'(u)(= g_u(u,0)), \quad \zeta(u,0) = \mathbf{x}(u), \tag{3.10}$$

on I, where A^T denotes the transpose of a 3×3-matrix A and

$$\mathbf{x}(u) := \cos\theta(u)\mathbf{n}(u) \mp \sin\theta(u)\mathbf{b}(u), \quad \cos\theta(u) := \frac{\kappa_s(u)}{\kappa(u)}. \tag{3.11}$$

Remark 3.11. Since $g_v = v\zeta$ and $\xi_v = v\zeta_u$, we have $\xi_v = v\zeta_u = g_{uv}$. Thus, the initial condition $\xi(u,0) = g_u(u,0)$ yields $\xi(u,v) = g_u(u,v)$.

Proof of Lemma 3.10. Since ds^2 is real analytic, E and G are real analytic functions. Since $g_v(u,0) = \mathbf{0}$, we can write

$$g_v(u,v) = v\zeta(u,v),$$

where $\zeta(u,v)$ is a real analytic function defined on a neighborhood of $I \times \{0\}$ in $\boldsymbol{R}^2$. Then

$$\zeta_v \cdot \zeta = \frac{(\zeta \cdot \zeta)_v}{2} = \frac{(G_0)_v}{4}. \tag{3.12}$$

On the other hand, since

$$v g_u \cdot \zeta = g_u \cdot g_v = 0, \tag{3.13}$$

we have $g_u \cdot \zeta = 0$. Differentiating this, we have

$$0 = v(\zeta \cdot g_u)_v = v\zeta_v \cdot g_u + v\zeta \cdot g_{uv} = v\zeta_v \cdot g_u + g_v \cdot g_{uv} = v\zeta_v \cdot g_u + \frac{G_u}{2}.$$

Since $G = v^2 G_0/2$, we have

$$\zeta_v \cdot g_u = -\frac{v}{4}(G_0)_u. \tag{3.14}$$

We now obtain information on $\zeta_v \cdot g_{uu}$. It holds that

$$v\zeta \cdot g_{uu} = g_v \cdot g_{uu} = (g_v \cdot g_u)_u - g_{uv} \cdot g_u = -g_{uv} \cdot g_u = -\frac{E_v}{2},$$

that is, we obtain

$$\zeta \cdot g_{uu} = -\frac{E_v}{2v}. \tag{3.15}$$

On the other hand, we have that

$$\begin{aligned} \zeta \cdot g_{uu} + v\zeta_v \cdot g_{uu} &= g_{vv} \cdot g_{uu} = (g_{vv} \cdot g_u)_u - g_{vvu} \cdot g_u \\ &= \left\{(g_v \cdot g_u)_v - (g_v \cdot g_{uv})\right\}_u - (g_{uv} \cdot g_u)_v + g_{uv} \cdot g_{uv} \\ &= (-G_u/2)_u - (E_v/2)_v + g_{uv} \cdot g_{uv}. \end{aligned}$$

This, together with (3.15), gives the following identity

$$\zeta_v \cdot g_{uu} = \frac{E_v - vE_{vv}}{2v^2} - v\frac{(G_0)_{uu}}{4} + v\zeta_u \cdot \zeta_u. \tag{3.16}$$

Since $E_v(u, 0) = 0$, the function E_v/v is a real analytic function, and the function

$$r(u, v) := \frac{E_v - vE_{vv}}{v^2} = \left(\frac{-E_v}{v}\right)_v \tag{3.17}$$

is also real analytic. By (3.13), (3.14) and (3.16), we have the third equality of (3.9) under the assumption that the 3×3 matrix

$$M(u, v) := (\zeta, g_u, \xi_u)$$

is regular, where $\xi := g_u$. The map g must have the initial data (3.10), where

$$\mathbf{x}(u) = \zeta(u, 0) = \lim_{v \to 0} \frac{g_v(u, v)}{v} = g_{vv}(u, 0).$$

By (3.6), $\mathbf{x}(u)$ can be written in the form

$$(\mathbf{x}_+(u) :=)\mathbf{x}(u) = \cos\theta(u)\mathbf{n}(u) - \sin\theta(u)\mathbf{b}(u), \tag{3.18}$$

where $\theta(u)$ is the function defined by (3.11) and $\kappa(u)$ (resp. $\kappa_s(u)$) is the curvature function of $\mathbf{c}(u)$ (resp. the singular curvature function defined by (3.7)). In fact, since the singular curvature κ_s of ds^2 is less than κ on I, there exists a real analytic angular function $\theta : I \to \boldsymbol{R}$ satisfying (3.11) and

$$0 < |\theta(u)| < \frac{\pi}{2} \qquad (u \in I).$$

Moreover, such a θ is determined up to a $\pm$-ambiguity. In particular,

$$(\mathbf{x}_-(u) :=)\mathbf{x}(u) = \cos\theta(u)\mathbf{n}(u) + \sin\theta(u)\mathbf{b}(u) \tag{3.19}$$

is the other possibility. $\square$

We now return to the proof of Theorem 3.8. We have

$$\begin{aligned}(M(u,0)=)\ &(\zeta(u,0),\, g_u(u,0),\, g_{uu}(u,0))\\ &=(\cos\theta(u)\mathbf{n}(u)-\sin\theta(u)\mathbf{b}(u),\, \mathbf{e}(u),\, \kappa(u)\mathbf{n}(u))\,.\end{aligned}$$

Since the singular curvature of ds^2 satisfies $|\kappa_s|<\kappa$ on I, the function $\sin\theta$ does not vanish on I. Thus the matrix $M(u,0)$ is regular for each $u\in I$. We can then apply the Cauchy-Kowalevski theorem (cf. [9]) for the system of partial differential equations (3.9) with initial data (3.10) and obtain a unique real analytic solution (g,ξ,ζ) of (3.9) defined on a neighborhood of $I\times\{0\}$ in $\boldsymbol{R}^2$. Thus, we obtained the existence of real analytic generalized cuspidal edges $g_\pm(u,v)$ corresponding to the initial data $\mathbf{x}_\pm(u)$. By the above construction of these $g_\pm$, the functions $\pm\theta$ coincide with the cuspidal angles of $g_\pm$, respectively. To accomplish the proof of Theorem 3.8, we need to verify that the first fundamental forms of $g_\pm$ coincide with ds^2. To show this, we consider the case $g=g_+$ with initial condition $\mathbf{x}(u):=\mathbf{x}_+(u)$, without loss of generality. The third equation of (3.9) yields $\zeta_v\cdot\zeta=(G_0)_v/4$, and hence we have $(\zeta\cdot\zeta-G_0/2)_v=0$. Since

$$\zeta(u,0)\cdot\zeta(u,0)-\frac{G_0(u,0)}{2}=\mathbf{x}(u)\cdot\mathbf{x}(u)-1=0,$$

the Cauchy-Kowalevski theorem yields that

$$\zeta\cdot\zeta=\frac{G_0}{2}. \tag{3.20}$$

Hence, by the first equation of (3.9), we have

$$g_v\cdot g_v=\frac{v^2G_0}{2}=G. \tag{3.21}$$

On the other hand, using (3.9), we have

$$(\xi-g_u)_v=\xi_v-g_{uv}=v\zeta_u-(g_v)_u=v\zeta_u-(v\zeta)_u=0.$$

The initial condition $\xi(u,0)=g_u(u,0)$ yields that $g_u=\xi$. Then $g_{uv}=\xi_v=v\zeta_u$ and

$$g_{uv}\cdot\zeta=v\zeta_u\cdot\zeta=v\frac{(\zeta\cdot\zeta)_u}{2}=\frac{v(G_0)_u}{4}$$

hold. Using this, we have

$$(g_u\cdot\zeta)_v=g_{uv}\cdot\zeta+g_u\cdot\zeta_v=\frac{v(G_0)_u}{4}-\frac{v(G_0)_u}{4}=0.$$

Since $g_u(u,0)\cdot\zeta(u,0)=0$, we can conclude that $g_u\cdot\zeta=0$, that is,

$$g_u\cdot g_v=0 \tag{3.22}$$

is obtained. We now prepare the following:

Lemma 3.12. *Suppose that* (*which is one of the conditions in* (3.9))

$$\zeta_v\cdot\xi_u(=\zeta_v\cdot g_{uu})=\frac{2r-v(G_0)_{uu}+4v\zeta_u\cdot\zeta_u}{4}.$$

Then the initial condition (3.18) *implies the following identity*

$$\frac{E_v}{2}+v\zeta\cdot\xi_u=0. \tag{3.23}$$

Proof. Using (3.20), we have that

$$\begin{aligned}(\zeta\cdot\xi_u)_v &= \zeta_v\cdot\xi_u+\zeta\cdot\xi_{uv}=\zeta_v\cdot\xi_u+\zeta\cdot g_{uuv}=\zeta_v\cdot\xi_u+\zeta\cdot(v\zeta_{uu})\\ &=\frac{1}{4}\Bigl(2r-v(G_0)_{uu}+4v\zeta_u\cdot\zeta_u\Bigr)+\zeta\cdot(v\zeta_{uu})\\ &=\frac{r}{2}-\frac{v}{2}(G_0)_{uu}+v(\zeta_u\cdot\zeta_u+\zeta\cdot\zeta_{uu})\\ &=\frac{r}{2}-\frac{v}{4}(\zeta\cdot\zeta)_{uu}+\frac{v}{2}(\zeta\cdot\zeta)_{uu}=\frac{r}{2}.\end{aligned}$$

By (3.17),

$$\left(\zeta\cdot\xi_u+\frac{E_v}{2v}\right)_v=0$$

holds. On the other hand, we have

$$\begin{aligned}\zeta(u,0)\cdot\xi_u(u,0) &= \mathbf{x}(u)\cdot g_{uu}(u,0)=(\cos\theta(u)\mathbf{n}(u)-\sin\theta(u)\mathbf{b}(u))\cdot\mathbf{c}''(u)\\ &=\bigl(\cos\theta(u)\mathbf{n}(u)-\sin\theta(u)\mathbf{b}(u)\bigr)\cdot(\kappa(u)\mathbf{n}(u))=\kappa(u)\cos\theta(u)\\ &=\kappa(u)\frac{\kappa_s(u)}{\kappa(u)}=\kappa_s(u)=\frac{-E_{vv}(u,0)}{2}=\lim_{v\to 0}\frac{-E_v(u,v)}{2v}.\end{aligned}$$

So we obtain (3.23). □

We again return to the proof of Theorem 3.8. By (3.23), we have

$$\frac{1}{2}(g_u\cdot g_u)_v=g_{uv}\cdot g_u=(g_v\cdot g_u)_u-g_v\cdot g_{uu}=-g_v\cdot g_{uu}=\frac{E_v}{2}.$$

This, with the initial condition $g_u(u,0)\cdot g_u(u,0)=\mathbf{c}'(u)\cdot\mathbf{c}'(u)=1$ implies

$$g_u\cdot g_u=E. \tag{3.24}$$

By (3.24), (3.22) and (3.21), we can conclude that ds^2 coincides with the first fundamental form of $g=g_+$, which implies the existence and uniqueness of $g=g_+$. Replacing θ by $-\theta$, we also obtain the existence and uniqueness of $g=g_-$. Since the cuspidal angles of $g_\pm$ are distinct, the image of g_- does not coincide with g_+. Since the orientation of $u\mapsto g_-(u,0)$ is compatible with that of the curve $u\mapsto g_+(u,0)$, the map g_- is a faithful isomer of g_+.

Here, we suppose ds^2 is non-parabolic at $(u,0)$, then g_+ and g_- are wave fronts by [5, Proposition 4 (o)]. Since ds^2 is of type I, the criterion of cuspidal edges given in [5, Proposition 4 (i)] yields that g_+ and g_- are both cuspidal edges.

Finally, the last assertion of Theorem 3.8 follows from the uniqueness of the system of partial equations (3.9) as a consequence of the Cauchy-Kowalevski theorem, proving Theorem 3.8.

By the above proof of Theorem 3.8, we obtain the following:

Corollary 3.13. *The cuspidal angle of g_- is $-\theta$, where θ is the cuspidal angle of g_+. In particular, g_- is a faithful isomer of g_+ since $\sin\theta\neq 0$.*

We next prove the following:

Lemma 3.14. *Let U be an open subset of the uv-plane $\boldsymbol{R}^2$ containing $J\times\{0\}$, and let ds^2 be a real analytic Kossowski metric of type I defined on U satisfying (1)–(3) of Lemma 3.5. Suppose that the singular set of ds^2 consists only of non-parabolic points. If there exist open subsets $V_i(\subset U)$ $(i=1,2)$ containing $J\times\{0\}$ and a diffeomorphism $\varphi: V_1\to V_2$ such that $\varphi^*ds^2=ds^2$ and $\varphi(u,0)=(u,0)$ hold for $u\in J$, then $V_1=V_2$ and φ is the identity map.*

Proof. Let $\mathbf{c}(u)$ $(u \in J)$ be a space curve satisfying the assumption of Theorem 3.8, and let g_+ be one of cuspidal edges realizing ds^2 as in Theorem 3.8. Since $g_+ \circ \varphi$ and g_+ have the common first fundamental form ds^2, the last assertion of Theorem 3.8 yields that $g_+ \circ \varphi$ coincides with either g_+ or g_-. Since $g_+ \circ \varphi$ and g_+ have the same image, they have a common cuspidal angle at each point of C. So there exists a symmetry T of C such that $T \circ g_+ \circ \varphi = g_+$. Suppose T is not the identity map. Since $\varphi(u, 0) = (u, 0)$, φ maps the domain $D_+ := \{v > 0\}$ to $D_- := \{v < 0\}$. However, it is impossible, because $\varphi^* ds^2 = ds^2$ and the Gaussian curvature on D_+ takes the opposite sign of that on D_- (cf. [5, (1.14)]). Thus, T is the identity map and $g_+ \circ \varphi = g_+$ holds. Since the singular set of g_+ consists of cuspidal edge points, g_+ is injective, and φ must be the identity map. □

Proposition 3.15. *Let ds^2 be a real analytic Kossowski metric belonging to $\mathcal{K}^\omega_*(\boldsymbol{R}^2_o)$. Suppose that φ is a local C^ω-diffeomorphism satisfying $\varphi^* ds^2 = ds^2$ and $\varphi(o) = o$ which is not the identity map. Then φ is an involution which reverses the orientation of the singular curve. Moreover, such a φ is uniquely determined.*

Proof. We can take a local coordinate system satisfying (1)–(3) of Lemma 3.5. Since $\varphi(o) = o$, the fact that $u \mapsto (u, 0)$ is the arc-length parametrization with respect to ds^2 yields that either $\varphi(u, 0) = (u, 0)$ or $\varphi(u, 0) = (-u, 0)$ holds. If $\varphi(u, 0) = (u, 0)$, then by Lemma 3.14, φ is the identity map, a contradiction. So we have $\varphi(u, 0) = (-u, 0)$. This means that φ reverses the orientation of the singular curve. In this situation, we have $\varphi \circ \varphi(u, 0) = (u, 0)$. Applying Lemma 3.14 again, $\varphi \circ \varphi$ is the identity map, that is, φ is an involution. We next suppose that ψ is another local C^ω-diffeomorphism satisfying $\psi^* ds^2 = ds^2$ and $\psi(o) = o$. Then $\varphi \circ \psi(u) = (u, 0)$ holds, and Lemma 3.14 yields that $\varphi \circ \psi$ is the identity map. So ψ must coincide with φ. □

Corollary 3.16. *Let ds^2_f be a real analytic Kossowski metric as the first fundamental form of $f \in \mathcal{G}^\omega_{*,3/2}(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$. Suppose that φ is a C^ω-symmetry of ds^2_f, then it is effective and is an involution reversing the orientation of the singular curve.*

Proof. Without loss of generality, we may assume that the parameters (u, v) of $f(u, v)$ satisfy (1)-(3) of Lemma 3.5 for ds^2_f. Let P be the midpoint of C with respect to the arc-length parameter. Then there exists $c \in J$ such that $f(c, 0) = P$. Thinking $o := (c, 0)$, we may regard f belongs to $\mathcal{G}^\omega_{*,3/2}(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C)$. Since $f \in \mathcal{G}^\omega_{*,3/2}(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$, by restricting f to a neighborhood of o, the metric ds^2_f can be considered as an element of $\mathcal{K}^\omega_*(\boldsymbol{R}^2_o)$ (cf. [5, (2) of Theorem A]). So the symmetry φ of ds^2_f satisfies the desired property by Proposition 3.15. Since φ is real analytic, the property is extended on a tubular neighborhood of the singular curve. □

Moreover, the following important property for symmetries of Kossowski metrics is obtained:

Theorem 3.17. *Let p be a singular point of a real analytic Kossowski metric ds^2 which is an accumulation point of non-parabolic singular points of type I. Suppose that φ is a local C^ω-diffeomorphism fixing p satisfying $\varphi^* ds^2 = ds^2$. Then φ is an involution and reverses the orientation of the singular curve if it is not the identity map.*

Proof. Let $\gamma(t)$ be a real analytic parametrization of the singular curve of the real analytic Kossowski metric ds^2 such that $\gamma(0) = p$. We let $\{p_n\}_{n=1}^\infty$ be a sequence of non-parabolic points converging to p. Since γ is real analytic, the existence of such a sequence implies that, for sufficiently small $\varepsilon(> 0)$, $\gamma((-\varepsilon, 0) \cup (0, \varepsilon))$ consists of non-parabolic points of type I. Then

$$s(t) := \int_0^t \sqrt{ds^2(\gamma'(u), \gamma'(u))}\, du \qquad (t \in (-\varepsilon, \varepsilon))$$

is a monotone increasing function of t, giving a continuous parametrization of γ. Using this parameter s, either $\varphi\circ\gamma(s)=\gamma(s)$ or $\varphi\circ\gamma(s)=\gamma(-s)$ holds. If the former case happens, then applying Proposition 3.15 at a non-parabolic point $\gamma(s)$ ($s\neq 0$), φ must be the identity map on a neighborhood of $\gamma(s)$. Since φ is real analytic, it must be the identity map on a neighborhood of p.

We next consider the case that $\varphi\circ\gamma(s)=\gamma(-s)$. Then $\varphi\circ\varphi\circ\gamma(s)=\gamma(s)$, and the above argument implies that φ is an involution, proving the assertion. □

Proof of Theorem I. Let ds^2_f be the first fundamental form of f. Then ds^2_f is a Kossowski metric of type I, by Proposition 3.1. Since f belongs to $\mathcal{G}^\omega_*(\boldsymbol{R}^2_J,\boldsymbol{R}^3,C)$ (cf. (0.5)), the singular curvature κ_s of ds^2_f is less than κ on J. By Theorem 3.8, there exist two generalized cuspidal edges $g_+,g_-\in\mathcal{G}^\omega_*(\boldsymbol{R}^2_J,\boldsymbol{R}^3,C)$ whose first fundamental forms coincide with ds^2_f. Since ds^2_f is the first fundamental form of f, the last assertion of Theorem 3.8 yields that either $f=g_+$ or $f=g_-$ holds. Without loss of generality, we may set $f=g_+$, then $\check{f}:=g_-$ is the desired isometric dual of f. The remaining assertions for $f\in\mathcal{G}^\omega_*(\boldsymbol{R}^2_o,\boldsymbol{R}^3,C)$ follow from Lemma 0.1. □

Definition 3.18. For each $f\in\mathcal{G}^\omega_*(\boldsymbol{R}^2_o,\boldsymbol{R}^3,C)$ (resp. $f\in\mathcal{G}^\omega_*(\boldsymbol{R}^2_J,\boldsymbol{R}^3,C)$), we call the above $\check{f}\in\mathcal{G}^\omega_*(\boldsymbol{R}^2_o,\boldsymbol{R}^3,C)$ (resp. $\check{f}\in\mathcal{G}^\omega_*(\boldsymbol{R}^2_J,\boldsymbol{R}^3,C)$) the *isometric dual* of f.

4. A representation formula for generalized cuspidal edges

We set $J=[-l,l]$ ($l>0$). Let $\mathbf{c}:J\to\boldsymbol{R}^3$ be an embedding with arc-length parameter whose curvature function $\kappa(u)$ is positive everywhere. We denote by $\mathbf{e}(u):=\mathbf{c}'(u)$, and by C the image of $\mathbf{c}$. We let $\mathbf{n}(u)$ and $\mathbf{b}(u)$ be the unit principal normal vector field and unit binormal vector field of $\mathbf{c}(u)$, respectively. We fix a sufficiently small $\delta(>0)$ and consider a map given by

$$f(u,v):=\mathbf{c}(u)+(A(u,v),B(u,v))\begin{pmatrix}\cos\theta(u) & -\sin\theta(u)\\ \sin\theta(u) & \cos\theta(u)\end{pmatrix}\begin{pmatrix}\mathbf{n}(u)\\ \mathbf{b}(u)\end{pmatrix}, \tag{4.1}$$

where $u\in J$ and $|v|<\delta$. Here $A(u,v)$, $B(u,v)$ and $\theta(u)$ are C^r-functions, and satisfy

$$A(u,0)=A_v(u,0)=0,\quad A_{vv}(u,0)\neq 0,\quad B(u,0)=B_v(u,0)=B_{vv}(u,0)=0.$$

Then it can be easily checked that any generalized cuspidal edges along C are right equivalent to one of such a map. Moreover, if $B_{vvv}(u,0)\neq 0$, then f is a cuspidal edge along C. The function $\theta(u)$ is called the *cuspidal angle* at $\mathbf{c}(u)$. Let $\kappa(u)$ be the curvature of $\mathbf{c}(u)$. Then the C^r-functions defined by

$$\kappa_s(u)=\kappa(u)\cos\theta(u),\qquad \kappa_\nu(t)=\kappa(u)\sin\theta(u) \tag{4.2}$$

give the *singular curvature* and the *limiting normal curvature* respectively. The map germ f can be determined by

$$(\theta(u),\ A(u,v),\ B(u,v)).$$

We call these functions *Fukui's data*.

Definition 4.1. In the expression (4.1), if

- u is an arc-length parameter of $\mathbf{c}$,
- for each $u\in J$, the map $(-\delta,\delta)\ni t\mapsto(A(u,t),B(u,t))\in\boldsymbol{R}^2$ is a generalized cusp at $t=0$ (called a *sectional cusp at* u), and t gives a normalized half-arc-length parameter (see the appendix),

then the expression (4.1) of f by setting $v=t$ as the normalized half-arc-length parameter is called the *normal form* of a generalized cuspidal edge.

We now fix such a normal form f. We set

$$\begin{pmatrix}\mathbf{v}_2(u)\\ \mathbf{v}_3(u)\end{pmatrix}=\begin{pmatrix}\cos\theta(u) & -\sin\theta(u)\\ \sin\theta(u) & \cos\theta(u)\end{pmatrix}\begin{pmatrix}\mathbf{n}(u)\\ \mathbf{b}(u)\end{pmatrix}, \tag{4.3}$$

then we have

$$f(u,t)=\mathbf{c}(u)+A(u,t)\mathbf{v}_2(u)+B(u,t)\mathbf{v}_3(u). \tag{4.4}$$

Definition 4.2. Let (a,b) $(a<b)$ be an interval on $\boldsymbol{R}$, and $\delta\in(0,\infty]$ a positive number. A C^r-differentiable ($r=\infty$ or $r=\omega$) quadruple $(\kappa,\tau,\theta,\hat{\mu})$ is called a *fundamental data* (or a *modified Fukui-data*) if

- $\kappa:(a,b)\to\boldsymbol{R}$ is a C^r-function such that $\kappa>0$,
- $\tau,\theta:(a,b)\to\boldsymbol{R}$ and $\hat{\mu}:(a,b)\times(-\delta,\delta)\to\boldsymbol{R}$ are C^r-functions.

Summarizing the above discussions, one can easily show the following representation formula for generalized cuspidal edges, which is a mixture of Fukui's representation formula as in [3, (1.1)] for generalized cuspidal edges and a representation formula for cusps in the appendix (cf. Lemma A.1):

Proposition 4.3. *Let $(\kappa,\tau,\theta,\hat{\mu})$ be a given fundamental data and $\mathbf{c}(u)$ $(u\in J)$ the space curve with arc-length parameter whose curvature function and torsion function are $\kappa(u)$ and $\tau(u)$. Then,*

$$f(u,t):=\mathbf{c}(u)+(A(u,t),B(u,t))\begin{pmatrix}\cos\theta(u) & -\sin\theta(u)\\ \sin\theta(u) & \cos\theta(u)\end{pmatrix}\begin{pmatrix}\mathbf{n}(u)\\ \mathbf{b}(u)\end{pmatrix} \tag{4.5}$$

gives a generalized cuspidal edge written in a normal form along $C:=\mathbf{c}(J)$, where (A,B) is given by

$$(A(u,t),B(u,t))=\int_0^t v(\cos\lambda(u,v),\sin\lambda(u,v))dv,\ \ \lambda(u,t):=\int_0^t\hat{\mu}(u,v)dv. \tag{4.6}$$

Moreover,

(1) *θ gives the cuspidal angle of f along $\mathbf{c}$,*
(2) *$t\mapsto\hat{\mu}(u,t)$ is the function given in (A.2) for the sectional cusp of f at u.*

Furthermore, any generalized cuspidal edge along C is right equivalent to such an f constructed in this manner (see also Remark 0.5).

Remark 4.4. Let $\mathbf{c}_0(u)$ be a space curve parametrized by the arc-length parameter u defined on an interval $J:=[-l,l]$ $(l>0)$, whose curvature function and torsion function are $\kappa(u)$ and $\tau(u)$, respectively. We assume that $\mathbf{c}_0(0)=\mathbf{0}$. Suppose that $C:=\mathbf{c}_0(J)$ admits a non-trivial symmetry T. Since $\mathbf{0}$ is the midpoint of C and is fixed by T, we may assume that $T\in\mathrm{O}(3)$ and set $\sigma:=\det(T)\in\{1,-1\}$. Then $\mathbf{c}_1(u):=T\mathbf{c}_0(-u)$ is a space curve whose curvature function and torsion function are $\kappa(u)$ and $\sigma\tau(u)$ respectively. We denote by $\mathbf{e}_i(u)(:=\mathbf{c}_i'(u))$, $\mathbf{n}_i(u)$ and $\mathbf{b}_i(u)$ $(i=0,1)$ the unit tangent vector, unit principal normal vector and unit binormal vector of $\mathbf{c}_i(u)$, respectively. Differentiating $T\circ\mathbf{c}_0(u)=\mathbf{c}_1(u)$, we have

$$\begin{aligned}&T\mathbf{e}_0(-u)=T\circ\mathbf{c}_0'(-u)=-\mathbf{c}_1'(u)=-\mathbf{e}_1(u),\\ &\kappa_0(-u)T\mathbf{n}_0(-u)=T\circ\mathbf{c}_0''(-u)=\mathbf{c}_1''(u)=\kappa_1(u)\mathbf{n}_1(u).\end{aligned}$$

In particular, $T\mathbf{e}_0(-u)=-\mathbf{e}_1(u)$, $T\mathbf{n}_0(-u)=\mathbf{n}_1(u)$ and $\kappa_0(-u)=\kappa_1(u)$ hold, where κ_i $(i=1,2)$ is the curvature function of $\mathbf{c}_i$. Since $\sigma:=\det(T)\in\{1,-1\}$, we have

$$\mathbf{b}_0=\mathbf{e}_0\times\mathbf{n}_0=(-T\mathbf{e}_1)\times(T\mathbf{n}_1)=-T\left(\mathbf{e}_1\times\mathbf{n}_1\right)=-\sigma T\mathbf{b}_1.$$

Using this, one can also obtain the relation $-\sigma\tau_0(-u) = \tau_1(u)$, where τ_i $(i = 1, 2)$ is the torsion function of $\mathbf{c}_i$. We set

$$f_i := \mathbf{c}_i + (A_i, B_i)\begin{pmatrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{pmatrix}\begin{pmatrix} \mathbf{n}_i \\ \mathbf{b}_i \end{pmatrix} \qquad (i = 0, 1),$$

and suppose

$$A_0(-u,t) = A_1(u,t), \quad B_0(-u,t) = -\sigma B_1(u,t), \quad \theta_0(-u) = -\sigma\theta_1(u).$$

Then

$$\begin{aligned}
&T \circ f_0(-u,t) \\
&= T\mathbf{c}_0(-u) + (A_0(-u,t), B_0(-u,t))\begin{pmatrix} \cos\theta_0(-u) & -\sin\theta_0(-u) \\ \sin\theta_0(-u) & \cos\theta_0(-u) \end{pmatrix}\begin{pmatrix} T\mathbf{n}_0(-u) \\ T\mathbf{b}_0(-u) \end{pmatrix} \\
&= \mathbf{c}_1(u) + (A_1(u,t), -\sigma B_1(u,t))\begin{pmatrix} \cos(-\sigma\theta_1(u)) & -\sin(-\sigma\theta_1(u)) \\ \sin(-\sigma\theta_1(u)) & \cos(-\sigma\theta_1(u)) \end{pmatrix}\begin{pmatrix} \mathbf{n}_1(u) \\ -\sigma\mathbf{b}_1(u) \end{pmatrix} \\
&= \mathbf{c}_1(u) + (A_1(u,t), B_1(u,t))\begin{pmatrix} \cos\theta_1(u) & -\sin\theta_1(u) \\ \sin\theta_1(u) & \cos\theta_1(u) \end{pmatrix}\begin{pmatrix} \mathbf{n}_1(u) \\ \mathbf{b}_1(u) \end{pmatrix} = f_1(u,t).
\end{aligned}$$

Thus, we obtain the relation $f_1(u,t) = T \circ f_0(-u,t)$. In particular, f_1 has the same first fundamental form as f_0. Moreover,

(a) if $T \in \mathrm{SO}(3)$, then the cuspidal angle of f_1 takes opposite sign of that of f_0. By the uniqueness of the isometric dual of f_0 (cf. Theorem 3.8), $\check{f}_0(u,t) = f_1(u,t) = T \circ f_0(-u,t)$ holds, that is, f_1 is the faithful isomer (i.e. the isometric dual) of f_0.

(b) if $T \in \mathrm{O}(3) \setminus \mathrm{SO}(3)$, then the cuspidal angle of f_1 coincides with that of f_0. Then $f_0(u,t) = f_1(u,t) = T \circ f_0(-u,t)$ holds (cf. Theorem 3.8), that is, the image of f_0 is invariant by T.

Remark 4.5. Let $f(u,t)$ be a generalized cuspidal edge associated to the data

$$(\kappa(u), \tau(u), \theta(u), \hat{\mu}(u,t)).$$

Then $f_\#(u,t) := f(-u,t)$ is also a generalized cuspidal edge along the same space curve as f but with the reversed orientation. If we set $\mathbf{c}_\#(u) := \mathbf{c}(-u)$, then $\mathbf{c}_\#(u) = f_\#(u,0)$ holds. By a similar calculation like as in Remark 4.4, one can easily verify that $(\kappa(-u), -\tau(-u), -\theta(-u), \hat{\mu}(-u,t))$ gives the fundamental data of $f_\#(u,t)$.

We next prove Theorem II in the introduction.

Proof of Theorem II. We fix $f \in \mathcal{G}^\omega_{**}(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$ arbitrarily. We denote by ds^2_f the first fundamental form of f. Since f is admissible, the singular curvature $\kappa_s(u)$ satisfies (0.9), and so (0.7) holds. By Theorem 3.8, there exist two distinct generalized cuspidal edges $g_\pm$ whose first fundamental forms coincide with ds^2_f such that $g_+ = f$, and $u \mapsto g_-(u,0)$ has the same orientation as that of $u \mapsto f(u,0)$. Since f is admissible, the singular curvature κ_s is determined only by ds^2_f. Thus $g_\pm$ belong to $\mathcal{G}^\omega_{**}(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$. By the proof of Theorem I, we know that $\check{f} := g_-$ gives the isometric dual of f.

On the other hand, we replace u with $-u$ (that is, the orientation of C is reversed). Since f is admissible, it holds that

$$0 < |\kappa_s(u)| \le \min_{u \in J} \kappa(u) < \kappa(-u) \qquad (u \in J).$$

So, applying Theorem 3.8 again, there exist two distinct generalized cuspidal edges

$$h_\pm \in \mathcal{G}^\omega_{**}(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$$

such that $u \mapsto h_\pm(u,0)$ have the same orientation as that of $u \mapsto f(-u,0)$. Then ds_f^2 gives the common first fundamental form of the generalized cuspidal edges $h_\pm$. By (3.11), we may assume that the cuspidal angle $\theta_*(u)$ (resp. $-\theta_*(u)$) ($\theta_*(u)\theta(u) > 0$) of h_+ (resp. h_-) satisfies

$$\cos\theta_*(u) = \frac{\kappa_s(u)}{\kappa(-u)}.$$

Since the orientation of the singular curves of $h_\pm$ is opposite of that of f, the two maps $h_\pm$ are non-faithful isomers of f. We set

$$f_* := h_+ \text{ (the inverse), and } \check{f}_* := h_- \text{ (the inverse dual).}$$

By the above Remark 4.5, the cuspidal angle of $f_\#(u,v) := f(-u,v)$ is $-\theta(-u)$, the cuspidal angle $\theta_*(u)$ takes opposite sign of that of $f_\#(u,v)$. So the image of f does not coincide with that of f_*. Hence f_* is an isomer of f.

By our construction of f_*, (1), (2) and (3) are obvious. So we prove (4). We suppose that the first fundamental form of a generalized cuspidal edge $k \in \mathcal{G}^\omega_{**}(\boldsymbol{R}^2_I, \boldsymbol{R}^3, C)$ is isometric to ds_f^2. (The case that $k \in \mathcal{G}^\omega_*(\boldsymbol{R}^2_o, \boldsymbol{R}^3, C)$ is obtained by Lemma 0.1.) Since the first fundamental form is determined independently of a choice of local coordinate system, we have $\mathcal{J}_C(f\circ\varphi) = \mathcal{J}_C(f)\circ\varphi$, where φ is a diffeomorphism on a certain tubular neighborhood of $J \times \{0\}$. So we may assume that $ds_k^2 = ds_f^2$ without loss of generality. Then k must coincide with one of $\{g_+, g_-, h_+, h_-\}$, because of the uniqueness of the solution of (3.9) with initial condition (3.10). □

Definition 4.6. We call the above f_* and $\check{f}_*$ the *inverse* and the *inverse dual* of $f \in \mathcal{G}^\omega_{**}(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$, respectively.

We next give criteria of a given germ of generalized cuspidal edge to be a cuspidal edge, cuspidal cross cap or 5/2-cuspidal edge in terms of the extended half-cuspidal curvature function $\hat{\mu}$.

Proposition 4.7. *Let $f \in \mathcal{G}^r(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$ be the generalized cuspidal edge associated to a fundamental data $(\kappa, \tau, \theta, \hat{\mu})$. Then*

1. *f gives a cuspidal edge along the u-axis if $\hat{\mu}(u,0) \neq 0$,*
2. *f gives a cuspidal cross cap at o if $\hat{\mu}(0,0) = 0$ and $\hat{\mu}_u(0,0) \neq 0$,*
3. *f gives a 5/2-cuspidal edge along the u-axis if $\hat{\mu}(u,0) = 0$ and $\hat{\mu}_{vv}(u,0) \neq 0$.*

The first and the second assertions have been proved in [3, Proposition 1.6].

Proof. We may assume that f is written in a normal form. The first assertion follows from (1) of Proposition A.2. The second assertion follows from the criterion for cuspidal cross caps given in [2], but can be proved much easier using (2) of [3, Proposition 4.4]. The third assertion is a consequence of (2) of Proposition A.2. □

To compute the first and the second fundamental forms of f in terms of fundamental data, the following Frenet-type formula for singular curves is convenient.

Lemma 4.8 (Izumiya-Saji-Takeuchi [7] and Fukui [3]). *The following formula holds (cf. (4.3)):*

$$\begin{pmatrix} \mathbf{e}' \\ \mathbf{v}_2' \\ \mathbf{v}_3' \end{pmatrix} = \begin{pmatrix} 0 & \kappa\cos\theta & \kappa\sin\theta \\ -\kappa\cos\theta & 0 & \tau-\theta' \\ -\kappa\sin\theta & -(\tau-\theta') & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e} \\ \mathbf{v}_2 \\ \mathbf{v}_3 \end{pmatrix}. \tag{4.7}$$

This formula can be rewritten as (cf. (4.3))

$$\begin{pmatrix} \mathbf{e}' \\ \mathbf{v}_2' \\ \mathbf{v}_3' \end{pmatrix} = \begin{pmatrix} 0 & \kappa_s & \kappa_\nu \\ -\kappa_s & 0 & \kappa_t \\ -\kappa_\nu & -\kappa_t & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e} \\ \mathbf{v}_2 \\ \mathbf{v}_3 \end{pmatrix},$$

which is the one given in Izumiya-Saji-Takeuchi [7, Proposition 3.1], where κ_t is the *cusp-directional torsion* defined in [11] and has the expression (cf. [3, Page 7])

$$\kappa_t = \tau - \theta'. \tag{4.8}$$

Using Lemma 4.8, one can easily obtain the following by a straightforward computation:

Proposition 4.9 (Fukui [3])**.** *The first fundamental form* $ds_f^2 = Edu^2 + 2Fdudt + Gdt^2$ *of* f *as in* (4.5) *is given by*

$$\begin{aligned} E &= (1 - (A\cos\theta + B\sin\theta)\kappa)^2 + (A_u + (\theta' - \tau)B)^2 + (B_u - (\theta' - \tau)A)^2, \\ F &= A_t(A_u + (\theta' - \tau)B) + B_t(B_u - (\theta' - \tau)A), \quad G = t^2, \end{aligned} \tag{4.9}$$

where κ, τ, θ *are functions of* u *and* A, B *are functions of* (u,t).

Proof. Differentiating $f = \mathbf{c} + A\mathbf{v}_2 + B\mathbf{v}_3$, we have

$$\begin{aligned} f_u &= (1 - (A\cos\theta + B\sin\theta)\kappa)\mathbf{e} + (A_u + (\theta' - \tau)B)\mathbf{v}_2 + (B_u - (\theta' - \tau)A)\mathbf{v}_3, \\ f_t &= A_t\mathbf{v}_2 + B_t\mathbf{v}_3. \end{aligned}$$

Since $E = f_u \cdot f_u$, $F = f_u \cdot f_t$ and $G = f_t \cdot f_t$, we obtain the assertion. □

We can write

$$\hat{\mu}(u,t) = \mu_0(u) + \mu_1(u)t + \mu_2(u)t^2 + \mu_3(u,t)t^3,$$

and then Lemma A.1 yields that

$$A = \frac{t^2}{2} - \frac{\mu_0(u)^2}{8}t^4 - \frac{\mu_0(u)\mu_1(u)}{10}t^5 + t^6 a_6(t,u), \tag{4.10}$$

$$B = \frac{\mu_0(u)}{3}t^3 + \frac{\mu_1(u)}{8}t^4 + \frac{2\left(-\mu_0(u)^3 + 2\mu_2(u)\right)}{30}t^5 + t^6 b_6(t,u), \tag{4.11}$$

where $a_6(t,u)$ and $b_6(t,u)$ denote C^r-functions.

Corollary 4.10. *The Gaussian curvature* K *of* ds_f^2 *satisfies*

$$K(u,t) = \frac{K_0(u)}{t} + K_1(u) + K_2(u)t + K_3(u,t)t^2,$$

where

$$\begin{aligned} K_0 &:= \mu_0\kappa_\nu, \quad K_1 := -\kappa_s\mu_0^2 - \kappa_t^2 + \kappa_\nu\mu_1, \\ K_2 &:= -\frac{\kappa_\nu\mu_0^3}{2} + \frac{\kappa_s\kappa_\nu\mu_0}{2} - \frac{3\kappa_s\mu_0\mu_1}{2} + \kappa_\nu\mu_2 - 2\mu_0'\kappa_t + \frac{\mu_0}{2}\kappa_t', \end{aligned}$$

and $K_3(u,t)$ *is a* C^r*-function. Here* κ_s, κ_ν *and* κ_t *are defined in* (0.4) *and* (4.8)*. Moreover,* $\mu_0 = \kappa_c/2$ *(cf.* (1.5)*) and* $\kappa_t' = d\kappa_t(u)/du$.

Fukui [3, Theorem 1.8] has already determined the first two terms K_0 and K_1. So the essential part of the above corollary is the statement for K_2.

Proof. One can obtain this formula by computing the sectional curvature of ds_f^2, or alternatively, one can get it by computing the second fundamental form of f as Fukui did in [3]. In each approach, (4.10) and (4.11) play crucial roles. □

As a consequence of this corollary, the first term

$$K_0 := \mu_0\kappa_\nu = \frac{\kappa_c\kappa_\nu}{2}$$

defined in [12] is an intrinsic invariant, which is called the *product curvature*. The second term K_1 is an intrinsic invariant. We consider the term K_2. Since $K_0 = \kappa_c\kappa_\nu/2$, and since μ_0 is equal to the cuspidal curvature κ_c, the fact that κ_s and $\kappa_c\kappa_\nu$ are intrinsic yields that

$$\tilde{K}_2 := -\frac{\kappa_\nu \mu_0^3}{2} - \frac{3\kappa_s\mu_0\mu_1}{2} + \kappa_\nu\mu_2 - 2\mu_0'\kappa_t + \frac{\mu_0}{2}\kappa_t'$$

is also an intrinsic invariant. Using this, we can prove the following assertion:

Proposition 4.11. *Let $f \in \mathcal{G}^r(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$ be the generalized cuspidal edge associated to a fundamental data $(\kappa, \tau, \theta, \hat{\mu})$ satisfying $\sin\theta \neq 0$. Then*

(1) *f gives a cuspidal edge along the u-axis if $K_0(u) \neq 0$,*
(2) *f gives a cuspidal cross cap at $u = 0$ if $K_0(0) = 0$ and $dK_0(0)/du = 0$, and*
(3) *f gives a $5/2$-cuspidal edge along the u-axis if $K_0(u) = 0$ and $K_2(u) \neq 0$.*

In particular, these conditions depend only on the first fundamental form of f.

Proof. Since $\sin\theta(u) \neq 0$, we have $\kappa_\nu(u) \neq 0$. Since $K_0 = \mu_0\kappa_\nu$, $K_0(u) = 0$ if and only if $\mu_0(u) = 0$. Since $\mu_0(u) = \hat{\mu}(u, 0)(= \kappa_c(u))$, the first and second assertions follow from (1) and (2) of Proposition 4.7, respectively. On the other hand, if $\mu_0(= \kappa_c)$ is identically zero, then $K_2 = \kappa_\nu\mu_2$. So $K_2(u) \neq 0$ if and only if $\mu_2(u) \neq 0$. Thus, the third assertion immediately follows from (3) of Proposition 4.7. □

We now prove Fact 1.1 in the introduction.

Proof of Fact 1.1. Since $\sin\theta \neq 0$ if and only if $\kappa_\nu \neq 0$, the assertions (1) and (2) follow from Theorem 3.8. We next prove (3). We remark that

$$\begin{aligned}
\mathcal{K}^\omega_*(\boldsymbol{R}^2_o) &= \{ds^2_f \in \mathcal{K}^\omega_{\mathrm{I}}(\boldsymbol{R}^2_o)\,;\, K_0(0) \neq 0\}, \\
\mathcal{K}^\omega_{p,*}(\boldsymbol{R}^2_o) &= \{ds^2_f \in \mathcal{K}^\omega_{\mathrm{I}}(\boldsymbol{R}^2_o)\,;\, K_0(0) = 0,\ dK_0(0)/du \neq 0\}, \\
\mathcal{K}^\omega_{a,*}(\boldsymbol{R}^2_o) &= \{ds^2_f \in \mathcal{K}^\omega_{\mathrm{I}}(\boldsymbol{R}^2_o)\,;\, K_0(u) = 0,\ K_2(0) \neq 0\}
\end{aligned}$$

hold in terms of our coordinates (u, t). We have shown the following (cf. Propositions 4.7 and 4.11).

- $K_0(0) \neq 0$ if and only if $\mu_0(0)(= \kappa_c(0)) \neq 0$.
- $K_0(0) = 0$ and $dK_0(0)/du \neq 0$ if and only if $\mu_0(0)(= \kappa_c(0)) = 0$ and $d\mu_0(0)/du \neq 0$.
- $K_0(u) = 0$ and $K_2(0) \neq 0$ if and only if $\mu_0(u) = 0$ and $\mu_2(0) \neq 0$.

By Corollary 3.2, the following assertions hold:

- $\hat{K}(o) \neq 0$ if and only if $K_0(0) \neq 0$.
- $\hat{K}(o) = 0$ and $\partial\hat{K}(o)/\partial u \neq 0$ if and only if $K_0(0) = 0$ and $dK_0(0)/du \neq 0$.

So the first fundamental form ds^2_f of f belongs to $\mathcal{K}^\omega_*(\boldsymbol{R}^2_o)$ (resp. $\mathcal{K}^\omega_{p,*}(\boldsymbol{R}^2_o)$) if and only if $\mu_0(0)(= \kappa_c(0)) \neq 0$ (resp. $\mu_0(0)(= \kappa_c(0)) = 0$ and $d\mu_0(0)/du \neq 0$). On the other hand, ds^2_f belongs to $\mathcal{K}^\omega_{a,*}(\boldsymbol{R}^2_o)$ if and only if $\mu_0(u) = 0$ and $\mu_1(0) \neq 0$. In fact, $\eta := \partial/\partial t$ gives the null direction of f along the u-axis (as the singular curve of ds^2_f), and we have (cf. (1.2)) $dK(\eta) = K_t(u, 0) = K_2(u)$. □

Finally, we consider the cuspidal edges with vanishing limiting normal curvature: A cuspidal edge is called *asymptotic* if its first fundamental form is asymptotic (see Section 1), which is equivalent to the condition that the cuspidal angle $\theta(u)$ of f is constantly equal to 0 or π along its edge.

If f is an asymptotic cuspidal edge, the singular curvature κ_s, limiting normal curvature κ_ν and cusp-directional torsion κ_t satisfy

$$\kappa_s = \varepsilon\kappa, \quad \kappa_\nu = 0, \quad \kappa_t = \tau, \tag{4.12}$$

where $\varepsilon := \cos\theta\,(\in \{1, -1\})$. So we get the following:

Proposition 4.12. *Let $f \in \mathcal{G}^r_{3/2}(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$ be a cuspidal edge associated to a fundamental data $(\kappa, \tau, \theta, \hat{\mu})$. If $\sin\theta$ vanishes identically, then*

1. *the limiting normal curvature κ_ν vanishes identically,*
2. *the first fundamental form of f is an asymptotic Kossowski metric, and*
3. *the Gaussian curvature K of f can be extended across its singular set as a C^r-function.*

Moreover, the sign of K coincides with the sign of $(K_1 =) - \varepsilon\kappa\mu_0^2 - \tau^2$ whenever $K_1 \neq 0$, where $\varepsilon := \cos\theta$.

As an application, we first consider the case K vanishes identically.

Corollary 4.13. *Let $f \in \mathcal{G}^r_{3/2}(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$ be the cuspidal edge whose Gaussian curvature K vanishes identically. Then C is a regular space curve whose torsion function does not vanish, and f is the tangential developable of C. In particular, f has no isomers.*

Proof. Since K vanishes identically, the identity $-\varepsilon\kappa\mu_0^2 = \tau^2$ holds along C. Since f is a cuspidal edge, μ_0 has no zeros, and the left hand side does not vanish. Thus, the torsion function τ of C also has no zeros. Since f is a wave front, its principal directions along C are well-defined (cf. [13, Proposition 1.6]). Moreover, each singular point of f is disjoint from umbilical set (cf. [13, Proposition 1,10]), and the zero principal curvature direction is uniquely determined at each point of C. Moreover, it can be easily seen that this direction must be the tangential direction of C. Since K vanishes identically, f must be a ruled surface (cf. [13, Proposition 2.2]), so it must be the tangential developable of C. □

Remark 4.14. The standard cuspidal edge $f_0(t) = (u^2, u^3, v)$ does not satisfy the assumption of Corollary 4.13, since the singular set image is a line.

We next consider the case $K > 0$. If $\theta = \pi$ and μ_0 is sufficiently large, then the Gaussian curvature K near the singular set can be positive. So we can construct cuspidal edges with $K > 0$. The following assertion is an immediate consequence of Proposition 4.12.

Corollary 4.15. *Let $f \in \mathcal{G}^r_{3/2}(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$ be the cuspidal edge whose Gaussian curvature K is bounded near singular set and positive, then it is asymptotic satisfying $\theta = \pi$ and $\kappa_s < 0$.*

The negativity of κ_s has been pointed out in [16]. Although Theorem 3.8 does not cover the case $\kappa_\nu = 0$, Brander [1] showed the existence of cuspidal edges in the case of $K = 1$ along a given space curve C of $\kappa_\nu > 0$ using the loop group theory.

5. Relationships among isomers

In this section, we show several properties of isomers, and prove the last two statements in the introduction. We fix a space curve $\mathbf{c}(u)$ satisfying $\mathbf{c}(0) = \mathbf{0}$ which is parametrized by arc-length defined on a closed interval $J := [-l, l]$ $(l > 0)$ whose curvature function $\kappa(u)$ is positive everywhere. We prove the following:

Proposition 5.1. *Let $f \in \mathcal{G}^\omega_{*,3/2}(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$. Then $\check{f}$ is congruent (cf. Definition 0.2) to f if and only if*

(1) *C lies in a plane, or*

(2) *C has a positive non-trivial symmetry and the first fundamental form ds^2_f has an effective symmetry* (*cf. Definition 0.4*).

Proof. We suppose that $\check{f}$ is congruent to f. By Remark 4.5, it is sufficient to consider the case that C does not lie in any plane. By Remark 0.5, there exist an isometry T on $\boldsymbol{R}^3$ and a diffeomorphism φ defined on a neighborhood of the singular curve of f such that

$$T \circ f \circ \varphi = \check{f}. \tag{5.1}$$

We consider the case that T fixes each point of C. Then C must lie in a plane, a contradiction. So T is a non-trivial symmetry of C, that is, it reverses the orientation of C. We suppose that T is a negative symmetry. Then (b) of Remark 4.4 implies that the image of f coincides with that of $T \circ f$. Since the image of $\check{f}$ is different from that of f, this case never happens. So T must be a positive symmetry, and then φ gives an effective symmetry of ds^2_f.

Conversely, if C has a positive non-trivial symmetry and the first fundamental form ds^2_f has an effective symmetry φ, then $T \circ f \circ \varphi$ is a faithful isomer of f as seen in (a) of Remark 4.4. Since such an isomer is uniquely determined (cf. Theorem 3.8), we have (5.1). □

Remark 5.2. Suppose that C is planar and S is the reflection with respect to the plane containing C. For each $f \in \mathcal{G}^r_{*,3/2}(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$, $S \circ f$ gives a faithful isomer of f. Moreover, if f is real analytic (i.e. $r = \omega$), then we have $\check{f} = S \circ f$ (cf. Definition 3.18).

Example 5.3. Let $f \in \mathcal{G}^\infty_*(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$ be an admissible generalized cuspidal edge whose fundamental data is $(\kappa, \tau, \theta, \hat{\mu})$ $(\tau \neq 0)$. Suppose that κ, τ and θ are constant, and the extended half-cuspidal curvature function $\hat{\mu}$ does not depend on u. In this case, without assuming the real analyticity of f, we can show the existence of an isometry $T \in \mathrm{SO}(3)$ and an effective symmetry φ of ds^2_f such that $T \circ f \circ \varphi$ gives a faithful isomer of f as follows: In fact, in this case C has the constant curvature κ and the constant torsion τ. Since $\tau \neq 0$, C is a helix in $\boldsymbol{R}^3$ and there exists a 180°-rotation $T \in \mathrm{SO}(3)$ with respect to the principal normal line at $\mathbf{0} \in C$ such that $T(C) = C$. By the first part of Proposition 5.10, it is sufficient to show that the first fundamental form

$$ds^2_f = E(t)du^2 + 2F(t)dudt + G(t)dt^2$$

of f admits an effective symmetry φ as an involution. In fact, if such a φ exists, then $(\check{f} :=) T \circ f \circ \varphi$ gives the isometric dual of f. In this situation, two functions A, B can be expressed as (cf. (4.9) and (4.6)) $A(t) := t^2\alpha(t)$ and $B(t) := t^3\beta(t)$, where $\alpha(t)$ and $\beta(t)$ are C^r-functions. By Proposition 4.9,

- $E(t)$ is positive for each t,
- there exists a C^∞-function $F_0(t)$ such that $F(t) = t^4 F_0(t)$, and $G(t) = t^2$.

Setting

$$\omega_1 = \sqrt{E(t)}\left(du + \frac{F(t)}{E(t)}dt\right), \qquad \omega_2 = t\sqrt{\frac{E(t) - t^6F_0(t)^2}{E(t)}}dt,$$

we have $ds^2_f = (\omega_1)^2 + (\omega_2)^2$. Moreover, if we set

$$x(u,t) := u + \int_0^t \frac{F(v)}{E(v)}dv, \qquad y(t) := \int_0^t \sqrt{\frac{E(v) - v^6F_0(v)^2}{E(v)}}dv. \tag{5.2}$$

Then we can take (x, y) as a new local coordinate system centered at $(0,0)$, and t can be considered as a function of y. So we can write $t = t(y)$, and

$$ds^2_f = E(y)dx^2 + t(y)^2dy^2.$$

So the local diffeomorphism $\varphi : (x, y) \mapsto (-x, y)$ gives an effective symmetry of ds^2_f.

Regarding the fact that the fundamental data of f is $(\kappa, \tau, \theta, \mu)$, we show in later that $\check{f}$ is right equivalent to the cuspidal edge whose fundamental data of $(\kappa, \tau, -\theta, \mu)$, see Proposition 6.1.

Proof of Theorem III. Suppose that ds^2_f admits a symmetry φ. Then this symmetry is effective (cf. Corollary 3.16). So, $f \circ \varphi$ and $\check{f} \circ \varphi$ must be right equivalent to $\check{f}_*$ and f_*, respectively. In particular, the number of right equivalence classes of $f, \check{f}, f_*, \check{f}_*$ is two.

Conversely, we suppose that two of $\{f, \check{f}, f_*, \check{f}_*\}$ are right equivalent. Replacing f by $\check{f}$, f_*, $\check{f}_*$, we may assume that one of the right equivalent pair is f and the other is $g \in \{\check{f},\ f_*,\ \check{f}_*\}$. Without loss of generality, we may assume that f is written in a normal form. Since $\check{f}$ cannot be right equivalent to f, the map g must be right equivalent to f_* or $\check{f}_*$, that is, there exists a local diffeomorphism φ such that $g = f \circ \varphi$, which implies $\varphi^* ds^2_f = ds^2_f$. If φ is an identity map, then $g = f$ holds. However, it contradicts the fact that $u \mapsto f(u, 0)$ and $u \mapsto f_*(u, 0) = \check{f}_*(u, 0)$ give mutually distinct orientations to C. So, by Corollary 3.16, φ must be an effective symmetry of ds^2_f. □

Corollary 5.4. *Let $f \in \mathcal{G}^{\omega}_{**,3/2}(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$. Suppose that*

(1) *C is planar and does not admit any non-trivial symmetry at $\mathbf{0}$, and*
(2) *ds^2_f admits no effective symmetries (cf. Definition 0.4).*

Then

- *$\check{f} := S \circ f$ holds, where $S \in \mathrm{O}(3)$ is the reflection with respect to the plane containing C,*
- *the isometric dual, inverse and the inverse dual are given by $S \circ f$, f_* and $S \circ f_*$, respectively. Moreover, f_* is not congruent to f.*

In particular, the four maps consist of two congruence classes.

Proof. As seen in Remark 5.2, $\check{f} := S \circ f$ holds. We next prove the second assertion. Since C lies in a plane, $\mathcal{I}_C(f) = S \circ f$ holds. By applying Theorem II, the right equivalence classes of $\mathcal{J}_C^{-1}(\mathcal{J}_C(f))$ are represented by $\{f,\ S \circ f,\ f_*,\ S \circ f_*\}$. It is sufficient to show that f_* is not congruent to f. If not, then, by Remark 0.5, there exist $T \in \mathrm{O}(3)$ and a diffeomorphism φ defined on a neighborhood of the singular curve of f such that $T \circ f_* \circ \varphi = f$. In particular, $\varphi^* ds^2_f = ds^2_f$ holds. By (1), T is not non-trivial. So, φ must be an effective symmetry, contradicting (2). □

We next consider the case that ds^2_f has an effective symmetry.

Proposition 5.5. *Let $f \in \mathcal{G}^{\omega}_{**,3/2}(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$. Suppose that*

(1) *C is non-planar and does not admit any non-trivial symmetry at $\mathbf{0}$,*
(2) *ds^2_f admits an effective symmetry φ.*

Then $\check{f}(:= \mathcal{I}_C(f))$ is not congruent to f, and $\check{f}$, $\check{f} \circ \varphi$ and $f \circ \varphi$ give the isometric dual, inverse and inverse dual, respectively.

Proof. By Proposition 5.1, $\check{f}$ is not congruent to f. Since $\check{f} \circ \varphi$ (resp. $f \circ \varphi$) has the same first fundamental form as f, the fact that φ is effective yields that it coincides with either f_* or $\check{f}_*$. Since the cuspidal angle of $\check{f} \circ \varphi$ (resp. $f \circ \varphi$) takes the opposite sign (resp. the same sign) of that of f (cf. Remark 4.5), we have $f_* = \check{f} \circ \varphi$ (resp. $\check{f}_* = f \circ \varphi$). □

Corollary 5.6. *Let $f \in \mathcal{G}^{\omega}_{**,3/2}(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$. Suppose that*

(1) *C is planar and does not admit any non-trivial symmetry at the origin $\mathbf{0}$,*
(2) *ds^2_f admits an effective symmetry φ.*

Then

- $\check{f} = S \circ f$ *holds, where* $S \in \mathrm{O}(3)$ *is the reflection with respect to the plane containing* C.
- *Moreover,* $S \circ f$, $S \circ f \circ \varphi$, $f \circ \varphi$ *give the isometric dual, inverse and inverse dual, respectively.*

As a consequence, all of isomers are congruent to f.

Proof. As we have seen in Remark 5.2, $\check{f} = S \circ f$ holds. Since $S \circ f \circ \varphi$ (resp. $f \circ \varphi$) has the same first fundamental form as f, the fact that φ is effective yields it coincides with f_* or $\check{f}_*$. Since the sign of cuspidal angle of $S \circ f \circ \varphi$ (resp. $f \circ \varphi$) along the curve $\mathbf{c}_\#(u) := \mathbf{c}(-u)$ takes the opposite sign (resp. the same sign) of that of f, we have $f_* = S \circ f \circ \varphi$ (resp. $\check{f}_* = f \circ \varphi$). Finally, it is obvious that the four maps are congruent. So the proposition is proved. □

We then consider the case that C has a non-trivial symmetry.

Proposition 5.7. *Let* $f \in \mathcal{G}^\omega_{**,3/2}(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$. *Suppose that*

(1) C *is non-planar and admits a non-trivial symmetry* $T \in \mathrm{O}(3)$ *at* $\mathbf{0}$,
(2) ds^2_f *does not admit any effective symmetries.*

Then

- $\check{f} := \mathcal{I}_C(f)$ *is not congruent to* f, *and*
- $T \circ \check{f}$, $T \circ f$ *are the inverse and inverse dual, respectively.*

In particular, f, $\check{f}$, $T \circ \check{f}$ *and* $T \circ f$ *consist of two congruence classes.*

Proof. By Proposition 5.1, $\check{f}$ is not congruent to f. So the assertion can be shown easily. □

We get the following corollary.

Corollary 5.8. *Let* $f \in \mathcal{G}^\omega_{**,3/2}(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$. *Suppose that* C *lies in a plane and admits a non-trivial symmetry* T *at the origin* $\mathbf{0}$. *Then* $\check{f} = S \circ f$ *holds, and* $T \circ f$, $S \circ T \circ f$ *give the inverse and the inverse dual of* f, *where* S *is a reflection with respect to the plane. As a consequence,* $f, \check{f}, f_*, \check{f}_*$ *belong to a single congruence class.*

Proof. Obviously, $\check{f} = S \circ f$ holds (cf. Remark 5.2). On the other hand, $T \circ f$ gives a non-faithful isomer, and its isometric dual $S \circ T \circ f$ also gives another non-faithful isomer. □

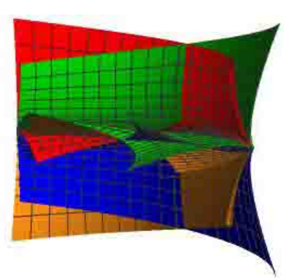

FIGURE 2. The four cuspidal edges given in Example 5.9

Example 5.9. We set

$$f(u,v) := \Big(\varphi(u,v)\cos u - 1,\ \varphi(u,v)\sin u,\ v^3 u + 2v^3 - v^2\Big),$$

where $\varphi(u,v) := -v^3u - 2v^3 - v^2 + 1$. Then, it has cuspidal edge singularities along

$$\mathbf{c}(u)\,(:= f(u,0)) = (\cos u - 1, \sin u, 0).$$

By setting,

$$S := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad T := \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$S \circ f$ is the faithful isomer, and $T \circ f$, $TS \circ f$ are non-faithful isomers. We remark that f is associated to Fukui's data (θ, A, B) given by

$$\theta = \frac{\pi}{4}, \quad A(u,v) := \sqrt{2}v^2, \quad B(u,v) := \sqrt{2}v^3(u+2).$$

Finally, we consider the case that C and ds^2_f admit a symmetry and an effective symmetry, respectively.

Proposition 5.10. *Let $f \in \mathcal{G}^{\omega}_{**,3/2}(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$. Suppose that*

(1) *C is non-planar and admits a non-trivial symmetry $T \in \mathrm{O}(3)$ at $\mathbf{0}$,*
(2) *ds^2_f admits an effective symmetry φ.*

Then any isomer of f is right equivalent to one of $\check{f}$, $\check{f} \circ \varphi$, $f \circ \varphi$. Moreover,

- *if T is positive (i.e. $T \in \mathrm{SO}(3)$), then $\check{f} = T \circ f \circ \varphi$, and*
- *if T is negative (i.e. $T \notin \mathrm{SO}(3)$), then $\check{f}$ is not congruent to f.*

Proof. We set $g := T \circ f \circ \varphi$. If T is positive, then g is a faithful isomer of f as shown in Remark 4.4. On the other hand, if T is negative, then $\check{f}$ is not congruent to f by Proposition 5.1 and so it not congruent to f. □

Proof of Theorem IV. We suppose that C has no non-trivial symmetries, and also ds^2_f has no symmetries. If two of $\{f, \check{f}, f_*, \check{f}_*\}$ are mutually congruent, replacing f by one of its isomers, we may assume that f is congruent to g, where g is one of $\{\check{f}, f_*, \check{f}_*\}$. By Proposition 5.1, we may assume that $g = f_*$ or $g = \check{f}_*$. Suppose that g is congruent to f. Then (cf. Remark 0.5) there exist a non-trivial symmetry $T \in \mathrm{O}(3)$ of C and a local diffeomorphism φ such that

$$T \circ g \circ \varphi = f.$$

Since C has no non-trivial symmetries, and ds^2_f has also no symmetries, φ is the identity map and T is not a non-trivial symmetry. However, this contradicts the fact that $u \mapsto f(u,0)$ and $u \mapsto f_*(u,0) = \check{f}_*(u,0)$ give mutually distinct orientations to C. So we obtained (1).

The assertion (2) follows from Corollaries 5.4, 5.6, 5.8 and Propositions 5.5, 5.7, and 5.10, by using the fact that any symmetries of ds^2_f are effective (cf. Corollary 3.16).

Finally, suppose that $N_f = 1$. We first consider the case that C lies in a plane. If C has no non-trivial symmetries and ds^2_f has also no symmetries, then $N_f = 2$ holds by Corollary 5.4. So either C or ds^2_f has a symmetry. If C has a symmetry, then $N_f = 1$ by Corollary 5.8 (this corresponds to the case (a)). On the other hand, if C has no non-trivial symmetries and ds^2_f also has a symmetry φ, then φ is effective (cf. Corollary 3.16). So, Corollary 5.6 yields that $N_f = 1$. (This corresponds to the case (b). In fact, we denote by T_0 the reflection with respect to the plane containing C. We let T_1 be a non-trivial symmetry of C. If T_1 is positive, then (b) holds obviously. On the other hand, if T_1 is negative, then $T_0 \circ T_1$ is a positive symmetry and (b) holds.)

So we may assume that C does not lie in any planes. The assumption $N_f = 1$ implies $\check{f}$ must congruent to f. By Proposition 5.1, this holds only when (c) happens, since C does not lie in any planes. □

6. Examples

One method to give a numerical approximation of a isometric dual g of a real analytic cuspidal edge f is to determine the Taylor expansion of $g(u,v)$ at $v=0$ along the u-axis as a singular set so that $g = \mathcal{I}_C(f)$. In [14, Page 85], we give a numerical approximation of the isometric dual of

$$f_0(u,v) = \left(u, -\frac{v^2}{2} + \frac{u^3}{6}, \frac{u^2}{2} + \frac{u^3}{6} + \frac{v^3}{6}\right).$$

We denote by C the image of singular curve $u \mapsto f_0(u,0)$. In the figure of the isometric dual $g_0 = \mathcal{I}_C(f_0)$ given in [14, Figure 2], the surface g_0 seems like it is lying on the almost opposite side of f_0. This is the reason why the cuspidal angle $\theta(u)$ of $f_0(u,v)$ is $\pi/2$ at $u=0$. The red lines of Figure 3 (left) indicates the section of f_0, g_0 at $u=-1/4$. The orange (resp. blue) surface corresponds to f_0 (resp. g_0). We can recognize that the cuspidal angle takes value less than $\pi/2$, that is, the normal direction of g_0 is linearly independent of that of f_0 at $(u,v)=(-1/4,0)$. On the other hand, Figure 3 (right) indicates the images of the numerical approximations of the two non-faithful isomers f_1, g_1 of f_0.

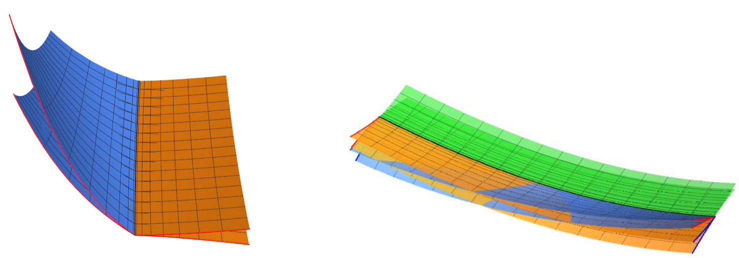

Figure 3. The images of f_0, g_0 (left), and the images of f_0, f_1, g_1 (right), where f_0 is indicated as the orange surfaces.

By Proposition 4.9, one can easily observe that the first fundamental form of $f_{-\theta}$ does not coincide with that of f_θ. This means that the image of $f_{-\theta}$ cannot coincide with that of f_θ nor $\check{f}_\theta$. However, one might expect the possibility that $f_{-\theta}$ is an isomer of f_θ. Here, we consider the case that the space curve C has a non-trivial symmetry T. In this case, we know that f, $\check{f}$, $T\circ f$, $T\circ \check{f}$ are only the possibilities of isomers. Thus, if $f_{-\theta}$ is an isomer of f_θ, then it must be congruent to either f or $\check{f}$. We give here the following two propositions which are related to one of these possibilities (by the following Proposition 6.1, Example 5.3 is just the case that $f_{-\theta}$ is right equivalent to $\check{f}$.)

Proposition 6.1. *Let C be a space curve which admits a non-trivial symmetry $T \in \mathrm{SO}(3)$ at $\mathbf{0}$, and let $f := f_\theta \in \mathcal{G}^\infty(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$ be a generalized cuspidal edge as in the formula (4.1) such that*

- *$T\circ f(-u,0) = f(u,0)$, and*
- *the cuspidal angle θ satisfies $\theta(u) = \sigma\theta(-u)$ where $\sigma \in \{+,-\}$.*

Suppose that $A(u,v)$ and $B(u,v)$ satisfy one of the following two conditions:

(1) *$A(-u,-v) = A(u,v)$ and $B(-u,-v) = -B(u,v)$ or*
(2) *$A(-u,v) = A(u,v)$ and $B(-u,v) = -B(u,v)$.*

Then $f_\theta = T \circ f_{-\sigma\theta} \circ \varphi$ holds, where $\varphi(u,v) = (-u,-v)$ (resp. $\varphi(u,v) = (-u,v)$) in the case of (1) (resp. (2)). In particular, $f_{-\theta}$ is a right equivalent to $\check{f}$ if $\sigma = +$, and the image of f is invariant under T if $\sigma = -$.

Proof. We consider the case $\sigma = +$, that is, $\theta(u) = \theta(-u)$. Since $T\circ\mathbf{c}(-u) = \mathbf{c}(u)$ and $T \in \mathrm{SO}(3)$ (cf. Remark 4.4),

$$-T\mathbf{e}(-u) = \mathbf{e}(u), \quad T\mathbf{n}(-u) = \mathbf{n}(u), \quad \mathbf{b}(u) = -T\mathbf{b}(-u).$$

In the case of (1) (resp. (2)), we set $\varphi(u,v) := (-u,-v)$ (resp. $\varphi(u,v) := (-u,v)$). Then $A\circ\varphi(u,v) = A(u,v)$ and $B\circ\varphi(u,v) = -B(u,v)$ hold, and so

$$\begin{aligned} T\circ f_\theta \circ \varphi &= \mathbf{c} + (A,-B)\begin{pmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\mathbf{n} \\ -\mathbf{b}\end{pmatrix} \\ &= \mathbf{c} + (A,B)\begin{pmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\mathbf{n} \\ \mathbf{b}\end{pmatrix} = f_{-\theta}, \end{aligned}$$

proving the relation $f_\theta = T\circ f_{-\sigma\theta}\circ\varphi$. The case $\theta(u) = -\theta(-u)$ is proved in the same way.

We then consider the case that $\sigma = 1$. In this case, $f_\theta = T\circ f_{-\theta}\circ\varphi$ holds. Since T is an isometry of $\boldsymbol{R}^3$, we have $\varphi^* ds^2_f = ds^2_g$, where $f := f_{-\theta}$ and $g = f_{-\theta}$. So g is isometric to f. Since the cuspidal angle of g takes the opposite sign of that of f, the image of g does not coincide with f. So g is a faithful isomer of f. Then the uniqueness of the faithful isomer of f (cf. Theorem 3.8) yields that g is right equivalent to $\check{f}$. □

Similarly, the following assertion holds.

Proposition 6.2. *Let C be a space curve which admits a non-trivial symmetry $T \in \mathrm{O}(3)\setminus\mathrm{SO}(3)$ at $\mathbf{0}$, and let $f := f_\theta \in \mathcal{G}^\infty(\boldsymbol{R}^2_J, \boldsymbol{R}^3, C)$ be a generalized cuspidal edge as in the formula (4.1) such that*

- *$T\circ f(-u,0) = f(u,0)$, and*
- *the cuspidal angle θ satisfies $\theta(u) = \sigma\theta(-u)$, where $\sigma \in \{+,-\}$.*

Suppose that $A(u,v)$ and $B(u,v)$ satisfy one of the following two conditions:

(1) *$A(-u,-v) = A(u,v)$ and $B(-u,-v) = B(u,v)$,*
(2) *$A(-u,v) = A(u,v)$ and $B(-u,v) = B(u,v)$.*

Then $f_\theta = T\circ f_{\sigma\theta}\circ\varphi$ holds, where $\varphi(u,v) = (-u,-v)$ (resp. $\varphi(u,v) = (-u,v)$) in the case of (1) (resp. (2)). In particular, $f_{-\theta}$ is right equivalent to $\check{f}$ if $\sigma = -$, and the image of f is invariant under T if $\sigma = +$.

Proof. Like as in the case of the proof of Proposition 6.1, $-T\mathbf{e}(-u) = \mathbf{e}(u)$ and $T\mathbf{n}(-u) = \mathbf{n}(u)$ hold. Since $\det(T) = -1$, we have $T\mathbf{b}(-u) = \mathbf{b}(u)$. In the case of (1) (resp. (2)), we set $\varphi(u,v) := (-u,-v)$ (resp. $\varphi(u,v) := (-u,v)$), then the relation $f_\theta = T\circ f_{\sigma\theta}\circ\varphi$ is obtained like as in the case of the proof of Proposition 6.1. One can also obtain the last assertion imitating the corresponding argument in the proof of Proposition 6.1. □

Example 6.3. Let a, b be real numbers so that $a > 0$ and $b \neq 0$. Then

$$\mathbf{c}(u) := \left(a\cos\left(\frac{u}{c}\right) - a,\ a\sin\left(\frac{u}{c}\right),\ \frac{bu}{c}\right) \qquad (u \in \boldsymbol{R})$$

gives a helix of constant curvature $\kappa := a/c^2$ and constant torsion $\tau := b/c^2$, where $c := \sqrt{a^2+b^2}$. At the point $\mathbf{0} := \mathbf{c}(0)$ on the helix, $\mathbf{c}$ satisfies $T(\mathbf{c}(\boldsymbol{R})) = \mathbf{c}(\boldsymbol{R})$, where $T \in \mathrm{SO}(3)$ is the $180°$-rotation with respect to the line passing through the origin $\mathbf{0}$ which is parallel to the principal normal vector $\mathbf{n}(0)$. We set $a = b = 1$, $\theta = \pi/4$. By setting

$$(A_1, B_1) := (v^2, v^3), \quad (A_2, B_2) := (v^2, v^5), \quad (A_3, B_3) := (v^2, uv^3).$$

The surfaces $g_{i,\pm} := f_{\pm\pi/4}$ $(i = 1, 2, 3)$ associated to the Fukui data $(\mathbf{c}, \pm\pi/4, A_i, B_i)$ correspond to cuspidal edges, 5/2-cuspidal edges, and cuspidal cross caps, respectively. The first two cases satisfy (1) of Proposition 6.1 and the third case satisfies (2) of Proposition 6.1. So $g_{i,-}$ $(i = 1, 2, 3)$ is a faithful isomer of $g_{i,+}$.

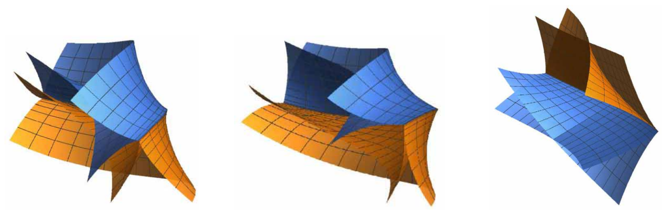

FIGURE 4. The images of cuspidal edges $g_{1,\pm}$ (left), 5/2-cuspidal edges $g_{2,\pm}$ (center) and cuspidal cross caps $g_{3,\pm}$ (right) given in Example 6.3.
(The orange surfaces correspond to $g_{i,+}$ and the blue surfaces correspond to $g_{i,-}$ for $i = 1, 2, 3$.)

Finally, we consider the case of fold singularities:

Example 6.4. We let $\mathbf{c}(u)$ be a C^∞-regular space curve with positive curvature κ and torsion τ. If we set

$$g_\pm(u, v) := \mathbf{c}(u) + \frac{v^2}{2}(\cos\theta\mathbf{n}(u) \mp \sin\theta\mathbf{b}(u)),$$

then it can be easily checked that g_- is a faithful isomer of g_+, where θ is a constant. These two surfaces can be extended to the following regular ruled surfaces:

$$\tilde{g}_\pm = \mathbf{c}(u) + \frac{v}{2}(\cos\theta\mathbf{n}(u) \mp \sin\theta\mathbf{b}(u)).$$

APPENDIX A. A REPRESENTATION FORMULA FOR GENERALIZED CUSPS

A plane curve $\sigma : J \to \boldsymbol{R}^2$ is said to have a singular point at $t = t_0$ if $\dot\sigma(t_0) = \mathbf{0}$ (the dot means d/dt). The singular point $t = t_0$ is called a *generalized cusp* if $\ddot\sigma(t_0) \neq \mathbf{0}$. In this situation, it is well-known that

(i) $t = t_0$ is a cusp if and only if $\ddot\sigma(t_0), \dddot\sigma(t_0)$ are linearly independent,
(ii) (cf. [15]) $t = t_0$ is a 5/2-cusp if and only if $\ddot\sigma(t_0), \dddot\sigma(t_0)$ are linearly dependent and

$$3\det(\ddot\sigma(t_0), \sigma^{(5)}(t_0))\ddot\sigma(t_0) - 10\det(\ddot\sigma(t_0), \sigma^{(4)}(t_0))\dddot\sigma(t_0) \neq \mathbf{0}.$$

From now on, we set $t_0 = 0$. The arc-length parameter $s(t)$ of σ given by

$$s(t) := \int_0^t |\dot\sigma(u)|du$$

is not smooth at $t = 0$, but if we set $w := \operatorname{sgn}(t)\sqrt{|s(t)|}$, then this gives a parametrization of σ near $t = 0$, which is called the *half-arc-length parameter* of σ near $t = 0$ in [17]. However, for our purpose, as Fukui [3] did, the parameter

$$v := \sqrt{2}w = \operatorname{sgn}(t)\left(2\int_0^t |\dot\sigma(u)|du\right)^{1/2} \tag{A.1}$$

called the *normalized half-arc-length parameter* is convenient, since it is compatible with the property $|f_{vv}| = 1$ for adapted coordinate systems (cf. Definition 3.4) of generalized cuspidal edges. This normalized half-arc-length parameter can be characterized by the property that $v^2/2$ gives the arc-length parameter of σ. Then by [17, Theorem 1.1], we can write

$$\sigma(v) = \int_0^v u(\cos\theta(u), \sin\theta(u))du, \qquad \theta(v) = \int_0^v \hat{\mu}(u)du. \tag{A.2}$$

We need the following lemma, which can be proved by a straightforward computation.

Lemma A.1. *Let v be the normalized half-arc-length parameter of the generalized cusp $\sigma(w)$ at $w=0$. Then there exists an orientation preserving isometry T of $\boldsymbol{R}^2$ such that*

$$T\circ\sigma(v) = \Big(\frac{v^2}{2} - \frac{\mu_0^2 v^4}{8} - \frac{\mu_0\mu_1 v^5}{10}, \frac{\mu_0 v^3}{3} + \frac{\mu_1 v^4}{8} + \frac{(-\mu_0^3 + 2\mu_2)v^5}{30}\Big) + o(v^5), \tag{A.3}$$

where

$$\hat{\mu}(v) = \sum_{j=0}^{2} \mu_j v^j + o(v^3),$$

and $o(v^5)$ (resp. $o(v^3)$) is a term higher than v^5 (resp. v^3).

Using this with (i) and (ii), one can easily obtain the following assertion:

Proposition A.2. *Let v be the normalized half-arc-length parameter of the generalized cusp $\sigma(w)$ at $w=0$. Then*

(1) *$w=0$ is a cusp of σ if and only if $\mu_0 \neq 0$, and*
(2) *$w=0$ is a $5/2$-cusp of σ if and only if $\mu_0 = 0$ and $\mu_2 \neq 0$.*

It is remarkable that the coefficient μ_1 does not affect the criterion for $5/2$-cusps. In this case, $\mu_0 = 0$ holds, and μ_1 and μ_2 are proportional to the "secondary cuspidal curvature" and the "bias" of $\sigma(t)$ at $t=0$, respectively. Geometric meanings for these two invariants for $5/2$-cusps can be found in [6, Proposition 2.2].

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References

[1] D. Brander, *Spherical surfaces*, Exp. Mathematics **25** (2016), 257–272.
[2] S. Fujimori, K. Saji, M. Umehara and K. Yamada, *Singularities of maximal surfaces*, Math. Z. **259** (2008), 827–848. DOI: 10.1007/s00209-007-0250-0
[3] T. Fukui, *Local differential geometry of cuspidal edge and swallowtail*, to appear in Osaka J. Math.
[4] M. Hasegawa, A. Honda, K. Naokawa, K. Saji, M. Umehara and K. Yamada, *Intrinsic properties of surfaces with singularities*, Internat. J. Math. **26** (2015), 1540008, 34pp.
[5] A. Honda, K. Naokawa, M. Umehara, and K. Yamada, *Isometric deformations of wave fronts at non-degenerate singular points*, to appear in Hiroshima Math. J.
[6] A. Honda, K. Saji, *Geometric invariants of $5/2$-cuspidal edges*, Kodai Math. J. **42** (2019), 496–525. DOI: 10.2996/kmj/1572487230
[7] S. Izumiya, K. Saji and N. Takeuchi, *Flat surfaces along cuspidal edges*, J. Singul. **16** (2017), 73–100. DOI: 10.5427/jsing.2017.16c
[8] M. Kossowski, *Realizing a singular first fundamental form as a nonimmersed surface in Euclidean 3-space*, J. Geom. **81** (2004), 101–113. DOI: 10.1007/s00022-004-2511-y
[9] S.G. Krantz and H.R. Parks, A Primer of Real Analytic Functions (Second Edition), Birkhäuser 2002, USA.
[10] M. Kokubu, W. Rossman, K. Saji, M. Umehara and K. Yamada, *Singularities of flat fronts in hyperbolic space*, Pacific J. Math. **221** (2005), 303–351, DOI: 10.2140/pjm.2005.221.303 Addendum: *Singularities of flat fronts in hyperbolic space*, Pacific J. Math. **294** (2018), 505–509. DOI: 10.2140/pjm.2018.294.505
[11] L. Martins, K. Saji, *Geometric invariants of cuspidal edges*, Canad. J. Math. **68** (2016), 455–462. DOI: 10.4153/cjm-2015-011-5

[12] L. Martins, K. Saji, M. Umehara and K. Yamada, *Behavior of Gaussian curvature and mean curvature near non-degenerate singular points on wave fronts*, Geometry and Topology of Manifolds, 247–281, Springer Proc. Math. Stat.,**154**, Springer (2016). DOI: 10.1007/978-4-431-56021-0_14

[13] S. Murata and M. Umehara, *Flat surfaces with singularities in Euclidean 3-space*, J. of Differential Geom. **82** (2009), 279–316. DOI: 10.4310/jdg/1246888486

[14] K. Naokawa, M. Umehara, and K. Yamada, *Isometric deformations of cuspidal edges*, Tohoku Math. J. **68** (2016), 73–90. DOI: 10.2748/tmj/1458248863

[15] I. R. Porteous, *Geometric Differentiation: For the Intelligence of Curves and Surfaces*, 2nd ed., Cambridge Univ. Press, 2001.

[16] K. Saji, M. Umehara, and K. Yamada, *The geometry of fronts*, Ann. of Math. **169** (2009), 491–529. DOI: 10.4007/annals.2009.169.491

[17] S. Shiba and M. Umehara, *The behavior of curvature functions at cusps and inflection points*, Differential Geom. Appl. **30** (2012), 285–299. DOI: 10.1016/j.difgeo.2012.04.001

[18] M. Umehara, *Differential geometry of surfaces with singular points*, (Japanese) Joy of Math. Eds. K. Ueno, T. Sunada and H. Arai, Nippon Hyoron Sha Co., Ltd. Publishers (2005), 50–64.

Atsufumi Honda, Department of Applied Mathematics, Faculty of Engineering, Yokohama National University, 79-5 Tokiwadai, Hodogaya, Yokohama 240-8501, Japan
Email address: `honda-atsufumi-kp@ynu.ac.jp`

Kosuke Naokawa, Department of Computer Science, Faculty of Applied Information Science, Hiroshima Institute of Technology, 2-1-1 Miyake, Saeki, Hiroshima, 731-5193, Japan
Email address: `k.naokawa.ec@cc.it-hiroshima.ac.jp`

Kentaro Saji, Department of Mathematics, Faculty of Science, Kobe University, Rokko, Kobe 657-8501
Email address: `saji@math.kobe-u.ac.jp`

Masaaki Umehara, Department of Mathematical and Computing Sciences, Tokyo Institute of Technology, Tokyo 152-8552, Japan
Email address: `umehara@is.titech.ac.jp`

Kotaro Yamada, Department of Mathematics, Tokyo Institute of Technology, Tokyo 152-8551, Japan
Email address: `kotaro@math.titech.ac.jp`

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UNLINKING SINGULAR LOCI FROM REGULAR FIBERS AND ITS APPLICATION TO SUBMERSIONS

OSAMU SAEKI

Dedicated to Professor Maria Aparecida Soares Ruas on the occasion of her 70th birthday

Abstract. Given a null-cobordant oriented framed link L in a closed oriented 3–manifold M, we study the condition for the existence of a generic smooth map of M to the plane that has L as an oriented framed regular fiber such that the singular point set is unlinked with L. As an application, we give a singularity theoretical proof to the theorem, originally proved by Hector, Peralta-Salas and Miyoshi, about the realization of a link in an open oriented 3–manifold as a regular fiber of a submersion to the plane.

1. Introduction

Let M be a smooth closed oriented 3–dimensional manifold and $f : M \to \mathbf{R}^2$ a smooth map. If $y \in f(M) \subset \mathbf{R}^2$ is a regular value, then $f^{-1}(y)$ is an oriented link in M and is naturally framed. Furthermore, if f is generic enough, then the singular point set $S(f)$ of f is an unoriented link in $M \setminus f^{-1}(y)$. In our previous paper [19], for an oriented framed link L in M, we characterized those unoriented links in $M \setminus L$ which arise as the singular point set of a generic map that has L as an oriented framed regular fiber. Such a characterization was given in terms of a relative Stiefel–Whitney class, or an obstruction to extending the trivialization of $TM|_L$ induced by the framing over the whole manifold M.

In this paper, we first study the obstruction class more in detail, and give a more practical characterization in terms of $\mathbf{Z}_2$ linking numbers. We also clarify the components of L which have non-trivial $\mathbf{Z}_2$ linking numbers with the singular point set. Then, as an application of such studies, we consider submersions of open oriented 3–manifolds to $\mathbf{R}^2$ that realize given oriented framed links as regular fibers. The idea is to consider a generic map f whose singular point set $S(f)$ is unlinked with a given oriented framed regular fiber and to delete a neighborhood of the singular point set $S(f)$ for obtaining a submersion. In this way, we get a singularity theoretical proof to the characterization theorem, originally due to Hector and Peralta-Salas [9] and Miyoshi [14], of those oriented (framed) links in $\mathbf{R}^3$ that arise as regular fibers of submersions. Recall that their proofs used the h-principle for submersions due to Phillips [16]. Instead, in this paper, we arrange the singular point set by using Levine's cusp elimination techniques [12] (see also [18, 19]) in a controlled way and push it to infinity, so that we get a submersion.

The paper is organized as follows. In §2, we recall several definitions and terminologies together with our main theorem in [19], which describes the characterization of singular point sets as unoriented links in terms of a certain obstruction class. In §3, we study the obstruction class more in detail, especially for closed oriented 3–manifolds M with $H_*(M; \mathbf{Z}) \cong H_*(S^3; \mathbf{Z})$. In such a case, we can identify the obstruction class in terms of $\mathbf{Z}_2$ linking numbers. Then, we can describe the condition for the obstruction class to vanish in terms of $\mathbf{Z}_2$ linking numbers. Finally

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in §4, we apply these results to submersions of open oriented 3–manifolds to $\mathbf{R}^2$. We will see that our singularity theoretical proof works well for punctured 3–manifolds, i.e. open 3–manifolds of the form $M^\circ = M \setminus D^3$ obtained from a closed 3–manifold M by removing a small closed 3–disk D^3 in M. For a general open oriented 3–manifold, we need to use an "absolute version" of the h-principle due to Phillips. Recall that the original proof due to Hector and Peralta-Salas [9] or Miyoshi [14] used the "relative version", stronger than the "absolute version", of the h-principle [7].

Throughout the paper, manifolds and maps are differentiable of class C^∞ unless otherwise indicated. All (co)homology groups are with $\mathbf{Z}_2$–coefficients unless otherwise indicated. The symbol "$\cong$" means an appropriate isomorphism between algebraic objects or a diffeomorphism between smooth manifolds.

2. Preliminaries

Let M (resp. N) be a closed 3–dimensional manifold (resp. a possibly noncompact surface) and consider a map $f : M \to N$. We denote by $S(f)$ the set of singular points of f. A point in $S(f)$ is a *fold singularity* (or a *cusp singularity*) of f if the map germ of f at that point is modeled on the map germ $(x, y, z) \mapsto (x, y^2 \pm z^2)$ (resp. $(x, y, z) \mapsto (x, y^3 + xy - z^2)$) at the origin. We say that a fold singularity is *definite* (resp. *indefinite*) if it is modeled on the map germ $(x, y, z) \mapsto (x, y^2 + z^2)$ (resp. $(x, y, z) \mapsto (x, y^2 - z^2)$). We say that f is *excellent* if $S(f)$ consists only of fold and cusp singularities. It is known that the set of excellent maps is always open and dense in the mapping space $C^\infty(M, N)$ endowed with the Whitney C^∞ topology (for example, see [6, 21]). If f is an excellent map, then $S(f)$ is an (unoriented) link in M, i.e. a finite disjoint union of smoothly embedded circles.

Let $f : M \to N$ be a map. For a regular value $y \in f(M) \subset N$, we call $L = f^{-1}(y)$ a *regular fiber*, which is a link in $M \setminus S(f)$. Note that L is naturally *framed*: its framing is given as the pull-back of the trivial normal framing of the point y in N. Furthermore, when M and N are oriented, L is naturally oriented.

In the following, we fix an orientation for $\mathbf{R}^2$ once and for all. For excellent maps of closed oriented 3–manifolds into $\mathbf{R}^2$, we have the following (for details, see [17, Proposition 5.1] and [19]).

Lemma 2.1. *Let L be an oriented framed link in a closed oriented 3–manifold M. Then, it is realized as an oriented framed regular fiber of an excellent map $f : M \to \mathbf{R}^2$ if and only if it is framed null-cobordant: i.e. there exists a compact oriented normally framed surface V embedded in M whose framed boundary coincides with L.*

Remark 2.2. Let L be an oriented link in a closed oriented 3–manifold M. Then, we can easily show that it bounds a compact oriented surface in M if and only if L represents zero in $H_1(M; \mathbf{Z})$. This can be proved by considering a certain map $M \setminus L \to S^1$. In particular, if $H_1(M; \mathbf{Z}) = 0$, then every oriented link bounds a compact oriented surface embedded in M.

Remark 2.3. It is known that every link in the 3–sphere is realized as a regular fiber of a restriction to S^3 of a certain polynomial map $\mathbf{R}^4 \to \mathbf{R}^2$ (see [1]). Furthermore, in [4], for a given link in the 3–sphere, the authors give an explicit algorithm to construct a quasi-holomorphic polynomial $\mathbf{C}^2 \to \mathbf{C}$ whose restriction to the unit sphere S^3 has the link as a regular fiber.

Now, let L be an oriented framed link in a closed oriented 3–manifold. If L is realized as a framed regular fiber of an excellent map $f : M \to \mathbf{R}^2$, then $S(f)$ is a link in $M \setminus L$. Thus, it is natural to ask the following.

QUESTION 2.4. Which links in $M \smallsetminus L$ appear as the singular point set $S(f)$ of an excellent map $f : M \to \mathbf{R}^2$ such that $f^{-1}(y)$ coincides with L as oriented framed links for some regular value $y \in \mathbf{R}^2$?

In order to answer to the above question, let us prepare some notations and terminologies. For an (unoriented) link J in $M \smallsetminus L$, we denote by $[J]_2 \in H_1(M \smallsetminus L)$ the $\mathbf{Z}_2$–homology class represented by J. Let $N(L)$ be a small tubular neighborhood of L in M disjoint from J. Since L is a framed link, we have a natural trivialization of $TM|_{N(L)}$. The obstruction to extending it over M is the relative Stiefel–Whitney class (see [10]), denoted by $w_2(M, L)$, which is an element of the $\mathbf{Z}_2$–cohomology group $H^2(M, N(L)) \cong H^2(M, L)$. Note that by excision and Poincaré–Lefschetz duality, we have

$$H^2(M, N(L)) \cong H^2(M \smallsetminus \operatorname{Int} N(L), \partial N(L)) \cong H_1(M \smallsetminus \operatorname{Int} N(L)) \cong H_1(M \smallsetminus L).$$

The following characterization, which answers to Question 2.4, has been proved in [19]. Recall that the proof was singularity theoretical in the sense that we used a result of Thom [20] about the homology class represented by the singular locus, and a cusp elimination result by Levine [12] for arranging the singular locus of an excellent map.

THEOREM 2.5. *Let L be an oriented null-cobordant framed link in a closed oriented 3–manifold M, and J an unoriented link in $M \smallsetminus L$. Then, there exist an excellent map $f : M \to \mathbf{R}^2$ and a regular value $y \in \mathbf{R}^2$ such that $f^{-1}(y)$ coincides with L as oriented framed links and that $S(f) = J$ if and only if $[J]_2 \in H_1(M \smallsetminus L)$ is Poincaré dual to $w_2(M, L) \in H^2(M, L)$.*

3. CASE OF INTEGRAL HOMOLOGY 3–SPHERES

In this section, we mainly consider closed oriented 3–manifolds M with

$$H_*(M; \mathbf{Z}) \cong H_*(S^3; \mathbf{Z})$$

and replace the condition described by the obstruction class $w_2(M, L)$ in Theorem 2.5 with that of $\mathbf{Z}_2$ linking numbers.

First, let M be an arbitrary closed oriented 3–manifold and L an oriented framed link in M. For the inclusion $j : (M, \emptyset) \to (M, L)$, the induced homomorphism $j^* : H^2(M, L) \to H^2(M)$ sends $w_2(M, L)$ to the second Stiefel–Whitney class $w_2(M)$ of M, which vanishes. By the cohomology exact sequence

$$H^1(L) \xrightarrow{\ \delta\ } H^2(M, L) \xrightarrow{\ j^*\ } H^2(M),$$

we have that $w_2(M, L) = \delta(\alpha)$ for some $\alpha \in H^1(L)$, although such an α may not be unique. In fact, such a class can be explicitly given as follows.

Set $L = L_1 \cup L_2 \cup \cdots \cup L_\mu$, where L_s are the components of L, $s = 1, 2, \ldots, \mu$. It is known that the tangent bundle TM of a closed oriented 3–manifold M is always trivial. Once a trivialization τ of TM is fixed, we can compare it with the specific trivialization of $TM|_{L_s}$ associated with the framing given for each component L_s of the framed link L. (We consider the trivialization given by the ordered vector fields v_1, v_2 and v_3, where v_1 is tangent to L_s consistent with the orientation, and v_2, v_3 are consistent with the framing.) This defines a well-defined element a_s in $\pi_1(SO(3)) \cong \mathbf{Z}_2$ for each s. Then, we have proved the following in [19].

LEMMA 3.1. *Let $\alpha \in H^1(L)$ be the unique cohomology class such that the Kronecker product $\langle \alpha, [L_s]_2 \rangle \in \mathbf{Z}_2$ coincides with a_s for each component L_s of L. Then, we have $\delta(\alpha) = w_2(M, L)$.*

Note that the trivialization τ of TM may not be unique. The set of homotopy classes of such trivializations is in one-to-one correspondence with the homotopy set $[M, SO(3)]$. If we consider the set of homotopy classes of trivializations on the 2–skeleton of M, then each such trivialization up to homotopy defines a *spin structure* on M, and the set of spin structures is in one-to-one correspondence with $H^1(M)$ (see [13]).

By the cohomology exact sequence,

$$H^1(M) \xrightarrow{i^*} H^1(L) \xrightarrow{\delta} H^2(M,L) \xrightarrow{j^*} H^2(M), \tag{3.1}$$

we see that for an arbitrary element $\beta \in \operatorname{Im} i^*$, we could choose $\alpha + \beta$ instead of α, where $i : L \to M$ is the inclusion map. The observation in the previous paragraph shows that this corresponds to choosing another trivialization which is "twisted along β".

The following proposition has also been proved in [19].

LEMMA 3.2. *Let L be an oriented framed link which bounds a compact oriented surface V consistent with the framing. Let $\alpha \in H^1(L)$ be an element such that $\delta(\alpha) = w_2(M,L)$. Then, we have*

$$\begin{aligned} \langle w_2(M,L), [V, \partial V]_2 \rangle &= \langle \delta(\alpha), [V, \partial V]_2 \rangle \\ &= \langle \alpha, [L]_2 \rangle \\ &\equiv \sharp L \pmod 2, \end{aligned}$$

where $\langle \cdot \, , \cdot \rangle$ is the Kronecker product, $[V, \partial V]_2 \in H_2(M,L)$ is the fundamental class of V in $\mathbf{Z}_2$–coefficients, and $\sharp L$ denotes the number of components of L.

Note that the above lemma is applicable for an arbitrary null-cobordant framed link L and that the value $\langle \alpha, [L]_2 \rangle \in \mathbf{Z}_2$ does not depend on a particular choice of α. Furthermore, if L has an odd number of components, then the obstruction $w_2(M,L)$ never vanishes.

Let us now consider the case of a local knot component. Suppose that the oriented framed link L contains a component L_s that lies in the interior of a closed 3–disk D embedded in M. Set $U = \operatorname{Int} D$, which is an open set of M diffeomorphic to $\mathbf{R}^3$. In the following, let us identify U with $\mathbf{R}^3$. In this case, up to homotopy, we may assume that the trivialization τ of TM over U is given by the standard one of $T\mathbf{R}^3$.

Let $\pi : \mathbf{R}^3 \to H$ be the orthogonal projection onto a generic hyperplane $H \cong \mathbf{R}^2$ in the sense that $\pi|_{L_s}$ is an immersion with normal crossings. Recall that the first vector field defining the trivialization $TM|_{L_s}$ associated with the framing on L_s is tangent to L_s consistent with the orientation. Since $\pi|_{L_s}$ is an immersion, we may assume that at each point x of L_s the remaining two vector fields give a 2–framing that is a basis for a 2–plane $N_x \subset T_x\mathbf{R}^3$ transverse to T_xL_s containing the direction $H^\perp$ perpendicular to H. Then, we count the number of times modulo 2 the 2–framing rotates in N_x with respect to a fixed positive direction of $H^\perp$ while x goes once around L_s. This number is denoted by $t_v(L_s)$, which is an element in $\mathbf{Z}_2$. Then, we have proved the following in [19].

LEMMA 3.3. *Let $\alpha \in H^1(L)$ be an arbitrary element such that $\delta(\alpha) = w_2(M,L)$. Then, we have*

$$\langle \alpha, [L_s]_2 \rangle \equiv t_v(L_s) + c(L_s) + 1 \pmod 2,$$

where $c(L_s)$ denotes the number of crossings of the immersion $\pi|_{L_s} : L_s \to H$ with normal crossings.

From now on, we will consider integral homology 3–spheres for M in this section. Let us start with the following.

DEFINITION 3.4. For an oriented link L in a closed oriented 3–manifold M with $H_1(M;\mathbf{Z}) = 0$, we always have a *Seifert surface*, i.e. a compact oriented surface V embedded in M such that $\partial V = L$. Such a Seifert surface is not unique; however, it is known that the induced framing on L is uniquely determined (for example, see [9, §3.6.1]). In the following, such a framing is said to be *preferred*.

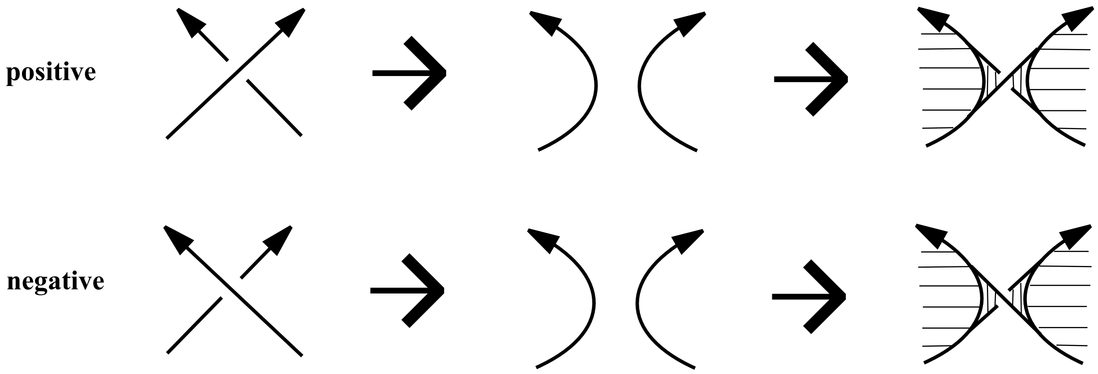

FIGURE 1. Seifert algorithm for positive and negative crossings

Then, for oriented links with preferred framings in the 3–sphere S^3, we have the following. In the following, we fix an orientation for S^3 once and for all.

PROPOSITION 3.5. *Let $L = L_1 \cup L_2 \cup \cdots \cup L_\mu$ be an oriented link in S^3, on which a preferred framing is given. Then $w_2(S^3, L) = 0$ if and only if for each s with $1 \leq s \leq \mu$, we have*

$$\sum_{t \neq s} \mathrm{lk}(L_s, L_t) \equiv 1 \pmod 2,$$

where lk *denotes the linking number.*

Proof. First, note that by the exact sequence (3.1) with $M = S^3$, we see that δ is injective and that $\alpha \in H^1(L)$ with $\delta(\alpha) = w_2(S^3, L)$ is uniquely determined. Therefore, $w_2(S^3, L) = 0$ if and only if $\langle \alpha, [L_s]_2 \rangle = 0$ for all s.

Now, we may assume that L is contained in $U \subset S^3$ as above, and let us consider the generic projection $\pi|_L : L \to H$. By the so-called Seifert algorithm, we can construct a compact oriented surface $V \subset S^3$ with $\partial V = L$ (see Fig. 1). Then, by construction, we see that when $\pi(x)$ goes once around $\pi(L_s)$, each time it goes through a positive (resp. negative) crossing point, it contributes $+1/2$ (resp. $-1/2$) to $t_v(L_s)$. Since the number of crossing points of $\pi(L_s)$ and $\pi(L_t)$ is even for each $t \neq s$, and $\pi(x)$ goes through each self-crossing point of $\pi(L_s)$ twice, we have

$$t_v(L_s) \equiv \frac{1}{2} \sum_{t \neq s} \widetilde{c}(L_s, L_t) + \widetilde{c}(L_s) \pmod 2$$

for each s, where $\widetilde{c}(L_s, L_t)$ is the sum of the signs of crossing points of $\pi(L_s)$ and $\pi(L_t)$, and $\widetilde{c}(L_s)$ is the sum of the signs of self-crossing points of $\pi(L_s)$. Then, since $\widetilde{c}(L_s) \equiv c(L_s) \pmod 2$, by Lemma 3.3, we have

$$\begin{aligned} \langle \alpha, [L_s]_2 \rangle &\equiv \frac{1}{2} \sum_{t \neq s} \widetilde{c}(L_s, L_t) + 1 \pmod 2 \\ &\equiv \sum_{t \neq s} \mathrm{lk}(L_s, L_t) + 1 \pmod 2, \end{aligned}$$

by the definition of linking numbers. Hence, the result follows. $\square$

Remark 3.6. The condition that appears in the statement of Proposition 3.5 is very similar to that in [9, Theorem 3.6.11]. In fact, in §4 we will prove the theorem obtained in [9] as an application of our Proposition 3.5.

In fact, we have the following more general result.

Proposition 3.7. *Let M be a closed connected oriented 3–manifold with*

$$H_1(M; \mathbf{Z}) = 0$$

and $L = L_1 \cup L_2 \cup \cdots \cup L_\mu$ be an oriented link in M, on which a preferred framing is given. Then, $w_2(M, L) = 0$ if and only if for each s with $1 \leq s \leq \mu$, we have

$$\sum_{t \neq s} \mathrm{lk}(L_s, L_t) \equiv 1 \pmod 2. \tag{3.2}$$

Proof. Since $H_1(M; \mathbf{Z}) = 0$, there exists a Seifert surface V for L, which is a compact oriented surface embedded in M with $\partial V = L$. By definition, this is consistent with the framing of L. Set $V' = V \smallsetminus \mathrm{Int}\, N(L)$ and $\widetilde{L}_s = V' \cap N(L_s)$ for each s, where $N(L)$ is a small tubular neighborhood of L in M, $N(L_s)$ is the component of $N(L)$ containing L_s, $\partial N(L)$ intersects V transversely, and $V \cap N(L)$ is a collar neighborhood of ∂V in V. Note that $\widetilde{L}_s$ is a knot parallel to L_s, and we orient $\widetilde{L}_s$ consistently with L_s. Then, the oriented link $\widehat{L}_s = L \smallsetminus L_s$ is $\mathbf{Z}$–homologous to $-\widetilde{L}_s$ in $M \smallsetminus L_s$, where $-\widetilde{L}_s$ denotes $\widetilde{L}_s$ with the opposite orientation.

Now, suppose $w_2(M, L) = 0$. In this case, the given framing of L extends over M. Let us suppose that a Seifert surface V_s for L_s is consistent with the given framing of L_s for some s. Then, by Lemma 3.2 applied to L_s, $w_2(M, L_s) \in H^2(M, L_s)$ does not vanish, as we obviously have $\sharp L_s = 1$. This implies that $a_s \in \mathbf{Z}_2$ as appears in Lemma 3.1 does not vanish. This contradicts our assumption that the framing of L extends over M. Therefore, an arbitrary Seifert surface V_s for L_s is not consistent with the given framing of L_s for each s. Since V is consistent with the framing of L_s, the linking number of L_s and $\widetilde{L}_s$ must be an odd integer. Since $-\widetilde{L}_s$ is $\mathbf{Z}$–homologous to $\widehat{L}_s$ in $M \smallsetminus L_s$, we have the congruence (3.2).

Conversely, suppose (3.2) holds for each s. Then, by the above argument we see that $a_s = 0$ for each s. Hence, by Lemma 3.1, we have $w_2(M, L) = 0$. This completes the proof. □

In fact, the above argument implies the following.

Proposition 3.8. *Let M be a closed connected oriented 3–manifold with*

$$H_1(M; \mathbf{Z}) = 0$$

and $L = L_1 \cup L_2 \cup \cdots \cup L_\mu$ be an oriented link in M, on which a preferred framing is given. For each s with $1 \leq s \leq \mu$, define $a_s \in \mathbf{Z}_2$ by

$$a_s = \sum_{t \neq s} \mathrm{lk}(L_s, L_t) + 1 \pmod 2.$$

Let $\alpha \in H^1(L)$ be the unique cohomology class such that $\langle \alpha, [L_s]_2 \rangle = a_s$ for all s. Then, we have $\delta(\alpha) = w_2(M, L)$.

When $H_1(M; \mathbf{Z}) = 0$, we have $H^1(M) = 0 = H^2(M)$, and hence the exact sequence (3.1) implies that we have the isomorphism $\delta : H^1(L) \to H^2(M, L)$. We easily see that its composition with the isomorphism $H^2(M, L) \to H_1(M \smallsetminus L)$ corresponds to the Alexander duality whose inverse isomorphism is given by taking $\mathbf{Z}_2$ linking numbers. This observation together with Theorem 2.5 leads to the following, which answers to Question 2.4 for oriented framed links in integral homology 3–spheres.

THEOREM 3.9. *Let M be a closed connected oriented 3–manifold with*

$$H_1(M; \mathbf{Z}) = 0,$$

$L = L_1 \cup L_2 \cup \cdots \cup L_\mu$ be an oriented link in M, and J be an unoriented link in $M \setminus L$. Then, there exists an excellent map $f : M \to \mathbf{R}^2$ such that $L = f^{-1}(y)$ for a regular value $y \in \mathbf{R}^2$ and $J = S(f)$ if and only if for each s with $1 \leq s \leq \mu$, the $\mathbf{Z}_2$ linking number of J with L_s coincides with

$$\sum_{t \neq s} \mathrm{lk}(L_s, L_t) + 1 \pmod 2.$$

Proof. By the above observations, we see that $[J]_2 \in H_1(M \setminus L)$ is Poincaré dual to

$$w_2(M, L) \in H^2(M, L)$$

if and only if it satisfies the condition on $\mathbf{Z}_2$ linking numbers in the theorem. Thus, the result follows from Theorem 2.5. $\square$

Let us observe the following.

LEMMA 3.10. *If the congruence (3.2) holds, then the number of components of L must be even.*

Proof. Consider the sum of all linking numbers

$$\sum_{s=1}^{\mu} \sum_{t \neq s} \mathrm{lk}(L_s, L_t) \in \mathbf{Z}$$

over all s and t with $s \neq t$. Since $\mathrm{lk}(L_s, L_t) = \mathrm{lk}(L_t, L_s)$, the above sum must be even. On the other hand, the congruence (3.2) implies that the above sum has the same parity as the number of components of L. Thus the result follows. $\square$

The above lemma together with Theorem 3.9 implies that for an integral homology 3–sphere M and an excellent map $f : M \to \mathbf{R}^2$, if $L = f^{-1}(y)$ has an odd number of components for a regular value $y \in \mathbf{R}^2$, then $S(f)$ has a non-trivial linking number with a component of L.

In order to get a more general result, let us introduce the following definition.

DEFINITION 3.11. Let M be a closed connected oriented 3–manifold and L, L' be non-empty disjoint closed sets in M. We say that L and L' are *not linked* if there exists an embedded 2–sphere in $M \setminus (L \cup L')$ which separates M into two components in such a way that one of them contains L and the other contains L'. If such a 2–sphere does not exist, then we say that L and L' are *linked*.

LEMMA 3.12. *Let M be a closed connected oriented 3–manifold containing an embedded 2–sphere S which separates M into two components M_1 and M_2, where M_1 and M_2 are the closures of the connected components of $M \setminus S$. If a framed link L is contained in $\mathrm{Int}\, M_1$ and is framed null-cobordant in M, then it is also framed null-cobordant in $\mathrm{Int}\, M_1$.*

Proof. Let V be a compact oriented normally framed surface in M which bounds L and is consistent with the framing of L. We may assume that V and S intersect each other transversely. Then, $V \cap S$ consists of a finite number of simple closed curves in the 2–sphere S. By considering $V \cap M_1$, adding 2–disks bounded by the simple closed curves in S, and by slightly translating the 2–disks in a parallel manner using the inner-most argument, we get a compact oriented surface embedded in $\mathrm{Int}\, M_1$. This gives a desired framed null-cobordism for L in $\mathrm{Int}\, M_1$. $\square$

We have the following as a result of Lemma 3.12.

Proposition 3.13. *Let M be a closed connected oriented 3–manifold and $f : M \to \mathbf{R}^2$ a smooth map. For a regular value $y \in \mathbf{R}^2$, if $L = f^{-1}(y)$ is non-empty and has an odd number of connected components, then L is necessarily linked with $S(f)$.*

Proof. Suppose that there exists a 2–sphere S that separates L and $S(f)$. Let M_1 and M_2 be the closures of the two components of $M \setminus S$ such that $L \subset \operatorname{Int} M_1$ and $S(f) \subset \operatorname{Int} M_2$. Since L is framed null-cobordant in M, it is also framed null-cobordant in $\operatorname{Int} M_1$ by Lemma 3.12. Therefore, there exists a compact oriented normally framed surface in $\operatorname{Int} M_1$ that bounds L. Let $\widehat{M_1}$ be the closed oriented 3–manifold obtained by attaching a 3–disk to M_1 along the boundary S. Then, since $f|_{M_1}$ is a submersion and $\pi_2(SO(3))$ vanishes, we see that the trivialization of $T\widehat{M_1}|_L$ extends to $\widehat{M_1}$, and hence $w_2(\widehat{M_1}, L)$ vanishes. Then, by Lemma 3.2 applied to $L \subset \widehat{M_1}$, this leads to a contradiction, since $\sharp L$ is odd by our assumption. Therefore, L and $S(f)$ are necessarily linked. This completes the proof. □

Note that the above proposition holds not only for excellent maps, but also for smooth maps.
In the case of integral homology 3–spheres, by Theorem 3.9 we have the following.

Proposition 3.14. *Let M be a closed connected oriented 3–manifold with*

$$H_1(M; \mathbf{Z}) = 0$$

and $L = L_1 \cup L_2 \cup \cdots \cup L_\mu$ be an oriented link in M. For an arbitrary excellent map $f : M \to \mathbf{R}^2$ such that $L = f^{-1}(y)$ for a regular value $y \in \mathbf{R}^2$, $S(f)$ necessarily links with each component L_s of L with

$$\sum_{t \neq s} \operatorname{lk}(L_s, L_t) \equiv 0 \pmod 2. \tag{3.3}$$

Compare the above proposition with [19, Problem 5.1]. For example, if the congruence (3.3) holds for all s, then for an excellent map $f : M \to \mathbf{R}^2$ such that $f^{-1}(y) = L$ for a regular value $y \in \mathbf{R}^2$, each component of L links with at least one component of $S(f)$.

We do not know if the results in this section for M with $H_1(M; \mathbf{Z}) = 0$ also hold for M with $H_1(M) = 0$ in $\mathbf{Z}_2$–coefficients.

Remark 3.15. In fact, Proposition 3.14 holds not only for excellent maps, but also for smooth maps, which can be proved as follows. Suppose that there exists a smooth map $g : M \to \mathbf{R}^2$ such that $L = g^{-1}(y)$ for a regular value $y \in \mathbf{R}^2$ and that $S(g)$ does not link with L_s. Then, we can approximate g by an excellent map f such that $S(f) \subset N(S(g))$ and $f|_{M \setminus N(S(g))} = g|_{M \setminus N(S(g))}$ for a sufficiently small neighborhood $N(S(g))$ of $S(g)$. Then, such an f leads to a contradiction.

4. Submersions of open 3–manifolds to $\mathbf{R}^2$

In this section, as an application of our results in [19] and in the previous sections of the present paper, we consider submersions of open orientable 3–manifolds to $\mathbf{R}^2$.

First, let us recall the following fundamental theorem for submersions of $\mathbf{R}^3$ to $\mathbf{R}^2$ obtained in [9].

Theorem 4.1 (Hector and Peralta-Salas, 2012). *Let $L = L_1 \cup L_2 \cup \cdots \cup L_\mu \subset \mathbf{R}^3$ be an oriented link in $\mathbf{R}^3$. Then, there exists a submersion $f : \mathbf{R}^3 \to \mathbf{R}^2$ such that $f^{-1}(y) = L$ for some $y \in \mathbf{R}^2$ if and only if for each s with $1 \leq s \leq \mu$, we have*

$$\sum_{t \neq s} \operatorname{lk}(L_s, L_t) \equiv 1 \pmod 2.$$

Recall that in [9], the authors used the h-principle for submersions [7, 16] for the proof. Here, we give a new proof to the above theorem using our singularity theoretical techniques.

Proof of Theorem 4.1. Let L be an oriented link in $\mathbf{R}^3$ which satisfies the condition about the linking numbers as in the theorem. By identifying the interior of an embedded 3–disk D in S^3 with $\mathbf{R}^3$, we may assume that $L \subset \operatorname{Int} D \subset S^3$. Then, by Proposition 3.5, we have $w_2(S^3, L) = 0$ with respect to the preferred framing on L. Therefore, for an arbitrary non-empty link J in $S^3 \setminus D$, there exists an excellent map $g : S^3 \to \mathbf{R}^2$ and a regular value $y \in \mathbf{R}^2$ such that $L = g^{-1}(y)$ and $J = S(g)$. By restricting g to $\mathbf{R}^3 = \operatorname{Int} D$, we get a submersion $f : \mathbf{R}^3 \to \mathbf{R}^2$ which has L as a regular fiber.

Conversely, suppose that we have a submersion $f : \mathbf{R}^3 \to \mathbf{R}^2$ and a regular value $y \in \mathbf{R}^2$ such that $f^{-1}(y) = L$. Then, we can find an embedded 3–disk $D \subset \mathbf{R}^3$ whose interior contains L. Note that $f|_D : D \to \mathbf{R}^2$ is a submersion which has L as a regular fiber. By embedding D into S^3, we can extend $f|_D$ to a smooth map $g_1 : S^3 \to \mathbf{R}^2$. Here, $f(\partial D)$ misses $y \in \mathbf{R}^2$, and since the second homotopy group of $\mathbf{R}^2 \setminus \{y\}$ is trivial, $f|_{\partial D}$ is null-homotopic inside $\mathbf{R}^2 \setminus \{y\}$. Therefore, we can arrange the smooth map g_1 in such a way that g_1 has $y \in \mathbf{R}^2$ as a regular value and that $g_1^{-1}(y) = L \subset \operatorname{Int} D$. Then, by slightly perturbing g_1 on a neighborhood of $S^3 \setminus \operatorname{Int} D$, we get an excellent map $g_2 : S^3 \to \mathbf{R}^2$ such that $y \in \mathbf{R}^2$ is a regular value, that $g_2^{-1}(y) = L$, and that $S(g_2)$ is contained in $S^3 \setminus \operatorname{Int} D$. In particular, $S(g_2)$ is $\mathbf{Z}_2$ null-homologous in $S^3 \setminus L$, and hence we have $w_2(S^3, L) = 0$. Then, by Proposition 3.5, we get the result. □

Remark 4.2. More generally, instead of $\mathbf{R}^3$, the above theorem holds also for an arbitrary open 3–manifold of the form $M \setminus D^3$ for a closed connected orientable 3–dimensional manifold M with $H_1(M; \mathbf{Z}) = 0$, where D^3 is a small closed 3–disk embedded in M.

In the case of a link with an odd number of components, we have the following.

Remark 4.3. Let $f : \mathbf{R}^3 \to \mathbf{R}^2$ be a smooth map, and suppose that $y \in \mathbf{R}^2$ is a regular value such that $L = f^{-1}(y)$ is compact and has an odd number of components. Then, by Proposition 3.13 together with an argument similar to the above, we see that the singular point set $S(f)$ necessarily links with L (see also the paragraph just after [15, Theorem 10]): in other words, we can find no 2–sphere embedded in $\mathbf{R}^3$ that separates L and $S(f)$. This implies, in particular, that such an f can never be a submersion.

In fact, we have the following.

Proposition 4.4. *Let M be a closed connected orientable 3–manifold with*

$$H_1(M; \mathbf{Z}) = 0$$

and set $M^\circ = M \setminus D^3$. Let $L = L_1 \cup L_2 \cup \cdots \cup L_\mu \subset M^\circ$ be an oriented link such that $f^{-1}(y) = L$ for some excellent map $f : M^\circ \to \mathbf{R}^2$ and a regular value $y \in \mathbf{R}^2$. Then, each component L_s of L with

$$\sum_{t \neq s} \operatorname{lk}(L_s, L_t) \equiv 0 \pmod 2 \tag{4.1}$$

links with at least one component of $S(f)$. In particular, such an f can never be a submersion.

Compare the above proposition with [19, Problem 5.1]. See also [2, 3, 5, 11] for related physical results.

Proof of Proposition 4.4. First note that each component of $S(f)$ is diffeomorphic to a circle or a real line. Furthermore, $S(f)$ is a closed submanifold of M° which may have infinitely many connected components.

Let V_s be a Seifert surface for L_s in M, where L_s satisfies (4.1). We may assume that $L_s \subset M^\circ$ and that $S(f)$ intersects V_s transversely at finitely many points. We have only to show that there are an odd number of intersection points.

Let $\widetilde{D}$ be a 3–disk in M such that $\operatorname{Int}\widetilde{D} \supset D^3$, $L \cap \widetilde{D} = \emptyset$, $V_s \cap \widetilde{D} = \emptyset$, and that $\partial\widetilde{D}$ intersects $S(f)$ transversely at finitely many points. Then, by an argument similar to that in the proof of Theorem 4.1, we can construct an excellent map $g : M \to \mathbf{R}^2$ such that $g|_{M \setminus \operatorname{Int}\widetilde{D}} = f|_{M \setminus \operatorname{Int}\widetilde{D}}$ and that $g^{-1}(y) = L$. By our assumption (4.1), we have that L_s has a non-trivial $\mathbf{Z}_2$ linking number with $S(g)$ by Theorem 3.9. Therefore, $S(g)$ intersects V_s transversely at an odd number of points. By construction of g, this implies that $S(f)$ also intersects V_s transversely at an odd number of points. This completes the proof. □

Remark 4.5. In fact, the above proposition holds not only for excellent maps, but also for smooth maps if we replace the statement "L_s links with at least one component of $S(f)$" by "L_s links with $S(f)$". This can be proved by an argument similar to that in Remark 3.15.

The following is a special case of a theorem proved by Miyoshi [14], who used a relative version of the h-principle for submersions [7]. Here, we use our singularity theoretical arguments in order to prove the theorem for punctured 3–manifolds.

Theorem 4.6. *Let M be a closed orientable 3–manifold and L a compact oriented framed link in $M^\circ = M \setminus D^3$. Then, there exists a submersion $f : M^\circ \to \mathbf{R}^2$ such that $f^{-1}(y)$ coincides with L as oriented framed links for some $y \in \mathbf{R}^2$ if and only if L bounds a proper normally framed surface in M° and the trivialization of $TM^\circ|_L$ induced by the framing of L extends over M°.*

Proof. If there exists a submersion f as in the theorem, then the inverse image by f of the half line $[y_1, \infty) \times \{y_2\} \subset \mathbf{R}^2$ is a proper normally framed surface in M° that bounds L, where $y = (y_1, y_2)$. Furthermore, since f is a submersion, we can pull-back the natural trivialization of $T\mathbf{R}^2$ to M° by f in such a way that the pull-back naturally extends the trivialization of $TM^\circ|_L$ induced by the framing of L.

Conversely, suppose that L bounds a proper normally framed surface V in M° and the trivialization of $TM^\circ|_L$ induced by the framing of L extends over M°. Let $\widetilde{D}$ be a small 3–disk neighborhood of D^3 whose interior contains D^3 such that $\widetilde{D} \subset M \setminus N(L)$ for a small tubular neighborhood $N(L)$ of L in M. Then, we may assume that V intersects $\partial\widetilde{D}$ transversely along finitely many embedded oriented circles. Note that then $V \cap \partial\widetilde{D}$ bounds a compact oriented surface V' in $\widetilde{D}$. Then, by replacing $V \cap \widetilde{D}$ by V', we see that L is framed null-cobordant in M. Furthermore, by our assumption, the trivialization of $TM^\circ|_L$ induced by the framing of L extends over M°. Since $\pi_2(SO(3))$ vanishes, this implies that it also extends over M. Therefore, we have that the obstruction $w_2(M, L)$ vanishes. Hence, by Theorem 2.5, there exists an excellent map $f : M \to \mathbf{R}^2$ and a regular value $y \in \mathbf{R}^2$ such that $f^{-1}(y)$ coincides with L as oriented framed links and that $S(f)$ is contained in $\operatorname{Int} D^3$. Then, f restricted to $M^\circ = M \setminus D^3$ is a desired submersion. □

In fact, if we use the "absolute version" of the h-principle [16] in order to treat the end of an open 3–manifold, we can prove the following. Note again that the following theorem was originally proved by Miyoshi [14] by using a "relative version" of the h-principle [7].

Theorem 4.7. *Let M be an open orientable 3–manifold and L a compact oriented framed link in M. Then, there exists a submersion $f : M \to \mathbf{R}^2$ such that $f^{-1}(y)$ coincides with L as oriented framed links for some $y \in \mathbf{R}^2$ if and only if L bounds a proper normally framed surface in M and the trivialization of $TM|_L$ induced by the framing of L extends over M.*

Proof. Necessity can be proved by the same argument as in the proof of Theorem 4.6.

Conversely, suppose that there exists a proper normally framed surface V in M that bounds L as described in the theorem. Let Q be a compact 3–dimensional submanifold of M with boundary such that $\operatorname{Int} Q \supset L$ and that ∂Q intersects V transversely along finitely many embedded circles.

Let us first construct a smooth map $g_1 : M \to \mathbf{R}^2$ as follows. Let $h : V \to [0, \infty)$ be a smooth function such that $h^{-1}(0) = \partial V = L$ and that h is non-singular near ∂V. Let $N(V) \cong V \times I$ be a tubular neighborhood of V in M, where $I = [-1, 1]$ and the I–factor is consistent with the normal orientation of V. Then, we define g_1 on $N(V)$ by

$$N(V) \cong V \times I \xrightarrow{h \times \mathrm{id}_I} [0, \infty) \times I \subset \mathbf{R}^2,$$

where id_I is the identity map of I. We can extend $g_1|_{N(V)}$ to $N(V) \cup N(L)$ in such a way that $g_1|_{N(L)}$ is a submersion, that the origin 0 is a regular value, and that the framed regular fiber $g_1^{-1}(0)$ coincides with L. Then, since $\mathbf{R}^2 \smallsetminus g_1(N(V) \cup N(L))$ is contractible, we can extend g_1 to the whole manifold M in such a way that 0 is still a regular value and that the framed regular fiber $g_1^{-1}(0)$ coincides with L.

Set $Q' = Q \smallsetminus \mathrm{Int}\, N(L)$, which is a compact 3–manifold with boundary $\partial Q \cup \partial N(L)$. Note that $g_1(Q') \subset \mathbf{R}^2 \smallsetminus \mathrm{Int}\, D$, where D is a small 2–disk neighborhood of the origin.

By our assumption, the framing on L extends over M. Using such a framing, we can construct a bundle epimorphism $T(M \smallsetminus \mathrm{Int}\, Q) \to T(\mathbf{R}^2 \smallsetminus \mathrm{Int}\, D)$ covering $g_1|_{M \smallsetminus \mathrm{Int}\, Q}$. Then, by the h-principle for submersions, g_1 is homotopic to a smooth map $g_2 : M \to \mathbf{R}^2$ such that

(1) g_2 is a submersion over $M \smallsetminus \mathrm{Int}\, Q$,
(2) $g_2 = g_1$ over $N(L)$,
(3) $g_2(M \smallsetminus \mathrm{Int}\, N(L)) \subset \mathbf{R}^2 \smallsetminus \mathrm{Int}\, D$.

Then, we can approximate g_2 by an excellent map g_3 that enjoys the same properties as g_2 described above. Then, $S(g_3)$ is a closed subset of Q, which is compact. Therefore, $S(g_3)$ is an unoriented link in $Q \smallsetminus \mathrm{Int}\, N(L)$. Furthermore, as we started with a framing that extends over M, the obstruction to extending the framing on $\partial(Q \smallsetminus \mathrm{Int}\, N(L))$ induced by g_3 to the whole Q vanishes. This implies that the $\mathbf{Z}_2$–homology class represented by $S(g_3)$ vanishes in Q. Then, by our techniques developed in [19] using Levine's cusp eliminations (see [12, 18]), we can homotope g_3 to an excellent map g_4 that satisfies the properties described above such that $S(g_4)$ is unlinked from L: more precisely, there exists an embedded 3–disk $B \subset \mathrm{Int}\, Q \smallsetminus N(L)$ such that $\mathrm{Int}\, B \supset S(g_4)$. Then, for an appropriate embedded arc $A \subset M \smallsetminus N(L)$ that "connects" B to infinity, we see that M is diffeomorphic to $M \smallsetminus (A \cup B)$ by a diffeomorphism that is the identity on $N(L)$ (for example, see [14]). Then, the restriction of g_4 to $M \smallsetminus (A \cup B)$ gives the desired submersion. This completes the proof. $\square$

Remark 4.8. It is known that there exist open 3–manifolds that cannot be embedded in compact 3–manifolds [8].

We finish this paper by posing an open problem.

Problem 4.9. Is there a polynomial map $\mathbf{R}^3 \to \mathbf{R}^2$ that is a submersion and has a compact regular fiber as in Theorem 4.1?

Compare the above problem with Remark 2.3.

One can find some relevant open problems in [9, §4] as well.

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References

[1] S. Akbulut and H. King, *All knots are algebraic*, Comment. Math. Helv. **56** (1981), 339–351.

[2] M.V. Berry and M.R. Dennis, *Knotted and linked phase singularities in monochromatic waves*, Proc. R. Soc. Lond. A **457** (2001), 2251–2263. DOI: 10.1098/rspa.2001.0826

[3] M.V Berry and M.R. Dennis, *Topological events on wave dislocation lines: birth and death of loops, and reconnection*, J. Physics A: Mathematical and Theoretical **40** (2007), 65–74. DOI: 10.1088/1751-8113/40/1/004

[4] B. Bode and M.R. Dennis, *Constructing a polynomial whose nodal set is any prescribed knot or link*, J. Knot Theory Ramifications **28**, (2019) 1850082 (31 pages). DOI: 10.1142/s0218216518500827

[5] M.R. Dennis, *Local phase structure of wave dislocation lines: twist and twirl*, J. Opt. A: Pure Appl. Opt. **6** (2004), S202–S208. DOI: 10.1088/1464-4258/6/5/011

[6] M. Golubitsky and V. Guillemin, *Stable mappings and their singularities*, Grad. Texts in Math., Vol. 14, Springer–Verlag, New York, Heidelberg, Berlin, 1973.

[7] A. Haefliger, *Lectures on the theorem of Gromov*, Proceedings of Liverpool Singularities Symposium, II (1969/1970) (Berlin), Lecture Notes in Math., vol. 209, Springer-Verlag, 1971, pp. 128–141. DOI: 10.1007/bfb0068900

[8] W. Haken, *Some results on surfaces in* 3*–manifolds*, in "1968 Studies in Modern Topology", pp. 39–98, Math. Assoc. Amer. (distributed by Prentice-Hall, Englewood Cliffs, N.J.).

[9] G. Hector and D. Peralta-Salas, *Integrable embeddings and foliations*, Amer. J. Math. **134** (2012), 773–825. DOI: 10.1353/ajm.2012.0018

[10] M.A. Kervaire, *Relative characteristic classes*, Amer. J. Math. **79** (1957), 517–558.

[11] H. Larocque, D. Sugic, D. Mortimer, A.J. Taylor, R. Fickler, R.W. Boyd, M.R. Dennis, and E. Karimi , *Reconstructing the topology of optical polarization knots*, Nature Physics (2018), https://doi.org/10.1038/s41567-018-0229-2. DOI: 10.1038/s41567-018-0229-2

[12] H.I. Levine, *Elimination of cusps*, Topology **3**, Suppl. 2 (1965), 263–296.

[13] J. Milnor, *Spin structures on manifolds*, Enseignement Math. (2) **9** (1963), 198–203.

[14] S. Miyoshi, *Links and submersions to the plane on an open* 3*-manifold*, J. Topol. Anal. **8** (2016), 677–690. DOI: 10.1142/s1793525316500266

[15] D. Peralta-Salas, *Realization problems in the theory of foliations*, In: S. Ibáñez, J.S. Pérez del Río, A. Pumariño, J.A. Rodríguez (eds.), Progress and Challenges in Dynamical Systems, Springer Proceedings in Mathematics & Statistics, vol. 54, pp. 309–323, Springer, Berlin, Heidelberg, 2013. DOI: 10.1007/978-3-642-38830-9_19

[16] A. Phillips, *Submersions of open manifolds*, Topology **6** (1967), 171–206. DOI: 10.1016/0040-9383(67)90034-1

[17] O. Saeki, *Stable maps and links in* 3*–manifolds*, Kodai Math. J. **17** (1994), 518–529. DOI: 10.2996/kmj/1138040047

[18] O. Saeki, *Constructing generic smooth maps of a manifold into a surface with prescribed singular loci*, Ann. Inst. Fourier (Grenoble) **45** (1995), 1135–1162. DOI: 10.5802/aif.1489

[19] O. Saeki, *Linking between singular locus and regular fibers*, J. of Singularities, vol. 21 (2020), 234–248. DOI: 10.5427/jsing.2020.21n

[20] R. Thom, *Les singularités des applications différentiables*, Ann. Inst. Fourier, Grenoble **6** (1955–1956), 43–87.

[21] H. Whitney, *On singularities of mappings of Euclidean spaces. I. Mappings of the plane into the plane*, Ann. of Math. (2) **62** (1955), 374–410. DOI: 10.2307/1970070

Osamu Saeki, Institute of Mathematics for Industry, Kyushu University, Motooka 744, Nishi-ku, Fukuoka 819-0395, Japan

Email address: `saeki@imi.kyushu-u.ac.jp`

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LOOPS IN GENERALIZED REEB GRAPHS ASSOCIATED TO STABLE CIRCLE-VALUED FUNCTIONS

ERICA BOIZAN BATISTA, JOÃO CARLOS FERREIRA COSTA, AND JUAN J. NUÑO-BALLESTEROS

Abstract. Let N be a smooth compact, connected and orientable 2-manifold with or without boundary. Given a stable circle-valued function $\gamma : N \to S^1$, we introduced a topological invariant associated to γ, called generalized Reeb graph. It is a generalized version of the classical and well known Reeb graph. The purpose of this paper is to investigate the number of loops in generalized Reeb graphs associated to stable circle-valued functions $\gamma : N \to S^1$. We show that the number of loops depends on the genus of N, the number of boundary components of N, and the number of open saddles of γ. In particular, we show a class of functions whose generalized Reeb graphs have the maximal number of loops.

1. Introduction

The Reeb graph was introduced by Reeb in [13] and it is well known that it is a complete topological invariant for Morse functions from S^2 to $\mathbb{R}$, where S^2 is the standard sphere in $\mathbb{R}^3$ (see [1, 14]).

Although originally introduced as a tool in Morse theory, the Reeb graphs have several applications in Computational Geometry, Computer Graphics, Engineering, Applied Mathematics, etc. A more extensive discussion of Reeb graphs and their variations in geometric modeling and visualization applications can be found in [4, 7].

An interesting problem related to Reeb graphs in the context of computational geometry is to investigate the number of loops of such graphs. The number of loops in a Reeb graph of a Morse function over a 2-manifold (orientable or non-orientable) with and without boundary was investigated in [5]. Later, some of these results were generalized in [8].

In this paper we study a similar problem. We investigate the number of loops in a graph associated to a stable circle-valued function $\gamma : N \to S^1$, where N is a smooth compact, connected and orientable 2-manifold with or without boundary and S^1 is the standard sphere in $\mathbb{R}^2$. The study of stable circle-valued functions was initiated by S.P. Novikov in the early 1980's related with a hydrodynamic problem [11, 12]. Today we can find applications and connections to many geometrical problems. Recently, an interesting connection with Singularity theory was obtained by the authors related to the topological classification of finitely determined map germs from $(\mathbb{R}^3, 0)$ to $(\mathbb{R}^2, 0)$ (see [2, 3]).

A stable circle-valued function is defined as follows:

Definition 1.1. Let N be a smooth compact, connected and orientable 2-manifold with boundary ∂N (including the case when $\partial N = \emptyset$), and let P be a smooth 1-manifold. We say that $\gamma : N \to P$ is *stable* if:

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(1) γ is Morse with distinct critical values;
(2) γ does not have critical points in ∂N;
(3) $\gamma|_{\partial N}$ is regular.

If $P = \mathbb{R}$ and $\gamma : N \to \mathbb{R}$ is stable, we can consider the following equivalence relation in N: given x, $y \in N$, $x \sim y$ if and only if $\gamma(x) = \gamma(y)$ and furthermore, x and y are in the same connected component of $\gamma^{-1}(\gamma(x))$. Reeb [13] showed that the quotient set $N/\sim$ admits a graph structure which is called *Reeb graph* associated to γ.

Intuitively, the Reeb graph associated to γ is obtained by contracting each connected component of the level curves of γ to points, where the vertices correspond to connected components of level curves containing critical points. Consider the following example, where $\gamma : N \to \mathbb{R}$ is the height function and N is a closed 2-manifold:

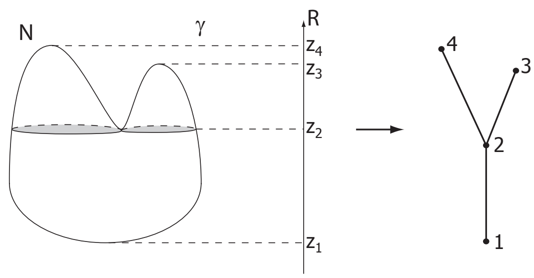

FIGURE 1. Reeb graph associated to the height function

When N is diffeomorphic to the sphere S^2, the Reeb graph is a tree (see [13]).

Since the Reeb graph gives the topological information about N, it is interesting to investigate the relation of its structure with topological elements such as Euler characteristic, Betti numbers, genus, etc. For instance, as motivation for this work, we can cite the following results:

Proposition 1.2. *([5, 8]) The Reeb graph of a Morse function over a connected orientable 2-manifold of genus g without boundary has g loops.*

Proposition 1.3. *([5, 8]) The Reeb graph of a Morse function over a connected orientable 2-manifold of genus g with $h \geqslant 1$ boundary components has between g and $2g + h - 1$ loops.*

Notice that the number of loops in the Reeb graph is given by the first Betti number of the graph, which is the rank of the first homology group. Also, it follows that the first Betti number of the 2-manifold N bounds from above the first Betti number of the graph, i.e.,

$$\text{number of loops} \leq \beta_1(\text{N}).$$

Figure 2 provides an example of a Reeb graph associated to $\gamma : N \to S^1$, where N is a 2-manifold with $h = 4$ boundary components and genus $g = 1$. The Reeb graph in this case has 3 loops, with $3 \leq 2g + h - 1 = 5 = \beta_1(N)$.

Remark 1.4. In the Reeb graph given in Figure 2, the slim traces indicate circle fibers and the bold traces arc fibers of γ, respectively. In Section 2, these different kind of traces in a Reeb graph are defined with more details.

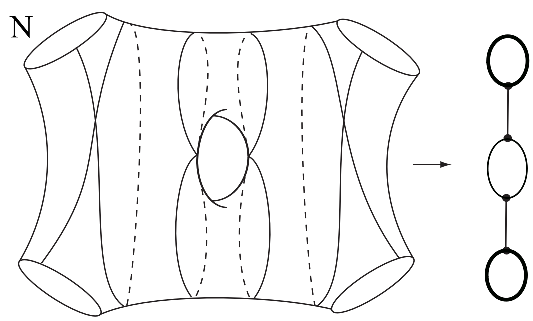

FIGURE 2. Reeb graph of a circle-valued Morse function γ.

In this work we obtain a similar relation to the number of loops, but now in a more general context, using stable circle-valued functions $\gamma : N \to S^1$ and the notion of generalized Reeb graphs.

2. The generalized Reeb graph

The generalized Reeb graph was introduced by the authors in [2, 3]. It is a generalized version of the classical Reeb graph, and it was inspired in Maksymenko's work [10].

Let $\gamma : N \to S^1$ be a stable circle-valued function, where N is a smooth connected, compact and orientable 2-manifold with or without boundary. Consider the following equivalence relation in N, analogous to the one given in the previous Section: given x, $y \in N$, $x \sim y$ if and only if $\gamma(x) = \gamma(y)$, where x and y are in the same connected component of $\gamma^{-1}(\gamma(x))$. The following result shows the structure of $N/\sim$:

Proposition 2.1. *Let N be a smooth connected, compact and orientable 2-manifold with or without boundary. Let $\gamma : N \to S^1$ be a stable circle-valued function. Then, the quotient space $N/\sim$ admits a graph structure as follows:*

(1) *The vertices are the connected components of level curves $\gamma^{-1}(v)$, where $v \in S^1$ is a critical value;*
(2) *Each edge is formed by points that correspond to connected components of level curves $\gamma^{-1}(v)$, where $v \in S^1$ is a regular value.*

Proof. Since γ is stable its critical points are isolated and N being compact, γ has a finite number of critical points. Moreover, N connected implies $N/\sim$ connected.

Let $v_1, \dots, v_r$ be the critical values of γ. Then,

$$\gamma|N - \gamma^{-1}(\{v_1, \dots, v_r\}) : N - \gamma^{-1}(\{v_1, \dots, v_r\}) \to S^1 - \{v_1, \dots, v_r\}$$

is regular, and the induced map

$$\tilde{\gamma} : (N - \gamma^{-1}(\{v_1, \dots, v_r\}))/\sim \to S^1 - \{v_1, \dots, v_r\}$$

is a local homeomorphism. Each connected component of $S^1 - \{v_1, \dots, v_r\}$ is homeomorphic to an open interval, so each connected component of $(N - \gamma^{-1}(\{v_1, \dots, v_r\}))/\sim$ is also homeomorphic to an open interval.

□

Remark 2.2. (1) *Let C_i be the connected components of ∂N, with $i = 1, \dots, n$. Then $\gamma|C_i : C_i \to S^1$ is a diffeomorphism.*
(2) *The level curves of γ intersect ∂N transversely.*

The possible topological types of the level curves of $\gamma : N \to S^1$ are:

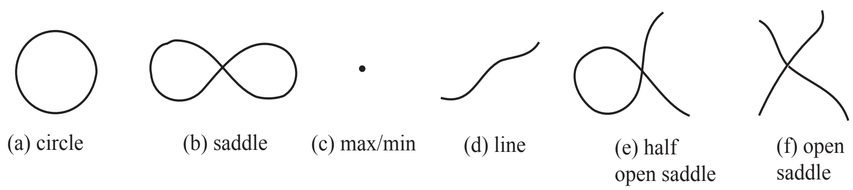

FIGURE 3. Topological types of level curves

By Remark 2.2 item (2), the level curves of γ that can intersect ∂N are only the types (d), (e) and (f). Furthermore, by item (1), each level curve of γ can intersects at most once a connected component C_i of ∂N, and these intersections happen in regular points.

The graph structure of $N/\sim$ given in Proposition 2.1 associated to a stable function $\gamma : N \to S^1$ will be denoted by Γ_γ. Each edge of Γ_γ can be of two types: one corresponds to connected components of circle type and will be denoted by a slim trace; another corresponds to connected components of interval type and will be denoted by a bold trace. We denote by Γ the subgraph of Γ_γ given by the slim edges with their respective vertices, and by Γ' the subgraph of Γ_γ given by the bold edges with their respective vertices (i.e., $\Gamma_\gamma = \Gamma \cup \Gamma'$).

Each vertex of the graph can be of six types, depending if the connected component has a maximum/minimum critical point, a saddle point, a half open saddle point, an open saddle point or a regular point. Then, the possible incidence rules of edges and vertices when $\gamma : N \to S^1$ is stable are given in Figure 4.

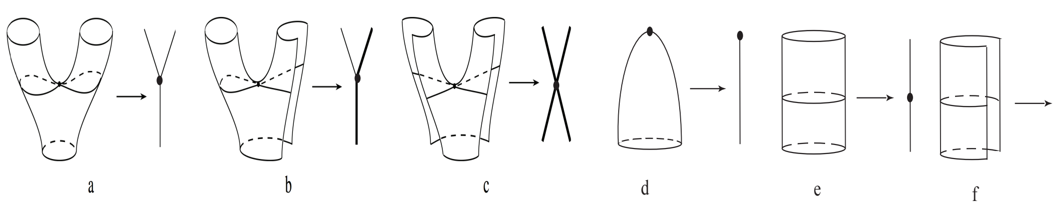

FIGURE 4. Incidence rules

We denote by S, S', S'', M, C and I the number of vertices of type (a) through (f), respectively.

Figure 5 represents some possible structures of the graph $N/\sim$ for stable maps from N to S^1. Notice that Γ and Γ' are not necessarily connected graphs.

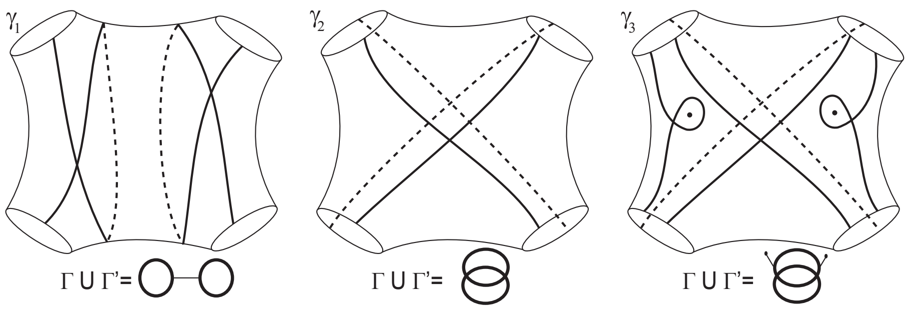

FIGURE 5. Graphs $N/\sim$ for stable maps $\gamma_i : N \to S^1$, $i = 1, 2, 3$

Let $v_1, \dots, v_k \in S^1$ be the critical values of $\gamma : N \to S^1$. We choose a base point $v_0 \in S^1$ and an orientation. We can reorder the critical values such that $v_0 < v_1 < \dots < v_k$ and we label each vertex with values $i \in \{1, \dots, k\}$, if it corresponds to critical values v_i.

Definition 2.3. Let $\gamma : N \to S^1$ be a stable circle-valued function. The graph given by $N/\sim$ together with the types of edges and the labels of the vertices, as previously defined is called the *generalized Reeb graph* associated to γ.

Example 2.4. Consider the stable circle-valued functions $\gamma_1 : S^2 \to S^1$, $\gamma_2 : N \to S^1$, where N is a 2-manifold with boundary, as appear in Figure 5. The respective generalized Reeb graphs, Γ_{γ_1} and Γ_{γ_2}, are exhibited in Figure 6.

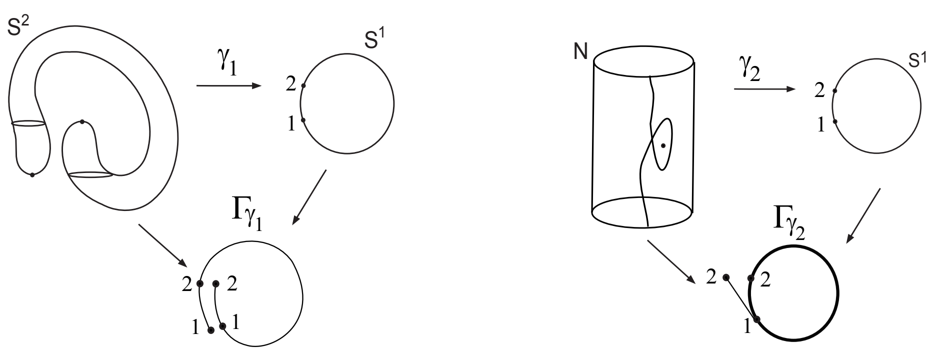

FIGURE 6. Generalized Reeb graphs

As previously stated, the main goal of this work is to investigate the number of loops in generalized Reeb graphs. This number is defined as follows:

Definition 2.5. Let Γ_γ be the generalized Reeb graph associated to the stable function $\gamma : N \to S^1$. The first Betti number of Γ_γ, denoted by $\beta_1(\Gamma_\gamma)$, is called the *number of loops* of Γ_γ.

In what follows, the notation β_i will indicate the ith Betti number.

3. Number of loops and other properties of Γ_γ

In this section we investigate the number of loops in generalized Reeb graphs and present some other properties of these graphs.

From now on, N will be a smooth connected, orientable and closed 2-manifold or N will be a 2-manifold with boundary obtained by taking a closed 2-manifold and removing h-disks. In the last case, by simplicity, we will simply say that N is a 2-manifold with boundary.

Theorem 3.1. *Let N be a closed 2-manifold of genus g and let $\gamma : N \to S^1$ be a non regular stable circle-valued function. Then the generalized Reeb graph Γ_γ of γ has g loops.*

Proof. First notice that Γ_γ is connected and $\chi(\Gamma_\gamma) = V - E$, where V, E denote the number of vertices and edges of Γ_γ, respectively.

On one hand, $V = M + S + I$ where M, S, I are the numbers of vertices of type: max/min, saddle or regular, respectively. Since γ is non regular, $V \neq 0$.

On the other hand, by Euler's formula $E = \frac{1}{2}\sum_{i=1}^{V} \deg(v_i)$ where $v_i \in V$ and $\deg(v_i)$ (the degree of v_i) is the number of edges incident to v_i. As γ is stable, the degree of each vertex of max/min type is 1, while of regular type is 2 and of saddle type is 3. Hence,

$$\chi(\Gamma_\gamma) = V - E = M + S + I - \frac{1}{2}(M + 2I + 3S) = \frac{M - S}{2} = \frac{2-2g}{2} = 1 - g.$$

Since Γ_γ is connected, it follows that $\beta_1(\Gamma_\gamma) = g$, i.e., Γ_γ has g loops.

□

Remark 3.2. If $\gamma : N \to S^1$ is a stable circle-valued function, where N is a closed 2-manifold with $\chi(N) \neq 0$, then γ is always non regular. In fact, suppose γ is regular. Then, γ should be surjective and from Ehresmann's fibration theorem [6], γ should be a locally trivial fibration. In particular, since F is a fiber of this fibration, it should happen that $0 \neq \chi(N) = \chi(S^1)\chi(F) = 0$, which is an absurd.

Corollary 3.3. *(Proposition 3.4 [2]) Let $\gamma : S^2 \to S^1$ be a stable circle-valued function. Then the generalized Reeb graph of γ is a tree.*

Remark 3.4. (1) Notice that the definition of generalized Reeb graph differs from the classical Reeb graph with respect to the vertices. In the classical case, the vertices are related just with the connected components of level curves $\gamma^{-1}(v)$ which contain a critical point. Hence, our generalized Reeb graph contains some extra vertices corresponding to the regular connected components of $\gamma^{-1}(v)$, where v is a critical value. Of course the classical Reeb graph can be obtained from the generalized one just by eliminating the extra vertices and joining the two adjacent edges. But in general, the generalized Reeb graph provides more information.

(2) The Figure 7 shows two stable functions $\gamma_1, \gamma_2 : S^2 \to S^1$ with their respective generalized Reeb graphs. Both functions share the same classical Reeb graph, but the generalized Reeb graphs are different. The stable function γ_1 is non surjective while γ_2 is surjective. Then γ_1 and γ_2 could not be topologically equivalent, i.e., there are no homeomorphisms $\phi : S^2 \to S^2$ and $\psi : S^1 \to S^1$ such that $\gamma_1 = \psi \circ \gamma_2 \circ \phi^{-1}$. This shows that the classical Reeb graph is not sufficient to distinguish between these two examples.

(3) If $\gamma : S^2 \to S^1$ is not surjective, then γ may be regarded as a Morse function from S^2 to $\mathbb{R}$ (via stereographic projection). In this case, the generalized Reeb graph can be obtained from the classical one just by adding the extra vertices each time that one passes through a critical value.

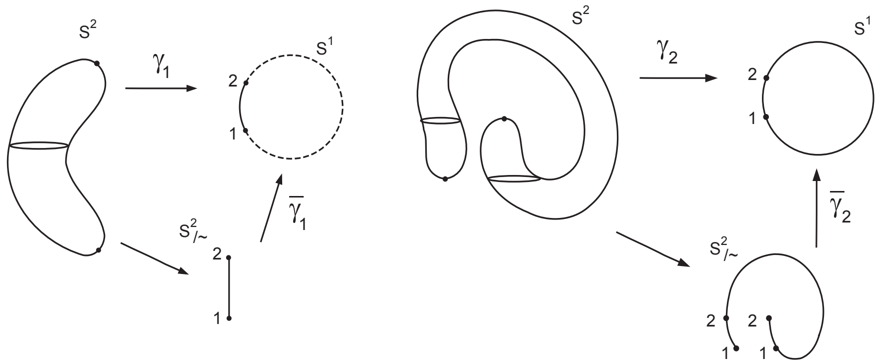

FIGURE 7. Stable functions and their generalized Reeb graphs

It is obvious that the labeling of vertices of the generalized Reeb graph is not uniquely determined, since it depends on the chosen orientations and the base points on each S^1. Different choices will produce either a cyclic permutation or a reversal of the labeling in the generalized Reeb graph.

The following result shows that the number of open saddles together with the genus and the number of boundary components of N, determine the number of loops in the generalized Reeb graph associated to $\gamma : N \to S^1$:

Theorem 3.5. *Let N be a 2-manifold with boundary and let $\gamma : N \to S^1$ be a stable circle-valued function. Then, the number of loops in Γ_γ is given by $g + \dfrac{h + S''}{2}$, where g is the genus of N, h is the number of connected components of ∂N and S'' is the number of vertices of open saddle type.*

Proof. Since Γ_γ is connected we have $\beta_0(\Gamma_\gamma) = 1$. The Euler characteristic of Γ_γ is given by $\chi(\Gamma_\gamma) = \beta_0(\Gamma_\gamma) - \beta_1(\Gamma_\gamma) = 1 - \beta_1(\Gamma_\gamma)$, where $\beta_1(\Gamma_\gamma)$ represents the number of loops in Γ_γ.

We also have that $\chi(\Gamma_\gamma) = V - E$, where V, E denote the number of vertices and edges of Γ_γ, respectively. Moreover, $V = M + S + S' + S'' + C + I$ where M, S, S', S'', C, I denote the numbers of vertices of each type listed in Section 2. On the other hand, by Euler's formula

$$E = \frac{1}{2} \sum_{i=1}^{V} deg(v_i)$$

where $v_i \in V$.

Since γ is stable, the degree of each vertex of max/min type is 1, while of regular type is 2 and saddle type is 3. Hence,

$$\chi(\Gamma_\gamma) = V - E = M + S + S' + S'' + C + I - \frac{1}{2}(M + 2C + 2I + 3S + 3S' + 4S'')$$

$$\Rightarrow \chi(\Gamma_\gamma) = \frac{M - S - S' - 2S''}{2} = \frac{\chi(N) - S''}{2} = 1 - g - \frac{(S'' + h)}{2}.$$

Therefore, the number of loops is given by $\beta_1(\Gamma_\gamma) = g + \dfrac{(h + S'')}{2}$.

∎

The next proposition shows that the first Betti number of N bounds the number of loops in a generalized Reeb graph, similar to what happens with the classical Reeb graph (see Section 1):

Proposition 3.6. *Let N be a 2-manifold with boundary and let $\gamma : N \to S^1$ be a stable circle-valued function. Then, the number of loops $= \beta_1(\Gamma_\gamma) \leq \beta_1(N)$.*

Proof. In the proof of Theorem 3.5 we showed that $2\chi(\Gamma_\gamma) = \chi(N) - S''$. Then,

$$\beta_1(N) = 2\beta_1(\Gamma_\gamma) - 1 - S''.$$

Note that

$$\chi(\Gamma_\gamma) = \chi(\Gamma \cup \Gamma') = \chi(\Gamma) + \chi(\Gamma') - \chi(\Gamma \cap \Gamma') = \chi(\Gamma) - S'' - S',$$

because

$$\chi(\Gamma') = V - E = S' + S'' + I - \frac{1}{2}(2S' + 4S'' + 2I) = -S''$$

and $\chi(\Gamma \cap \Gamma') = S'$.

However, since N is a 2-manifold with boundary, the number of connected components of Γ is at most S', which means that $\chi(\Gamma) \leq S' - \beta_1(\Gamma)$.

Then,

$$\chi(\Gamma_\gamma) = \chi(\Gamma) - S'' - S' \leq -\beta_1(\Gamma) - S'' \leq -S''.$$

Therefore,

$$\beta_0(\Gamma_\gamma) - \beta_1(\Gamma_\gamma) = \chi(\Gamma_\gamma) \leq -S'' \;\;\Leftrightarrow\;\; \beta_1(\Gamma_\gamma) \geq 1 + S''.$$

Consequently,

$$\beta_1(N) = 2\beta_1(\Gamma_\gamma) - (1 + S'') \geq \beta_1(\Gamma_\gamma) \;\;\Rightarrow\;\; \beta_1(\Gamma_\gamma) \leq \beta_1(N).$$

□

A consequence of Theorem 3.5 and Proposition 3.6 is the following relation

$$g + \frac{(h + S'')}{2} \leq 2g + h - 1 \;\;\Rightarrow\;\; S'' \leq 2g + h - 2.$$

The next result shows a class of functions whose generalized Reeb graphs have the maximal number of loops:

Theorem 3.7. *Let N be a 2-manifold with boundary and let $\gamma : N \to S^1$ be a stable circle-valued function. If $\beta_0(\Gamma) = S'$ then Γ_γ has the maximal number of loops, i.e., $\beta_1(\Gamma_\gamma) = 2g + h - 1$.*

Proof. Since γ is stable and $h \neq 0$, then $\Gamma' \neq \emptyset$. We divide the proof in two cases:

Case 1: $S' = 0$.

Since $\Gamma_\gamma = \Gamma \cup \Gamma'$ is connected, $\Gamma \cap \Gamma'$ is the set of vertices that correspond to the half open saddles type and $\Gamma' \neq \emptyset$, we have that $\Gamma = \emptyset$.

Consequently, $M = 0$ and $S = 0$. By the Poincaré-Hopf Theorem it follows that

$$2 - 2g - h = M - S - S' - S'' = -S'' \;\;\Rightarrow\;\; S'' = 2g + h - 2.$$

As

$$1 - \beta_1(\Gamma_\gamma) = \chi(\Gamma_\gamma) = \chi(\Gamma') = -S'' = -(2g + h - 2),$$

then $\beta_1(\Gamma_\gamma) = 2g + h - 1$.

Case 2: $S' \neq 0$.

Notice that the level curves of half open saddle type divide N in two connected components. Consider $\alpha_1, \dots, \alpha_{S'}$ the level curves of half open saddle type, and let v_i be the vertex corresponding to α_i in $\Gamma_\gamma = \Gamma \cup \Gamma'$, with $i = 1, \dots, S'$. Then, for each vertex v_i there are 3 incident edges, 2 bold traced edges and 1 slim traced edge.

Let B_i be the connected component of N determined by α_i that contains the level curves corresponding to the slim traced edges arriving at v_i. Since $\Gamma \cap \Gamma' = \{v_i, i = 1, \dots, S'\}$, $\Gamma_\gamma = \Gamma \cup \Gamma'$ is connected and $\beta_0(\Gamma) = S'$, then each connected component of Γ contains exactly one vertex v_i, $i = 1, \dots, S'$.

Assume that $B_i \cap \partial N \neq \emptyset$ for some $i = 1, \dots, S'$. Then, B_i contains the level curves of interval type. Consequently, it contains a level curve of half open saddle type. Hence, there is a connected component of Γ which contains two vertices corresponding to half open saddles. But this is a contradiction, therefore $B_i \cap \partial N = \emptyset$.

Since $\gamma|B_i$ is Morse for every $i = 1, \dots, S'$, it follows that B_i contains only level curves of saddle type, circle type and max/min type. Also, the subgraph $\Gamma_{\gamma|B_i}$ satisfies $1 - \beta_1(\Gamma_{\gamma|B_i}) = M_i - S_i$, where M_i is the number of vertices of max/min type and S_i is the number of vertices of saddle type of $\Gamma_{\gamma|B_i}$, respectively. It follows that

$$\sum_{i=1}^{S'} \left(1 - \beta_1(\Gamma_{\gamma|B_i})\right) = \sum_{i=1}^{S'} (M_i - S_i) \Rightarrow S' - \beta_1(\Gamma) = M - S \Rightarrow \beta_1(\Gamma) = -M + S + S'.$$

Also, notice that $\beta_0(\Gamma) = S'$ implies $\beta_0(\Gamma') = 1$, then

$$\chi(\Gamma') = -S'' \Rightarrow \beta_1(\Gamma') = 1 + S''.$$

Consequently,

$$\begin{aligned} \chi(\Gamma_\gamma) &= \chi(\Gamma) + \chi(\Gamma') - \chi(\Gamma \cap \Gamma') = \beta_0(\Gamma) - \beta_1(\Gamma) + \beta_0(\Gamma') - \beta_1(\Gamma') - S' \\ &= S' - (-M + S + S') + 1 - (1 + S'') - S' = M - S - S' - S'' = \chi(N). \end{aligned}$$

Therefore, $\beta_1(\Gamma_\gamma) = 2g + h - 1$.

□

The next picture illustrates a stable circle-valued function under the conditions of Theorem 3.7.

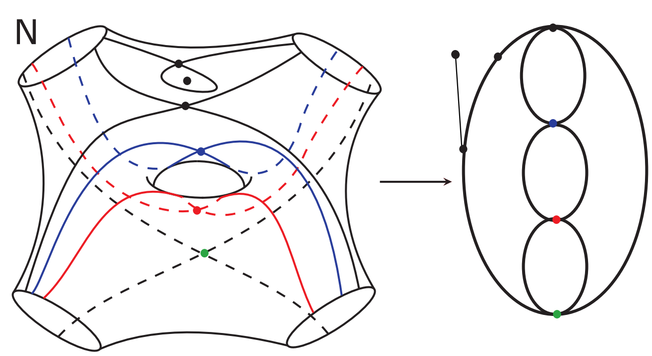

FIGURE 8. Stable circle-valued function with maximal number of loops in the generalized Reeb graph

Remark 3.8. Consider $\gamma : N \to S^1$ a stable circle-valued function, where N is a 2-manifold with boundary and genus zero. Notice that since $\beta_0(\Gamma) \leq S'$, if $\beta_0(\Gamma') = 1$ then $\beta_0(\Gamma) = S'$. Consequently, the number of loops of Γ_γ is maximal.

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References

[1] V.I. Arnold, *Topological classification of Morse functions and generalisations of Hilbert's 16-th problem*, Math. Phys. Anal. Geom. Vol. 10 (2007) 227–236. DOI: 10.1007/s11040-007-9029-0

[2] E.B. Batista, J.C.F. Costa, J.J. Nuño-Ballesteros, *The Reeb graph of a map germ from $\mathbb{R}^3$ to $\mathbb{R}^2$ with isolated zeros*, Proc. Edinb. Math. Soc. (2) 60 (2017), no. 2, 319-348.

[3] E.B. Batista, J.C.F. Costa, J.J. Nuño-Ballesteros, *The Reeb graph of a map germ from $\mathbb{R}^3$ to $\mathbb{R}^2$ with non isolated zeros*, Bull. Braz. Math. Soc. (N.S.) 49 (2018), no. 2, 369-394. DOI: 10.1007/s00574-017-0058-4

[4] S. Biasotti, D. Giorgi, M. Spagnuolo, B. Falcidieno, *Reeb graphs for shape analysis and applications*, Theoret. Comput. Sci. 392 (13) (2008) 5-22. DOI: 10.1016/j.tcs.2007.10.018

[5] K. Cole-McLaughlin, H. Edelsbrunner, J. Harer, V. Natarajan, V. Pascucci, *Loops in Reeb graphs of 2-manifolds*. Discrete Comput. Geom. 32 (2004), no. 2, 231–244. DOI: 10.1007/s00454-004-1122-6

[6] C. Ehresmann, *Les connexions infinitésimales dans un espace fibré différentiable*, Colloque de Topologie, Bruxelles (1950) 29–55.

[7] A.T. Fomenko, T.L. Kunii, *Topological Methods for Visualization*. Springer-Verlag, Tokyo, 1997.

[8] I. Gelbukh, *Loops in Reeb graphs of n-manifolds*, Discrete Comput. Geom. 59 (2018), no. 4, 843-863. DOI: 10.1007/s00454-017-9957-9

[9] M. Golubitsky and V. Guillemin, *Stable mappings and their singularities*, Graduate Texts in Mathematics 14, Springer, New York, 1973. DOI: 10.1007/978-1-4615-7904-5

[10] S.I. Maksymenko, *Deformations of circle-valued Morse functions on surfaces*. Ukrainian Math. J. 62 (2011), no. 10, 1577–1584. DOI: 10.1007/s11253-011-0450-y

[11] S.P. Novikov, *Multivalued functions and functionals. An analogue of the Morse theory*, In Soviet Math. Dokl, vol. 24, (1981) 222–226.

[12] S.P. Novikov, *The Hamiltonian formalism and a multivalued analogue of Morse theory,(Russian)*, Uspekhi Mat. Nauk, 37, (1982) 3–49.

[13] G. Reeb, *Sur les points singuliers d'une forme de Pfaff completement intégrable ou d'une fonction numérique*, C. R. Acad. Sci. Paris 222 (1946) 847–849.

[14] O. Saeki, *Topology of Singular Fibers of Differentiable Maps*, Lecture Notes in Mathematics, 1854. Springer-Verlag, Berlin (2004).

Erica Boizan Batista, Centro de Ciências e Tecnologia, Universidade Federal do Cariri - UFCA, Campus de Juazeiro do Norte-CE, Brazil

Email address: erica.batista@ufca.edu.br

João Carlos Ferreira Costa, Universidade Estadual Paulista (Unesp), Instituto de Biociências, Letras e Ciências Exatas, Campus de São José do Rio Preto, Brazil

Email address: joao.costa@unesp.br

Juan J. Nuño-Ballesteros, Departament de Matemàtiques, Universitat de València, Campus de Burjassot 46100 Spain

Email address: Juan.Nuno@uv.es

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APPARENT CONTOURS OF STABLE MAPS OF SURFACES WITH BOUNDARY INTO THE PLANE

TAKAHIRO YAMAMOTO

Dedicated to Professor Takashi Nishimura on the occasion of his 60th birthday.

Abstract. Let M be a connected compact surface with boundary. A C^∞ map $M \to \mathbb{R}^2$ is admissible if it is non-singular on a neighborhood of the boundary. For a C^∞ stable map $f\colon M \to \mathbb{R}^2$, denote by $c(f)$ and $n(f)$, $i(f)$ the number of cusps and nodes, connected components of the set of singular points respectively. In this paper, we introduce the notion of admissibly homotopic among C^∞ maps $M \to \mathbb{R}^2$, and we will determine the minimal number $c + n$ for each admissibly homotopy class.

1. Introduction

Let M be a connected compact surface with boundary ∂ and P a surface without boundary. Denote by $C^\infty(M, P)$ the set of C^∞ maps $M \to P$ equipped with the Whitney C^∞ topology. A C^∞ map $f\colon M \to P$ is called a C^∞ *stable map*, (or *stable map* for short), if there exists a neighborhood $N(f) \subset C^\infty(M, P)$ of f such that every map $g \in N(f)$ is C^∞ right-left equivalent[1] to f. A C^∞ map $f\colon M \to P$ is stable if and only if f has fold, cusp and B_2 as its singularities, and $f|_{(S(f)\cup\partial)\setminus(C(f)\cup B(f))}$ is an immersion with normal crossings, where $C(f)$ and $B(f)$ denote the set of cusp points and B_2 points of f respectively, see Proposition 2.2 for details.

Note that if a C^∞ map $f\colon M \to P$ is stable, then $f|_\partial\colon \partial \to P$ is stable. Note also that a B_2 point is a fold point (or regular point) if we ignore the boundary (resp. we restrict f to boundary).

A C^∞ map $f\colon M \to P$ is called *admissible* if it is submersive on an open neighborhood of the boundary. Note that a C^∞ stable map $f\colon M \to P$ is admissible if and only if it has no B_2 points.

For a C^∞ stable map $f\colon M \to P$, denote by $c(f)$ and $n(f)$, $i(f)$ the numbers of cusps and nodes, connected components of the set singular points of f respectively.

Denote by M_k a connected compact surface with exactly k boundary components. A connected compact and orientable (or non-orientable) surface of genus g with exactly k boundary components is denoted by $\Sigma_{g,k}$ (resp. $N_{g,k}$). The 2-dimensional sphere and the plane are denoted by S^2 and $\mathbb{R}^2$ respectively.

For a C^∞ map $f\colon M \to P$, define the set of singular points of f as

$$S(f) = \{p \in M \mid \operatorname{rank} d_pf < 2\}.$$

We call $f(S(f))$ the *apparent contour* (or *contour* for short) of f and denote it by $\gamma(f)$. For a closed surface M, the apparent contour of a stable map $M \to P$ ($P = \mathbb{R}^2$, S^2) relates the topology of M as classical result of Thom [11] and a formula obtained by Pignoni [9] show.

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[1]Two maps f, $g \in C^\infty(M, P)$ are C^∞ *right-left equivalent* if there exist a diffeomorphism $\Phi\colon M \to M$ preserving the boundary and a diffeomorphism $\psi\colon P \to P$ such that $f \circ \Phi = \psi \circ g$.

Pignoni [9] introduced the notion of a minimal contour of a closed surface: The contour $\gamma(f)$ of a stable map $f\colon M \to \mathbb{R}^2$ is called a *minimal contour* of M if the number $c(f)+n(f)$ is the smallest among stable maps $g\colon M \to \mathbb{R}^2$ which satisfy $i(g)=1$. Then, Demoto [2] introduced the notion of a minimal contour of a C^∞ map $f_0\colon M \to P$ between surfaces and studied that of a C^∞ map $S^2 \to S^2$: Let $f_0\colon M \to P$ be a C^∞ map and $f\colon M \to P$ a C^∞ stable map which is homotopic to f_0 and satisfies $i(f)=1$. Call $\gamma(f)$ a *minimal contour* of f_0 if the number $c(f)+n(f)$ is the smallest among C^∞ stable maps $g\colon M \to P$ which are homotopic to f_0 and $i(g)=1$. Then, Kamenosono and the author [7] studied minimal contours of C^∞ maps $M \to S^2$ of closed surfaces M. Apparent contours of stable maps between surfaces were also studied in [15, 16, 3, 17]. Studying minimal contours of C^∞ maps make the very first step toward classifying generic C^∞ maps of surfaces up to right-left equivalence.

In this paper, we study minimal contour of C^∞ maps of surfaces with boundary. More precisely, for a surface M with boundary and a surface P without boundary, we introduce the notion of *admissibly homotopic* which is an equivalence relation among admissible C^∞ maps $M \to P$, and *admissible minimal contour* of an admissible C^∞ map $M \to P$. Then, we study admissible minimal contours of admissible C^∞ maps $M_1 \to \mathbb{R}^2$.

This paper is organized as follows. In §2, we prepare some notions and introduce the main-theorems (Theorems 2.3 and 2.5). In §3, we prepare some notions concerning stable maps $f\colon M_k \to \mathbb{R}^2$ ($k \geq 1$) and introduce the formula as an application of formulas obtained by Pignoni [9] and Imai [6]. In §4, we construct admissible stable maps $\Sigma_{g,1} \to \mathbb{R}^2$ ($g \geq 0$) and $N_{g,1} \to \mathbb{R}^2$ ($g \geq 1$) which are in the lists of Theorem 2.3 and 2.5 respectively. In §5, we show the contours of stable maps constructed in § 4 are admissible minimal contours. In §6, we pose a problem which concerns the apparent contours of stable fold maps $f\colon M_k \to \mathbb{R}^2$, where a stable map $f\colon M_k \to \mathbb{R}^2$ of a surface with boundary is called *fold map* if it has no cups as its singularities.

Throughout this paper, all surfaces are connected and smooth of class C^∞, and all maps are smooth of class C^∞ unless stated otherwise. The symbols r and $g \geq 0$ denote integers. For a topological space X, id_X denotes the identity map of X.

2. Main-Theorem

In this section, we introduce some notions and introduce the main-theorems (Theorems 2.3 and 2.5).

Let M_k be a compact and connected surface with exactly k boundary components $\partial_1 \cup \cdots \cup \partial_k$. Then, admissible C^∞ maps $f_0, f_1\colon M_k \to \mathbb{R}^2$ are said *admissibly homotopic* if there exists a C^∞ map $H\colon M_k \times [0,1] \to \mathbb{R}^2$ such that $H_t = H(\cdot, t)\colon M_k \to \mathbb{R}^2$ is an admissible C^∞ map for each $t \in [0,1]$, and $H_0 = f_0$ and $H_1 = f_1$.

Let $f\colon M_k \to \mathbb{R}^2$ be an admissible C^∞ map. Then, for each component ∂_j, orient the regular curve $f(\partial_j) \subset \mathbb{R}^2$ so that at each point, the inner of $f(M_k)$ is in the left hand side. Note that the definition of the orientation for $f(\partial_j) \subset \mathbb{R}^2$ is well-defined by virtue of the assumption that f is admissible. Then, call the rotation number of $f(\partial_j) \subset \mathbb{R}^2$ the *boundary rotation number* of ∂_j (or *rotaion number* of ∂_j for short) with respect to f and denote it by $W(f; \partial_j)$. If $k=1$, then call the rotation number of $f(\partial) \subset \mathbb{R}^2$ the *boundary rotation number* of f and denote it by $W(f)$. Furthermore, in the case that $M = \Sigma_g$ and $k=1$, define $s(f)=+1$ (or -1) if there exists a neighborhood of $N(\partial)$ of ∂ such that $f|_{N(\partial)}$ preserves (resp. reverses) the orientation of $N(\partial)$.

Proposition 2.1. (1) *Admissible stable maps $f_0, f_1\colon \Sigma_{g,1} \to \mathbb{R}^2$ are admissibly homotopic if and only if $W(f_0)=W(f_1)$ and $s(f_0)=s(f_1)$.*

(2) *Admissible stable maps* $f_0, f_1\colon N_{g,1} \to \mathbb{R}^2$ *are admissibly homotopic if and only if* $W(f_0) = W(f_1)$.

Proof. (1) If f_0 and f_1 are admissibly homotopic, then $s(f_0) = s(f_1)$ and regular curves $f_0(\partial)$ and $f_1(\partial)$ are regularly homotopic. It implies that $W(f_0) = W(f_1)$.

We consider the opposite direction. If $W(f_0) = W(f_1)$, then regular curves $f_0(\partial)$ and $f_1(\partial)$ with the canonical orientation are regularly homotopic. Thus, there exists a C^∞ map

$$H'\colon \partial \times [0,1] \to \mathbb{R}^2$$

so that $H'(\cdot, 0) = f_0|_\partial$ and $H'(\cdot, 1) = f_1|_\partial$. Then, we can extend H' to a C^∞ map

$$H''\colon N(\partial) \times [0,1] \to \mathbb{R}^2$$

on a neighborhood of ∂ so that $H''|_{\partial\times[0,1]} = H'$ and $H''_t = H''(\cdot, t)\colon N(\partial) \to \mathbb{R}^2$ is a submersion for any $t \in [0, 1]$. Note that if $s(f_0) = s(f_1) = +1$ (or $s(f_0) = s(f_1) = -1$), then $H''_t = H''(\cdot, t)$ is an immersion which preserves (resp. reverses) orientation of a neighborhood of ∂ for each $t \in [0, 1]$. On the other hand, we decompose $\Sigma_{g,1}$ into a simplicial complex. We also decompose $\Sigma_{g,1} \times [0, 1]$ into a simplicial complex which is compatible with the simplicial decomposition of $\Sigma_{g,1}$. We define a map $H\colon \Sigma_{g,1} \times [0, 1] \to \mathbb{R}^2$ by the following manner:

- 0**-simplex:** If a 0-simplex $\sigma =< a_0 >$ is in $N(\partial)\times[0,1]$ (or $\Sigma_{g,1}\times\{0\}$, $\Sigma_{g,1}\times\{1\}$), then we define $H(a_0) = H''(a_0)$ (resp. $H(a_0) = f_0(a_0)$, $H(a_0) = f_1(a_0)$). Otherwise, we define $H(a_0) = 0 \in \mathbb{R}^2$.
- 1**-simplex:** If a 1-simplex $\sigma =< a_0, a_1 >$ is in $N(\partial) \times [0, 1]$, (or $\Sigma_{g,1} \times \{0\}$, $\Sigma_{g,1} \times \{1\}$), then $H|_\sigma$ is defined by $H|_\sigma = H''|_\sigma$ (resp. $H|_\sigma = f_0|_\sigma$, $H|_\sigma = f_1|_\sigma$). Otherwise, we define $H|_\sigma$ by $H(x) = \lambda_0 H(a_0)+\lambda_1 H(a_1)$, where $x = \lambda_0 a_0+\lambda_1 a_1 \in \sigma$ with the property that $\lambda_i \in \mathbb{R}_{\geq 0}$ and $\lambda_0 + \lambda_1 = 1$.
- 2**-simplex:** If a 2-simplex $\sigma =< a_0, a_1, a_2 >$ is in $N(\partial) \times [0, 1]$ (or $\Sigma_{g,1} \times \{0\}$, $\Sigma_{g,1} \times \{1\}$), then $H|_\sigma$ is defined by $H|_\sigma = H''|_\sigma$ (resp. $H|_\sigma = f_0|_\sigma$, $H|_\sigma = f_1|_\sigma$). Otherwise, we define $H|_\sigma$ by $H(x) = \lambda_0 H(a_0)+\lambda_1 H(a_1)+\lambda_2 H(a_2)$, where $x = \lambda_0 a_0+\lambda_1 a_1+\lambda_2 a_2 \in \sigma$ with the property that $\lambda_i \in \mathbb{R}_{\geq 0}$ $(i = 0, 1, 2)$, and $\lambda_0 + \lambda_1 + \lambda_2 = 1$.
- 3**-simplex:** If a 3-simplex $\sigma =< a_0, a_1, a_2, a_3 >$ is in $N(\partial) \times [0, 1]$, then $H|_\sigma$ is defined by $H|_\sigma = H''|_\sigma$. Otherwise, we define $H|_\sigma$ by

  $$H(x) = \lambda_0 H(a_0) + \lambda_1 H(a_1) + \lambda_2 H(a_2) + \lambda_3 H(a_3),$$

  where $x = \lambda a_0 + \lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3 \in \sigma$ with the property that $a_i \in \mathbb{R}$, $a_i > 0$ $(i = 0, 1, 2, 3)$, and $a_0 + a_1 + a_2 + a_3 = 1$.

Then, by perturbing H slightly, if necessary, we obtain a desired C^∞ map $\Sigma_{g,1} \times [0, 1] \to \mathbb{R}^2$. Namely, f_0 and f_1 are admissibly homotopic.

(2) The case of C^∞ maps $N_{g,1} \to \mathbb{R}^2$ is also proved by similar way of (1). We omit the proof here. □

C^∞ stable maps of compact and connected surfaces with boundary into surfaces without boundary are characterized by the following way.

Proposition 2.2 (Bluce and Giblin [1])**.** *Let* M *be a compact and connected surface possibly with boundary* ∂ *and* P *a surface without boundary. A* C^∞ *map* $f\colon M \to P$ *is* C^∞ *stable if and only if it satisfies the following conditions.*

(1) (*Local conditions*) *In the following, for* $p \in \partial$, *we use local coordinates* (x, y) *around* p *such that* $\mathrm{Int}M$ *and* ∂ *correspond to the sets* $\{y > 0\}$ *and* $\{y = 0\}$ *respectively.*

(1a) *For $p \in \mathrm{Int}M$, the germ of f at p is right-left equivalent to one of the following:*

$$(x,y) \mapsto \begin{cases} (x,y), & p\text{: regular point,} \\ (x,y^2), & p\text{: fold point,} \\ (x,y^3+xy), & p\text{: cusp point.} \end{cases}$$

(1b) *For $p \in \partial$, the germ of f at p is right-left equivalent to one of the following:*

$$(x,y) \mapsto \begin{cases} (x,y) & p\text{: regular point of } f|_{N(\partial M)}, \\ (x,y^2+xy) & p\text{: } B_2 \text{ point.} \end{cases}$$

(2) (*Global conditions*) *For each $q \in f(S(f) \cup \partial)$, the multi-germ*

$$(f|_{S(f)\cup\partial}, f^{-1}(q) \cap (S(f) \cup \partial))$$

is right-left equivalent to one of the four multi-germs whose images are as depicted in Figure 1, where blue curves and gray curves represent $f(S(f))$ and $f(\partial)$ respectively: (1) represent immersion mono-germs $(\mathbb{R},0) \ni t \mapsto (t,0) \in (\mathbb{R}^2,0)$ which correspond to a single fold point or a single boundary point respectively, and (2) represents cusp mono-germ $(\mathbb{R},0) \ni t \mapsto (t^2,t^3) \in (\mathbb{R}^2,0)$ which correspond to a cusp point, (3) represents B_2 multi-germ which corresponds to a single point in $\partial \cap S(f)$, (4) represent normal crossings of two immersion germs, each of which corresponds to a fold point or a boundary point.

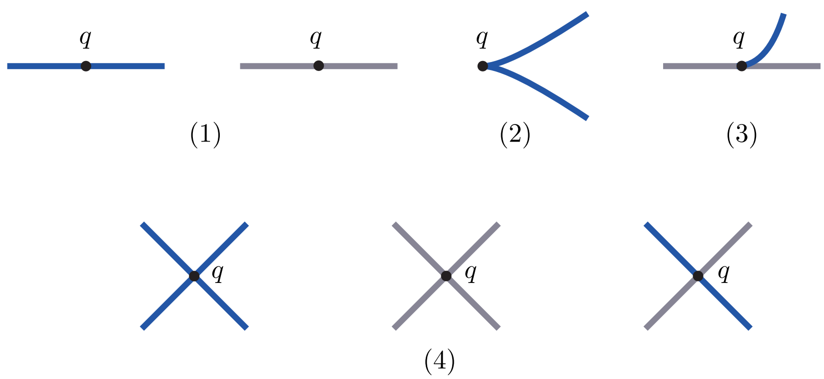

FIGURE 1. The images of multi-germs of $f|_{S(f)\cup S(f|_{\partial M})}$

Let $f_0\colon M_1 \to P$ be an admissible C^∞ map and $f\colon M_1 \to P$ an admissible C^∞ stable map which is admissibly homotopic to f_0. Call $\gamma(f)$ an *admissible minimal contour* of f_0 if the number $c(f)+n(f)$ is the smallest among stable maps $g\colon M_1 \to \mathbb{R}^2$ which are admissibly homotopic to f_0 and $i(g)=1$. Note that the number of connected components of the set of singular points is allowed to vary during admissible homotopy.

Theorem 2.3. *Let $g \geq 0$ be an integer and $f\colon \Sigma_{g,1} \to \mathbb{R}^2$ be a rotation number r admissible stable map. The contour $\gamma(f)$ is an admissible minimal contour if and only if the pair $(c(f),n(f))$ is one of the pairs below:*

$g = 0$**:**

$$(c(f),n(f)) = \begin{cases} (r+1,0) & \text{if } r \geq 0, \\ (-r-1,-r-1) & \text{if } r \leq -1. \end{cases}$$

$g = 1$**:**

$$(c(f), n(f)) = \begin{cases} (r+3, 0) \ \textit{or} \ (r-1, 4) & \textit{if } r \geq 1, \\ (r+3, 0) & \textit{if } -2 \leq r \leq 0, \\ (-r-3, -r-3) & \textit{if } r \leq -3, \end{cases}$$

$g = 2$**:**

$$(c(f), n(f)) = \begin{cases} (r-3, 6) & \textit{if } r \geq 3, \\ (1, 5) & \textit{if } r = 2, \\ (r+1, 4) \ \textit{or} \ (r+5, 0) & \textit{if } -1 \leq r \leq 1, \\ (r+5, 0) & \textit{if } -4 \leq r \leq -2, \\ (-r-5, -r-5) & \textit{if } r \leq -5, \end{cases}$$

$g \geq 3$**:**

$$(c(f), n(f)) =$$

$$\begin{cases} (r-2g+1, 2g+2) & \textit{if } r \geq 2g-1, \\ (2, 6+2k) & \textit{if } r = 9-2g+4k, \ k = 0, \ldots, g-3, \\ (1, 6+2k) & \textit{if } r = 8-2g+4k, \ k = 0, \ldots, g-3, \\ (0, 6+2k) & \textit{if } r = 7-2g+4k, \ k = 0, \ldots, g-3, \\ (1, 5+2k) & \textit{if } r = 6-2g+4k, \ k = 0, \ldots, g-2, \\ (r+2g-3, 4) \ \textit{or} \ (r+2g+1, 0) & \textit{if } 3-2g \leq r \leq 5-2g, \\ (r+2g+1, 0) & \textit{if } -2g \leq r \leq 2-2g, \\ (-r-2g-1, -r-2g-1) & \textit{if } r \leq -1-2g. \end{cases}$$

Remark that the number $c+n$ of an admissible minimal contour of a C^∞ map $f_0 \colon \Sigma_{g,1} \to \mathbb{R}^2$ depend only on the boundary rotation number $W(f_0)$. It does not depend on the sign $s(f_0)$.

Corollary 2.4. *The number $c+n$ of an admissible minimal contour of a rotation number r admissible stable map $\Sigma_{g,1} \to \mathbb{R}^2$ is one of the items below:*

$$c + n = \begin{cases} r+3 & \textit{if } r \geq 2g-1, \\ (r+2g+5)/2 & \textit{if } 3-2g \leq r < 2g-1 \ \textit{and} \ r \equiv 3-2g \mod 4, \\ (r+2g+6)/2 & \textit{if } 2-2g \leq r < 2g-1 \ \textit{and} \ r \equiv 2-2g \ \textit{or} \ -2g \mod 4, \\ (r+2g+7)/2 & \textit{if } 1-2g \leq r < 2g-1 \ \textit{and} \ r \equiv 1-2g \mod 4, \\ r+2g+1 & \textit{if } -2g \leq r \leq 2-2g, \\ -2(r+1+2g) & \textit{if } r \leq -1-2g. \end{cases}$$

Theorem 2.5. *Let $g \geq 1$ be an integer and $h \colon N_{g,1} \to \mathbb{R}^2$ be a rotation number r admissible stable map. The contour $\gamma(h)$ is an admissible minimal contour if and only if the pair $(c(h), n(h))$ is one of the items below:*

$$(c(h), n(h)) = \begin{cases} (1, |g+r-4|/2) & \textit{if } r \geq 2-g \ \textit{and} \ r \equiv g \mod 2, \\ (0, |g+r-3|/2) & \textit{if } r \geq 1-g \ \textit{and} \ r \not\equiv g \mod 2, \\ (1, -(g+r)/2) & \textit{if } r \leq -g \ \textit{and} \ r \equiv g \mod 2, \\ (0, -(g+r+1)/2) & \textit{if } r \leq -1-g \ \textit{and} \ r \not\equiv g \mod 2. \end{cases}$$

3. Topological formula of apparent contour

In this section, we introduce topological formula of apparent contours of admissible stable maps $M \to \mathbb{R}^2$ of surfaces with boundary.

Let us recall some notions introduced by Pignoni [9]. Let M_k be a compact and connected surface with exactly k boundary components $\partial = \partial_1 \cup \cdots \cup \partial_k$ and $f\colon M_k \to \mathbb{R}^2$ an admissible stable map whose contour is non-empty. Then, for each component ∂_j, orient the regular curve $f(\partial_j) \subset \mathbb{R}^2$ so that at each point, the inner of $f(M_k)$ is in the left hand side. Note that the definition of the orientation for $f(\partial_j)$ is well-defined by virtue of the assumption that f is admissible. Let $S(f) = S_1 \cup \cdots \cup S_\ell$ be the decomposition of $S(f)$ into the connected components and set $\gamma_i = f(S_i)$ $(i = 1, \dots, \ell)$. Note that $\gamma(f) = \gamma_1 \cup \cdots \cup \gamma_\ell$. For each γ_i, denote by U_i the unbounded component of $\mathbb{R}^2 \setminus \gamma_i$. Note that $\partial U_i \subset \gamma_i$.

Orient γ_i so that at each fold point image, the surface is "folded to the left hand side". More precisely, for a point $y \in \gamma_i$ which is not a cusp or a node, choose a normal vector v of γ_i at y such that $f^{-1}(y')$ contains more elements than $f^{-1}(y)$, where y' is a regular value of f close to y in the direction of v. Let τ be a tangent vector of γ_i at y such that the ordered pair (τ, v) is compatible with the given orientation of $\mathbb{R}^2$. It is easy to see that τ gives a well-defined orientation for γ_i.

Definition 3.1. A point $y \in \partial U_i \setminus \{\text{cusps, nodes}\}$ is said to be *positive* if the normal orientation v at y points toward U_i. Otherwise, it is said to be *negative*.

A component γ_i is said to be *positive* if all points of $\partial U_i \setminus \{\text{cusps, nodes}\}$ are positive; otherwise, γ_i is said to be *negative*. The numbers of positive and negative components are denoted by i^+ and i^- respectively.

By the geometrical condition of the surface $\Sigma_{g,1}$, we obtain the following lemma.

Lemma 3.2. *Let $f\colon \Sigma_{g,1} \to \mathbb{R}^2$ be an admissible stable map whose singular points set consists of one component. Then, the contour is a negative component.*

Definition 3.3. A point $y \in \partial U_i \setminus \{\text{cusps, nodes}\}$ is called an *admissible starting point* if y is a positive (or negative) point of a positive (resp. negative) component γ_i. Note that for each i, there always exists an admissible starting point on γ_i.

Definition 3.4. Let $y \in \gamma_i$ be an admissible starting point and $Q \in \gamma_i$ a node. Let $\alpha\colon [0,1] \to \gamma_i$ be a parameterization consistent with the orientation which is singular only when the image is a cusp such that $\alpha^{-1}(y) = \{0, 1\}$. Then, there are two numbers $0 < t_1 < t_2 < 1$ satisfying $\alpha(t_1) = \alpha(t_2) = Q$.

We say that Q is *positive* if the orientation of $\mathbb{R}^2$ at Q defined by the ordered pair $(\alpha'(t_1), \alpha'(t_2))$ coincides with that of $\mathbb{R}^2$ at Q; *negative*, otherwise.

The number of positive (or negative) nodes on γ_i is denoted by N_i^+ (resp. N_i^-). The definition of a positive (or negative) node on γ_i depends on the choice of an admissible starting point y. However, it is known that the algebraic number $N_i^+ - N_i^-$ does not depend on the choice of y, see [12] for details. Thus, the algebraic number $N^+ - N^- = \sum_{i=1}^{k} (N_i^+ - N_i^-)$ is well defined. Note that nodes arising from $\gamma_i \cap \gamma_j$ $(i \neq j)$ play no role in the computation.

Then, we have the following formula as an application of the formula of Pignoni [9] and Imai [6].

Proposition 3.5. *For an admissible stable map $f\colon M_k \to \mathbb{R}^2$, we have*

$$g = \varepsilon(M_k) \left((N^+ - N^-) + \frac{c(f)}{2} + (1 + i^+ - i^-) - \frac{1}{2} \sum_{j=1}^{k} (r_j + 1) \right) \tag{3.1}$$

where $\varepsilon(M_k)$ is equal to 1 if M_k is orientable or 2 if M_k is non-orientable, and r_j denotes the rotation number of $f|_{\partial_i}$.

Proof. To compute the Euler characteristic $\chi(M_k)$, apply a result of Levine [8]: For an admissible stable map $f\colon M_k \to \mathbb{R}^2$, we have

$$\chi(M_k) = \sum_{i=1}^{\ell} \tau(\gamma_i) + \frac{1}{2}\sum_{j=1}^{k} \tau(e_j),$$

where γ_i and e_j denote $f(S_i)$ and $f(\partial_j)$ respectively, and $\tau(\gamma_i)$ and $\tau(e_j)$ denote the double tangent turning number of γ_i and e_j with respect to the canonical orientation respectively. For an oriented closed curve α, the double tangent turning number $\tau(\alpha)$ is defined as the degree of the map $\alpha \to \mathbb{R}P^1$ assigning to each point on the curve its tangent line. This map is also defined at cusp points. If α has no cusps, then $\tau(\alpha) = 2r(\alpha)$ where $r(\alpha)$ denotes the normal degree of α. To compute $\tau(\alpha)$, apply a result of Quine [10]: For a closed plane curve α, we have

$$\tau(\alpha) = 2\eta(\alpha) + 2n^+ - 2n^- + c^+ - c^-,$$

where $\eta(\alpha) = \pm 1$ is defined according to the orientation of the curve α, c^+ (or c^-) denotes the number of positive (resp. negative) cusps of α, and n^+ (or n^-) the number of positive (resp. negative) nodes of α, see [10] for details. Comparing the definitions of the items in the Quine's formula with the ones introduced in this paper, we see: (a) the sign of the double points is the opposite of that defined by Quine; (b) when the contour is endowed with its canonical orientation, each cusp is negative. Thus,

$$\tau(\gamma_i) = 2\eta(\gamma_i) + 2N_i^- - 2N_i^+ - c_i,$$

where c_i denotes the number of cusps of γ_i. $\eta(\gamma_i) = +1$ if and only if γ_i is negative.

$$\sum_{i=1}^{k} \tau(\gamma_i) = 2i^- - 2i^+ + 2N^- - 2N^+ - c(f).$$

Each $f(\partial_j)$ is a closed curve with no cusp: $\tau(f(\partial_j)) = 2r_j$. Hence, by applying the formula of Levine to f, we obtain

$$\chi(M_k) = 2i^- - 2i^+ + 2N^- - 2N^+ - c(f) + \sum_{j=1}^{k} r_j. \tag{3.2}$$

Then, the result follows immediately. □

Corollary 3.6. *Let $f\colon \Sigma_{g,1} \to \mathbb{R}^2$ be an admissible stable map of rotation number r. Then, the number of cusps of f and the rotation number r never have the same parity.*

Lemma 3.7. *Let $f\colon \Sigma_{g,1} \to \mathbb{R}^2$ be an admissible stable map. If $\gamma(f)$ has a node, then it has at least one negative node.*

4. Admissible stable maps $M_1 \to \mathbb{R}^2$

In this section, we construct boundary rotation number $r \in \mathbb{Z}$ stable maps $f_{r,g}\colon \Sigma_{g,1} \to \mathbb{R}^2$ $(g \geq 0)$ and $h_{r,g}\colon N_{g,1} \to \mathbb{R}^2$ $(g \geq 1)$ whose singular points sets consist of one component and whose pairs (c, n) are in the lists of Theorems 2.3 and 2.5 respectively. Note that constructing such stable maps is a part of a proof of Theorem 2.3 (or Theorem 2.5).

Note that in Figures, boundary curves are drawn in gray and the image of boundary curves are also drawn in gray.

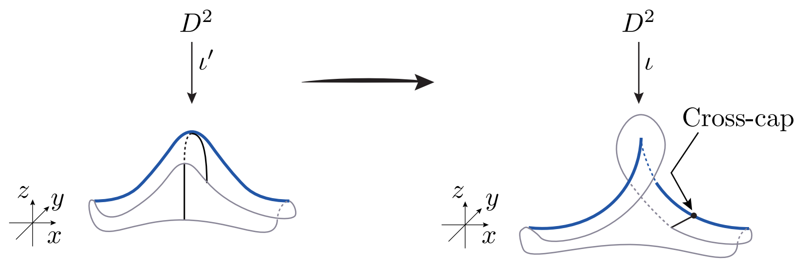

FIGURE 2. Modification I: By applying this modification, the rotation number increase by one.

4.1. **Admissible stable maps $\Sigma_{0,1} \to \mathbb{R}^2$.** For a boundary rotation number r' admissible stable map $f' \colon \Sigma_{g',1} \to \mathbb{R}^2$ whose singular points set consists of i' components and have c' cusps and n' nodes, by applying modifications I (or II, III) defined by Figure 2 (resp. Figures 3, 4), we obtain a boundary rotation number r admissible stable map $f \colon \Sigma_{g,1} \to \mathbb{R}^2$ whose singular points set consists of i components and has c cusps and n nodes. Note that a C^∞ map $\Sigma_{g,1} \to \mathbb{R}^2$ is locally defined by the projection $\mathbb{R}^3 \to \mathbb{R}^2$ into the xz-plane composed with a C^∞ map $\iota' \colon D^2 \to \mathbb{R}^3$ of the 2-dimensional disc. Figures 2, 3 and 4 represent modifications for a C^∞ map $\iota' \colon D^2 \to \mathbb{R}^3$. Note that the modified maps $\iota \colon D^2 \to \mathbb{R}^3$ in Figure 2 and 3, 4 have one cross-cap:

(1) Modification I (Figure 2):

$$(r, g, i, c, n) = (r' + 1, g', i', c' + 1, n')$$

(2) Modification II (Figures 3):

$$(r, g, i, c, n) = (r' - 1, g', i', c' + 1, n' + 1)$$

(3) Modification III (Figure 4):

$$(r, g, i, c, n) = (r' - 2, g' + 1, i', c', n')$$

Figure 5 define a rotation number -1 admissible stable map $f_{-1,0} \colon \Sigma_{0,1} \to \mathbb{R}^2$ whose triple (i, c, n) is equal to $(1, 0, 0)$. More precisely, $f_{-1,0}$ is defined by $f_{-1,0} = \pi_{xz} \circ \iota$.

By applying modification I inductively to $f_{-1,0}$, we obtain an admissible stable map

$$f_{r,0} \colon \Sigma_{0,1} \to \mathbb{R}^2$$

whose triple (i, c, n) is equal to $(1, r + 1, 0)$ for each integer $r \geq -1$.

By applying modification II inductively to $f_{-1,0}$, we obtain an admissible stable map

$$f_{r,0} \colon \Sigma_{0,1} \to \mathbb{R}^2$$

whose triple (i, c, n) is equal to $(1, -r - 1, -r - 1)$ for each integer $r \leq -1$.

4.2. **Admissible stable maps $\Sigma_{1,1} \to \mathbb{R}^2$.** For each integer $r' \leq 2$, by applying modification III to $f_{r',0}$, we obtain boundary rotation number $r \leq 0$ admissible stable maps $f_{r,1}$ whose triples (i, c, n) are one of the items below:

$$(i, c, n) = \begin{cases} (1, r + 3, 0) & \text{if } -2 \leq r \leq 0, \\ (1, -r - 3, -r - 3) & \text{if } r \leq -3. \end{cases}$$

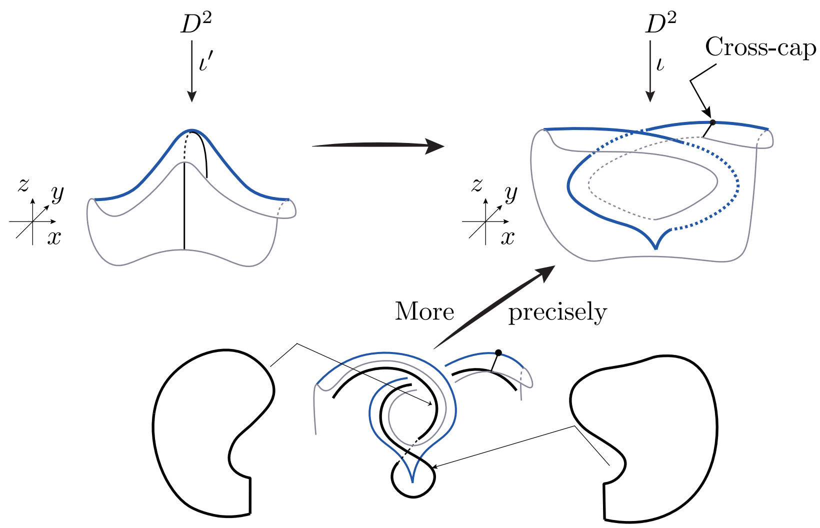

FIGURE 3. Modification II: By applying this modification, the rotation number decrease by one.

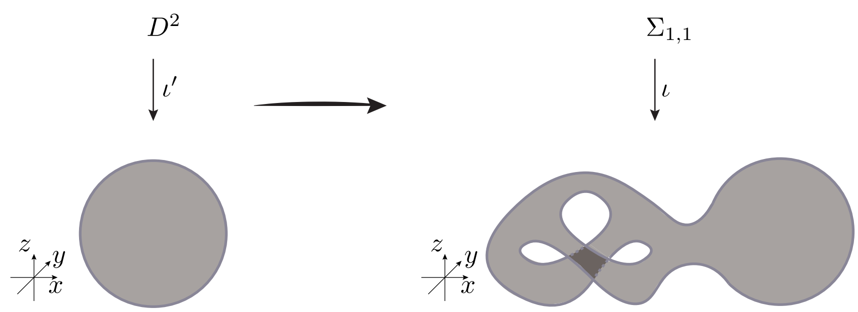

FIGURE 4. Modification III: By applying this modification, the rotation number decrease by two and the genus of the source surface increase by one.

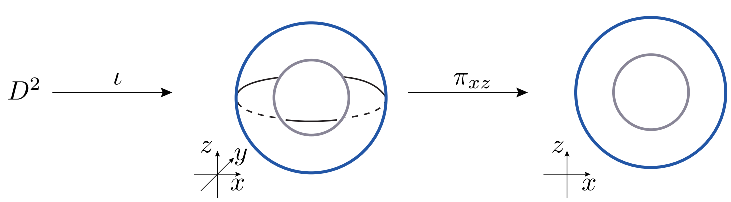

FIGURE 5. Admissible stable map $D^2 \to \mathbb{R}^2$ of rotation number -1.

Let us construct stable maps $f_{r,1}$ $(r \geq 1)$. Figures 6 and 7 show degree one stable maps $f'_1, f'_2 \colon \Sigma_1 \to S^2$ obtained by Kamenosono and the author [7]. Note that the contours of these

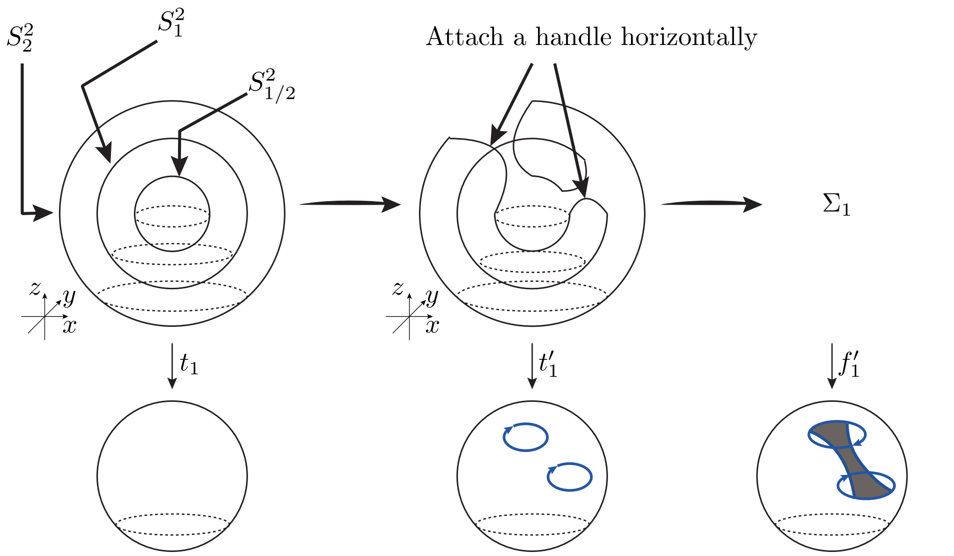

FIGURE 6. A degree one stable map $f_1\colon \Sigma_g \to S^2$: f_1' is obtained by the following manner: (1) Define $S_r^2 = \{(x,y,z) \in \mathbb{R}^3 \mid z^2 + y^2 + z^2 = r^2\}$ and put $M = S_{1/2}^2 \cup S_1^2 \cup S_2^2$. Define $t_1\colon M \to S_1^2$ by $x \mapsto x/|x|$. (2) By attaching two handles vertically between $S_{1/2}^2$ and S_1^2, $S_{1/2}^2$ and S_2^2, we obtain a degree one stable map $t_1'\colon S^2 \to S^2$ whose triple is equal to $(2,0,0)$. (3) By attaching a handle horizontally as the Figure, we obtain a degree one stable map $f_1'\colon \Sigma_g \to S^2$ whose triple $(i,,c,n)$ is equal to $(1,0,4)$.

maps are minimal contours. Stable maps $f_1', f_2'\colon \Sigma_1 \to S^2$ induce rotation number one admissible stable maps $f_{1,1}^1, f_{1,1}^2 \Sigma_{1,1} \to \mathbb{R}^2$ whose contours are as depicted in right-hand side of Figures 8 and 9 respectively. By applying modification I inductively to $f_{1,1}^1$ and $f_{1,1}^2$, we obtain rotation number $r \geq 1$ admissible stable maps $f_{r,1}^1$, $f_{r,1}^2\colon \Sigma_{1,1} \to \mathbb{R}^2$ whose triples (i,c,n) are equal to $(1, r-1, 4)$, $(1, r+3, 0)$ respectively.

4.3. **Admissible stable maps $\Sigma_{2,1} \to \mathbb{R}^2$.** For each $r' \leq 0$ (or $r' = 1,2,3$), by applying modification III to $f_{r',1}$ (resp. $f_{r',1}^1$, $f_{r',1}^2$), we obtain boundary rotation number $r \leq -2$ (resp. $r = -1,0,1$) admissible stable maps $f_{r,2}$ (resp. $f_{-1,2}^1$, $f_{0,2}^1$, $f_{1,2}^1$, $f_{-1,2}^2$, $f_{0,2}^2$, $f_{1,2}^2$) whose triples (i,c,n) are one of the items below:

$$(i,c,n) = \begin{cases} (1, r+1, 4) \text{ or } (r+5, 0) & \text{if } -1 \leq r \leq 1, \\ (1, r+5, 0) & \text{if } -4 \leq r \leq -2, \\ (1, -r-5, -r-5) & \text{if } r \leq -5. \end{cases}$$

Let us construct rotation number $r \geq 2$ admissible stable maps $\Sigma_{2,1} \to \mathbb{R}^2$.

Proposition 4.1. *For each $g \geq 2$, there are rotation numbers $2g-2$ and $2g-1$ admissible stable maps $f_{2g-2,g}$ and $f_{2g-1,g}\colon \Sigma_{g,1} \to \mathbb{R}^2$ whose triples (i,c,n) are equal to $(1,1,2g+1)$ and $(1,0,2g+2)$ respectively.*

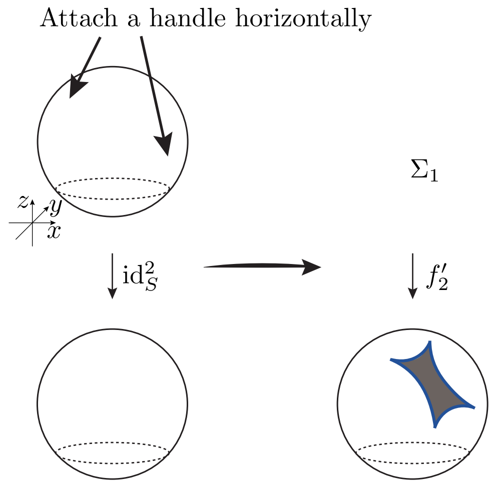

FIGURE 7. A degree one stable map $f_2'\colon \Sigma_g \to S^2$: f_2 is obtained by attaching a handle horizontally to the source sphere of the identity map on S^2.

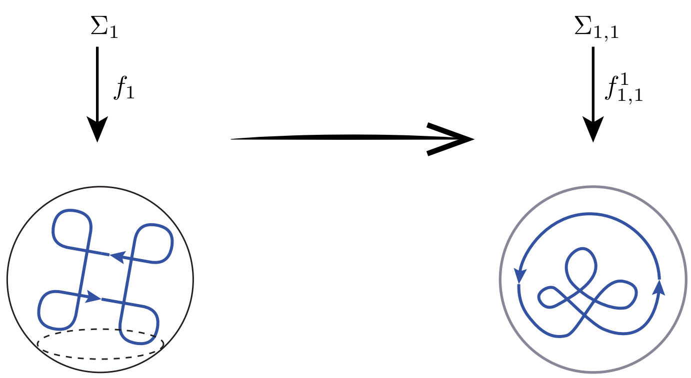

FIGURE 8. Admissible stable map $f_{1,1}^1\colon \Sigma_{1,1} \to \mathbb{R}^2$.

Proof. Figures 10 and 11 define boundary rotation number two and three admissible stable maps $f_{2,2}$ and $f_{3,2}\colon \Sigma_{2,1} \to \mathbb{R}^2$ whose triples (i,c,n) are equal to $(1,1,5)$ and $(1,0,6)$ respectively. More precisely, to define $f_{2,2}$ (or $f_{3,2}$), we decompose $\Sigma_{2,1}$ into three pieces. Then, define inclusions of each pieces into $\mathbb{R}^3$ as depicted in Figure 10 (resp. Figure 11). Note that $\Sigma_{2,1}$ is restored by attaching the three pieces along bold curves and dotted lines which are labeled in Figure 10 (resp. Figure 11). An admissible stable map $f_{2,2}$ (resp. $f_{3,2}$) is defined by the projection π_{xz} composed with the inclusion.

We can construct such admissible stable maps $f_{2g-2,g}$ and $f_{2g-1,g}$ as well as the cases $f_{2,2}$ and $f_{3,2}$. $\square$

By applying modification I inductively to $f_{3,2}$, we obtain a rotation number r admissible stable map $f_{r,2}\colon \Sigma_{2,1} \to \mathbb{R}^2$ whose triple (i,c,n) is equal to $(1,r-3,6)$ for each $r \geq 3$.

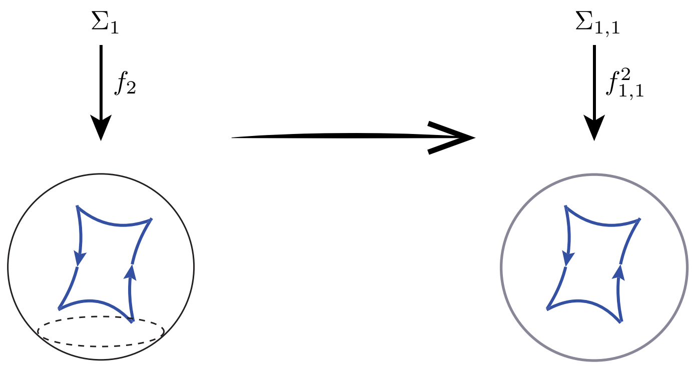

FIGURE 9. Admissible stable map $f^2_{1,1}\colon \Sigma_{1,1} \to \mathbb{R}^2$

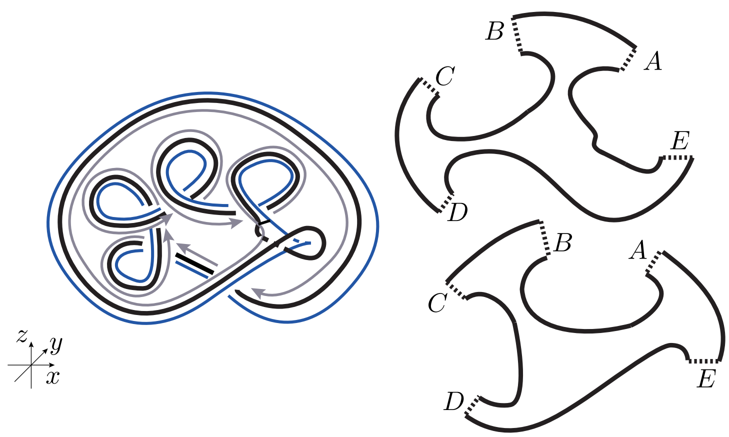

FIGURE 10. Admissible stable map $\Sigma_{2,1} \to \mathbb{R}^2$.

4.4. **Admissible stable maps $\Sigma_{g,1} \to \mathbb{R}^2$ ($g \geq 3$).** Let us consider the case $g = 3$. In this case, we already have admissible stable maps $f_{4,3}$ and $f_{5,3}$ whose triples (i, c, n) are equal to $(1, 1, 7)$ and $(1, 0, 8)$ respectively by Proposition 4.1.

By applying modification III to $f_{r',2}$ where $2 \leq r' \leq 5$ or $r' \leq -2$ (or $f^1_{r',2}$, $f^2_{r',2}$ where $-1 \leq r' \leq 1$), we obtain boundary rotation number $0 \leq r \leq 3$ or $r \leq -4$ (resp. $-3 \leq r \leq -1$) admissible stable maps $f_{r,3}$ (resp. $f^1_{r,3}$, $f^2_{r,3}$) whose triples (i, c, n) are one of the items below:

$$(i, c, n) = \begin{cases} (1, r-1, 6) & \text{if } 1 \leq r \leq 3, \\ (1, 1, 5) & \text{if } r = 0, \\ (1, r+3, 4) \text{ or } (r+7, 0) & \text{if } -3 \leq r \leq -1, \\ (1, r+7, 0) & \text{if } -6 \leq r \leq -4, \\ (1, -r-7, -r-7) & \text{if } r \leq -7. \end{cases}$$

Then, by applying modification I inductively to $f_{5,3}$, we obtain a boundary rotation number r admissible stable map $f_{r,3}\colon \Sigma_{3,1} \to \mathbb{R}^2$ whose triple (i, c, n) is equal to $(1, r-5, 8)$ for each $r \geq 5$.

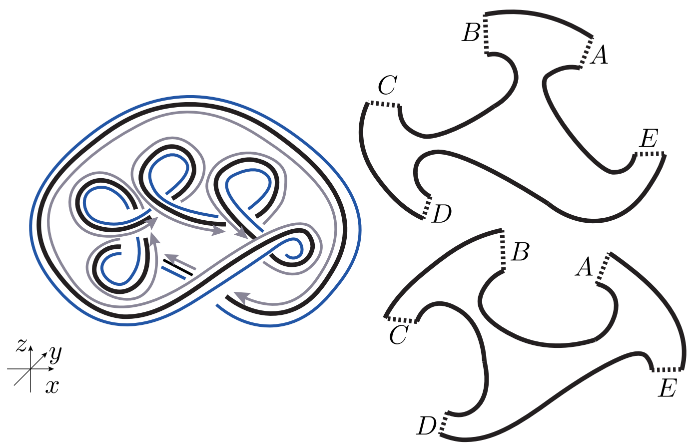

FIGURE 11. Admissible stable map $\Sigma_{2,1} \to \mathbb{R}^2$.

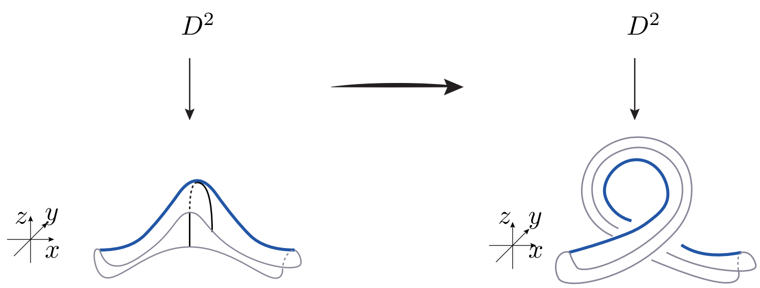

FIGURE 12. Modification IV: By applying this modification, the rotation number increases by two.

Similarly, for each $g \geq 4$ and $r \leq 2g-3$, we construct $f_{r,g}$ where $5-2g \leq r \leq 2g-3$ or $r \leq 2-2g$ (or $f^1_{r,g}$, $f^2_{r,g}$ where $3-2g \leq r \leq 5-2g$) by applying modification III to $f_{r'+2,g'-1}$ (resp. $f^1_{r'+2,g'-1}$, $f^2_{r'+2,g'-1}$ where $5-2g' \leq r' \leq 7-2g'$). Then, by applying modification I inductively to $f_{2g-1,g}$, we obtain an admissible stable map $f_{r,g}$ for each $r \geq 2g-1$. Note that we already have $f_{2g-2,g}$ in Proposition 4.1.

4.5. **Admissible stable maps** $N_{g,1} \to \mathbb{R}^2$**.** By applying modification IV (or V, VI) defined by Figure 12 (resp. Figures 13, 14) for a boundary rotation number r' admissible stable map $h\colon N_{g',1} \to \mathbb{R}^2$ whose singular points set consists of i' components and has c' cusps and n' nodes, we obtain a boundary rotation number r admissible stable map $h\colon N_{g,1} \to \mathbb{R}^2$ whose singular points set consists of i components and has c cusps and n nodes:

(4) Modification IV

$$(r, g, i, c, n) = (r'+2, g', i', c', n'+1)$$

(5) Modification V

$$(r, g, i, c, n) = (r'-2, g', i', c', n'+1)$$

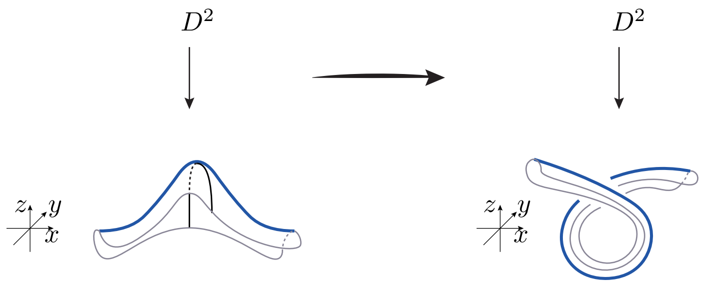

FIGURE 13. Modification V: By applying this modification, the rotation number increases by two.

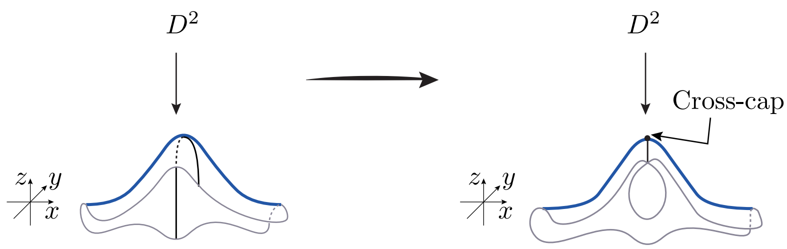

FIGURE 14. Modification VI: By applying this modification, the rotation number decreases by one.

(6) Modification VI

$$(r, g, i, c, n) = (r' - 1, g' + 1, i', c', n')$$

Note that the modified map $\iota' \colon D^2 \to \mathbb{R}^3$ have one cross-cap.

Furthermore, by applying modification III to a boundary rotation number r' admissible stable map $h' \colon N_{g',1} \to \mathbb{R}^2$, we obtain a boundary rotation number $r' - 2$ admissible stable map $h' \colon N_{g'+2,1} \to \mathbb{R}^2$.

Figure 15 defines C^∞ maps $\iota_i \colon N_{1,1} \to \mathbb{R}^3$ ($i = -2, -1, 2$ and 3). Then, the projection π_{xz} composed with ι_{-2}, ι_{-1}, ι_2 and ι_3 define boundary rotation number -2, -1, 2 and 3 admissible stable maps $h_{-2,1}$, $h_{-1,1}$, $h_{2,1}$ and $h_{3,1} \colon N_{1,1} \to \mathbb{R}^2$ whose triples (i, c, n) are equal to $(1, 0, 0)$, $(1, 1, 0)$, $(1, 0, 0)$ and $(1, 1, 0)$ respectively.

By applying modification IV to $h_{-2,1}$ and $h_{-1,1}$, we obtain boundary rotation number zero and one admissible stable maps $h_{0,1}$ and $h_{1,1} \colon N_{1,1} \to \mathbb{R}^2$ whose triples (i, c, n) are equal to $(1, 0, 1)$ and $(1, 1, 1)$ respectively.

By applying modification IV inductively to $h_{2,1}$ and $h_{3,1}$, we obtain a boundary rotation number $r \geq 2$ admissible stable map $h_{r,1} \colon N_{1,1} \to \mathbb{R}^2$ whose triple (i, c, n) is equal to $(1, 0, (r - 2)/2)$ if $r \geq 2$ is even, $(1, 1, (r - 3)/2)$ otherwise.

Similarly, by applying modification V inductively to $h_{-2,1}$ and $h_{-1,1}$, we obtain a boundary rotation number $r \leq -1$ admissible stable map $h_{r,1} \colon N_{1,1} \to \mathbb{R}^2$ whose triple (i, c, n) is equal to $(1, 0, (-r - 2)/2)$ if $r \leq -1$ is even, $(1, 1, (-r - 1)/2)$ otherwise.

Thus, we see that for each triple (i, c, n) in the list of Theorem 2.5 ($g = 1$), there exists an admissible stable map $N_{1,1} \to \mathbb{R}^2$ whose triple (i, c, n) is the triple.

Then, by applying modification III inductively to $h_{r',1} \colon N_{1,1} \to \mathbb{R}^2$, we obtain a boundary rotation number r admissible stable map $h_{r,g} \colon N_{g,1} \to \mathbb{R}^2$ whose triples (i, c, n) are in the list of Theorem 2.5 for each odd number $g \geq 1$ and each $r \in \mathbb{Z}$. Furthermore, by applying modification

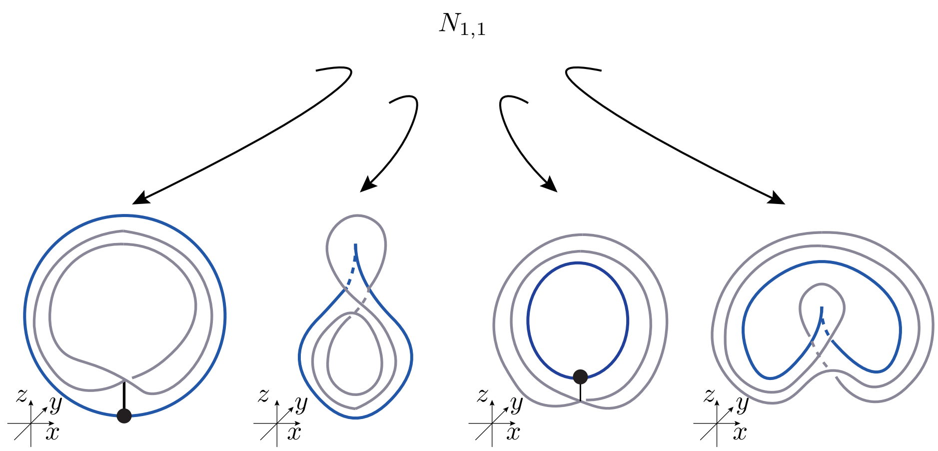

FIGURE 15. Admissible stable maps $N_{1,1} \to \mathbb{R}^2$ of rotation numbers $-2, -1, 2$ and 3, respectively

VI inductively to $h_{r',g'}\colon N_{g',1} \to \mathbb{R}^2$ with odd $g' \geq 1$, we obtain $h_{r,g}\colon N_{g,1} \to \mathbb{R}^2$ whose triples (i, c, n) are in the list of Theorem 2.5 for each even $g \geq 2$ and $r \in \mathbb{Z}$.

Thus, we see that for each (i, c, n) in the list of Theorem 2.5, there is a boundary rotation number r admissible stable map $h\colon N_{g,1} \to \mathbb{R}^2$ whose triple (i, c, n) is equal to the triple.

5. Proof of minimum of $c + n$ in Theorem 2.3

Let $g \in \mathbb{Z}_{\geq 0}$ and $r \in \mathbb{Z}$. To prove Theorem 2.3 we need the following Lemmas.

Lemma 5.1 (M. Yamamoto [14])**.** *Let $f\colon \Sigma_{g,1} \to \mathbb{R}^2$ be a rotation number r admissible stable map whose singular points set consists of one component. Then, $c(f) \geq |r+1| - 2g$ and $c(f) \not\equiv r$* mod 2.

Lemma 5.2. *Let $f\colon \Sigma_{g,1} \to \mathbb{R}^2$ be a rotation number r admissible stable map whose singular points set consists of one component.*

1. *If f has no cusps, then $r \equiv 2g - 1 \mod 4$.*
2. *If $r \equiv 2g + 1 \mod 4$, then $\gamma(f)$ has at least two cusps.*

Proof. (1) For such stable map $f\colon \Sigma_{g,1} \to \mathbb{R}^2$, $\Sigma_{g,1}$ is decomposed into three pieces as

$$\Sigma_{g,1} = \Sigma_{g-t,1} \sqcup N(S(f)) \sqcup \Sigma_{t,2}, \quad 0 \leq t \leq g,$$

where $N(S(f))$ denote a tubular neighborhood of $S(f)$. Note that $f_1; = f|_{\Sigma_{g-t,1}}$ and $f_2 := f|_{\Sigma_{t,2}}$ are immersions. Then, by applying a result of Heafliger:

For an immersed surface $M_k \subset \mathbb{R}^2$, the Euler-Poincare characteristic
$\chi(M_k)$ is equal to the normal degree of ∂M_k.

If $W(f_1) = k$, then we have $\chi(\Sigma_{g-t,1}) = k$ and $\chi(\Sigma_{t,2}) = k + r$. This shows that $2g = 1 + r + 4t$.

(2) Put $r = 2g + 1 + 4k$. Then, formula (3.1) implies the conclusion. ☐

Let us divide a proof into two cases $g = 0$ and $g \geq 1$.

5.1. $g = 0$**.** Lemma 5.1 shows that the contour $\gamma(f_{r,0})$ is an admissible minimal contour for each $r \geq 0$.

Let us consider the case $r \leq -1$. Let $f\colon \Sigma_{0,1} \to \mathbb{R}^2$ be an admissible stable map of rotation number r whose singular points set consists of one component. Then, Lemma 5.1 implies that $c(f) \geq -(r+1)$. In this case, (3.1) and Lemma 3.2 show that

$$\frac{r+1}{2} = (N^+ - N^-) + \frac{c(f)}{2}.$$

Then, we have

$$\frac{r+1}{2} = (N^+ - N^-) + \frac{c(f)}{2} \geq (N^+ - N^-) - \frac{r+1}{2}.$$

This implies that $(r+1) \geq (N^+ - N^-)$. Note that $(r+1)$ is negative. Thus, we have $N^- \geq -(r+1)$. Then,

$$c(f) + n(f) \geq \frac{c(f)}{2} + \frac{r+1}{2} + 2N^- \geq -2(r+1).$$

Thus, for such admissible stable maps, we have $c(f) + n(f) \geq -2(r+1)$. This shows that the contour $\gamma(f_{r,0})$ $(r \leq -1)$ is an admissible minimal contour.

5.2. $g \geq 1$**.** At first, let us consider the case $r \geq 2g-1$. Let $f\colon \Sigma_{g,1} \to \mathbb{R}^2$ be an admissible stable map of rotation number r whose singular points set consists of one component. Then the formula (3.1) and Lemma 3.2 show that

$$g + \frac{r+1}{2} = (N^+ - N^-) + \frac{c(f)}{2}. \tag{5.1}$$

If $\gamma(f)$ has no node, then $c(f) = 2g + r + 1$. If $\gamma(f)$ hsa a node, then Lemma 3.7 and Lemma 5.1 yeild that

$$c(f) + n(f) \geq \frac{c(f)}{2} + g + \frac{r+1}{2} + 2N^- \geq r+3.$$

This shows that the contour $\gamma(f_{r,g})$ $(r \geq 2g-1)$ is an admissible minimal contour.

The case $-2g \leq r \leq 2g$ is also proved by using Lemmas 5.1, 5.2 and the similarly argument as the above case.

Then, let us consider the case $r \leq -2g-1$. Let $f\colon \Sigma_{g,1} \to \mathbb{R}^2$ be a rotation number r admissible stable map whose singular points set consists of one component. The formula (5.1) and Lemma 5.1 imply

$$g + \frac{r+1}{2} \geq (N^+ - N^-) + \frac{-r-1-2g}{2}.$$

Thus, we have

$$2g + r + 1 \geq (N^+ - N^-).$$

Note that $2g + r + 1$ is negative. Thus, $N^- \geq -(2g + r + 1)$. Then,

$$c(f) + n(f) \geq \frac{c(f)}{2} + g + \frac{r+1}{2} + 2N^- \geq -2(r + 2g + 1).$$

Therefore, the contour $\gamma(f_{r,g})$ $(r \leq -2g-1)$ is admissible minimal contour.

It completes the proof of Theorem 2.3.

6. Proof of minimum of $c + n$ in Theorem 2.5

Let $g \in \mathbb{Z}_{\geq 1}$ and $r \in \mathbb{Z}$. Proposition 3.5 yeilds the following lemma.

Lemma 6.1. *Let $h\colon N_{g,1} \to \mathbb{R}^2$ be a boundary rotation number r admissible stable map whose singular points set consists of one component. Then, the numbers $g+r$ and $c(h)$ never have the same parity. In particular, if $g+r$ is an even number, then h has at least one cusp.*

Proof. Let $h\colon N_{g,1} \to \mathbb{R}$ be a such stable map. Then, formula (3.1) induces the following modulo two equation

$$g \equiv c(h) - (r+1).$$

It implies the conclusion. $\square$

We divide a proof into two cases $g = 1$ and $g \geq 2$.

6.1. $g = 1$.

Lemma 6.1 shows that the contours $\gamma(h_{r,1})$ $(r = -2, -1, 2, 3)$ are admissible minimal contours.

At first, let us consider the case $r \geq 4$. Let $h\colon N_{1,1} \to \mathbb{R}^2$ be a boundary rotation number r admissible stable map whose singular points set consists of one component.

(i1) $i^+ = 1$. Then, the formula (3.1) implies $2(N^+ - N^-) + c(h) = r - 2$. If $\gamma(h)$ has no nodes, then $c(h) = r - 2$. If $\gamma(h)$ has a node, then

$$c(h) + n(h) = \frac{r - 2 + c(h)}{2} + 2N^- \geq \frac{r - 2 + c(h)}{2}.$$

This yeilds that if $r \geq 4$ is odd (or even), then

$$c(h) + n(h) \geq (r-1)/2$$

(resp. $c(h) + n(h) \geq (r-2)/2$).

(i2) $i^- = 1$. Then, the formula (3.1) implies $2(N^+ - N^-) + c(h) = r + 2$. If $\gamma(h)$ has no nodes, then $c(h) = r + 2$. If $\gamma(h)$ has a node, then

$$c(h) + n(h) = \frac{r + c(h) + 2}{2} + 2N^- \geq \frac{r + c(h) + 2}{2}.$$

This yields that if $r \geq 4$ is odd (or even), then

$$c(h) + n(h) \geq (r+3)/2$$

(resp. $c(h) + n(h) \geq (r+2)/2$).

(i1) and (i2) show that if $r \geq 4$ is odd (or even), then $c(h) + n(h) \geq (r-1)/2$ (resp. $c(h) + n(h) \geq (r-2)/2$). This implies that the contour $\gamma(h_{r,1})$ $(r \geq 4)$ is an admissible minimal contour.

Then, let us consider the case $r \leq -3$. Let $h\colon N_{1,1} \to \mathbb{R}^2$ be a boundary rotation number r admissible stable map whose singular points set consists of one component.

(i1) $i^+ = 1$. Then, the formula (3.1) induces $2(N^+ - N^-) = r - c(h) - 2$. Note that $r - c(h) - 2 \leq 0$. Thus, we have $N^- \geq -(r - c(h) - 2)/2$. Then,

$$c(h) + n(h) = \frac{r + c(h) - 2}{2} + 2N^- \geq \frac{3c(h) - r + 2}{2}.$$

Lemma 6.1 yields that if $r \leq -3$ is odd (or even), then $c(h) + n(h) \geq (-r+5)/2$ (resp. $c(h) + n(h) \geq (-r+2)/2$).

(i2) $i^- = 1$. Then, the formula (3.1) induces $2(N^+ - N^-) = r - c(h) + 2$. If $\gamma(h)$ has no nodes, then $c(h) = r + 2$. If $\gamma(h)$ has a node, then $(N^+ - N^-) = (r - c(h) + 2)/2 \leq 0$. Thus, we have $N^- \geq -(r - c(h) + 2)/2$. Then,

$$c(h) + n(h) = \frac{r - c(h) + 2}{2} + 2N^- \geq \frac{3c(h) - r - 2}{2}.$$

Lemma 6.1 shows that if $r \leq -3$ be odd (or even), then $c(h) + n(h) \geq (-r+1)/2$ (resp. $(-r-2)/2$).

(i1) and (i2) show that $\gamma(h_{r,1})$ ($r \leq -3$) is an admissible minimal contour.

Formula (3.1) implies the following.

Lemma 6.2. *Let $h\colon N_{1,1} \to \mathbb{R}^2$ be a boundary rotation number 0 admissible stable map whose singular points set consists of one component. Then, $c(h) + n(h) \geq 1$.*

Therefore, $\gamma(h_{0,1})$ is an admissible minimal contour.

We can show that $\gamma(h_{1,1})$ is minimal as the above case.

Thus, we complete the proof of the Theorem 2.5 for $g = 1$.

6.2. $g \geq 2$**.** Lemma 6.1 shows that the contours $\gamma(h_{-g,g})$ and $\gamma(h_{-g-1,g})$ are admissible minimal contours.

At first, let us consider $r \geq -g+1$. Let $h\colon N_{g,1} \to \mathbb{R}^2$ be a boundary rotation number r admissible stable map whose singular points set consists of one component.

(i1) $i^+ = 1$. Then, formula (3.1) shows that $2(N^+ - N^-) + c = g + r - 3$. If $\gamma(h)$ has no nodes, then $c(h) = g + r - 3$. If $\gamma(h)$ has a node, then

$$c(h) + n(h) = c(h) + \frac{g + r - c(h) - 3}{2} + 2N^- \geq \frac{g + r + c(h) - 3}{2}.$$

Lemma 6.1 shows that if $g + r$ is even (or odd), then $c(h) + n(h) \geq (g + r - 2)/2$ (resp. $c(h) + n(h) \geq (g + r - 3)/2$).

(i2) $i^- = 1$. Then, formula (3.1) shows that $2(N^+ - N^-) + c(h) = g + r + 1$. If $\gamma(h)$ has no nodes, then $c(h) = g + r + 1$. If $\gamma(h)$ has a node, then

$$c(h) + n(h) = c(h) + \frac{g + r - c(h) + 1}{2} + 2N^- \geq \frac{g + r + c(h) + 1}{2}.$$

Lemma 6.1 shows that if $g + r$ is even (or odd), then $c(h) + n(h) \geq (g + r + 2)/2$ (resp. $c(h) + n(h) \geq (g + r + 1)/2$).

(i1) and (i2) implies that the conturs $\gamma(h_{r,g})$ ($r \geq -g+1$) are an admissible minimal contours.

Then, let $r \leq -g-2$. Let $h\colon N_{g,1} \to \mathbb{R}^2$ be a boundary rotation number r admissible stable map whose singular points set consists of one component.

(i1) $i^+ = 1$. Formula (3.1) shows that $2(N^+ - N^-) = g + r - c(h) - 3 \leq 0$. Thus, we have $N^- \geq -(g + r - c(h) - 3)/2$, Then,

$$c(h) + n(h) = c(h) + \frac{g + r - c(h) - 3}{2} + 2N^- \geq \frac{-g - r + c(h) + 3}{2}.$$

Lemma 6.1 shows that if $g + r$ is even (or odd), then $c(h) + n(h) \geq (-g - r + 4)/2$ (resp. $c(h) + n(h) \geq (-g - r + 3)/2$).

(i2) $i^- = 1$. Formula (3.1) shows that $2(N^+ - N^-) = g + r - c(h) + 1 \le 0$. Thus, we have $N^- \ge -(g + r - c(h) + 1)/2$, Then,

$$c(h) + n(h) = c(h) + \frac{g + r - c(h) + 1}{2} + 2N^- \ge \frac{-g - r + 3c(h) - 1}{2}.$$

Lemma 6.1 shows that $g + r$ is even (or odd), then

$$c(h) + n(h) \ge (-g - r + 2)/2$$

(resp. $c(h) + n(h) \ge (-g - r - 1)/2$).

(i1) and (i2) implies that $\gamma(h_{r,g})$ ($r \ge -g - 2$) is an admissible minimal contour.

It completes the proof of Theorem 2.5.

7. Problem

Let M be a compact connected surface with boundary and P a surface without boundary. A C^∞ map $f\colon M \to P$ is called a *fold map* if f has only fold points as its singularities.

Let $f\colon M \to \mathbb{R}^2$ be a boundary rotation number r admissible stable fold map. Then, call the contour $\gamma(f)$ an $\mathcal{F}$-(i,n)-*minimal contour* of boundary rotation number r maps $M \to \mathbb{R}^2$ if the pair $(i(f), n(f))$ is the smallest among rotation number r admissible stable fold maps $M \to \mathbb{R}^2$ with respect to the lexicographic order.

Problem 7.1. Let $M = \Sigma_{g,1}$ or $N_{g,1}$. Study an $\mathcal{F}$-(i,n)-minimal contour of boundary rotation number r maps $M \to \mathbb{R}^2$.

References

[1] J. W. Bruce, P. J. Giblin, *Projections of surfaces with boundary*, Proc. London Math. Soc. (3) 60 (1990), no. 2, 392–416.

[2] S. Demoto, *Stable maps between 2-spheres with a connected fold curve*. Hiroshima Math. J. **35** (2005), no. 1, 93–113. DOI: 10.32917/hmj/1150922487

[3] T. Fukuda and T. Yamamoto, *Apparent contours of stable maps into the sphere*, J. Sing. **3** (2011), 113–125. DOI: 10.5427/jsing.2011.3g

[4] M. Golubitsky and V. Guillemin, *Stable mappings and their singularities*, Grad. Texts in Math., Vol. 14, Springer, New York-Heidelberg, 1973.

[5] A. Haefliger, *Quelques remarques sur les applications différentiables d'une surface dans le plan*, (French) Ann. Inst. Fourier. Grenoble 10 1960 47–60. DOI: 10.5802/aif.97

[6] N. Imai, *Discriminant set of a stable map and the Euler characteristic of a surface with boundary* (in Japanese), Master Thesis, Hiroshima Univ., March 1999.

[7] A. Kamenosono and T. Yamamoto, *The minimal numbers of singularities of stable maps between surfaces*, Topology Appl. **156** (2009), pp. 2390-2405. DOI: 10.1016/j.topol.2009.06.010

[8] H. Levine, *Computing the Euler charactristic of a manifold with boundary*, Proc. Amer. Math. Soc. 123 (1995), no. 8, 2563–2567.

[9] R. Pignoni, *Projections of surfaces with a connected fold curve*, Topology Appl. **49** (1993), no. 1, 55–74. DOI: 10.1016/0166-8641(93)90129-2

[10] J. R. Quine, *Plűcker equations for curves*. Amer. Math. Monthly **88** (1981), no. 1, 21–29.

[11] R. Thom. *Les singularités des applications différentiables*. (French) Ann. Inst. Fourier (Grenoble) 6 (1955/56), 43–87.

[12] H. Whitney, *On regular families of curves*, Bull. Amer. Math. Soc. 47, (1941). 145–147. DOI: 10.1090/s0002-9904-1941-07395-7

[13] H. Whitney, *On singularities of mappings of euclidean spaces. I. Mappings of the plane into the plane*. Ann. of Math. (2) **62** (1955), 374–410. DOI: 10.2307/1970070

[14] M. Yamamoto, *Pseudo-immersions of oriented surfaces with one boundary component into the plane*. Proc. Roy. Soc. Edinburgh Sect. A **139** (2009), no. 6, 1327–1335.

[15] T. Yamamoto, *Apparent contours with minimal number of singularities*, Kyushu J. Math. **64**(2010), no. 1, 1–16. DOI: 10.2206/kyushujm.64.1

[16] T. Yamamoto, *Apparent contours of stable maps between closed surfaces*, Kodai Math. J. 40 (2017), no. 2, 358–378. DOI: 10.2996/kmj/1499846602

[17] T. Yamamoto, *Number of singularities of stable maps on surfaces*, Pacific J. Math., Vol. 280 (2016), No. 2, 489–510. DOI: 10.2140/pjm.2016.280.489

Takahiro Yamamoto, Department of Mathematics, Tokyo Gakugei University, Koganei, Tokyo, 184-8501, Japan

Email address: `yamush@u-gakugei.ac.jp`

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A QUICK TRIP THROUGH FIBRATION STRUCTURES

A. A. DO ESPIRITO SANTO, D. DREIBELBIS, M. F. RIBEIRO, AND R. N. ARAÚJO DOS SANTOS

Abstract. In this article we review the classical results about the existence of fibered structures for real and complex singularities in the local setting, commonly known in the literature as Milnor's fibration structures. After reviewing the classical studies, we describe some generalizations in two main directions, namely, the existence of open book structures on semialgebraic manifolds, and the existence of the Milnor fibration in a stratified sense.

1. Introduction

The existence of a fibration near an isolated singularity is fundamental to the understanding of the local structure of the pair space-function.

In the famous Princeton notes of 1968 [Mi], J. Milnor established the foundations for studying fibration structures for germs of complex analytic functions $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ with $\dim \operatorname{Sing} f \geq 0$. In this setting, it was shown that given a representative $f : U \subset \mathbb{C}^{n+1} \to \mathbb{C}$ with U an open set in $\mathbb{C}^{n+1}$, $f(0) = 0$, there exists a small enough real number $\varepsilon_0 > 0$ such that for any $0 < \varepsilon \leq \varepsilon_0$,

$$\phi := \frac{f}{\|f\|} : S_{\varepsilon}^{2n+1} \setminus K_{\varepsilon} \to S^1 \tag{1}$$

is a locally trivial smooth fibration, where $K_{\varepsilon} = f^{-1}(0) \cap S_{\varepsilon}^{2n+1}$ is called the *link* of the singularity at the origin.

In chapters 5, 6 and 7 of [Mi], Milnor gave differentiable and topological descriptions of the link and the fibers $F_{\theta} = \phi^{-1}\left(e^{i\theta}\right)$, where $e^{i\theta} \in S^1$, showing that independent of the dimension of the singular locus, the fiber is a $(2n)$-dimensional smooth parallelizable manifold with the homotopy type of a k-dimensional CW-complex, with $k \leq n$.

In addition, whenever $\operatorname{Sing} f = \{0\}$, Milnor associated to the singular point of f a multiplicity denoted by $\mu(f)$, later named by several authors as *the Milnor number of the singularity*, given by the topological degree of the map

$$\varepsilon \frac{\nabla f}{\|\nabla f\|} : S_{\varepsilon}^{2n+1} \to S_{\varepsilon}^{2n+1}.$$

In this case it was also shown that the fiber F_{θ} has the same homotopy type of a bouquet of n-dimensional spheres $\bigvee_{i=1}^{\mu(f)} S_i^n$, with $\mu(f)$ spheres in the bouquet.

In 1976, Lê Dũng Tráng in his article [Le] proved the existence of a general fibration structure on a complex analytic set, as follows.

Let X be an analytic set in an open neighborhood U of the origin $0 \in \mathbb{C}^{n+1}$. Let $f : (X, 0) \to (\mathbb{C}, 0)$ be a germ of a holomorphic function.

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Theorem 1.1. [Le, Milnor-Lê Fibration] *For any small enough $\varepsilon > 0$, there exists η, $0 < \eta \ll \varepsilon$, such that*

$$f_| : B_\varepsilon^{2n+2} \cap X \cap f^{-1}(D_\eta \setminus \{0\}) \to D_\eta \setminus \{0\} \tag{2}$$

is a locally trivial topological fibration.

An important point to notice here is that this topological fibration structure becomes a smooth fibration if $X \setminus V_f$ is a non-singular analytic set in $\mathbb{C}^{n+1}$ (see details in [Ham, Le]).

As a particular case of the previous theorem, one can state:

Corollary 1.2. [Le, Existence of Milnor-Lê (tube) fibration] *Let $f : \left(\mathbb{C}^{n+1}, 0\right) \to (\mathbb{C}, 0)$ be a holomorphic function germ. Then there exists small enough $\varepsilon > 0$, such that for any $0 < \delta \ll \varepsilon$, the map*

$$f_| : \overline{B}_\varepsilon^{2n+2} \cap f^{-1}\left(D_\delta \setminus \{0\}\right) \to D_\delta \setminus \{0\} \tag{3}$$

is the projection of a locally trivial smooth fibration. In addition, for any small enough ε, there exists η, $0 < \eta \ll \varepsilon$, such that

$$f_| : B_\varepsilon^{2n+2} \cap f^{-1}\left(S_\eta^1\right) \to S_\eta^1 \tag{4}$$

is the projection of a locally trivial smooth fibration. Moreover, the fibrations (1) and (4) are equivalent[1].

Milnor also explained how to extend the study to a real analytic map germ

$$G : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0),\ m > p \geq 2,$$

with isolated singular point at the origin, i.e., $\operatorname{Sing} G = \{0\}$ as a germ of a set. In this case he observed that, for any small enough $\varepsilon > 0$, there exists a projection map

$$S_\varepsilon^{n-1} \setminus K_\varepsilon \to S_1^{p-1}$$

that is a smooth locally trivial fibration, induced by G, but which in general fails to be the canonical map $G/\|G\|$ like (1) (see section 2.2). However, one gets that G always induces a trivial fibration structure over a neighborhood of the *link* K_ε, and consequently an *open book structure* (or $NS-$pair) on S_ε^{n-1} for some extension of the projection $G/\|G\|$ (see Section 3).

More recently in [ACT1, AT1, AT2], the authors have defined and proved the existence of *singular higher open book structures* on spheres of small enough radius, which extends the real and complex fibrations results previously proved by Milnor.

In another direction, the authors in [DACA] have shown how it is possible to extend these results to the class of semi-algebraic maps, in such a way that it is possible to derive, as a particular case, the existence of fibration structures mentioned above. More precisely, let $G : \mathbb{R}^m \to \mathbb{R}^p$, $m > p \geq 2$, be a C^2 semi-algebraic map and $W \hookrightarrow \mathbb{R}^N$ an embedded compact and connected semi-algebraic manifold. The authors adapted some conditions used in [ACT1, ACT2, AT1, AT2, Ma] to ensure that the restriction map

$$\overline{G} = \frac{G}{\|G\|} : W \setminus V_G \to S^{p-1}$$

with $V_G := G^{-1}(0)$, gives a higher open book structure on W and consequently a locally trivial smooth fibration. In this case, the link of the structure is $V_W(G) = W \cap V_G$.

[1]Two locally trivial smooth fibrations $p : E \to B$ and $p' : E' \to B$ are said to be *equivalent* if there is a smooth diffeomorphism $h : E \to E'$ such that $p' \circ h = p$.

In the past few years the study of the existence of fibration structures in the real setting has concentrated on real maps with isolated singularities and on classes of singular maps with the property $\operatorname{Sing} G \subset V_G$, which in this work will be denoted by $\operatorname{Disc} G = \{0\}$ (cf [ACT1, AT1, AT2, C, CSS3, DA, Ma, Mi, PT, RSV]).

The complementary case, when $\operatorname{Disc} G$ is larger than $\{0\}$, has been studied, for instance, by Hamm in [Ham]. Hamm studied the case where the germs of holomorphic maps

$$G : (\mathbb{C}^{n+p}, 0) \to (\mathbb{C}^p, 0)$$

are also an ICIS - *Isolated Complete Intersection Singularity* [2]. This means the map defines a local complete intersection germ V_G such that V_G has an isolated singularity at the origin, i.e., the ICIS condition amounts to the condition $\operatorname{Sing} G \cap V_G = \{0\}$. Hamm proved the following result.

Theorem 1.3. *Let $G := (G_1, \ldots, G_p) : (\mathbb{C}^{n+p}, 0) \to (\mathbb{C}^p, 0)$, $p \geq 1$, be an ICIS at 0. Then,*

$$G_| : B_\varepsilon^{2(n+p)} \cap G^{-1}(B_\eta^{2p} \setminus \operatorname{Disc} G) \to B_\eta^{2p} \setminus \operatorname{Disc} G \tag{5}$$

is a locally trivial smooth fibration.

This fibration was also called *the Milnor fibration* and it generalizes the previous isolated singular case for holomorphic functions. The discriminant set $\operatorname{Disc} G$ is a complex hypersurface of $\mathbb{C}^p$. Hence, it does not disconnect the complement $B_\eta^{2p} \setminus \operatorname{Disc} G$ and the topological type of the fibers of (5) does not change. Moreover, the fiber F is a real $2n$-dimensional smooth manifold with the homotopy type of a bouquet of n-dimensional spheres $\bigvee_{i=1}^{\mu} S_i^n$, where now $\mu := \operatorname{rank} H_n(F, \mathbb{Z})$, the rank of the homology in the middle dimension of the fiber with integer coefficients.

For a real analytic map germ $G : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$ with positive dimensional discriminant set, i.e. $\dim \operatorname{Disc} G > 0$, the existence of fibration structures was pointed out theoretically in [ACT1, Theorem 1.3] and [MS], but no concrete families of examples have been studied. In [CGS], the authors presented a Milnor-Lê type result over the complement of the image $G(\operatorname{Sing} G)$, under assumptions of Thom regularity.

In [ART1] the authors have considered this general situation and have introduced two local fibrations structures. The first one was over the complement of the discriminant, which was called a *Milnor-Hamm tube fibration*. The second was a general notion of *stratified tube fibration* by considering in addition all singular fibers over the stratified discriminant. In the latter case, the tube fibration, which was called a *singular Milnor tube fibration*, is actually a collection of finitely many fibrations over path-connected subanalytic sets.

In [ART2], the authors considered again the setting $\dim \operatorname{Disc} G > 0$ and introduced the *Milnor-Hamm sphere fibration*. They gave natural sufficient conditions for which this fibration exists, and they presented several classes of maps which satisfies these conditions. Moreover, they have shown that the Milnor-Hamm tube and Milnor-Hamm sphere fibrations are extensions of the previous ones treated in [ACT1, AT1, AT2, CGS, CSS2, Ma, Mi].

In this work we present a brief survey about the results described above, as well as some comparisons between the main results found in the literature. This paper complements the nice survey paper [S2], recently published.

[2] One of the richest sources of information on ICIS is Looijenga's classical book [Lo2]. See also the reedited version [Lo3].

2. 0-dimensional discriminant set

In this section we consider the fibration on the so-called *Milnor's tube*, and the fibration on a sphere of radius small enough for the case where the classical discriminant set is 0-dimensional. Classically, this case was studied in two approaches: isolated critical point and isolated critical value.

2.1. **Isolated critical point: tube fibration.** Given a representative of $G : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$, $m > p \geq 2$, in the first part of the proof of [Mi, Theorem 11.2], Milnor proved that if G has an isolated critical point at the origin $0 \in \mathbb{R}^m$, then for any small enough $\varepsilon > 0$, there exists η, $0 < \eta \ll \varepsilon$, such that the restriction map

$$G_{|} : \overline{B}^m_\varepsilon \cap G^{-1}(S^{p-1}_\eta) \to S^{p-1}_\eta \tag{6}$$

is the projection of a locally trivial smooth fibration. More precisely, Milnor proved the following result:

Theorem 2.1. [Mi] *Let $G : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$ be a real analytic map germ such that* $\operatorname{Sing} G = \{0\}$ *as a germ of an analytic set at the origin. Then there exists $\varepsilon_0 > 0$ such that, for each ε, $0 < \varepsilon \leq \varepsilon_0$, there exists η, $0 < \eta \ll \varepsilon$, such that (6) is a smooth fiber bundle.*

Geometrically, a standard picture for the total space $\overline{B}^m_\varepsilon \cap G^{-1}(S^{p-1}_\eta)$ is as in the Figure 1 below[3]. The boundary manifold $\overline{B}^m_\varepsilon \cap G^{-1}(S^{p-1}_\eta)$ looks like a "tube" surrounding the special fiber V_G. For this reason several authors called this space "the Milnor tube".

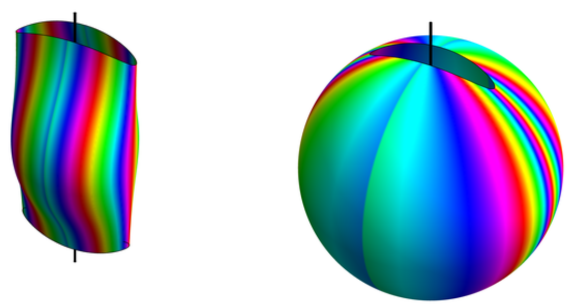

Figure 1. $G(x, y, z) = (x, y(x^2 + y^2 + z^2))$ Milnor tube and Milnor sphere fibrations.

Remark 2.2. It is not hard to see that the structure of the fibration (6) does not change up to isotopy for any $\varepsilon > 0$ and $\eta > 0$ small enough. Consequently, we will denote the Milnor tube as M_G.

2.2. **Sphere fibration: Milnor's example.** Concerning the sphere fibration in this real setting, Milnor guaranteed the existence of a diffeomorphism between the Milnor tube M_G and the complement $S^{m-1}_\varepsilon \setminus \operatorname{int}(T)$ of an open tubular neighborhood $\operatorname{int}(T)$ of the link K_ε in S^{m-1}_ε, where $T := \left\{x \in S^{m-1}_\varepsilon \mid \|G(x)\| \leq \eta\right\}$. This diffeomorphism is the identity on the boundary of

[3] In the case the link $K_\varepsilon = V_G \cap S^{m-1}_\varepsilon$ is not empty for any small enough ε.

the tube, which allows one to extend it to an open book structure (see Section 3). This diffeomorphism and the locally trivial smooth fibration (6) guaranteed by Theorem 2.1, can be composed to get a map

$$\zeta : S_\varepsilon^{m-1} \setminus \mathrm{int}(T) \to S_\eta^{p-1}$$

which is a fibration, as stated in the following result:

Theorem 2.3. [Mi, Theorem 11.2, p. 97] *Let $G : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$, $m \geq p \geq 2$, be a real analytic map germ such that* $\mathrm{Sing}\, G = \{0\}$ *as a germ of an analytic set at the origin. Then there exists $\varepsilon_0 > 0$ such that, for each ε, $0 < \varepsilon \leq \varepsilon_0$, there exists η, $0 < \eta \ll \varepsilon$, such that*

$$\zeta : S_\varepsilon^{m-1} \setminus int(T) \to S_\eta^{p-1} \tag{7}$$

is a smooth fiber bundle.

Moreover, Milnor showed that each fiber F_ζ of the fibration ζ is a smooth compact $(m-p)$-dimensional manifold bounded by a copy of K_ε. If the link K_ε is not empty for any small enough $\varepsilon > 0$, it is a $(m-p-1)$-dimensional closed smooth submanifold of the sphere and the fiber is $(p-2)$-connected. On the other hand, if the link K_ε is empty, then the manifold $\overline{B}_\varepsilon^m \cap G^{-1}(S_\eta^{p-1})$ is diffeomorphic to the sphere S_ε^{m-1}. Moreover, when $m > p$ the fibration (7) given in Theorem 2.3 becomes a *Hopf fibration*[4] $G_| : S^{2t-1} \to S^t$, with $t = 2, 4, 8$.

Next, Milnor presented the following remark without a proof [Mi, remark on p.99]:

"*with a little more effort one can prove that the entire complement $S_\varepsilon^{m-1} \setminus K_\varepsilon$ also fibers on S_η^{p-1}*".

In order to make this more precise, in [AT1, AT2] and [ACT1], the authors gave a complete proof for this remark.

Milnor also noted that in general the map projection of the fibration (7) fails to be the canonical map $G/\|G\|$, like it is for the above cited case of holomorphic function germs. In particular, in [Mi, p. 99], Milnor considered the mapping $G := (G_1, G_2) : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ given by $G(x, y) = (x, x^2 + y(x^2 + y^2))$ which satisfies $\mathrm{Sing}\, G = V_G = \{0\}$ and consequently has an isolated singular point at the origin. Theorem 2.3 gives the existence of the fibration in the sphere. However, the map $G/\|G\|$ cannot be the projection of a locally trivial smooth fibration on S_ε^1, because it is not a submersion for ε small enough.

In fact, considering $\mathbf{v} := (x, y)$ and the matrix

$$A(\mathbf{v}) = \begin{pmatrix} G_1(\mathbf{v})\nabla G_2(\mathbf{v}) - G_2(\mathbf{v})\nabla G_1(\mathbf{v}) \\ \mathbf{v} \end{pmatrix}$$

one can see that there exists a curve C (see Figure 2) of tangency points between the fibers of the map

$$G/\|G\| : B_\varepsilon^2 \setminus V_G \to S^1$$

and the small spheres [5]. The curve C contains the origin in its closure, hence the intersection $C \cap S_\varepsilon^1$ provides the critical locus of the map $G/\|G\| : S_\varepsilon^1 \to S^1$ for any small enough $\varepsilon > 0$.

As we will see in more details in the next section, the curve C represents the set of ρ-nonregular points of $G/\|G\|$ (see Lemma 2.10 and Remark 2.11). Consequently (c.f. Definition 2.9), the map $G/\|G\|$ is not ρ-regular and this is precisely the reason why the map $G/\|G\|$ fails to be the projection of a locally trivial smooth fibration.

[4]It is well known that this case is only possible for the pairs of dimensions $(m, p) \in \{(4, 3), (8, 5), (16, 9)\}$, according to [CL, Lemma 1, p. 151], and $G : A \times A \to A \times \mathbb{R}$ is given by $G(x, y) = (2x\bar{y}, |y|^2 - |x|^2)$, where A denotes the complex numbers, the quaternions, or the Cayley numbers.

[5]It is also known as the polar curve.

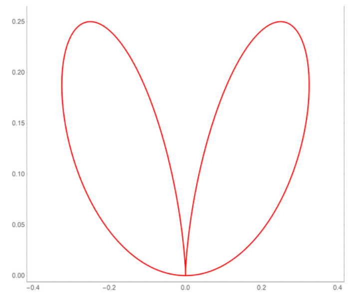

FIGURE 2. Curve of tangencies between the fibers of $G/\|G\|$ and spheres centered at the origin, for $G(x, y) = (x, x^2 + y(x^2 + y^2))$

REMARK 2.4. The phenomenon described above in the Milnor example can be reproduced in higher dimensions using the isolated singularity map $G : (\mathbb{R}^{m+2}, 0) \to (\mathbb{R}^2, 0)$ given by

$$G(x, y, z_1, \ldots, z_m) = (x, x^2 + y(x^2 + y^2 + z_1^2 + \cdots + z_m^2)).$$

2.3. **Non-isolated singular case: tube fibration.** Both fibrations, the Milnor tube fibration and the sphere fibration, in the real case were extended later for non-isolated singular map germs under the assumption that the discriminant set is 0-dimensional . In order to state properly these results we need to provide new definitions and notations.

Let us consider $U \subset \mathbb{R}^m$ an open subset such that $0 \in U$ and let $\rho : U \to \mathbb{R}_{\geq 0}$ be a non-negative proper function which defines the origin.

Definition 2.5. Let $G : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$ be an analytic map germ. We denote by

$$M_\rho(G) := \{x \in U \mid \rho \not\pitchfork_x G\}$$

the set of *ρ-nonregular points* of G, sometimes also called the *Milnor set* of G.

The transversality of the fibers of a map G to the levels of ρ is called *ρ-regularity* and we will see below that it is a condition for the existence of a locally trivial smooth fibration. It was used in the local (stratified) setting by Thom, Milnor, Mather, Looijenga, Bekka, e.g. [Be, Lo1, Mi, Th1, Th2] and more recently in [ACT1, AT1, AT2], and [CSS1, CSS3] under a different name *d-regularity*, as well as at infinity in the references [ACT2, DRT, NZ, Ti1, Ti2].

It follows from Definition 2.5 that the Milnor set $M_\rho(G)$ is the set of points $x \in U$ such that the vectors $\{\nabla\rho(x), \nabla G_1(x), \ldots, \nabla G_p(x)\}$ are linearly dependent over $\mathbb{R}$, i.e., $M_\rho(G)$ is the singular locus $\mathrm{Sing}\,(G, \rho)$ of the pair of map $(G, \rho) : U \to \mathbb{R}^p \times \mathbb{R}$. Hence, the singular set $\mathrm{Sing}\, G$ is included in $M_\rho\,(G)$.

For the sake of simplicity, in what follows ρ is the square of the Euclidean distance function $\rho(x) = \|x\|^2$, and we write $M(G) := M_\rho(G)$ for short. However, all results carry out easily over any other function ρ as considered above.

Consider the following condition:

$$\overline{M(G) \setminus V_G} \cap V_G \subseteq \{0\} \tag{8}$$

where the closure of the set $\overline{M(G) \setminus V_G}$ is thought as a germ of a set at the origin. See Figure 3 for an example.

Condition (8) was used in [ACT1, AT1, AT2], where it was shown that it insures the existence of the Milnor tube fibration. More recently, this condition was adapted by the authors in [ART1] and used in a stratified sense to ensure the existence of a singular Milnor tube fibration (see

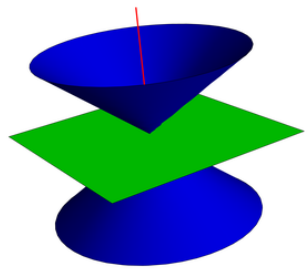

FIGURE 3. From Example 2.8, $M(G)$ is the cone and the plane, while V_G is the plane and the line. Hence G satisfies Condition (8).

Section 5.1 below). Note that this condition is equivalent to saying that for all small enough $\varepsilon > 0$ and $0 < \eta \ll \varepsilon$, the map:

$$G_| : S_\varepsilon^{m-1} \cap G^{-1}(\overline{B}_\eta^p \setminus \{0\}) \to \overline{B}_\eta^p \setminus \{0\}$$

is a locally trivial smooth fibration.

In [Ma] D. Massey considered Condition (8) but with different notation and called it the *Milnor condition (b)*. Massey used the condition to prove the existence of the Milnor tube fibration in the local setting, as in Theorem 2.6 below. Here we shall use the same notation of [ACT1] and [ART1].

Theorem 2.6. [Ma, Existence of the (full) Milnor's tube fibration] *Let $G : U \to \mathbb{R}^p$ be as above and assume that it has isolated critical value at origin, i.e.* $\operatorname{Disc} G = \{0\}$, *and satisfies Condition* (8)*. Then there exists $\varepsilon_0 > 0$ such that, for each ε, $0 < \varepsilon \le \varepsilon_0$, there exists η, $0 < \eta \ll \varepsilon$, such that*

$$G_| : \overline{B}_\varepsilon^m \cap G^{-1}(\overline{B}_\eta^p \setminus \{0\}) \to \overline{B}_\eta^p \setminus \{0\} \tag{9}$$

is the projection of a locally trivial smooth fibration.

Corollary 2.7. [Ma, Existence of the tube fibration] *Given G with the conditions of Theorem 2.6, for any small enough $\varepsilon > 0$, there exists η, $0 < \eta \ll \varepsilon$, such that*

$$G_| : \overline{B}_\varepsilon^m \cap G^{-1}(S_\eta^{p-1}) \to S_\eta^{p-1}$$

is the projection of a locally trivial smooth fibration.

In this case we also denote $M_G = \overline{B}_\varepsilon^m \cap G^{-1}(S_\eta^{p-1})$ and also call it the Milnor tube.

EXAMPLE 2.8. Let $G : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$ given by $G(x, y, z) = (xy, xz)$. Consider $\mathbf{v} := (x, y, z)$. One has that

$$\mathrm{J}G(\mathbf{v}) = \begin{bmatrix} y & x & 0 \\ z & 0 & x \end{bmatrix}$$

and

$$\mathrm{J}G(\mathbf{v})[\mathrm{J}G(\mathbf{v})]^t = \begin{bmatrix} x^2 + y^2 & yz \\ yz & x^2 + z^2 \end{bmatrix}$$

where $\mathrm{J}G(\mathbf{v})$ and $[\mathrm{J}G(\mathbf{v})]^t$ denote the Jacobian matrix of G in $\mathbf{v}$ and its transpose, respectively. We know that $\operatorname{Sing} G = \{\det(\mathrm{J}G(\mathbf{v})[\mathrm{J}G(\mathbf{v})]^t) = 0\}$ thus $\operatorname{Sing} G = \{x = 0\}$. Since

$$V_G = \{x = 0\} \cup \{y = z = 0\}$$

one gets that $\operatorname{Disc} G = \{0\}$. Now to compute the Milnor set $M(G)$ let us consider the matrix

$$B(\mathbf{v}) := \begin{bmatrix} y & x & 0 \\ z & 0 & x \\ x & y & z \end{bmatrix}.$$

The Milnor set $M(G) = \{\mathbf{v} \in \mathbb{R}^3 \mid \det(B(\mathbf{v})) = 0\}$. Consequently,

$$M(G) = \{x = 0\} \cup \{x^2 - y^2 - z^2 = 0\},$$

and G satisfies Condition (8). Therefore, by Theorem 2.6, G has a Milnor tube fibration.

In Figure 4 below one can see that the Milnor tube M_G consists of two connected components. Compare with Figure 1.

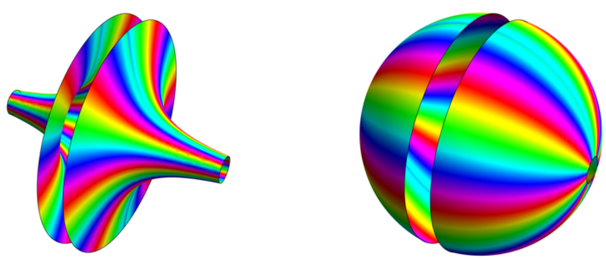

FIGURE 4. Milnor tube and Milnor sphere fibrations for $G(x, y, z) = (xy, xz)$.

2.4. **Existence of the Sphere fibration.** Several authors have worked on the problem of fibration over spheres in the real setting, for isolated and non-isolated singularities, e.g. [A1, ACT1, AT1, CSS1, CSS3, RA, RSV]. In [ACT1, AT1, AT2] the authors generalized all previous results as we describe below. In order to explain their main results, define the map $\Psi : \mathbb{R}^m \setminus V_G \to S^{p-1}$ through the diagram:

$$\begin{tikzcd} \mathbb{R}^m \setminus V_G \arrow[r, "G"] \arrow[dr, dashed, "\Psi"'] & \mathbb{R}^p \setminus \{0\} \arrow[d, "\pi_1"] \\ & S^{p-1} \end{tikzcd}$$

where π_1 is radial projection: $\pi_1(x) = x/\|x\|$. Given a neighborhood $U \in \mathbb{R}^m$ of 0, define the set of ρ-nonregular points of Ψ as the set

$$M(\Psi) = \{x \in U \setminus V_G \mid \rho \not\pitchfork_x \Psi\}.$$

Definition 2.9. The map germ Ψ *is ρ-regular* when $M(\Psi) = \emptyset$, as a germ of a set at the origin.

The set $M(\Psi)$ was characterized as follows.

Lemma 2.10. [AT1, AT2, ACT1, S] *Let $G := (G_1, \ldots, G_p) : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$ be an analytic map germ. Then on the open set $\{G_1(x) \neq 0\}$[6] one has that*

$$M(\Psi) = \left\{ x \in U \setminus V_G \,|\, rank \begin{bmatrix} \Omega_2(x) \\ \vdots \\ \Omega_p(x) \\ \nabla\rho(x) \end{bmatrix} < p \right\},$$

where $\Omega_k = G_1 \nabla G_k - G_k \nabla G_1$, for $k = 2, \ldots, p$.

Remark 2.11. We notice that for any $x \notin V_G$, if $\rho \pitchfork_x G$ then $\rho \pitchfork_x \Psi$. Hence, $M(\Psi) \subset M(G) \setminus V_G$.

Since the $\rho-$regularity is a measurement of transversality between the normal spaces of the fibers of ρ and Ψ, the set $M(\Psi)$ does not depend on the particular choice of the open set $\{G_1(x) \neq 0\}$. In general, for $G_i(x) \neq 0$, $1 \leq i \leq p$, one can find appropriate generators for the normal space of the fibers $X_y = \Psi^{-1}(y)$, $y = \Psi(x)$, considering the collection of vectors $\Omega_{i,k}(x) = G_i \nabla G_k(x) - G_k \nabla G_i(x)$, $k = 1, 2, 3, \ldots, \hat{i}, \ldots, p$, where $\hat{i}$ means that the index i is omitted. See [DACA, Lemma 3.3 and Remark 3.4] for more details.

It also follows from [AT1] that the condition $M(\Psi) = \emptyset$ is equivalent to saying that for small enough $\varepsilon > 0$, the projection $\Psi : S_\varepsilon^{m-1} \setminus K_\varepsilon \to S^{p-1}$ is a smooth submersion. However, since the map is not proper (unless the link is empty), it might not be a fibration.

In [ACT1] the authors used Condition (8) to ensure that the map Ψ is a projection of a locally trivial smooth fibration. In this setting where $\operatorname{Disc} G = \{0\}$ their result can be read as:

Theorem 2.12. [ACT1, Theorem 1.3] *Let $G : U \to \mathbb{R}^p$, $m > p \geq 2$ be an analytic map germ such that $codim\, V_G = p$. Suppose G satisfies Condition (8), i.e.,*

$$\overline{M\,(G) \setminus V_G} \cap V_G \subseteq \{0\}\,.$$

If Ψ is ρ-regular, then for any ε, $0 < \varepsilon \leq \varepsilon_0$, the map projection

$$\Psi : S_\varepsilon^{m-1} \setminus K_\varepsilon \to S^{p-1} \tag{10}$$

is a locally trivial smooth fibration, independent (up to isotopies) of small enough $\varepsilon > 0$.

Example 2.13 ([Han], p. 35). Let $G : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$, $G(x, y, z) = (x^2 + y^2, (x^2 + y^2)z)$. By hand calculations, one can see that $\operatorname{Sing} G = V_G = \{x = y = 0\}$, hence $\operatorname{Disc} G = \{0\}$. Moreover, by Lemma 2.10, $M(\Psi) = \emptyset$ and therefore Ψ is ρ-regular. Also, $M(G) = \mathbb{R}^3$,

$$\overline{M\,(G) \setminus V_G} \cap V_G = V_G \neq \{0\}$$

and Condition (8) fails. Therefore we cannot prove that Ψ is a locally trivial fibration. Indeed, the topological type of the fibers of Ψ changes along S^1; sometimes the fiber is a circle, sometimes the fiber is empty (see Figure 5). This shows that the hypothesis in Theorem 2.12 (or, Theorem 1.3 of [ACT1]) can not be weakened and therefore it is sharp!

Example 2.14 (Revising the sphere fibration for holomorphic functions). Let

$$f : \left(\mathbb{C}^{n+1}, 0\right) \to (\mathbb{C}, 0)$$

be a germ of a holomorphic function. We see that the hypothesis of Theorem 2.12 are naturally satisfied if we consider f as a real map germ from $\mathbb{R}^{2n+2}$ to $\mathbb{R}^2$. Indeed, it is well known that any holomorphic function satisfies the Łojasiewicz inequality

$$\|f(z)\|^\theta \leq c\|\nabla f(z)\|,$$

[6] Here, this set means $\{x \in U \setminus V_G \,|\, G_1(x) \neq 0\}$.

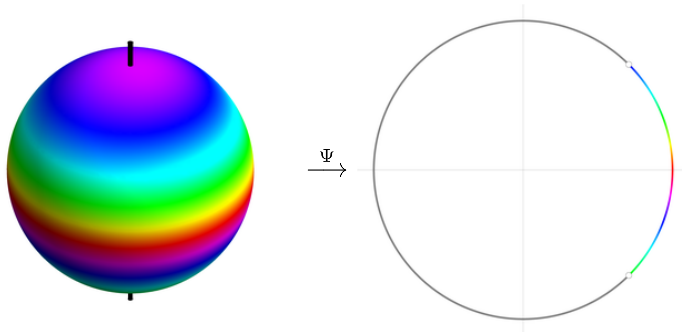

FIGURE 5. Ψ for $G(x, y, z) = ((x^2 + y^2), (x^2 + y^2)z)$. Colored points on S^1 have circles for fibers, while gray points have empty fibers.

where $0 < \theta < 1$, $c > 0$, and for any z in a small open neighborhood of the origin. So the isolated critical value condition is already satisfied. Moreover, Hamm and Lê in [HL, Theorem 1.2.1 p. 322] have proved that the Łojasiewicz inequality implies that f is Thom regular at V_f and hence f satisfies Condition (8). Finally, by [Mi, Lemma 4.3], one gets that for all $\varepsilon > 0$ small enough, $M(f/\|f\|) = \emptyset$, as a germ of a set. Therefore, from Theorem 2.12 the Milnor fibration on the sphere follows.

Let us point out some important facts.

In the paper [S1] published in 1997, the author used the method known as *Pencil* to construct examples of real analytic map germs with isolated singular point at the origin, which induces the so-called "Open book decomposition on the sphere" (see Definition 3.3), and hence the Milnor fibration on the sphere. Such construction was also used by the authors in [RSV]. In the paper [RA] published in 2005, the authors used this technique and tools from Stratification theory to ensure the existence of the Milnor fibration for real map germs $G : (\mathbb{R}^m, 0) \to (\mathbb{R}^2, 0)$ with $m > 2$. Inspired by [RA], in the paper [AT1] on arXiv (2008) and in the paper [AT2] published in 2010, the authors used the technique of blow-up to provide a generalization of the method for map germs $G : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$ with $m > p \geq 2$, and with that, they were able to prove two results which were generalized later in [ACT1].

In order to produce a new class of purely real examples, the authors in [ACT1] used the theory of mixed functions (see [Oka1, Oka2, Oka3] and Chapter 3 of [Ri] for definitions and properties), and proved Theorem 2.16 below. Before stating the theorem, let us consider the following definition.

Definition 2.15. [CT, CT1, CSS3, Oka2, Oka3, PT] A mixed polynomial function $f : \mathbb{C}^n \to \mathbb{C}$ is called *polar weighted-homogeneous* if there are non-zero integers $p_1, ..., p_n$ and d, such that $gcd(p_1, ..., p_n) = 1$ and

$$\sum_{j=1}^{n} p_j \left(\nu_j - \mu_j\right) = d$$

for any monomial of the expansion $f(\mathbf{z}, \bar{\mathbf{z}}) = \sum_{\nu,\mu} c_{\nu,\mu} \mathbf{z}^{\nu} \bar{\mathbf{z}}^{\mu}$. We call $(p_1, ..., p_n)$ the polar weight of f and d the polar degree of f. More precisely, f is polar weighted homogeneous of type $(p_1, ..., p_n; d)$ if and only if it satisfies the following equation for all $\lambda \in S^1$:

$$f(\lambda \cdot (\mathbf{z}, \bar{\mathbf{z}})) = \lambda^d f(\mathbf{z}, \bar{\mathbf{z}}),$$

where the corresponding S^1-action on $\mathbb{C}^n$ is:

$$\lambda \cdot (\mathbf{z}, \bar{\mathbf{z}}) = \left(\lambda^{p_1} z_1, ..., \lambda^{p_n} z_n, \lambda^{-p_1} \bar{z}_1, ..., \lambda^{-p_n} \bar{z}_n\right), \ \lambda \in S^1.$$

Theorem 2.16. [ACT1, Theorem 1.4] *Let $f : \mathbb{C}^n \to \mathbb{C}$ be a non-constant mixed polynomial which is polar weighted-homogeneous, $n \geq 2$, such that codim$_{\mathbb{R}} V_f = 2$. Then for any $\varepsilon > 0$ small enough, the projection*

$$f/\|f\| : S^{2n-1}_{\varepsilon} \setminus K_{\varepsilon} \to S^1$$

is a locally trivial smooth fibration, independent (up to isotopies) of small enough $\varepsilon > 0$.

Moreover, they proved the result below where now no control on the projection of the fibration is required outside a neighborhood of the link in the sphere.

Theorem 2.17. [ACT1, Theorem 2.1] *Let $G : U \to \mathbb{R}^p$, $m > p \geq 2$ be an analytic map such that codim $V_G = p$ and* $\operatorname{Disc} G = \{0\}$ *which satisfies Condition (8). Then there exists a locally trivial smooth fibration*

$$S^{m-1}_{\varepsilon} \setminus K_{\varepsilon} \to S^{p-1}$$

which is independent of small enough $\varepsilon > 0$, up to isotopies.

The control of the projection of the fibration is directly related to the ρ-regularity of the map Ψ, as has been seen in Theorem 2.12 and in the discussion that precedes it. This point is the main difference between Theorem 2.12 and Theorem 2.17 (for further details see [ACT1, Section 2]).

2.5. **Fibration on sphere under Thom regularity condition.** In the sequence of papers [CSS1, CSS3], the authors considered maps germs $G : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$, $m > p \geq 2$, with isolated critical value and satisfying a condition called *d-regularity* which, together with the Thom regularity, ensured the existence of the sphere fibrations. To do that, the authors associated to G *a pencil*, as we explain below. We follow the notations and the construction as described in the paper [CSS1], published in 2010.

For each $l \in \mathbb{RP}^{p-1}$ consider the line $\mathcal{L}_l \subset \mathbb{R}^p$ through the origin and set

$$X_l = \{x \in U | G(x) \in \mathcal{L}_l\}.$$

In particular, if we consider the commutative diagram

$$\begin{tikzcd}
\mathbb{R}^m \setminus V_G \arrow[r, "G"] \arrow[dr, "\Psi"] \arrow[ddr, dashed, "\Psi^*"'] & \mathbb{R}^p \setminus \{0\} \arrow[d, "\pi_1"] \\
& S^{p-1} \arrow[d, "\pi"] \\
& \mathbb{RP}^{p-1}
\end{tikzcd}$$

where π_1 is radial projection and π is the canonical double covering, then $X_l = (\Psi^*)^{-1}(l) \cup V_G$.

Each X_l is a real analytic variety that contains V_G, and since G has an isolated critical value, then each $X_l \setminus V_G$ is either empty or it is an $(m-p+1)$-dimensional smooth submanifold of U. The family $\left\{X_l : l \in \mathbb{RP}^{p-1}\right\}$ is called *the canonical pencil* of G.

Definition 2.18. [CSS1, Definition of d-regularity] The map G is said to be *d-regular* at 0 if there exist a metric d induced by some positive-definite quadratic form and an $\varepsilon > 0$ such that every sphere (for the metric d) of radius $\leq \varepsilon$ centered at 0 meets each $X_l \setminus V_G$ transversely, whenever the intersection is not empty. We shall also say that G is d-regular with respect to the metric d.

In order to study the existence of Milnor fibrations associated to a map G, the authors introduced an auxiliary function $\mathfrak{G} : B_\varepsilon^m \setminus V_G \to B_\varepsilon^p$ called the *Spherification map* of G. This function was defined by

$$\mathfrak{G}(x) = \|x\| \frac{G(x)}{\|G(x)\|}$$

and it was used to characterize the d-regularity as follows.

Proposition 2.19. [CSS1, Proposition 3.2] *Let $G : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$ be an analytic map germ with an isolated critical value at the origin. The following statements are equivalent:*

(i) The map G is d-regular at 0.

(ii) For each sphere S_ε^{m-1} of small enough radius $\varepsilon > 0$, the restriction map

$$\mathfrak{G} : S_\varepsilon^{m-1} \setminus V_G \to S_\varepsilon^{p-1}$$

is a submersion.

(iii) The spherification map $\mathfrak{G}$ is a submersion at each $x \in B_\varepsilon^m \setminus V_G$.

(iv) The map $\Psi_| : S_\varepsilon^{m-1} \setminus K_\varepsilon \to S^{p-1}$ is a submersion for any small enough sphere S_ε^{m-1}.

This proposition shows that when d is the square of the Euclidean metric, then d-regularity of G is equivalent to ρ-regularity of Ψ. The main result of [CSS1] is the following.

Theorem 2.20. [CSS1, Theorem 5.3] *Assume either V_G is a point or $\dim V_G > 0$ and G has the Thom regularity. The following statements are equivalent:*

(i) The map G is d-regular at 0.

(ii) One has a commutative diagram of smooth fiber bundles on $S_\varepsilon^{m-1} \setminus K_\varepsilon$ for any small enough sphere S_ε^{m-1}:

$$\begin{tikzcd} S_\varepsilon^{m-1} \setminus K_\varepsilon \arrow[rr, "\phi"] \arrow[dr, "\psi"'] & & S^{p-1} \arrow[dl, "\pi"] \\ & \mathbb{RP}^{p-1} & \end{tikzcd}$$

where $\psi := (G_1(x) : \cdots : G_p(x))$ and $\phi := G/\|G\| : S_\varepsilon^{m-1} \setminus K_\varepsilon \to S^{p-1}$ is the Milnor fibration on G.

(iii) For any small enough sphere S_ε^{m-1}, the restriction $\mathfrak{G} : S_\varepsilon^{m-1} \setminus V_G \to S_\varepsilon^{p-1}$ is a smooth fiber bundle and this is the Milnor fibration ϕ up to multiplication by a constant.

2.6. **Comparing the fibration structure on spheres under Thom regularity at V_G and Condition (8).** One can show that if a map germ G is Thom regular at V_G then G satisfies Condition (8). Example 2.21 below shows that the converse in not true in general. Therefore, Theorem 2.12 is more general than Theorem 2.20.

EXAMPLE 2.21. [Han, Example 1.4.9] Consider $G(x, y, z) = (x, y(x^2 + y^2) + xz^2)$ in three real variables. One has that $\operatorname{Sing} G = V_G = \{x = y = 0\}$ and $M(G) = \{x = y = 0\} \cup \{z = 0\}$. Hence, $\overline{M(G) \setminus V_G} \cap V_G = \{0\}$ and Condition (8) holds. We claim that $M(\Psi) = \emptyset$. Indeed, let $\mathbf{v} = (x, y, z) \in \mathbb{R}^3$ and consider the matrix

$$B(\mathbf{v}) := \begin{bmatrix} \Omega_2(\mathbf{v}) \\ \mathbf{v} \end{bmatrix},$$

where

$$\Omega_2(\mathbf{v}) = (x(2xy + z^2) - y(x^2 + y^2) - xz^2, x(x^2 + 3y^2), 2x^2 z).$$

By Lemma 2.10,

$$M(\Psi) = \{\mathbf{v} \in B^3_\varepsilon \setminus V_G \mid \det\left(B(\mathbf{v})[B(\mathbf{v})]^t\right) = 0\}.$$

Since

$$\det\left(B(\mathbf{v})[B(\mathbf{v})]^t\right) = (x^2 + y^2)(x^6 + 3x^4y^2 + 5x^4z^2 - 8x^3yz^2 + 3x^2y^4 + 6x^2y^2z^2 + y^6 + y^4z^2)$$

and $M(\Psi) \subset M(G) \setminus V_G$, then $M(\Psi) = \emptyset$. By Theorem 2.12, we get the sphere fibration $\Psi : S^{m-1}_\varepsilon \setminus K_\varepsilon \to S^{p-1}$.

On the other hand, for any value $z \neq 0$, consider the point $p = (0, 0, z)$, $\mathrm{T}_p V_G = \operatorname{span}\{(0, 0, 1)\}$, and the sequence $p_n = (\frac{1}{n}, 0, z)$ which converges to p. One has that $\mathrm{T}_{p_n} G^{-1}(G(p_n)) = \operatorname{span}\{v_n\}$, where

$$v_n = \left(0, \frac{-2z}{\sqrt{4z^2 + \frac{1}{n^2}}}, \frac{1}{\sqrt{4z^2n^2 + 1}}\right);$$

hence $v_n \to (0, \pm 1, 0)$, where plus and minus depends on the sign of z. Therefore,

$$\lim_n (T_{p_n} G^{-1}(G(p_n))) = \operatorname{span}\{(0, 1, 0)\}$$

and G is not Thom regular at V_G.

Remark 2.22. Another source of examples of maps with Milnor tube and sphere fibration without the Thom regularity can be found in the recent paper [Ri2].

3. Open Book Structures on semialgebraic sets

The *classical open book structures* with smooth binding appear in the literature relative to 3-manifolds and in different branches of mathematics under many names like *Lefschetz pencils* (Algebraic and Symplectic Geometry), *fibered links*, *Neuwirth-Stallings pairs*, or *spinnable structures* (Topology).

As explained by the authors in [AT1], this consists of a pair (K, θ) where $K \subset M$ is a 2-codimensional submanifold of a real manifold M and $\theta : M \setminus K \to S^1$ with $S^1 := \partial B^2$, is a locally trivial smooth fibration such that K admits a neighborhood N diffeomorphic to $B^2 \times K$ for which K is identified with $\{0\} \times K$ and the restriction $\theta_{|N\setminus K}$ is the following composition with the natural projections:

$$N \setminus K \overset{\text{diffeo}}{\simeq} (B^2 \setminus \{0\}) \times K \overset{\text{proj}}{\to} B^2 \setminus \{0\} \overset{s/\|s\|}{\to} S^1. \tag{11}$$

In that case, K is the *binding* and the closure of the fibers of θ are the pages of the open book.

As described in the introduction, an important example of classical open book structure on a small sphere S^{2n-1}_ε can be obtained if we consider a germ of a holomorphic function $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$, under the condition that $\operatorname{Sing} f = \{0\}$.

Milnor noted that if $G : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$, $m \geq p \geq 2$, has an isolated critical point at $0 \in \mathbb{R}^m$, then for any small enough $\varepsilon > 0$, the complement $S^{m-1}_\varepsilon \setminus K_\varepsilon$ of the link K_ε is the total space of a smooth fiber bundle over the unit sphere S^{p-1}. In such a case, one can conclude from Milnor's comment that the sphere S^{m-1}_ε is endowed with an open book structure with binding K_ε, where now the binding is of *higher codimension* $p \geq 2$ instead of 2.

These structures were extended later, as follows:

Definition 3.1. [AT2, Definition 2.1] A *higher open book structure* of a real manifold M is a pair (K, θ), where K is a p-codimensional non-empty submanifold of M and $\theta : M \setminus K \to S^{p-1}$ is a locally trivial smooth fibration over the sphere $S^{p-1} = \partial B^p$, such that K admits a neighborhood N diffeomorphic to $B^p \times K$ for which K is identified to $\{0\} \times K$ and the restriction $\theta_{|N\setminus K}$ is the composition

$$N \setminus K \overset{\text{diffeo}}{\simeq} (B^p \setminus \{0\}) \times K \overset{\text{proj}}{\to} B^p \setminus \{0\} \overset{s/\|s\|}{\to} S^{p-1}.$$

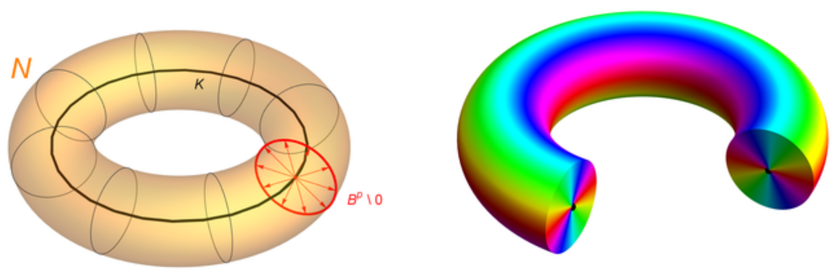

FIGURE 6. Left: an example of N and K from Definition 3.1. Right: a cross section of the corresponding open book structure.

REMARK 3.2. In this case E. Looijenga in [Lo1] called this structure a Neuwirth-Stallings pair, or NS-pair, and denoted them by $(S_\varepsilon^{m-1}, K_\varepsilon)$.

In [AT1], the authors presented a general criterion for the existence of these structures associated to a real map germ G with isolated critical point at $0 \in \mathbb{R}^m$ and with $\theta = G/\|G\|$ (see [AT1, Theorem 1.1]). In [AT2], they focused on the existence of higher open book structures defined by map germs which satisfies the condition $\operatorname{Sing} G \cap V_G \subset \{0\}$, which is the most general one under which open book structures with non-singular binding K may exist. Finally, in [ACT1], the authors introduced the notion of *singular open book structure* as follows.

Definition 3.3. [ACT1, Definition 1.1]. The pair (K, θ) is a *higher open book structure with singular binding* on an analytic manifold M of dimension $m - 1 \geq p \geq 2$, if $K \subset M$ is a singular real subvariety of codimension p and $\theta : M \setminus K \to S^{p-1}$ is a locally trivial smooth fibration such that K admits a neighborhood N for which the restriction $\theta_{|N\setminus K}$ is the composition $N \setminus K \overset{h}{\to} B^p \setminus \{0\} \overset{s/\|s\|}{\to} S^{p-1}$, where h is a locally trivial fibration.

They investigated the case when V_G contains non-isolated singularities and thus the link K_ε is not a manifold. Under the hypothesis of Theorem 2.12, they ensured the pair (K_ε, Ψ) is an open book structure with singular binding on S_ε^{m-1} having extended all previous results related to the existence of open book structures of [AT1] and [AT2]. In addition, they found important classes of genuine real analytic mappings which yield such structures (see for instance Theorem 2.16).

REMARK 3.4. Based on the results obtained in [ACT1], the authors in [ACT2] considered polynomial maps $G : \mathbb{R}^m \to \mathbb{R}^p$, $m \geq p \geq 1$. Under certain adapted conditions defined in terms of the Milnor sets $M(G)$ and $M(\Psi)$, they ensured the existence of an *open book decomposition at infinity with singular binding* (i.e., on spheres of large enough radius R).

Motivated by recent techniques developed in [ACT1, AT1, AT2] and [ACT2], the authors in [DACA] guaranteed the existence of a fibration structure associated to a more general class of maps and sets. Actually, they have considered C^2-semi-algebraic maps $G : \mathbb{R}^m \to \mathbb{R}^p$ and embedded compact semi-algebraic manifolds without boundary $W \subset \mathbb{R}^m$ of dimension $n-1 \geq p$. In this new setting, they introduced sufficient conditions in order to ensure the existence of an open book structure on W and, as a consequence, extended both previous open book structures on local and global cases. For that, the first step was to consider an appropriate extension of the Milnor set as below.

Definition 3.5. [DACA]
Let $G : \mathbb{R}^m \to \mathbb{R}^p$ be a C^2-semi-algebraic map, $W \subset \mathbb{R}^m$ a compact semi-algebraic $(n-1)$-dimensional submanifold embedded in $\mathbb{R}^m$ and

$$\bar{G} := \frac{G}{\|G\|} : \mathbb{R}^m \setminus V_G \to S^{p-1}.$$

Consider $\bar{G}_{|W} : W \setminus V_W(G) \to S^{p-1}$ where $V_W(G) = V_G \cap W$, and

(i) Σ_G the set of critical points of G;

(ii) $\Sigma_{\bar{G}}$ the set of critical points of $\bar{G}$;

(iii) Σ_G^W the set of critical points of $G_{|W}$;

(iv) $\Sigma_{\bar{G}}^W$ the set of critical points of $\bar{G}_{|W}$.

The map G satisfies the *generalized Milnor condition* (b) whenever $\overline{\Sigma_{\bar{G}}^W \setminus V_W(G)} \cap V_W(G) = \emptyset$. Moreover, G satisfies the *generalized Milnor condition* (a) when $\Sigma_{\bar{G}}^W = \emptyset$.

With the notations above, the authors in [DACA] stated and proved the following result.

Theorem 3.6 (Structural Theorem). *Let $G : \mathbb{R}^m \to \mathbb{R}^p$ be a C^2-semi-algebraic map such that G satisfies the generalized Milnor condition (a). Then the following statements are equivalent:*

(i) $\bar{G}_{|W}$ is a locally trivial smooth fibration induced by G on W;

(ii) The map G satisfies the generalized Milnor condition (b).

Let us point out that the proof of Theorem 3.6 follows similar arguments used in [ACT1, ACT2, AT2], and consequently also guarantee the existence of an open book structure on W. The Structural Theorem generalizes the analogues for local and global cases.

In addition, considering the canonical projection $\pi_j : \mathbb{R}^p \to \mathbb{R}^{p-1}$ for $p \geq 2$, and

$$\pi_j(x_1, \dots, x_p) = (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_p),$$

where $j = 1, \dots, p$, the authors also have shown that the composition $\hat{G}_j := \pi_j \circ G : \mathbb{R}^m \to \mathbb{R}^{p-1}$ provides a new open book structures for W, (see [DACA, Lemma 3.5]). Moreover, the fibers of new and old structure are related as follows: if F_G and $F_{\hat{G}_j}$ are the fibers of locally trivial smooth fibrations induced by G and $\hat{G}_j$ on W, respectively, then $F_{\hat{G}_j}$ is homotopically equivalent to the product $F_G \times [0, 1]$. This ensures that one can, without loss of generality, reduce the study of the topology of the fibers of a C^2-semi-algebraic map $G = (G_1, \dots, G_p) : \mathbb{R}^m \to \mathbb{R}^p$ satisfying generalized Milnor conditions to the study of the singularity type of $G_i, i = 1, \dots, p$, i.e., any coordinate function.

4. Positive dimensional discriminant set

Let

$$G : U \subset \mathbb{R}^m \to \mathbb{R}^p, \quad m > p \geq 2,$$

be a representative of a map germ $G : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$ with positive dimensional discriminant set $\operatorname{Disc} G$. Consider a Whitney stratification $\mathbb{W} = \{\mathcal{C}_j\}_{j=1}^r$ of $\operatorname{Disc} G$ with the origin a single stratum. Let us assume that the complement $\mathbb{R}^p \setminus \operatorname{Disc} G$ is equal to union $\cup_{i=1}^k \mathcal{D}_i$, where on each connected component $\mathcal{D}_i$ the topology of the fibers of G does not change.

Let us consider the following situation: for $i \neq j$ such that $\mathcal{C}_k \subset \overline{\mathcal{D}_i} \cap \overline{\mathcal{D}_j} \setminus \{0\}$, let $p_i \in \mathcal{D}_i$ and $p_j \in \mathcal{D}_j$ and let $l_{i,j}$ be a path connecting them, with $l_{i,j}$ intersecting $\mathcal{C}_k$ once and is in general position[7] (see Figure 7).

The problem is: How do we describe the topological changes of the topology of the fibers over p_i and over p_j as we travel along $l_{i,j}$?

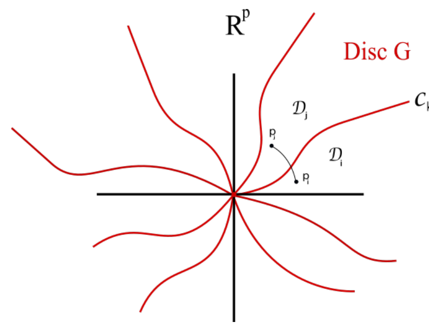

Figure 7. Positive dimensional discriminant set and the complementary set $\mathbb{R}^p \setminus \operatorname{Disc} G$.

Maybe this problem is too hard to approach as it is stated. However, it motivates one to think of a natural way to extend the Milnor fibrations for map germs with positive dimensional discriminant sets as done by H. Hamm in [Ham] (see Theorem 1.3).

As explained in detail in [ART1] and [ART2], in this new setting the following problems have to be taken into account so that the fibration problem can be well posed:

a) The local fibration must be independent of the small enough neighborhood data, like in Equations (1) and (5). This does not come automatically for map germs with positive dimensional discriminant set outside the ICIS case (see Examples 4.2 and 2.13).
b) The image of the map germ G may not be a neighborhood of $\{0\}$ in $\mathbb{R}^p$ (see Example 5.9). Moreover, it may not be independent of the radius ε of the ball $B_\varepsilon^m \subset \mathbb{R}^m$, and thus the image of G may not be well defined as a set germ in $(\mathbb{R}^p, 0)$ (see Examples 4.2 and 2.13).
c) The set $G(\operatorname{Sing} G)$ may not be well defined as a set germ. In case the image $G(\operatorname{Sing} G)$ of the singular locus is a set germ, and when the image $\operatorname{Im} G$ is a set germ too and has a

[7] It means that the tangent vector of $l_{i,j}$ at the point of intersection is not contained in the tangent space of the stratum $\mathcal{C}_k$

boundary[8] which contains the origin $\{0\}$, then in this new setting it seems appropriate that the "discriminant set" $\operatorname{Disc} G$ should contain this boundary (see Definition 4.7).

Recall that, given subsets $V, W \subset \mathbb{R}^p$ containing the origin and denoting $(V, 0)$ and $(W, 0)$ their respective germs at $\{0\}$, then one has $(V, 0) = (W, 0)$ as a germ of a set if and only if there exists some open ball $B_\varepsilon \subset \mathbb{R}^p$ centered at 0 and of radius $\varepsilon > 0$ such that $V \cap B_\varepsilon = W \cap B_\varepsilon$.

Definition 4.1. [ART1] Let $G : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$, $m \geq p > 0$, be a continuous map germ. We say that the image $G(K)$ of a set $K \subset \mathbb{R}^m$ containing 0 is a *well-defined set germ* at $0 \in \mathbb{R}^p$ if, for any open balls $B_\varepsilon, B_{\varepsilon'}$ centered at 0, with $\varepsilon, \varepsilon' > 0$, we have the equality of germs $[G(B_\varepsilon \cap K)]_0 = [G(B_{\varepsilon'} \cap K)]_0$.

Whenever the images $\operatorname{Im} G$ and $G(\operatorname{Sing} G)$ are well-defined as germs, we say that G is a *nice map germ*.

Example 4.2. [ART1, Example 2.1] Let $G : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$, $G(x, z) = (x, xz)$. For the 2-disks

$$D_t := \{|x| < t, |z| < t\}$$

as a basis of open neighborhoods of 0 for $t > 0$, we get that the image $A_t := G(D_t)$ is the full angle with vertex at 0, having the horizontal axis as bisector, and of slope $< t$. Since the relations defining A_t depend of t, it means that the image of G is not well-defined as a germ (see Figure 8). A similar behavior happens over $\mathbb{C}$ instead of $\mathbb{R}$.

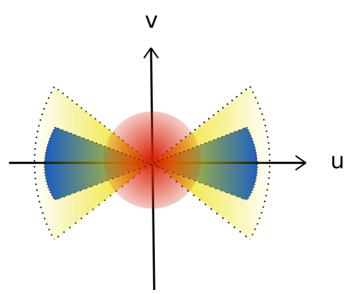

Figure 8. Images A_{t_1} and A_{t_2} with $t_1 \neq t_2$ in the yellow and blue color, respectively.

Remark 4.3. The authors in [ART1] point out that even if the image $\operatorname{Im} G$ of a map G is well-defined as a germ, the restriction of G to some subset might not be (see [ART1, Remark 2.3]). Therefore, in the definition of a nice map germ, it is necessary to ask that the set $G(\operatorname{Sing} G)$ is well-defined as a germ as well.

Example 4.4. Given $G : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$, $m \geq p \geq 2$ with $\operatorname{Disc} G = \{0\}$. If Condition (8) holds true, then G is a nice map germ (see [Ma, Corollary 4.7]). In particular, any non-constant germ of a holomorphic function is nice.

Remark 4.5. One can do similar calculations as in Example 4.2 on the map germ

$$G : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0), \quad G(x, y, z) = (x^2 + y^2, (x^2 + y^2)z)$$

[8][ART1]: Whenever $\operatorname{Im} G$ is well-defined as a set germ, its boundary $\partial \overline{\operatorname{Im} G} := \overline{\operatorname{Im} G} \setminus \operatorname{int}(\operatorname{Im} G)$ is a closed subanalytic proper subset of $\mathbb{R}^p$, where $\operatorname{int} A := \mathring{A}$ denotes the p-dimensional interior of a subanalytic set $A \subset \mathbb{R}^p$ (hence it is empty whenever $\dim A < p$), and $\overline{A}$ denotes the closure of it. One considers here $\partial \overline{\operatorname{Im} G}$ as a set germ at $0 \in \mathbb{R}^p$; this is of course empty if (and only if) the equality $(\operatorname{Im} G, 0) = (\mathbb{R}^p, 0)$ holds.

(Example 2.13), and find that $\operatorname{Im} G$ is not well-defined as a set germ, and thus G is not nice. Note that while $\operatorname{Disc} G = \{0\}$, Condition (8) is not satisfied, so we cannot conclude that G is nice (like we could in Example 4.4).

EXAMPLE 4.6. In [ART1] the authors found sufficient conditions for an analytic map germ with positive dimensional discriminant set to be a nice germ and have introduced a good class of maps with this property, namely the map germs of type

$$f\bar{g} : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0),$$

where $f, g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ are holomorphic germs such that the meromorphic function f/g is irreducible.

The authors in [ART1] gave an appropriate definition of the discriminant set as the locus where the topology of the fibers may change.

Definition 4.7. For a nice map germ G, the *discriminant* is the following set

$$\operatorname{Disc}^* G := \overline{G(\operatorname{Sing} G)} \cup \partial \overline{\operatorname{Im} G} \tag{12}$$

which is a closed subanalytic set of dimension strictly less than p, well-defined as a germ since G is nice.

Usually the discriminant set $\operatorname{Disc} G$ is just $G(\operatorname{Sing} G)$. However, in this new setting where $\dim \operatorname{Disc} G > 0$, the complement of the discriminant set may consist of several connected components through the origin (see Figure 7), and hence the base space of the fibration may not be a connected space and the topological type of the fibers may not be unique. Consequently, the classical definition of discriminant is not sufficient to detect the change of the topological type of the fibers. We also note that when $\operatorname{Disc} G = \{0\}$ (like in the previous sections) and G satisfies Condition (8), then $\operatorname{Disc}^* G = \operatorname{Disc} G$.

5. Singular Milnor tube fibration

Definition 5.1. Let $G : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$, $m \geq p > 0$, be a non-constant analytic nice map germ. We say that G has a *Milnor-Hamm* (*tube*) *fibration* if, for any $\varepsilon > 0$ small enough, there exists $0 < \eta \ll \varepsilon$ such that the restriction:

$$G_| : B^m_\varepsilon \cap G^{-1}(B^p_\eta \setminus \operatorname{Disc}^* G) \to B^p_\eta \setminus \operatorname{Disc}^* G \tag{13}$$

is a locally trivial fibration over each connected component $\mathcal{C}_i$ included in $B^p_\eta \setminus \operatorname{Disc}^* G$, such that it is independent of the choices of ε and η up to diffeomorphisms.

In order to guarantee the existence of fibration (13), the authors in [ART1] considered the following condition

$$\overline{M(G) \setminus G^{-1}(\operatorname{Disc}^* G)} \cap V_G \subseteq \{0\} \tag{14}$$

where the closure of the analytic set $M(G) \setminus G^{-1}(\operatorname{Disc}^* G)$ is considered as a set germ at the origin. Condition (14) is a direct extension of Condition (8). Therefore, the next result is a natural extension of Theorem 2.6 for the case where $\dim \operatorname{Disc}^* G > 0$.

Theorem 5.2. [ART1, Lemma 3.3] *Let $G : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$ be a non-constant nice analytic map germ, $m \geq p > 0$. If G satisfies Condition (14), then G has a Milnor-Hamm (tube) fibration (13).*

A similar type of fibration but with the stronger assumptions of Thom regularity have been studied in [CGS]. In the article, the authors considered a real analytic map germ $G : (U, 0) \to (\mathbb{R}^p, 0)$, where $U \subset \mathbb{R}^m$ is an open set, $m > p \geq 2$, G has a critical point at 0, and V_G has dimension ≥ 2. They considered a fixed closed ball $\bar{B}^m_\varepsilon$ as a stratified set with strata the interior B^m_ε and the boundary $S^{m-1}_\varepsilon = \partial \bar{B}^m_\varepsilon$, the restriction map $G_| : \bar{B}^m_\varepsilon \to \mathbb{R}^p$ and its discriminant set as $\Delta^\varepsilon_G := G(\mathcal{C}(B^m_\varepsilon) \cup \mathcal{C}(S^{m-1}_\varepsilon))$, where $\mathcal{C}(B^m_\varepsilon)$ and $\mathcal{C}(S^{m-1}_\varepsilon)$ stand for the set of critical points of G on the open ball and on the sphere, respectively. With these notations, they used the Thom Isotopy Theorem to get that the map

$$G_| : \bar{B}^m_\varepsilon \cap G^{-1}(\mathbb{R}^p \setminus \Delta^\varepsilon_G) \to \mathbb{R}^p \setminus \Delta^\varepsilon_G$$

is a locally trivial fibration (see [CGS, Proposition 2.1]). As a consequence for each fixed $\varepsilon > 0$ and $\eta > 0$ they obtained the following locally trivial fibration [CGS, Corollary 2.2]:

$$G_| : \bar{B}^m_\varepsilon \cap G^{-1}(B^p_\eta \setminus \Delta^\varepsilon_G) \to B^p_\eta \setminus \Delta^\varepsilon_G. \tag{15}$$

In order to ensure that the fibration (15) does not depend on $\varepsilon > 0$, they considered Whitney stratifications $\mathbb{W}$ and $\mathbb{S}$ of U and $G(U)$, respectively, such that V_G is a union of strata and both stratifications give the stratification of G. They further assume that G satisfies the Thom a_f-property with respect to such stratification of G i.e., $(\mathbb{W}, \mathbb{S}, G)$ is a Thom stratified mapping (see [CGS, Proposition 2.4]).

Since the Thom a_f-property implies Condition (14), the examples below show that [CGS, Proposition 2.4] under the nice condition is a particular case of Theorem 5.2.

Example 5.3. [ART1, Example 5.3] Let F be one of the mixed functions:

1) $F_1(x, y) = xy\bar{x}$ from [ACT1],
2) $F_2(x, y, z) = (x + z^k)\bar{x}y$ for a fixed $k \geq 2$ from [PT],
3) $F_3(w_1, \ldots, w_n) = w_1 \left(\sum_{j=1}^{k} |w_j|^{2a_j} - \sum_{t=k+1}^{n} |w_t|^{2a_t}\right)$ from [Oka4].

They are all polar weighted-homogeneous and thus, by [ACT1, Theorem 1.4], one obtains that $\operatorname{Disc}^* F_j = \{0\}$ and that F_j is nice and has Milnor tube fibration. It was also proved in the respective papers that F_j is not Thom regular.

Let $G_j := (F_j, g)$, where $g(v) = v$ and note that $\operatorname{Disc}^* G_j = \{0\} \times \mathbb{C}$. By [ART1, Lemma 5.1] the map G_j satisfies Condition (14) and therefore, by Theorem 5.2, G_j has a Milnor-Hamm (tube) fibration. However, again by [ART1, Lemma 5.1] G_j is not a Thom stratified mapping.

Summing up, the authors in [ART1] have shown that the Thom regularity of the map G may fail whereas the Milnor-Hamm (tube) fibration still exists. Moreover, they present several classes of map germs with Milnor-Hamm fibration by introducing a weaker type of Thom regularity condition called *∂-Thom regularity condition*.

Remark 5.4. In article [MS], the authors defined a type of tube fibration in a more general setting and presented a necessary and sufficient condition on the fibers of coordinate functions to ensure its existence [MS, Proposition 2.5]. However, since their main objective was to study the topology of real analytic map germs with isolated critical value, i.e., $\operatorname{Disc} G = \{0\}$, they did not present examples in the more general case.

5.1. **Singular Milnor tube fibration.** In [ART1] the authors have defined a general notion of *stratified tube fibration* by considering all singular fibers over the stratified discriminant, and they have shown that such structure is a natural generalization of Milnor-Hamm fibration. In that case, the tube fibration is actually a collection of finitely many fibrations over path-connected subanalytic sets. In order to make this notion more precise, they made use of the classical stratification theory (see e.g. [GLPW]), and they considered the following definitions.

Definition 5.5. [ART1] Let $G : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$ be a non-constant analytic map germ, $m \geq p > 1$. Let $G_\varepsilon : B^m_\varepsilon \to \operatorname{Im} G_\varepsilon$ denote the restriction of G to a small ball. Consider a locally finite subanalytic Whitney stratifications $(\mathbb{W}, \mathbb{S})$ of the source of G_ε and of its target, respectively, such that $\overline{\operatorname{Im} G_\varepsilon}$ is a union of strata, that $\operatorname{Disc}^* G_\varepsilon$ is a union of strata, and that G_ε is a stratified submersion. In particular every stratum is a non-singular, open and connected subanalytic set at the respective origin, and moreover:

(i) The image by G_ε of a stratum of $\mathbb{W}$ is a single stratum of $\mathbb{S}$,

(ii) The restriction $G_| : W_\alpha \to S_\beta$ is a submersion, where $W_\alpha \in \mathbb{W}$, and $S_\beta \in \mathbb{S}$.

One calls $(\mathbb{W}, \mathbb{S})$ a *regular stratification of the map germ G.*

We say that G is *S-nice* whenever all the above subsets of the target are well-defined as subanalytic germs, independent of the radius ε.

Definition 5.6. [ART1] Let $G : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$ be a non-constant S-nice analytic map germ. We say that G has a *singular Milnor tube fibration* relative to some regular stratification $(\mathbb{W}, \mathbb{S})$, which is well-defined as a germ at the origin by our assumption, if for any small enough $\varepsilon > 0$ there exists $0 < \eta \ll \varepsilon$ such that the restriction:

$$G_| : B^m_\varepsilon \cap G^{-1}(B^p_\eta \setminus \{0\}) \to B^p_\eta \setminus \{0\} \tag{16}$$

is a stratified locally trivial fibration which is independent, up to stratified homeomorphisms, of the choices of ε and η.

The authors clarified the notion of stratified fibration by saying that *stratified locally trivial fibration* meant that for any stratum S_β, the restriction $G_{|G^{-1}(S_\beta)}$ is a locally trivial fibration.

In order to ensure the existence of stratified fibration (16), they defined the *stratwise Milnor set* $M(G)$ with respect to the stratifications $\mathbb{W}$ and $\mathbb{S}$, as the union of the Milnor sets of the restrictions of G to each stratum. Namely, $M(G) := \sqcup_\alpha M(G_{|W_\alpha})$, where

$$M(G_{|W_\alpha}) := \left\{ x \in W_\alpha \mid \rho_{|W_\alpha} \not\pitchfork_x G_{|W_\alpha} \right\},$$

with $W_\alpha \in \mathbb{W}$ the germ at the origin of some stratum, and $\rho_{|W_\alpha}$ the restriction of the distance function ρ to the subset W_α (see [ART1, Definition 6.4]). They then considered the following condition:

$$\overline{M(G) \setminus V_G} \cap V_G \subset \{0\}. \tag{17}$$

which restricted to $M(G) \setminus G^{-1}(\operatorname{Disc}^* G)$ is just Condition (14). Finally, with the notations and definitions above, the main result in this new setting is the following:

Theorem 5.7. [ART1] *Let $G : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$ be a non-constant S-nice analytic map germ. If G satisfies Condition (17), then G has a singular Milnor tube fibration (16).*

The corollary below says that the singular Milnor tube fibration (16) generalizes the previous Milnor-Hamm fibration.

Corollary 5.8. [ART1] *Under the hypotheses of Theorem 5.7, the map G has a Milnor-Hamm fibration over $B^p_\eta \setminus \operatorname{Disc}^* G$, with nonsingular Milnor fiber over each connected component.*

Example 5.9. [ART1] Let $G : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$, $G(x, y, z) = (xy, z^2)$. One has:

$V_G = \{x = z = 0\} \cup \{y = z = 0\}$	$\operatorname{Im} G = \mathbb{R} \times \mathbb{R}_{\geq 0} \subsetneq \mathbb{R}^2$
$\operatorname{Sing} G = \{x = y = 0\} \cup \{z = 0\}$	$G(\operatorname{Sing} G) = \{0\} \times \mathbb{R}_{\geq 0} \cup \mathbb{R} \times \{0\}$
$\operatorname{Disc}^* G = \{(0, \beta) \mid \beta \geq 0\} \cup \{(\lambda, 0) \mid \lambda \in \mathbb{R}\}$	$G^{-1}(\operatorname{Disc}^* G) = \{x = 0\} \cup \{y = 0\} \cup \{z = 0\}$
$M(G) = \{x = \pm y\} \cup \{z = 0\}$	$\overline{M(G) \setminus G^{-1}(\operatorname{Disc}^* G)} = \{x = \pm y\}.$

It follows that G is nice and satisfies Condition (14). Indeed to check this, consider

$$p_0 = (x_0, y_0, z_0) \in \overline{M(G) \setminus G^{-1}(\mathrm{Disc}^* G)} \cap V_G.$$

Hence, there exists a sequence $p_n := (x_n, y_n, z_n) \in M(G) \setminus G^{-1}(\mathrm{Disc}^* G)$ such that $p_n \to p_0$ with $p_0 \in V_G$. Consequently, $z_0 = 0$ and $x_n = \pm y_n \neq 0$ because $p_n \notin G^{-1}(\mathrm{Disc}^* G)$. Since $x_0 = \lim x_n = \pm \lim y_n = y_0 = 0$, one concludes that $p_0 = (0, 0, 0)$. Thus G has a Milnor-Hamm fibration by Theorem 5.2. In particular, each fiber consists of four open segments, consisting of hyperbolas sitting in two planes parallel and equal distance to the xy-plane, (see Figure 9).

The complement $\mathbb{R}^2 \setminus \mathrm{Disc}^* G$ consists of 3 connected components. We have: the fiber over $\mathbb{R} \times \mathbb{R}_{<0}$ is empty; the fiber over $\mathbb{R}_{>0} \times \mathbb{R}_{>0}$ and the fiber over $\mathbb{R}_{<0} \times \mathbb{R}_{>0}$ are two non-intersecting hyperbolas, with 4 connected components.

Moreover, it follows that G is S-nice and satisfies Condition (17), thus it has a singular tube fibration by Theorem 5.7. The singular tube fibration fibers over three of the strata of the discriminant as follows: over the positive vertical axis, the fibers are two disconnected components each of which being two intersecting lines; over the positive and the negative horizontal axis, the fibers are both hyperbolas with two components (see Figure 9).

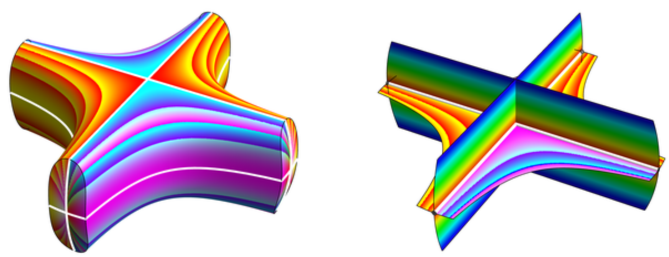

FIGURE 9. The Milnor-Hamm tube fibration (left) and the singular Milnor tube fibration over $\mathrm{Disc}^* G$ (right) for $G(x, y, z) = (xy, z^2)$. Each color scheme is a fibration over a connected component of the codomain.

In order to find good class singularities with the singular Milnor tube fibrations, the authors considered the following condition of regularity which does not require $\mathbb{W}$ to be a Thom regular stratification.

Definition 5.10. [ART1] Let $G : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$ be a non-constant analytic map germ. We say that G is *Thom regular at* V_G if there exists a Whitney stratification $(\mathbb{W}, \mathbb{S})$ like in Definition 5.5 such that 0 is a point stratum in $\mathbb{S}$, that V_G is a union of strata of $\mathbb{W}$, and that the Thom a_g-regularity condition is satisfied at any stratum of V_G.

Then they proved the following result

Theorem 5.11. [ART1] *Let $G : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$ be a non-constant S-nice analytic map germ. If G is Thom regular at V_G, $\dim V_G > 0$, then G has a singular Milnor tube fibration (16). In particular, if $V_G \cap \mathrm{Sing}\, G = \{0\}$ and $\dim V_G > 0$, then G has a Milnor-Hamm fibration (13).* □

Example 5.12. Let $f, g : \mathbb{C}^2 \to \mathbb{C}$ given by

$$f(x,y) = x^2 + y^2 \quad \text{and} \quad g(x,y) = x^2 - y^2.$$

One has $V_{(f,g)} = \{(0,0)\}$ and

$$\operatorname{Sing}(f,g) = \{x=0\} \cup \{y=0\};$$

hence (f,g) is obviously Thom regular at $V_{(f,g)}$. It then follows from [ART1, Theorem 4.3] that $f\bar{g}$ is Thom regular at $V_{f\bar{g}}$ hence, by Theorem 5.11, it has a Milnor-Hamm fibration, and also a singular Milnor tube fibration.

6. Milnor-Hamm sphere fibration

Inspired by the techniques developed by Milnor [Mi] and detailed in [AT2], the authors in [ART2] considered the problem of existence of a fibration structure over small spheres under a general situation when the discriminant $\operatorname{Disc}^* G$ has positive dimension. They introduced the *Milnor-Hamm sphere fibration*, gave natural sufficient conditions of singular maps that shows the fibration exists, and exhibited several such classes of singular maps. They then stated the problem of equivalence with the corresponding tube fibration and they showed how to solve it for some class of maps in the general setting under natural supplementary conditions.

First, the authors introduced a natural condition for a nice map germ G under which it was possible to define the sphere fibrations whenever $\operatorname{Disc}^* G$ is positive dimensional.

Definition 6.1. [ART2] Let $G : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$ be a real analytic map germ. We say that its discriminant $\operatorname{Disc}^* G$ is *radial* if, as a set germ at the origin, it is a union of real half-lines or the origin only.

The next example is a natural way of building map germs with radial discriminants.

Example 6.2. [ART2] Let $f : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$ be a real analytic map germ and let $g : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ be a germ of a diffeomorphism, such that f and g are in separable variables, and consider the pair of map germs

$$G := (f, g) : (\mathbb{R}^m \times \mathbb{R}, 0) \to (\mathbb{R}^p \times \mathbb{R}, 0).$$

Since $\operatorname{Sing} G = \operatorname{Sing} f \times \mathbb{R}$, one has that if $\operatorname{Disc}^* f$ is radial, then $\operatorname{Disc}^* G$ is radial.

Let $G : U \to \mathbb{R}^p$ be a representative of the map germ G for some open set $U \ni 0$ and recall the definition of Ψ:

$$\Psi := \frac{G}{\|G\|} : U \setminus V_G \to S^{p-1}. \tag{18}$$

In order to define a new fibration structure associated to the nice map germ G under assumption of radial discriminant, the authors have shown [ART2] that the restriction

$$\Psi_| : S^{m-1}_\varepsilon \setminus G^{-1}(\operatorname{Disc}^* G) \to S^{p-1} \setminus \operatorname{Disc}^* G \tag{19}$$

is well defined for $\varepsilon > 0$ small enough.

Definition 6.3. [ART2] We say that the map germ $G : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$ with radial discriminant has a *Milnor-Hamm sphere fibration* whenever the restriction (19) is a locally trivial smooth fibration which is independent, up to diffeomorphisms, of the choice of ε provided it is small enough.

In this more general setting, in [ART2] the authors defined *ρ-regularity* of Ψ whenever the following inclusion of germs is satisfied: $M(\Psi) \subset G^{-1}(\mathrm{Disc}^* G)$.

Finally with the notations and definitions above, the most general result regarding the existence of fibration structures on a sphere associated to non-constant nice map germs has been enunciated and demonstrated in [ART2]. It is the direct extension of [ACT1, Theorem 1.3] and its proof follows from the case $\mathrm{Disc}^* G = \{0\}$.

Theorem 6.4. *Let $G : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$, $m > p \geq 2$, be a non-constant nice analytic map germ with radial discriminant, satisfying Condition (14). If Ψ is ρ-regular then G has a Milnor-Hamm sphere fibration.*

EXAMPLE 6.5. [ART1, ART2] Let $G : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$ given by $G(x, y, z) = (xy, z^2)$. It follows from Example 5.9 that $G^{-1}(\mathrm{Disc}^* G)$ is the union of the coordinates planes in $\mathbb{R}^3$, hence it intersects the sphere S^2_ε on three great circles. Since $M(\Psi_G) = \mathrm{Sing}\, G$, it follows that Ψ is ρ-regular. Therefore, by Theorem 6.4 G has a Milnor-Hamm sphere fibration (see Figure 10).

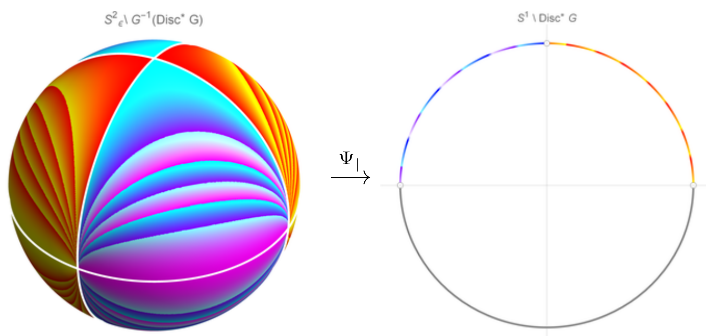

FIGURE 10. Milnor-Hamm sphere fibration for G. Each color scheme is a fibration over a connected component of the $S^1 \setminus \mathrm{Disc}^* G$.

REFERENCES

[A1] R. N. Araújo dos Santos, *Uniform (m)-condition and strong Milnor fibrations.* Singularities II, 189-198, Contemp. Math., 475, Amer. Math. Soc., Providence, RI, 2008. DOI: 10.1090/conm/475/09283

[ACT2] R. N. Araújo dos Santos, Y. Chen, M. Tibăr, *Real polynomial maps and singular open books at infinity.* Math. Scand. 118 (2016), no. 1, 57-69. DOI: 10.7146/math.scand.a-23296

[ACT1] R. N. Araújo dos Santos, Y. Chen, M. Tibăr, *Singular open book structures from real mappings.* Cent. Eur. J. Math. 11 (2013) no. 5, 817-828. DOI: 10.2478/s11533-013-0212-1

[ART1] R. N. Araújo dos Santos, M.F. Ribeiro, M. Tibăr, *Fibrations of highly singular map germs.* Bull. Sci. Math. 155 (2019), 92-111. DOI: 10.1016/j.bulsci.2019.05.001

[ART2] R. N. Araújo dos Santos, M.F. Ribeiro, M. Tibăr, *Milnor-Hamm sphere fibrations and the equivalence problem*, J. Math. Soc. Japan 72 (2020), no. 3, 945–957 DOI: 10.2969/jmsj/82278227

[AT1] R. N. Araújo dos Santos, M. Tibăr, *Real map germs and higher open books.* January 2008. arχiv: 0801.3328

[AT2] R. N. Araújo dos Santos, M. Tibăr, *Real map germs and higher open book structures.* Geom. Dedicata 147 (2010), 177-185. DOI: 10.1007/s10711-009-9449-z

[Be] K. Bekka, *C-régularité et trivialité topologique.* Singularity theory and its applications, Part I (Coventry, 1988/1989), 42-62, Lecture Notes in Math., 1462, Springer, Berlin, 1991. DOI: 10.1007/bfb0086373

[CT] Chen, Y. Bifurcation Values of Mixed Polynomials and Newton Polyhedra. Ph.D. thesis, Université de Lille 1, Lille, 2012.

[CT1] Chen, Y. Milnor fibration at infinity for mixed polynomials. Cent. Eur. J. Math. 12, (2014) 28-38.

[CL] P. T. Church, K. Lamotke, *Non-trivial polynomial isolated singularities.* Nederl. Akad. Wetensch. Proc. Ser. A 78. Indag. Math. 37 (1975), 149–154. DOI: 10.1016/1385-7258(75)90027-x

[C] J-L. Cisneros-Molina, *Join theorem for polar weighted homogeneous singularities*. Singularities II, 43-59, Contemp. Math., 475, Amer. Math. Soc., Providence, RI, 2008. DOI: 10.1090/conm/475/09274

[CGS] J-L. Cisneros, N. Grulha, J. Seade, *On the topology of real analytic maps*. Internat. J. Math. 25 (2014), no. 7, 145-154.

[CSS2] J-L. Cisneros-Molina, J. Seade, J. Snoussi, *Milnor fibrations and d-regularity for real analytic singularities*. Internat. J. Math. 21 (2010), no. 4, 419-434. DOI: 10.1142/s0129167x10006124

[CSS3] J-L. Cisneros-Molina, J. Seade, J. Snoussi, *Milnor fibrations and the concept of d-regularity for analytic map germs*. Real and complex singularities, 1-28, Contemp. Math., 569, Amer. Math. Soc., Providence, RI, 2012. DOI: 10.1090/conm/569/11241

[CSS1] J-L. Cisneros-Molina, J. Seade, J. Snoussi, *Refinements of Milnor's fibration theorem for complex singularities*. Advanced in Mathematics, 222, 2009, 937–970. DOI: 10.1016/j.aim.2009.05.010

[DRT] L. R. G. Dias, M. A. S. Ruas, M. Tibăr, *Regularity at infinity of real mappings and a Morse-Sard theorem*. J. Topol. 5 (2012), no. 2, 323-340. DOI: 10.1112/jtopol/jts005

[DA] N. Dutertre, R. N. Araújo dos Santos, *Topology of real Milnor fibrations for non-isolated singularities*. Int. Math. Res. Not. IMRN 2016, no. 16, 4849–4866.

[DACA] N. Dutertre, R. N. Araújo dos Santos, Y. Chen, A. A. do Espirito Santo, *Open book structures on semi-algebraic manifolds*. Manuscripta Mathematica, 149 (2016) (1-2), 205-222.

[GLPW] C. G. Gibson, E. Looijenga, A. du Plessis, K. Wirthmüller, *Topological stability of smooth mappings*. Lecture Notes in Mathematics, Vol. 552. Springer-Verlag, Berlin-New York, 1976. DOI: 10.1007/bfb0095245

[Ham] H. Hamm, *Lokale topologische Eigenschaften komplexer Räume*. Math. Ann. 191 (1971), 235-252. DOI: 10.1007/bf01578709

[Han] N. B. Hansen, *Milnor's Fibration Theorem for Real Singularities*. Master Thesis, Faculty of Mathematics and Natural Sciences University of Oslo. 2014.

[HL] H. Hamm, Lê D. Tráng, *Un théorème de Zariski du type de Lefschetz*. Ann. Sci. École Norm. Sup. (4) 6 (1973), 317-355. DOI: 10.24033/asens.1250

[Le] Lê D. Tráng, *Some remarks on relative monodromy*. Real and complex singularities (Proc. Ninth Nordic Summer School/NAVF Sympos. Math., Oslo, 1976), pp. 397-403. Sijthoff and Noordhoff, Alphen aan den Rijn, 1977. DOI: 10.1007/978-94-010-1289-8_11

[Lo1] E. J. N. Looijenga, *A note on polynomial isolated singularities*. Nederl. Akad. Wetensch. Proc. Ser. A 74. Indag. Math. 33 (1971), 418-421.

[Lo2] E. J. N. Looijenga, *Isolated Singular Points on Complete Intersections*. London Mathematical Society Lecture Note Series, 77. Cambridge University Press, Cambridge, 1984.

[Lo3] E. J. N Looijenga. *Isolated Singular Points on Complete Intersections*. Second edition. Surveys of Modern Mathematics, 5. International Press, Somerville, MA; Higher Education Press, Beijing, 2013. xiv+136 pp.

[Ma] D. B. Massey, *Real analytic Milnor fibrations and a strong Łojasiewicz inequality*. Real and Complex Singularities, 268-292, London Math. Soc. Lecture Note Ser., 380, Cambridge Univ. Press, Cambridge, 2010. DOI: 10.1017/cbo9780511731983.020

[MS] A. Menegon, J. Seade, *On the Lê-Milnor fibration for real analytic maps*. Math. Nachr. 290 (2016), no. 2-3, 382–392. DOI: 10.1002/mana.201600066

[Mi] J. W. Milnor, *Singular points of complex hypersurfaces*. Annals of Mathematics Studies, No. 61 Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo 1968. DOI: 10.1017/s0013091500027097

[NZ] A. Némethi, A. Zaharia, *Milnor fibration at infinity*. Indag. Math. (N.S.) 3 (1992), no. 3, 323–335. DOI: 10.1016/0019-3577(92)90039-n

[Oka1] M. Oka, *On the bifurcation of the multiplicity and topology of the Newton Boundary*. J. Math. Soc. Japan 31 (1979), no. 3, 435-450. DOI: 10.2969/jmsj/03130435

[Oka2] M. Oka, *Topology of polar weighted homogeneous hypersurface*. Kodai Math. J. 31 (2008), no. 2, 163–182. DOI: 10.2996/kmj/1214442793

[Oka3] M. Oka, *Non degenerate mixed functions*. Kodai Math. J. 33 (2010), no. 1, 1–62. DOI: 10.2996/kmj/1270559157

[Oka4] M. Oka, *On Milnor fibrations of mixed functions, a_f-condition and boundary stability*. Kodai Math. J. 38 (2015), no. 3, 581-603. DOI: 10.2996/kmj/1446210596

[PT] A. J. Parameswaran, M. Tibăr, *Thom regularity and Milnor tube fibrations*. Bull. Sci. Math. 143 (2018), 58-72. DOI: 10.1016/j.bulsci.2017.12.001

[Ri] M. F. Ribeiro, *Singular Milnor Fibrations*. PhD Thesis, Instituto de Ciências Matemáticas e de Computação, São Carlos. University of São Paulo, February 2018. DOI: 10.11606/T.55.2018.tde-06072018-115031

[Ri2] M. F. Ribeiro, *New classes of mixed functions without Thom regularity*. Bulletin of the Brazilian Mathematical Society, New Series (154) 2019. DOI: 10.1007/s00574-019-00154-z

[RA] M. A. S. Ruas, R.N. Araújo dos Santos, *Real Milnor Fibrations and (C)-regularity.* Manuscripta Math. 117 (2005), no. 2, 207-218. DOI: 10.1007/s00229-005-0555-4

[RSV] M. A. S. Ruas, J. Seade, A. Verjovsky, *On Real Singularities with a Milnor Fibration,* Trends in singularities, 191-213, Trends Math., Birkhäuser, Basel, 2002. DOI: 10.1007/978-3-0348-8161-6_9

[S] A. A. E. Santo, *Decomposição open book generalizada em conjuntos semi-algébricos.* PhD thesis, Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos, 2014. DOI: 10.11606/t.55.2014.tde-06072015-144158

[S2] J. Seade, *On Milnor's fibration theorem and its offspring after 50 years* . Bulletin of the American Mathematical Society, v. 56, (2019) n. 2, pp.281–348. DOI: 10.1090/bull/1654

[S1] J. Seade, *Open book decompositions associated to holomorphic vector fields.* Bol. Soc. Mat. Mexicana (3), vol.3, pp.323–336, 1997.

[Th1] R. Thom, *Les structures différentiables des boules et des sphéres.* Colloque Géom. Diff. Globale (Bruxelles, 1958) pp. 27-35 Centre Belge Rech. Math., Louvain, 1959.

[Th2] R. Thom, *Ensembles et morphismes stratifiés.* Bull. Amer. Math. Soc. 75 1969 240-284. DOI: 10.1090/s0002-9904-1969-12138-5

[Ti2] M. Tibăr, *Polynomials and vanishing cycles.* Cambridge Tracts in Mathematics, 170. Cambridge University Press, Cambridge, 2007.

[Ti1] M. Tibăr, *Regularity at infinity of real and complex polynomial functions.* Singularity theory (Liverpool, 1996), 249-264, London Math. Soc. Lecture Note Ser., 263, Cambridge Univ. Press, Cambridge, 1999. DOI: 10.1017/cbo9780511569265.016

A. A. do Espirito Santo, UFRB, Universidade Federal do Recôncavo da Bahia, Centro de Ciências Exatas e Tecnológicas, Campus de Cruz das Almas, Rua Rui Barbosa, 710, Centro, CEP 44.380-000 - Cruz das Almas, Ba, Brazil
Email address: andrade@ufrb.edu.br

D. Dreibelbis, University of North Florida, Department of Mathematics and Statistics, 1 UNF Drive, Jacksonville, FL 32224.
Email address: ddreibel@unf.edu

M. F. Ribeiro, UFES, Universidade Federal do Espirito Santo, Av. Fernando Ferrari, 514 - Goiabeiras - CP 29.075-910 Vitória, Espirito Santo, Brazil.
Email address: maico.ribeiro@ufes.br

R. N. Araújo dos Santos, ICMC, Universidade de São Paulo, Av. Trabalhador São-Carlense, 400 - CP Box 668, 13560-970 São Carlos, São Paulo, Brazil.
Email address: raimundo.icmc@gmail.com, rnonato@icmc.usp.br

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ON THE TOPOLOGY OF NON-ISOLATED REAL SINGULARITIES

NICOLAS DUTERTRE

Dedicated to Terence Gaffney and Maria Ruas, on the occasions of their 70th birthdays, and to Marcelo Saia, on the occasion of his 60th birthday

Abstract. Khimshiashvili proved a topological degree formula for the Euler characteristic of the Milnor fibres of a real function-germ with an isolated singularity. We give two generalizations of this result for non-isolated singularities. As corollaries we obtain an algebraic formula for the Euler characteristic of the fibres of a real weighted-homogeneous polynomial and a real version of the Lê-Iomdine formula. We have also included some results of the same flavor on the local topology of locally closed definable sets.

1. Introduction

Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be an analytic function-germ with an isolated critical point at the origin. Khimshiashvili [16] proved the following formula for the Euler characteristic of the real Milnor fibres of f:

$$\chi\big(f^{-1}(\delta) \cap B_\epsilon\big) = 1 - \mathrm{sign}(-\delta)^n \mathrm{deg}_0 \nabla f,$$

where $0 < |\delta| \ll \epsilon \ll 1$, B_ϵ is the closed ball centered at the origin of radius ϵ and $\mathrm{deg}_0 \nabla f$ is the topological degree of the mapping $\frac{\nabla f}{|\nabla f|} : S_\epsilon \to S^{n-1}$ (here S_ϵ is the boundary of B_ϵ). Later Fukui [15] generalized this result for the fibres of a one-parameter deformation of f. A corollary of the Khimshiashvili formula due to Arnol'd [1] and Wall [37] states that

$$\chi(\{f \leq 0\} \cap S_\epsilon) = 1 - \mathrm{deg}_0 \nabla f,$$
$$\chi(\{f \geq 0\} \cap S_\epsilon) = 1 + (-1)^{n-1} \mathrm{deg}_0 \nabla f,$$

and if n is even,

$$\chi(\{f = 0\} \cap S_\epsilon) = 2 - 2\ \mathrm{deg}_0 \nabla f.$$

In [31] Szafraniec extended the results of Arnold and Wall to the case of an analytic function-germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ with non-isolated singularities. Namely he constructed two function-germs g_- and g_+ with isolated critical points and proved that

$$\chi\left(\{f \leq 0\} \cap S_\epsilon\right) = 1 - \mathrm{deg}_0 \nabla g_+ \text{ and } \chi\left(\{f \geq 0\} \cap S_\epsilon\right) = 1 - \mathrm{deg}_0 \nabla g_-.$$

In [32] he improved this result for weighted homogeneous polynomials. If $f : \mathbb{R}^n \to \mathbb{R}$ is a weighted homogeneous polynomial then he constructed two polynomials g_1 and g_2 with an algebraically isolated critical point at 0 such that

$$\chi\left(\{f \leq 0\} \cap S^{n-1}\right) = 1 - \mathrm{deg}_0 \nabla g_1 \text{ and } \chi\left(\{f \geq 0\} \cap S^{n-1}\right) = 1 - \mathrm{deg}_0 \nabla g_2.$$

Thanks to the Eisenbud-Levine-Khimshiashvili formula [14, 16],

$$\chi\left(\{f \leq 0\} \cap S^{n-1}\right) \quad \text{and} \quad \chi\left(\{f \geq 0\} \cap S^{n-1}\right)$$

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Key words and phrases. Topological degree, Euler characteristic, Real Milnor fibres.

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can be computed algebraically.

The aim of this paper is to extend the Khimshiashvili formula for function-germs with arbitrary singularities. We will work in the more general framework of definable functions. Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be a definable function-germ of class C^r, $r \geq 2$. Our first new result is Lemma 2.5 where we give a relation between the Euler characteristic of $f^{-1}(\delta) \cap B_\epsilon$ (resp. $f^{-1}(-\delta) \cap B_\epsilon$), with $0 < \delta \ll \epsilon \ll 1$, and the Euler characteristic of the link at the origin of $\{f \leq 0\}$ (resp. $\{f \geq 0\}$). Applying the results of Szafraniec, we obtain our first generalization of the Khimshiashvili formula (Corollary 2.6) for polynomially bounded structures and an algebraic formula for the Euler characteristic of a regular fibre of a weighted homogeneous polynomial (Corollary 2.7). We note that the paper [7] presents a different approach for the computation of this Euler characteristic.

Our second generalization of the Khimshiashvili formula is an adaptation to the real case of the methods based on the generic polar curve, introduced in the complex case by Lê [19] and Teissier [34, 35] and developed later by Massey [24, 25, 26]. For $v \in S^{n-1}$, we denote by Γ_v the following relative polar set:

$$\Gamma_v = \{x \in \mathbb{R}^n \setminus \Sigma_f \mid \operatorname{rank}(\nabla f(x), v) < 2\},$$

where $\Sigma_f = \{x \in \mathbb{R}^n \mid \nabla f(x) = 0\}$ is the critical locus of f. For v generic in S^{n-1}, Γ_v is a curve. Let $\mathcal{B}$ be the set of its connected components. For each $\mathbf{b} \in \mathcal{B}$, we denote by $\sigma(\mathbf{b})$ the sign of $\det\big[\nabla f_{x_1}, \ldots, \nabla f_{x_n}\big]$ on $\mathbf{b}$, where for $i = 1, \ldots, n$, f_{x_i} denotes the partial derivative $\frac{\partial f}{\partial x_i}$. Morevover on $\mathbf{b}$ the partial derivative $\frac{\partial f}{\partial v}$ does not vanish so we can decompose $\mathcal{B}$ into the disjoint union $\mathcal{B}^+ \sqcup \mathcal{B}^-$, where $\mathcal{B}^+$ (resp. $\mathcal{B}^-$) is the set of half-branches on which $\frac{\partial f}{\partial v} > 0$ (resp. $\frac{\partial f}{\partial v} < 0$). This enables to define the following indices (Definition 4.8):

$$\lambda^+ = \sum_{\mathbf{b} \in \mathcal{B}^+} \sigma(\mathbf{b}) \text{ and } \lambda^- = \sum_{\mathbf{b} \in \mathcal{B}^-} \sigma(\mathbf{b}).$$

Then we define the following four indices (Definition 4.11):

$$\begin{aligned}
\gamma^{+,+} &= \chi\big(f^{-1}(0) \cap \{x_1 = a\} \cap B_\epsilon\big) - \chi\big(f^{-1}(\alpha) \cap \{x_1 = a\} \cap B_\epsilon\big),\\
\gamma^{+,-} &= \chi\big(f^{-1}(0) \cap \{x_1 = -a\} \cap B_\epsilon\big) - \chi\big(f^{-1}(\alpha) \cap \{x_1 = -a\} \cap B_\epsilon\big),\\
\gamma^{-,+} &= \chi\big(f^{-1}(0) \cap \{x_1 = a\} \cap B_\epsilon\big) - \chi\big(f^{-1}(-\alpha) \cap \{x_1 = a\} \cap B_\epsilon\big),\\
\gamma^{-,-} &= \chi\big(f^{-1}(0) \cap \{x_1 = -a\} \cap B_\epsilon\big) - \chi\big(f^{-1}(-\alpha) \cap \{x_1 = -a\} \cap B_\epsilon\big),
\end{aligned}$$

where $0 < \alpha \ll a \ll \epsilon$. Our second generalization of the Khimshiashvili formula relates the Euler characteristic of the real Milnor fibres to these new indices. Namely in Theorem 4.12 we show that

$$\chi\big(f^{-1}(-\delta) \cap B_\epsilon\big) = 1 - \lambda^- - \gamma^{-,-} = 1 - \lambda^+ - \gamma^{-,+},$$

and that

$$\chi\big(f^{-1}(\delta) \cap B_\epsilon\big) = 1 - (-1)^n \lambda^- - \gamma^{+,+} = 1 - (-1)^n \lambda^+ - \gamma^{+,-},$$

where $0 < \delta \ll \epsilon \ll 1$. Then we apply this result to the case where Σ_f has dimension one. In this case, we denote by $\mathcal{C}$ the set of connected components of $\Sigma_f \setminus \{0\}$. For $v \in S^{n-1}$ generic, the function v^* does not vanish on any half-branch $\mathbf{c} \in \mathcal{C}$, so we can decompose $\mathcal{C}$ into the disjoint union $\mathcal{C}^+ \sqcup \mathcal{C}^-$, where $\mathcal{C}^+$ (resp. $\mathcal{C}^-$) is the set of half-branches on which $v^* > 0$ (resp. $v^* < 0$). For each $\mathbf{c} \in \mathcal{C}$, let $\tau(\mathbf{c})$ be the value that the function $a \mapsto \deg_q \nabla f_{|x_1^{-1}(a)}$, $\{q\} = \mathbf{c} \cap \{x_1 = a\}$, takes close to the origin. Then we set $\gamma^+ = \sum_{\mathbf{c} \in \mathcal{C}^+} \tau(\mathbf{c})$ and $\gamma^- = \sum_{\mathbf{c} \in \mathcal{C}^-} \tau(\mathbf{c})$. In this situation, Theorem 4.12 takes the following form (Theorem 5.4):

$$\chi\big(f^{-1}(-\delta) \cap B_\epsilon\big) = 1 - \lambda^- - \gamma^- = 1 - \lambda^+ - \gamma^+,$$

$$\chi\big(f^{-1}(\delta) \cap B_\epsilon\big) = 1 - (-1)^n(\lambda^+ - \gamma^-) = 1 - (-1)^n(\lambda^- - \gamma^+),$$

where $0 < \delta \ll \epsilon \ll 1$. Hence the indices λ^+, λ^-, γ^+ and γ^- appear to be real versions of the first two Lê numbers defined by Massey in [24]. We note that the paper [36] contains also formulas for the Euler characteristic of the real Milnor fibres of a function-germ with a one-dimensional critical locus.

In the complex case, the Lê-Iomdine formula ([20, 18], see also [24, 27, 28, 30] for improved versions) relates the Euler characteristic of the Milnor fibre of an analytic function-germ with one-dimensional singular set to the Milnor fibre of an analytic function-germ with an isolated singularity, given as the sum of the initial function and a sufficiently big power of a generic linear form. As a corollary of Theorem 5.4, we establish a real version of this formula (Theorem 5.12), i.e., a relation between the Euler characteristic of the real Milnor fibres of f and the real Milnor fibres of a function of the type $f + v^{*k}$, for $v \in S^{n-1}$ generic and $k \in \mathbb{N}$ big enough.

We have also included some results on the local topology of locally closed definable sets. More precisely, we consider a locally closed definable set X equipped with a Whitney stratification such that $0 \in X$, and a definable function $g : (X, 0) \to (\mathbb{R}, 0)$ with an isolated critical point at the origin. In Lemma 3.1 we extend to this setting the results of Arnold and Wall mentioned above, i.e., we give relations between the Euler characteristics of the sets $X \cap \{g \ ? \ \pm \delta\} \cap B_\epsilon$, where $0 < \delta \ll \epsilon \ll 1$ and $? \in \{\leq, \geq\}$, and the Euler characteristics of the sets $X \cap \{g \ ? \ 0\} \cap S_\epsilon$, where $0 < \epsilon \ll 1$ and $? \in \{\leq, \geq\}$. We give two corollaries (Corollaries 3.3 and 3.4) when the stratum that contains 0 has dimension greater than or equal to 1.

The paper is organized as follows. In Section 2, we prove the first generalization of the Khimshiashvili formula based on Szafraniec's methods. In Section 3, we give the results on the local topology of locally closed definable sets. Section 4 contains the second generalization of the Khimshiashvili formula, based on the study of generic relative polar curves. In Section 5, we establish the real version of the Lê-Iomdine formula.

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2. Some general results on the real Milnor fibre

Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be a definable function-germ of class C^r, $r \geq 2$. By Lemma 10 in [2] or by the main theorem of [21], we can equip $f^{-1}(0)$ with a finite Whitney stratification that satisfies the Thom (a_f)-condition.

Lemma 2.1. *There exists $\epsilon_0 > 0$ such that for $0 < \epsilon \leq \epsilon_0$, there exists δ_ϵ such that for $0 < \delta \leq \delta_\epsilon$, the topological type of $f^{-1}(\delta) \cap B_\epsilon$ does not depend on the choice of the couple (ϵ, δ).*

Proof. Let $\epsilon_0 > 0$ be such that for $0 < \epsilon \leq \epsilon_0$, the sphere S_ϵ intersects $f^{-1}(0)$ transversally. Then there exists a neighborhood U_ϵ of 0 in $\mathbb{R}$ such that for each $\delta \in U_\epsilon$, the fibre $f^{-1}(\delta)$ intersects the sphere S_ϵ transversally. If it is not the case, then we can find a sequence of points $(p_m)_{m \in \mathbb{N}}$ in S_ϵ such that the vectors $\frac{p_m}{|p_m|}$ and $\frac{\nabla f(p_m)}{|\nabla f(p_m)|}$ are collinear, and such that the sequence converges to a point p in $S_\epsilon \cap f^{-1}(0)$. If S denotes the stratum of $f^{-1}(0)$ that contains p then, applying the Thom (a_f)-condition, there exists a unit vector V normal to $T_p S$ such that $\frac{p}{|p|}$ and V are collinear. This contradicts the fact that S_ϵ intersects $f^{-1}(0)$ transversally.

Now let us fix $\epsilon > 0$ with $\epsilon \leq \epsilon_0$. Let us choose $\delta_\epsilon > 0$ such that $[0, \delta_\epsilon]$ is included in U_ϵ and δ is a regular value of f for $0 < \delta \leq \delta_\epsilon$. Let (ϵ_1, δ_1) and (ϵ_2, δ_2) be two couples with $0 < \epsilon_i \leq \epsilon$ and $0 < \delta_i \leq \delta_{\epsilon_i}$ for $i = 1, 2$. If $\epsilon_1 = \epsilon_2$ then the Thom-Mather first isotopy lemma implies that the fibres $f^{-1}(\delta_1)\cap B_{\epsilon_1}$ and $f^{-1}(\delta_2)\cap B_{\epsilon_2}$ are homeomorphic. Now assume that $\epsilon_1<\epsilon_2$. By the same arguments as above, there exists a neighborhood U of 0 in $\mathbb{R}$ such that for each $\delta\neq 0$ in U, the distance function to the origin has no critical point on $f^{-1}(\delta)\cap(B_{\epsilon_2}\setminus \mathring{B}_{\epsilon_1})$. Let us choose $\delta_3\neq 0$ in U such that $0<\delta_3\leq \min\{\delta_1,\delta_2\}$. By the first case, $f^{-1}(\delta_3)\cap B_{\epsilon_1}$ is homeomorphic to $f^{-1}(\delta_1)\cap B_{\epsilon_1}$ and $f^{-1}(\delta_3)\cap B_{\epsilon_2}$ is homeomorphic to $f^{-1}(\delta_2)\cap B_{\epsilon_2}$. But, since the distance function to the origin has no critical points on $f^{-1}(\delta_3)\cap(B_{\epsilon_2}\setminus \mathring{B}_{\epsilon_1})$, the fibres $f^{-1}(\delta_3)\cap B_{\epsilon_2}$ and $f^{-1}(\delta_3)\cap B_{\epsilon_1}$ are homeomorphic. $\square$

Of course a similar result is true for negative values of f.

Definition 2.2. The (real) Milnor fibres of f are the sets $f^{-1}(\delta)\cap B_\epsilon$ and $f^{-1}(-\delta)\cap B_\epsilon$, where $0<\delta\ll\epsilon\ll 1$.

Sometimes we call $f^{-1}(\delta)\cap B_\epsilon$ (resp. $f^{-1}(-\delta)\cap B_\epsilon$) the positive (resp. negative) Milnor fibre of f. The Khimshiashvili formula [16] relates the Euler characteristic of the Milnor fibres to the topological degree of ∇f at the origin, when f has an isolated singularity.

Theorem 2.3 (The Khimshiashvili formula)**.** *If f has an isolated critical point at the origin then*

$$\chi\big(f^{-1}(\delta)\cap B_\epsilon\big)=1-\mathrm{sign}(-\delta)^n\mathrm{deg}_0\nabla f,$$

where $0<|\delta|\ll\epsilon\ll 1$.

Proof. We give a proof for we will need a similar argument later. Let U be a small open subset of $\mathbb{R}^n$ such that $0\in U$ and f is defined in U. We perturb f in a Morse function $\tilde{f}:U\to\mathbb{R}$. Let $p_1,\dots,p_k$ be the critical points of $\tilde{f}$, with respective indices $\lambda_1,\dots,\lambda_k$. Let $\delta>0$, by Morse theory we have:

$$\chi\big(f^{-1}([-\delta,\delta])\cap B_\epsilon\big)-\chi\big(f^{-1}(-\delta)\cap B_\epsilon\big)=\sum_{i=1}^{k}(-1)^{\lambda_i}.$$

Actually we can choose $\tilde{f}$ sufficiently close to f so that the p_i's lie in $f^{-1}([-\frac{\delta}{4},\frac{\delta}{4}])$. Now the inclusion $f^{-1}(0)\cap B_\epsilon\subset f^{-1}([-\delta,\delta])\cap B_\epsilon$ is a homotopy equivalence (Durfee [8] proved this result in the semi-algebraic case, but his argument holds in the C^r definable case, see also [6, 17]) and $f^{-1}(0)\cap B_\epsilon$ is the cone over $f^{-1}(0)\cap S_\epsilon$, so $\chi\big(f^{-1}([-\delta,\delta])\cap B_\epsilon\big)=1$. This gives the result for the negative Milnor fibre. To get the result for the positive one, it is enough to replace f with $-f$. $\square$

The following formulas are due to Arnol'd [1] and Wall [37].

Corollary 2.4. *With the same hypothesis on f, we have:*

$$\chi(\{f\leq 0\}\cap S_\epsilon)=1-\mathrm{deg}_0\nabla f,$$

$$\chi(\{f\geq 0\}\cap S_\epsilon)=1+(-1)^{n-1}\mathrm{deg}_0\nabla f.$$

If n is even, we have:

$$\chi(\{f=0\}\cap S_\epsilon)=2-2\ \mathrm{deg}_0\nabla f.$$

Proof. By a deformation argument due to Milnor [23], $f(-\delta)\cap B_\epsilon$, $\delta>0$, is homeomorphic to $\{f\leq-\delta\}\cap S_\epsilon$, which is homeomorphic to $\{f\leq 0\}\cap S_\epsilon$ if δ is very small. $\square$

We start our study of the general case with an easy lemma.

Lemma 2.5. *Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be a definable function germ of class C^r, $r \geq 2$, and let $0 < \delta \ll \epsilon$. If n is even then*

$$\chi\left(f^{-1}(-\delta) \cap B_\epsilon\right) = \chi\left(\{f \geq 0\} \cap S_\epsilon\right),$$

and

$$\chi\left(f^{-1}(\delta) \cap B_\epsilon\right) = \chi\left(\{f \leq 0\} \cap S_\epsilon\right).$$

If n is odd then

$$\chi\left(f^{-1}(-\delta) \cap B_\epsilon\right) = 2 - \chi\left(\{f \geq 0\} \cap S_\epsilon\right),$$

and

$$\chi\left(f^{-1}(\delta) \cap B_\epsilon\right) = 2 - \chi\left(\{f \leq 0\} \cap S_\epsilon\right).$$

Proof. If the Milnor fibre is empty or a compact manifold without boundary then the result is obvious.

Otherwise, if n is even then $f^{-1}(-\delta) \cap B_\epsilon$ is an odd-dimensional manifold with boundary and so

$$\chi\left(f^{-1}(-\delta) \cap B_\epsilon\right) = \frac{1}{2}\chi\left(f^{-1}(-\delta) \cap S_\epsilon\right) = \chi\left(\{f \geq -\delta\} \cap S_\epsilon\right).$$

But for δ small, the inclusion $\{f \geq 0\} \cap S_\epsilon \subset \{f \geq -\delta\} \cap S_\epsilon$ is a homotopy equivalence (see [8]).

If n is odd then $\{f \geq -\delta\} \cap B_\epsilon$ is an odd-dimensional manifold with corners. Rounding the corners, we get

$$\begin{aligned}\chi\left(\{f \geq -\delta\} \cap B_\epsilon\right) = \frac{1}{2}\Big(&\chi\left(f^{-1}(-\delta) \cap B_\epsilon\right) + \chi\left(\{f \geq -\delta\} \cap S_\epsilon\right) \\ &-\chi\left(f^{-1}(-\delta) \cap S_\epsilon\right)\Big) = \frac{1}{2}\left(\chi\left(f^{-1}(-\delta) \cap B_\epsilon\right) + \chi\left(\{f \geq -\delta\} \cap S_\epsilon\right)\right).\end{aligned}$$

But the inclusion $\{f \geq 0\} \cap B_\epsilon \subset \{f \geq -\delta\} \cap B_\epsilon$ is a homotopy equivalence and so

$$\chi\left(\{f \geq -\delta\} \cap B_\epsilon\right) = 1.$$

$\square$

For the rest of this section, we assume that the structure is polynomially bounded. The techniques developed and the results proved by Szafraniec [31] (see also [4]) are valid in this context. Let $\omega(x) = x_1^2 + \cdots + x_n^2$. Then there exists an integer $d > 0$ sufficiently big such that $g_+ = f - \omega^d$ and $g_- = -f - \omega^d$ have an isolated critical point at the origin. Moreover Szafraniec showed that

$$\chi\left(\{f \leq 0\} \cap S_\epsilon\right) = 1 - \deg_0 \nabla g_+ \text{ and } \chi\left(\{f \geq 0\} \cap S_\epsilon\right) = 1 - \deg_0 \nabla g_-.$$

Applying the previous lemma, we can state our first generalization of the Khimshiashvili formula.

Corollary 2.6. *If $0 < \delta \ll \epsilon$, we have:*

$$\chi\left(f^{-1}(-\delta) \cap B_\epsilon\right) = 1 - (-1)^n \deg_0 \nabla g_-,$$

and

$$\chi\left(f^{-1}(\delta) \cap B_\epsilon\right) = 1 - (-1)^n \deg_0 \nabla g_+.$$

In general, the exponent d is difficult to estimate. However, in the case of a weighted-homogeneous polynomial, Szafraniec [32] provided another method which is completely effective.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a real weighted homogeneous polynomial function of type $(d_1, \cdots, d_n; d)$ with $\nabla f(0) = 0$. Let p be the smallest positive integer such that $2p > d$ and each d_i divides p. Also let $a_i = \dfrac{p}{d_i}$ and

$$\omega = \frac{x_1^{2a_1}}{2a_1} + \cdots + \frac{x_n^{2a_n}}{2a_n}.$$

Now consider $g_1 = f - \omega$ and $g_2 = -f - \omega$. Szafraniec proved that g_1 and g_2 have an algebraically isolated critical point at the origin and that

$$\chi\left(\{f \leq 0\} \cap S^{n-1}\right) = 1 - \deg_0 \nabla g_1 \text{ and } \chi\left(\{f \geq 0\} \cap S^{n-1}\right) = 1 - \deg_0 \nabla g_2.$$

Applying Lemma 2.5, we obtain the following Khimshiashvili type formula for the fibres of a real weighted homogeneous polynomial.

Corollary 2.7. *We have*

$$\chi\left(f^{-1}(-1)\right) = 1 - (-1)^n \deg_0 \nabla g_2,$$

and

$$\chi\left(f^{-1}(1)\right) = 1 - (-1)^n \deg_0 \nabla g_1.$$

Note that $\deg_0 \nabla g_1$ and $\deg_0 \nabla g_2$ can be computed algebraically thanks to the Eisenbud-Levine-Khimshiashvili formula [14, 16] because they have an algebraically isolated zero at the origin.

Let us apply this corollary to the examples presented in [32].

(1) Let $f(x, y, z) = x^2 y - y^4 - yz^3$. By [32], we have that $\deg_0 \nabla g_1 = \deg_0 \nabla g_2 = 1$. So $\chi\left(f^{-1}(-1)\right) = \chi\left(f^{-1}(1)\right) = 2$.
(2) Let $f(x, y, z) = x^3 + x^2 z - y^2$. By [32], we have that $\deg_0 \nabla g_1 = 1$ and $\deg_0 \nabla g_2 = -1$. So $\chi\left(f^{-1}(-1)\right) = 0$ and $\chi\left(f^{-1}(1)\right) = 2$.
(3) Let $f(x, y, z) = x^3 - xy^2 + xyz + 2x^2 y - 2y^3 - y^2 z - xz^2 + yz^2$. Then by [32], $\deg_0 \nabla g_1 = 3$, so $\chi\left(f^{-1}(1)\right) = 4$.

3. Some results on the topology of locally closed definable sets

Let X be a locally closed definable set. We assume that 0 belongs to X. We equip X with a finite definable C^r, $r \geq 2$, Whitney stratification. The fact that such a stratification exists is due to Loi [22]. Recently Nguyen, Trivedi and Trotman [29] gave another proof of this result. We denote by S_0 the stratum that contains 0.

Let $g : (X, 0) \to (\mathbb{R}, 0)$ be a definable function that is the restriction to X of a definable function G of class C^r, $r \geq 2$, defined in a neighborhood of the origin. We assume that g has at worst an isolated critical point (in the stratified sense) at the origin. As in the previous section, the positive and the negative real Milnor fibres of g are the sets $g^{-1}(\delta) \cap X \cap B_\epsilon$ and $g^{-1}(-\delta) \cap X \cap B_\epsilon$, where $0 < \delta \ll \epsilon \ll 1$.

Lemma 3.1. *For $0 < \delta \ll \epsilon \ll 1$, we have*

$$\chi\big(X \cap g^{-1}(-\delta) \cap B_\epsilon\big) = \chi\big(X \cap \{g \leq 0\} \cap S_\epsilon\big),$$

and

$$\chi\big(X \cap g^{-1}(\delta) \cap B_\epsilon\big) = \chi\big(X \cap \{g \geq 0\} \cap S_\epsilon\big).$$

Proof. Using the methods developed in [11], we can assume that the critical points of g on $X \cap S_\epsilon$ are isolated, that they lie in $\{g \neq 0\}$ and that they are outwards-pointing (resp. inwards-pointing) in $\{g > 0\}$ (resp. $\{g < 0\}$). Let us denote them by $\{p_1, \ldots, p_s\}$.

We recall that if $Z \subset \mathbb{R}^n$ is a locally closed definable set, equipped with a Whitney stratification and p is an isolated critical point of a definable function $\phi : Z \to \mathbb{R}$, restriction to Z of a C^2-definable function Φ, then the index of ϕ at p is defined as follows:

$$\operatorname{ind}(\phi, Z, p) = 1 - \chi\big(Z \cap \{\phi = \phi(p) - \eta\} \cap B_r(p)\big),$$

where $0 < \eta \ll r \ll 1$ and $B_r(p)$ is the closed ball of radius r centered at p.

As in [11], Section 3, we can apply the results proved in [9]. Namely, by Theorem 3.1 in [9], we can write

$$\chi\big(\{g \leq 0\} \cap X \cap S_\epsilon\big) = \sum_{i \ \mid \ g(p_i)<0} \operatorname{ind}(g, X \cap S_\epsilon, p_i),$$

and for $0 < \delta \ll \epsilon$,

$$\chi\big(\{g \leq \delta\} \cap X \cap B_\epsilon\big) = \sum_{i \ \mid \ g(p_i)<0} \operatorname{ind}(g, X \cap B_\epsilon, p_i) + \operatorname{ind}(g, X, 0).$$

By Lemma 2.1 in [9], $\operatorname{ind}(g, X \cap S_\epsilon, p_i) = \operatorname{ind}(g, X \cap B_\epsilon, p_i)$ if $g(p_i) < 0$. Moreover,

$$\operatorname{ind}(g, X, 0) = 1 - \chi\big(g^{-1}(-\delta) \cap X \cap B_\epsilon\big)$$

and, as explained in the proof of Theorem 2.3, $\chi\left(\{g \leq \delta\} \cap X \cap B_\epsilon\right) = 1$ if δ is small enough. Combining these observations, we find that

$$\chi\big(X \cap g^{-1}(-\delta) \cap B_\epsilon\big) = \chi\big(X \cap \{g \leq 0\} \cap S_\epsilon\big).$$

$\square$

Remark 3.2. We believe that it is possible to establish these equalities applying a stratified version of the Milnor deformation argument mentionned in the proof of Corollary 2.4. This is done by Comte and Merle in [5] when X is conic and g is a generic linear form.

For the rest of this section, we will denote by $\operatorname{Lk}(Y)$ the link at the origin of a definable set Y.

Corollary 3.3. *Assume that* $\dim S_0 > 0$ *and that* $g_{|S_0}$ *has no critical point at* 0, *i.e.,* $g^{-1}(0)$ *intersects* S_0 *transversally at* 0. *Then the following equalities hold:*

$$\chi\big(\operatorname{Lk}(X \cap \{g \leq 0\})\big) = \chi\big(\operatorname{Lk}(X \cap \{g \geq 0\})\big) = 1,$$

and

$$\chi\big(\operatorname{Lk}(X)\big) + \chi\big(\operatorname{Lk}(X \cap \{g = 0\})\big) = 2.$$

Proof. If $g_{|S_0}$ has no critical point at 0, then $g : X \to \mathbb{R}$ is a stratified submersion in a neighborhood of 0. Furthermore for $0 < \epsilon \ll 1$, the sphere S_ϵ intersects $X \cap \{g = 0\}$ transversally, so 0 is a regular value of $g_{|X \cap B_\epsilon}$. Therefore if δ is small enough,

$$\chi\big(X \cap \{g = -\delta\} \cap B_\epsilon\big) = \chi\big(X \cap \{g = \delta\} \cap B_\epsilon\big) = \chi\big(X \cap \{g = 0\} \cap B_\epsilon\big) = 1.$$

It is enough to apply the previous lemma and then the Mayer-Vietoris sequence. $\square$

For $v \in S^{n-1}$, we denote by v^* the function $v^*(x) = \langle v, x \rangle$, where $\langle \ , \ \rangle$ is the standard scalar product. The previous corollary applies to a generic linear form v^*.

Corollary 3.4. *Assume that* $\dim S_0 > 0$. *If* $v \notin S^{n-1} \cap (T_0 S_0)^\perp$, *then*

$$\chi\big(\operatorname{Lk}(X \cap \{v^* \leq 0\})\big) = \chi\big(\operatorname{Lk}(X \cap \{v^* \geq 0\})\big) = 1,$$

and

$$\chi\big(\operatorname{Lk}(X)\big) + \chi\big(\operatorname{Lk}(X \cap \{v^* = 0\})\big) = 2.$$

Proof. If $v \notin (T_0 S_0)^\perp$, then $v^*_{|S_0}$ has no critical point at 0. $\square$

Let us relate this corollary to results that we proved in earlier papers. Combining Theorem 5.1 in [11] and the comments after Theorem 2.6 in [12], we can write that if dim $S_0 > 0$,

$$\chi\big(\mathrm{Lk}(X)\big) + \frac{1}{g_n^{n-1}} \int_{G_n^{n-1}} \chi\big(\mathrm{Lk}(X \cap H)\big) dH = 2,$$

where G_n^{n-1} is the Grassmann manifold of linear hyperplanes in $\mathbb{R}^n$ and g_n^{n-1} is its volume. This last equality is based on the study of the local behaviour of the generalized Lipschitz-Killing curvatures made in [5] and [11]. We see that it is actually a direct consequence of Corollary 3.4, which gives a more precise result on the local topology of locally closed definable sets. Similarly for $0 < k < \dim S_0$, we know that

$$-\frac{1}{g_n^{n-k-1}} \int_{G_n^{n-k-1}} \chi\big(\mathrm{Lk}(X \cap H)\big) dH + \frac{1}{g_n^{n-k+1}} \int_{G_n^{n-k+1}} \chi\big(\mathrm{Lk}(X \cap L)\big) dL = 0,$$

where G_n^{n-k} is the Grassmann manifold of k-dimensional vector spaces in $\mathbb{R}^n$ and g_n^{n-k} is its volume. In fact a recursive application of Corollary 3.4 shows that $\chi\big(\mathrm{Lk}(X \cap H)\big) = \chi\big(\mathrm{Lk}(X \cap L)\big)$ for H generic in G_n^{n-k-1} and L generic in G_n^{n-k+1}.

Let us give another application of Corollary 3.4 to the topology of real Milnor fibres. As in the previous section, $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ is the germ at the origin of a definable function of class C^r, $r \geq 2$. We assume that $f^{-1}(0)$ is equipped with a finite Whitney stratification that satisfies the Thom (a_f)-condition. Let S_0 be the stratum that contains 0.

Corollary 3.5. *If* dim $S_0 > 0$ *and if* $v \notin S^{n-1} \cap (T_0 S_0)^\perp$, *then for* $0 < \delta \ll \epsilon \ll 1$, *we have*

$$\chi\big(f^{-1}(\delta) \cap B_\epsilon\big) = \chi\big(f^{-1}(\delta) \cap \{v^* = 0\} \cap B_\epsilon\big),$$

and

$$\chi\big(f^{-1}(-\delta) \cap B_\epsilon\big) = \chi\big(f^{-1}(-\delta) \cap \{v^* = 0\} \cap B_\epsilon\big).$$

Proof. Applying Corollary 3.4 to the sets $\{f \geq 0\}$ and $\{f \leq 0\}$, we get that

$$\chi\big(\mathrm{Lk}(\{f \ ? \ 0\})\big) + \chi\big(\mathrm{Lk}(\{f \ ? \ 0\} \cap \{v^* = 0\})\big) = 2,$$

where $? \in \{\leq, \geq\}$. Lemma 2.5 applied to f and $f_{|\{v^*=0\}}$ gives the result. ☐

In the next section, we will give a generalization of this result based on generic relative polar curves.

4. Milnor fibres and relative polar curves

Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be a definable function-germ of class C^r, $r \geq 2$. We will give a second generalization of the Khimshiashvili formula in this setting. For this we need first to study the behaviour of a generic linear function on the fibres of f and the behaviour of f on the fibres of a generic linear function.

We start with a study of the critical points of $v^*_{|f^{-1}(\delta)}$ for δ small and v generic in S^{n-1}. Let

$$\Gamma_v = \{x \in \mathbb{R}^n \setminus \Sigma_f \mid \mathrm{rank}(\nabla f(x), v) < 2\}.$$

We will need a first genericity condition. We can equip $f^{-1}(0)$ with a finite Whitney stratification that satisfies the Thom (a_f)-condition.

Lemma 4.1. *There exists a definable set* $\Sigma_1 \subset S^{n-1}$ *of positive codimension such that if* $v \notin \Sigma_1$, *then* $\{v^* = 0\}$ *intersects* $f^{-1}(0) \setminus \{0\}$ *transversally in a neighborhood of the origin.*

Proof. It is a particular case of Lemma 3.8 in [10]. ☐

Lemma 4.2. *If* $v \notin \Sigma_1$ *then* $\Gamma_v \cap f^{-1}(0) = \emptyset$.

Proof. If it is not the case then we can find an arc $\alpha : [0, \nu[\to f^{-1}(0)$ such that $\alpha(0) = 0$ and for $0 < s < \nu$, $\nabla f(\alpha(s)) \neq 0$ and $\operatorname{rank}(\nabla f(\alpha(s)), v) < 2$. Let S be the stratum that contains $\alpha(]0, \nu[)$. Since $\nabla f(\alpha(s))$ is normal to $T_{\alpha(s)}S$, the points in $\alpha(]0, \nu[)$ are critical points of $v^*_{|S}$ and hence lie in $\{v^* = 0\}$. This contradicts Lemma 4.1. □

Corollary 4.3. *If $v \notin \Sigma_1$ then $\Gamma_v \cap \{v^* = 0\} = \emptyset$.*

Proof. As in the proof of the previous lemma, we see that

$$\text{if } \Gamma_v \cap \{v^* = 0\} \neq \emptyset, \text{ then } \Gamma_v \cap f^{-1}(0) \neq \emptyset.$$

□

Lemma 4.4. *There exists a definable set $\Sigma_2 \subset S^{n-1}$ of positive codimension such that if $v \notin \Sigma_2$, Γ_v is a curve (possibly empty) in the neighbourhood of the origin.*

Proof. Let

$$M = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \mid \operatorname{rank}(\nabla f(x), y) < 2\}.$$

Let $p = (x_0, y_0)$ be a point in $M \setminus (\Sigma_f \times \mathbb{R}^n)$. We can assume that $f_{x_1}(x_0) \neq 0$. Therefore locally $M \setminus (\Sigma_f \times \mathbb{R}^n)$ is given by the equations $m_{12}(x, y) = \cdots = m_{1n}(x, y) = 0$, where

$$m_{ij}(x, y) = \begin{vmatrix} f_{x_i}(x) & f_{x_j}(x) \\ y_i & y_j \end{vmatrix}.$$

The Jacobian matrix of the mapping $(m_{12}, \ldots, m_{1n})$ has the following form

$$\begin{pmatrix} * & \cdots & * & -f_{x_2} & f_{x_1} & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ * & \cdots & * & -f_{x_n} & 0 & \cdots & f_{x_1} \end{pmatrix}.$$

This implies that $M \setminus (\Sigma_f \times \mathbb{R}^n)$ is a C^{r-1} manifold of dimension $n+1$. The Bertini-Sard theorem ([3], 9.5.2) implies that the discriminant D of the projection

$$\begin{array}{cccc} \pi_y & : & M \setminus (\Sigma_f \times \mathbb{R}^n) & \to & \mathbb{R}^n \\ & & (x, y) & \mapsto & y \end{array}$$

is a definable set of dimension less than or equal to $n-1$. Hence for all $v \in S^{n-1} \setminus D$, the dimension of $\pi_y^{-1}(v)$ is less than or equal to 1. But $\pi_y^{-1}(v)$ is exactly Γ_v and we set $\Sigma_2 = D \cap S^{n-1}$. □

Corollary 4.5. *Let $v \in S^{n-1}$ be such that $v \notin \Sigma_2$. There exists δ'_v such that for $0 < |\delta| \leq \delta'_v$, the critical points of $v^*_{|f^{-1}(\delta)}$ are Morse critical points in a neighborhood of the origin.*

Proof. After a change of coordinates, we can assume that $v = e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^n$ and so that $v^*(x) = x_1$.

Let p be a point in $\Gamma_v = \Gamma_{e_1}$. If $f_{x_1}(p) = 0$ then, since the minors $m_{1i} = \frac{\partial(f, x_1)}{\partial(x_1, x_i)}$, $i = 2, \ldots, n$, vanish at p, $f_{x_i}(p) = 0$ for $i = 2, \ldots, n$ and so p belongs to Σ_f, which is impossible. Therefore $f_{x_1}(p) \neq 0$ and by the proof of Lemma 4.4, we conclude that Γ_{e_1} is defined by the vanishing of the minors m_{1i}, $i = 2, \ldots, n$, and that

$$\operatorname{rank}\big(\nabla m_{12}, \ldots, m_{1n}\big) = n - 1$$

along Γ_{e_1}. Let $\mathbf{a}$ be an arc (i.e., a connected component) of Γ_{e_1}, and let $\alpha : [0, \nu[\to \bar{\mathbf{a}}$ be a C^r definable parametrization such that $\alpha(0) = 0$ and $\alpha(]0, \nu[) \subset \mathbf{a}$. Since f does not vanish on $\mathbf{a}$, the function $f \circ \alpha$ is strictly monotone which implies that for $s \in]0, \nu[$,

$$(f \circ \alpha)'(s) = \langle \nabla f(\alpha(s)), \alpha'(s) \rangle \neq 0.$$

Hence the vectors

$$\nabla f(\alpha(s)), \nabla m_{12}(\alpha(s)), \ldots, \nabla m_{1n}(\alpha(s))$$

are linearly independent since the $\nabla m_{1i}(\alpha(s))$'s are orthogonal to $\alpha'(s)$. By Lemma 3.2 in [33], this is equivalent to the fact that the function $x_1 : f^{-1}(f(\alpha(s)) \to \mathbb{R}$ has a non-degenerate critical point at $\alpha(s)$. It is easy to conclude because Γ_{e_1} has a finite numbers of arcs. □

From now on, we will work with $v \in S^{n-1}$ such that $v \notin \Sigma_1 \cup \Sigma_2$. After a change of coordinates, we can assume that $v = e_1 = (1, 0, \ldots, 0)$ and so the conclusions of Lemma 4.1, Lemma 4.2, Corollary 4.3, Lemma 4.4 and Corollary 4.5 are valid for Γ_{x_1} and $\{x_1 = 0\}$. Let us study the points of Γ_{x_1} more accurately. By the previous results, we know that if p is a point of Γ_{x_1} close to the origin then p is a Morse critical point of $x_{1|f^{-1}(f(p))}$, $f_{x_1}(p) \neq 0$, $x_1(p) \neq 0$ and $f(p) \neq 0$.

Lemma 4.6. *Let p be a point in Γ_{x_1} close to the origin. Let $\mu(p)$ be the Morse index of $x_{1|f^{-1}(f(p))}$ at p. Then p is a Morse critical point of $f_{|x_1^{-1}(x_1(p))}$ and if $\theta(p)$ is the Morse index of $f_{|x_1^{-1}(x_1(p))}$ at p then*

$$(-1)^{\mu(p)} = (-1)^{n-1} \mathrm{sign}(f_{x_1}(p))^{n-1} (-1)^{\theta(p)}.$$

Proof. By Lemma 3.2 in [33], we know that

$$\det\big[\nabla f(p), \nabla f_{x_2}(p), \ldots, \nabla f_{x_n}(p)\big] \neq 0,$$

and that

$$(-1)^{\mu(p)} = (-1)^{n-1} \mathrm{sign}(f_{x_1}(p))^n \mathrm{sign}\ \det\big[\nabla f(p), \nabla f_{x_2}(p), \ldots, \nabla f_{x_n}(p)\big].$$

But $\nabla f(p) = f_{x_1}(p) e_1$ and so $\det\big[e_1, \nabla f_{x_2}(p), \ldots, \nabla f_{x_n}(p)\big] \neq 0$ and

$$(-1)^{\mu(p)} = (-1)^{n-1} \mathrm{sign}(f_{x_1}(p))^{n-1} \mathrm{sign}\ \det\big[e_1, \nabla f_{x_2}(p), \ldots, \nabla f_{x_n}(p)\big].$$

Still using Lemma 3.2 in [33], we see that p is a Morse critical point of $f_{|x_1^{-1}(x_1(p))}$ and that

$$(-1)^{\mu(p)} = (-1)^{n-1} \mathrm{sign}(f_{x_1}(p))^{n-1} (-1)^{\theta(p)}.$$

□

Lemma 4.7. *Let p be a point in Γ_{x_1} close to the origin. Then*

$$\det\big[\nabla f_{x_1}(p), \nabla f_{x_2}(p), \ldots, \nabla f_{x_n}(p)\big] \neq 0$$

and

$$(-1)^{\theta(p)} = \mathrm{sign}\,(x_1(p) f_{x_1}(p))\, \mathrm{sign}\ \det\big[\nabla f_{x_1}(p), \nabla f_{x_2}(p), \ldots, \nabla f_{x_n}(p)\big].$$

Proof. Since $\det\big[e_1, \nabla f_{x_2}(p), \ldots, \nabla f_{x_n}(p)\big] \neq 0$, we can write

$$\nabla f_{x_1}(p) = \beta(p) e_1 + \sum_{i=2}^{n} \beta_i(p) \nabla f_{x_i}(p),$$

and so,

$$\det\big[\nabla f_{x_1}(p), \nabla f_{x_2}(p), \ldots, \nabla f_{x_n}(p)\big] = \beta(p) \det\big[e_1, \nabla f_{x_2}(p), \ldots, \nabla f_{x_n}(p)\big].$$

Let $\alpha : [0, \nu[\to \overline{\Gamma_{x_1}}$ be a parametrization of the arc that contains p. We have

$$(f_{x_1} \circ \alpha)'(s) = \langle \nabla f_{x_1}(\alpha(s)), \alpha'(s) \rangle = \beta(\alpha(s)) \langle e_1, \alpha'(s) \rangle = \beta(\alpha(s))(x_1 \circ \alpha)'(s).$$

But since f_{x_1} and x_1 do not vanish on Γ_{x_1}, $(f_{x_1} \circ \alpha)'(s)$ and $(x_1 \circ \alpha)'(s)$ do not vanish for s small. Therefore for p close to the origin, $\beta(p) \neq 0$ and

$$\mathrm{sign}\ \beta(p) = \mathrm{sign}\,(x_1(p) f_{x_1}(p))\,.$$

□

Let $\mathcal{B}$ be the set of connected components of Γ_{x_1}. If $\mathbf{b} \in \mathcal{B}$ then $\mathbf{b}$ is a half-branch on which the functions f_{x_1} and $\det\big[\nabla f_{x_1}, \ldots, \nabla f_{x_n}\big]$ have constant sign. So we can decompose $\mathcal{B}$ into the disjoint union $\mathcal{B}^+ \sqcup \mathcal{B}^-$ where $\mathcal{B}^+$ (resp. $\mathcal{B}^-$) is the set of half-branches on which $f_{x_1} > 0$ (resp. $f_{x_1} < 0$). If $\mathbf{b} \in \mathcal{B}$, we denote by $\sigma(\mathbf{b})$ the sign of $\det\big[\nabla f_{x_1}, \ldots, \nabla f_{x_n}\big]$ on $\mathbf{b}$.

Definition 4.8. We set $\lambda^+ = \sum_{\mathbf{b} \in \mathcal{B}^+} \sigma(\mathbf{b})$ and $\lambda^- = \sum_{\mathbf{b} \in \mathcal{B}^-} \sigma(\mathbf{b})$.

Remark 4.9. If f has an isolated critical point at the origin then for $\eta \neq 0$ small enough, $(\nabla f)^{-1}(\eta, 0, \ldots, 0)$ is exactly $\Gamma_{x_1} \cap f_{x_1}^{-1}(\eta)$. Moreover if $p \in \Gamma_{x_1} \cap f_{x_1}^{-1}(\eta)$, then

$$\text{sign} \det\big[\nabla f_{x_1}(p), \nabla f_{x_2}(p), \ldots, \nabla f_{x_n}(p)\big] \neq 0.$$

Hence $(\eta, 0, \ldots, 0)$ is a regular value of ∇f and so

$$\deg_0 \nabla f = \sum_{p \in \Gamma_{x_1} \cap f_{x_1}^{-1}(\eta)} \text{sign} \det\big[\nabla f_{x_1}(p), \nabla f_{x_2}(p), \ldots, \nabla f_{x_n}(p)\big].$$

If $\eta > 0$ (resp. $\eta < 0$), this implies that $\deg_0 \nabla f = \lambda^+$ (resp. λ^-).

The following lemma will enable us to define other indices associated with f and x_1.

Lemma 4.10. *There exists $\epsilon_0 > 0$ such that for $0 < \epsilon \leq \epsilon_0$, there exists $a_\epsilon > 0$ such that for $0 < a \leq a_\epsilon$, there exists $\alpha_{a,\epsilon} > 0$ such that for $0 < \alpha \leq \alpha_{a,\epsilon}$, the topological type of $f^{-1}(\alpha) \cap \{x_1 = a\} \cap B_\epsilon$ does not depend on the choice of the triple (ϵ, a, α).*

Proof. For $a > 0$ small enough, we define $\beta(a)$ by

$$\beta(a) = \inf\big\{|f(p)| \mid p \in \Gamma_{x_1} \cap \{x_1 = a\}\big\}.$$

The function β is well defined because $\Gamma_{x_1} \cap \{x_1 = a\}$ is finite and $\beta(a) > 0$. Moreover it is definable and so it is continuous on a small interval of the form $]0, u[$. This implies that the set

$$\mathcal{O} = \{(a, \alpha) \in \mathbb{R} \times \mathbb{R}^* \mid a \in]0, u[\text{ and } 0 < \alpha < \beta(a)\}$$

is open and connected.

Since $\{x_1 = 0\}$ intersects $f^{-1}(0) \setminus \{0\}$ transversally (in the stratified sense),

$$\{x_1 = 0\} \cap f^{-1}(0) \setminus \{0\}$$

is Whitney stratified, the strata being the intersections of $\{x_1 = 0\}$ with the strata of $f^{-1}(0) \setminus \{0\}$.

Let $\epsilon_0 > 0$ be such that for $0 < \epsilon \leq \epsilon_0$, the sphere S_ϵ intersects $\{x_1 = 0\} \cap f^{-1}(0)$ transversally. Then there exists a neighborhood $\mathcal{U}_\epsilon$ of $(0,0)$ in $\mathbb{R}^2$ such that for each (a, α) in $(\mathbb{R} \times \mathbb{R}^*) \cap \mathcal{U}_\epsilon$, the fibre $f^{-1}(\alpha) \cap \{x_1 = a\}$ intersects S_ϵ transversally. If it is not the case, then we can find a sequence of points $(p_m)_{m \in \mathbb{N}}$ in S_ϵ such that the vectors e_1, $\frac{p_m}{|p_m|}$ and $\frac{\nabla f(p_m)}{|\nabla(p_m)|}$ are linearly dependent, and such that the sequence converges to a point p in $S_\epsilon \cap f^{-1}(0) \cap \{x_1 = 0\}$. If S denotes the stratum of $f^{-1}(0)$ that contains p then, applying the Thom (a_f)-condition and the method of Lemma 3.7 in [11], there exists a unit vector v normal to $T_p S$ such that the vectors e_1, $\frac{p}{|p|}$ and v are linearly dependent. But e_1 and v are linearly independent for $\{x_1 = 0\}$ intersects S transversally at p. Therefore S_ϵ does not intersect $S \cap \{x_1 = 0\}$ transversally at p, which is a contradiction. Moreover we can assume that $\mathcal{U}_\epsilon \cap \mathcal{O}$ is connected.

Now let us fix $\epsilon > 0$ with $\epsilon \leq \epsilon_0$. Let us choose $a_\epsilon > 0$ such that $a_\epsilon < u$ and the interval $]0, a_\epsilon]$ is included in $\mathcal{U}_\epsilon$. For each $a \in]0, a_\epsilon]$, there exists $\alpha'_{a,\epsilon}$ such that $\{a\} \times]0, \alpha'_{a,\epsilon}]$ is included in $\mathcal{U}_\epsilon$. We choose $\alpha_{a,\epsilon}$ such that $\alpha_{a,\epsilon} \leq \alpha'_{a,\epsilon}$ and $\alpha_{a,\epsilon} < \beta(a)$, which implies that (a, α) is a regular value of (x_1, f) for $0 < \alpha \leq \alpha_{a,\epsilon}$.

Let $(\epsilon_1, a_1, \alpha_1)$ and $(\epsilon_2, a_2, \alpha_2)$ be two triples with $0 < \epsilon_i \leq \epsilon$, $0 < a_i \leq a_{\epsilon_i}$ and $0 < \alpha_i \leq \alpha_{a_i, \epsilon_i}$ for $i = 1, 2$. If $\epsilon_1 = \epsilon_2$ then the Thom-Mather first isotopy lemma implies that the fibres

$f^{-1}(\alpha_1) \cap \{x_1 = a_1\} \cap B_{\epsilon_1}$ and $f^{-1}(\alpha_2) \cap \{x_1 = a_2\} \cap B_{\epsilon_2}$ are homeomorphic, because (a_1, α_1) and (a_2, α_2) belong to the connected set $\mathcal{U}_{\epsilon_1} \cap \mathcal{O}$.

Now assume that $\epsilon_1 < \epsilon_2$. By the same arguments as above, there exists a neighborhood $\mathcal{U}$ of $(0,0)$ in $\mathbb{R}^2$ such that for each $(a, \alpha) \in (\mathbb{R} \times \mathbb{R}^*) \cap \mathcal{U}$, the distance function to the origin has no critical point on

$$f^{-1}(\alpha) \cap \{x_1 = a\} \cap (B_{\epsilon_2} \setminus \mathring{B}_{\epsilon_1}).$$

Let us choose $(a_3, \alpha_3) \in (\mathbb{R} \times \mathbb{R}^*) \cap \mathcal{U}$ such that

$$0 < a_3 \leq \min\{a_{\epsilon_1}, a_{\epsilon_2}\} \quad \text{and} \quad \alpha_3 \leq \min\{\alpha_{a_3,\epsilon_1}, \alpha_{a_3,\epsilon_2}\}.$$

Then, by the first case, $f^{-1}(\alpha_3) \cap \{x_1 = a_3\} \cap B_{\epsilon_1}$ is homemorphic to $f^{-1}(\alpha_1) \cap \{x_1 = a_1\} \cap B_{\epsilon_1}$ and $f^{-1}(\alpha_3) \cap \{x_1 = a_3\} \cap B_{\epsilon_2}$ is homemorphic to $f^{-1}(\alpha_2) \cap \{x_1 = a_2\} \cap B_{\epsilon_2}$. But, since the distance function to the origin has no critical point on $f^{-1}(\alpha_3) \cap \{x_1 = a_3\} \cap B_{\epsilon_2} \setminus \mathring{B}_{\epsilon_1}$, the fibre $f^{-1}(\alpha_3) \cap \{x_1 = a_3\} \cap B_{\epsilon_1}$ is homeomorphic to $f^{-1}(\alpha_3) \cap \{x_1 = a_3\} \cap B_{\epsilon_2}$. $\square$

Similarly, there exists $\epsilon_0' > 0$ such that for $0 < \epsilon \leq \epsilon_0'$, there exists $b_\epsilon > 0$ such that for $0 < a \leq b_\epsilon$, the topological types of $f^{-1}(0) \cap \{x_1 = a\} \cap B_\epsilon$ and $f^{-1}(0) \cap \{x_1 = -a\} \cap B_\epsilon$ do not depend on the choice of (ϵ, a). Therefore we can make the following definition.

Definition 4.11. We set

$$\begin{aligned}
\gamma^{+,+} &= \chi\big(f^{-1}(0) \cap \{x_1 = a\} \cap B_\epsilon\big) - \chi\big(f^{-1}(\alpha) \cap \{x_1 = a\} \cap B_\epsilon\big), \\
\gamma^{+,-} &= \chi\big(f^{-1}(0) \cap \{x_1 = -a\} \cap B_\epsilon\big) - \chi\big(f^{-1}(\alpha) \cap \{x_1 = -a\} \cap B_\epsilon\big), \\
\gamma^{-,+} &= \chi\big(f^{-1}(0) \cap \{x_1 = a\} \cap B_\epsilon\big) - \chi\big(f^{-1}(-\alpha) \cap \{x_1 = a\} \cap B_\epsilon\big), \\
\gamma^{-,-} &= \chi\big(f^{-1}(0) \cap \{x_1 = -a\} \cap B_\epsilon\big) - \chi\big(f^{-1}(-\alpha) \cap \{x_1 = -a\} \cap B_\epsilon\big),
\end{aligned}$$

where $0 < \alpha \ll a \ll \epsilon$.

Now we are in position to state the generalization of the Khimshiashvili formula. Remember that e_1 satisfies the genericity conditions of Lemmas 4.1 and 4.4.

Theorem 4.12. *Assume that* $e_1 \notin \Sigma_1 \cup \Sigma_2$. *For* $0 < \delta \ll \epsilon \ll 1$, *we have*

$$\chi\big(f^{-1}(-\delta) \cap B_\epsilon\big) = 1 - \lambda^- - \gamma^{-,-} = 1 - \lambda^+ - \gamma^{-,+},$$

$$\chi\big(f^{-1}(\delta) \cap B_\epsilon\big) = 1 - (-1)^n \lambda^- - \gamma^{+,+} = 1 - (-1)^n \lambda^+ - \gamma^{+,-}.$$

Proof. The set of critical points of x_1 on $f^{-1}(-\delta) \cap \mathring{B}_\epsilon$ is exactly $\Gamma_{x_1} \cap f^{-1}(-\delta)$. Moreover we know that if $p \in \Gamma_{x_1} \cap f^{-1}(-\delta)$ then $x_1(p) \neq 0$. By Morse theory, we have

$$\chi\big(f^{-1}(-\delta) \cap B_\epsilon \cap \{x_1 \geq 0\}\big) - \chi\big(f^{-1}(-\delta) \cap B_\epsilon \cap \{x_1 = 0\}\big) = \sum_{\substack{p \in \Gamma_{x_1} \cap f^{-1}(-\delta) \\ x_1(p) > 0}} (-1)^{\mu(p)},$$

$$\chi\big(f^{-1}(-\delta) \cap B_\epsilon \cap \{x_1 \leq 0\}\big) - \chi\big(f^{-1}(-\delta) \cap B_\epsilon \cap \{x_1 = 0\}\big) = (-1)^{n-1} \cdot \sum_{\substack{p \in \Gamma_{x_1} \cap f^{-1}(-\delta) \\ x_1(p) < 0}} (-1)^{\mu(p)}.$$

Here we remark that $f^{-1}(\delta) \cap B_\epsilon$ is a manifold with boundary and x_1 may have critical points on the boundary. But by Lemma 3.7 in [11], these critical points lie in $\{x_1 \neq 0\}$ and are outwards-pointing (resp. inwards-pointing) in $\{x_1 > 0\}$ (resp. $\{x_1 < 0\}$). That is why they do not appear in the above two formulas. Adding the two equalities and applying the Mayer-Vietoris sequence, we obtain

$$\chi\big(f^{-1}(-\delta) \cap B_\epsilon\big) - \chi\big(f^{-1}(-\delta) \cap B_\epsilon \cap \{x_1 = 0\}\big) =$$
$$\sum_{\substack{p \in \Gamma_{x_1} \cap f^{-1}(-\delta) \\ x_1(p) > 0}} (-1)^{\mu(p)} + (-1)^{n-1} \sum_{\substack{p \in \Gamma_{x_1} \cap f^{-1}(-\delta) \\ x_1(p) < 0}} (-1)^{\mu(p)}.$$

Since $\nabla f = f_{x_1} e_1$ on Γ_{x_1}, it is easy to check that p belongs to $\Gamma_{x_1} \cap \{f < 0\} \cap \{x_1 > 0\}$ if and only if p belongs to $\Gamma_{x_1} \cap \{f_{x_1} < 0\} \cap \{x_1 > 0\}$ and p belongs to $\Gamma_{x_1} \cap \{f < 0\} \cap \{x_1 < 0\}$ if and only if p belongs to $\Gamma_{x_1} \cap \{f_{x_1} > 0\} \cap \{x_1 < 0\}$. Let us decompose $\mathcal{B}^+$ into the disjoint union $\mathcal{B}^+ = \mathcal{B}^{+,+} \sqcup \mathcal{B}^{+,-}$ where $\mathcal{B}^{+,+}$ (resp. $\mathcal{B}^{+,-}$) is the set of half-branches of $\mathcal{B}^+$ on which $x_1 > 0$ (resp. $x_1 < 0$). Similarly we can write $\mathcal{B}^- = \mathcal{B}^{-,+} \sqcup \mathcal{B}^{-,-}$. Combining Lemma 4.6 and Lemma 4.7, we can rewrite the above equality in the following form:

$$\chi\big(f^{-1}(-\delta) \cap B_\epsilon\big) - \chi\big(f^{-1}(-\delta) \cap B_\epsilon \cap \{x_1 = 0\}\big) = -\sum_{\mathbf{b} \in \mathcal{B}^{-,+}} \sigma(\mathbf{b}) - \sum_{\mathbf{b} \in \mathcal{B}^{+,-}} \sigma(\mathbf{b}). \qquad (1)$$

Since $(-\delta, 0)$ is a regular value of (f, x_1) then there exists $a_\delta > 0$ such that for $0 < a \leq a_\delta$, $(-\delta, \pm a)$ are regular value of (f, x_1) and

$$\chi\big(f^{-1}(-\delta) \cap \{x_1 = \pm a\} \cap B_\epsilon\big) = \chi\big(f^{-1}(-\delta) \cap \{x_1 = 0\} \cap B_\epsilon\big).$$

Let us fix a such that $0 < a \leq a_\delta$ and let us relate $\chi\big(f^{-1}(-\delta) \cap \{x_1 = -a\} \cap B_\epsilon\big)$ to $\chi\big(f^{-1}(\alpha) \cap \{x_1 = -a\} \cap B_\epsilon\big)$ where $0 < \alpha \ll a$. Note that the set of critical points of f on $\{x_1 = -a\} \cap \mathring{B}_\epsilon$ is exactly $\Gamma_{x_1} \cap \{x_1 = -a\}$. Moreover this set of critical points is included in $\{f > -\delta\}$. Indeed, if it is not the case, then there is a half-branch of Γ_{x_1} that intersects $\{x_1 = -a\}$ on $\{f \leq -\delta\}$. But since x_1 and f are negative on this branch, this would imply that Γ_{x_1} intersects $\{f = -\delta\}$ on $\{-a \leq x_1 < 0\}$, which is not possible for $a \leq a_\delta$.

Now let us look at the critical points of f on $\{x_1 = -a\} \cap S_\epsilon$. In the proof of Lemma 4.10, we established the existence of a neighborhood $\mathcal{U}_\epsilon$ of $(0,0)$ in $\mathbb{R}^2$ such that for each $(a, \alpha) \in (\mathbb{R} \times \mathbb{R}^*) \cap \mathcal{U}_\epsilon$, the fibre $f^{-1}(\alpha) \cap \{x_1 = a\}$ intersects S_ϵ transversally. Therefore we can choose δ such that the critical points of f on $\{x_1 = 0\} \cap S_\epsilon \cap \{f \neq 0\}$ lie in $\{|f| > \delta\}$. Moreover by a Curve Selection Lemma argument, they are outwards-pointing in $\{f > \delta\}$ and inwards-pointing in $\{f < -\delta\}$. So, if a is small enough, then the critical points of f on $\{x_1 = -a\} \cap S_\epsilon \cap \{f \neq 0\}$, lying in $\{|f| > \delta\}$, are outwards-pointing (resp. inwards-pointing) in $\{f > \delta\}$ (resp. $\{f < -\delta\}$). By Morse theory, we find that

$$\chi\big(\{f \leq -\delta\} \cap \{x_1 = -a\} \cap B_\epsilon\big) - \chi\big(\{f = -\delta\} \cap \{x_1 = -a\} \cap B_\epsilon\big) = 0$$

$$\chi\big(\{-\delta \leq f \leq -\alpha\} \cap \{x_1 = -a\} \cap B_\epsilon\big) - \chi\big(\{f = -\delta\} \cap \{x_1 = -a\} \cap B_\epsilon\big) = \sum_{\substack{p \in \Gamma_{x_1} \cap \{x_1 = -a\} \\ f(p) < 0}} (-1)^{\theta(p)},$$

$$\chi\big(\{f \geq \alpha\} \cap \{x_1 = -a\} \cap B_\epsilon\big) - \chi\big(\{f = \alpha\} \cap \{x_1 = -a\} \cap B_\epsilon\big) = \sum_{\substack{p \in \Gamma_{x_1} \cap \{x_1 = -a\} \\ f(p) > 0}} (-1)^{\theta(p)}.$$

By the Mayer-Vietoris sequence, we have that

$$\begin{aligned}
1 = \chi\big(\{x_1 = -a\} \cap B_\epsilon\big) &= \chi\big(\{f \leq -\delta\} \cap \{x_1 = -a\} \cap B_\epsilon\big) \\
&+ \chi\big(\{-\delta \leq f \leq -\alpha\} \cap \{x_1 = -a\} \cap B_\epsilon\big) - \chi\big(\{f = -\delta\} \cap \{x_1 = -a\} \cap B_\epsilon\big) \\
&- \chi\big(\{f = -\alpha\} \cap \{x_1 = -a\} \cap B_\epsilon\big) + \chi\big(\{-\alpha \leq f \leq \alpha\} \cap \{x_1 = -a\} \cap B_\epsilon\big) \\
&+ \chi\big(\{f \geq \alpha\} \cap \{x_1 = -a\} \cap B_\epsilon\big) - \chi\big(\{f = \alpha\} \cap \{x_1 = -a\} \cap B_\epsilon\big).
\end{aligned}$$

Using the fact that the inclusion

$$\{f = 0\} \cap \{x_1 = -a\} \cap B_\epsilon \subset \{-\alpha \leq f \leq \alpha\} \cap \{x_1 = -a\} \cap B_\epsilon$$

is a homotopy equivalence and applying the above equalities, we get

$$1 = \sum_{\substack{p \in \Gamma_{x_1} \cap \{x_1=-a\} \\ f(p)<0}} (-1)^{\theta(p)} + \sum_{\substack{p \in \Gamma_{x_1} \cap \{x_1=-a\} \\ f(p)>0}} (-1)^{\theta(p)} + \chi\big(f^{-1}(-\delta) \cap \{x_1=-a\} \cap B_\epsilon\big)$$

$$+\chi\big(f^{-1}(0) \cap \{x_1=-a\} \cap B_\epsilon\big) - \chi\big(f^{-1}(-\alpha) \cap \{x_1=-a\} \cap B_\epsilon\big).$$

By Lemma 4.7, we can rewrite this equality in the following form:

$$1 = -\sum_{\mathbf{b} \in \mathcal{B}^{+,-}} \sigma(\mathbf{b}) + \sum_{\mathbf{b} \in \mathcal{B}^{-,-}} \sigma(\mathbf{b}) + \chi\big(f^{-1}(-\delta) \cap \{x_1=-a\} \cap B_\epsilon\big) + \gamma^{-,-}. \quad (2)$$

Combining (1) and (2), we obtain the first equality of the statement. The second one is obtained replacing $-a$ with a in the above discussion. The third and fourth ones are obtained replacing f with $-f$. ☐

Remark 4.13. (1) If f has an isolated critical point at the origin then we recover the Khimshiashvili formula because

$$\gamma^{-,-} = \gamma^{+,+} = \gamma^{+,-} = \gamma^{-,+} = 0$$

and $\lambda^+ = \lambda^- = \deg_0 \nabla f$.

(2) If we denote by S_0 the stratum that contains 0 and if we assume that dim $S_0 > 0$, then by the Thom (a_f)-condition, the polar curve Γ_v is empty in a neighborhood of 0 if $v \notin S^{n-1} \cap (T_x S_0)^\perp$. Then applying Equality (1) of the previous proof, we recover Corollary 3.5. Actually, we can say more about the relation between the topologies of $f^{-1}(\pm\delta) \cap B_\epsilon$ and $f^{-1}(\pm\delta) \cap B_\epsilon \cap \{v^* = 0\}$. As mentionned in the proof of Theorem 4.12, the critical points of v^* restricted to $f^{-1}(\pm\delta) \cap S_\epsilon$ lie in $\{v^* \neq 0\}$ and are outwards-pointing (resp. inwards-pointing) in $\{v^* > 0\}$ (resp. $\{v^* < 0\}$). So we can apply the arguments of the proof of Theorem 6.3 in [13] to get that $f^{-1}(\pm\delta) \cap B_\epsilon$ is homeomorphic to $f^{-1}(\pm\delta) \cap B_\epsilon \cap \{v^* = 0\} \times [-1,1]$.

5. One dimensional critical locus and a real Lê-Iomdine formula

In this section, we apply the results of Section 4 to the case of a one-dimensional singular set, in order to establish a real version of the Lê-Iomdine formula.

Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be a definable function-germ of class C^r, $2 \leq r$. We assume that dim $\Sigma_f = 1$. In a neighborhood of the origin, the partition

$$\big(f^{-1}(0) \setminus \Sigma_f, \Sigma_f \setminus \{0\}, \{0\}\big)$$

gives a Whitney stratification of $f^{-1}(0)$ which satisfies the Thom (a_f)-condition, because the points where the Whitney conditions and the Thom (a_f)-condition may fail form a 0-dimensional definable set of $\Sigma_f \setminus \{0\}$. Let $\mathcal{C}$ be the set of half-branches of Σ_f, i.e., the set of connected components of $\Sigma_f \setminus \{0\}$.

Lemma 5.1. *There exists a definable set $\Sigma_3 \subset S^{n-1}$ of positive codimension such that if $v \notin \Sigma_3$, v^* does not vanish on $\Sigma_f \setminus \{0\}$ in a neighborhood of the origin.*

Proof. Let $\mathbf{c} \in \mathcal{C}$. If v^* vanishes on $\mathbf{c}$ in a neighborhood of the origin then, if $u \neq 0$ is on $C_0\mathbf{c}$ (the tangent cone at $\mathbf{c}$ at the origin) then $v^*(u) = 0$ and so $v \in u^\perp$. So if $v \notin \cup_{\mathbf{c} \in \mathcal{C}} (C_0\mathbf{c})^\perp$ then v^* does not vanish on $\Sigma_f \setminus \{0\}$. But $(\cup_{\mathbf{c} \in \mathcal{C}} (C_0\mathbf{c})^\perp) \cap S^{n-1}$ has dimension less than or equal to $n-2$. ☐

From now on, we assume that $e_1 \in S^{n-1}$ is generic, i.e., $e_1 \notin \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$. Since $e_1 \notin \Sigma_3$, there exists $a_1 > 0$ such that for $0 < a \leq a_1$, $x_1^{-1}(\pm a)$ intersects $\Sigma_f \setminus \{0\}$ transversally and so, the points in $x_1^{-1}(\pm a) \cap (\Sigma_f \setminus \{0\})$ are isolated critical points of $f_{|\{x_1=\pm a\}}$. For $q \in x_1^{-1}(\pm a) \cap (\Sigma_f \setminus \{0\})$, we denote by $\deg_q \nabla f_{|x_1^{-1}(\pm a)}$ the topological degree of the mapping

$$\frac{\nabla f_{|x_1^{-1}(\pm a)}}{|\nabla f_{|x_1^{-1}(\pm a)}|} : x_1^{-1}(\pm a) \cap S_{\epsilon'}(q) \to S^{n-2},$$

where $S_{\epsilon'}(q)$ is the sphere centered at q of radius ϵ' with $0 < \epsilon' \ll 1$.

Let us write $\mathcal{C} = \mathcal{C}^+ \sqcup \mathcal{C}^-$ where $\mathcal{C}^+$ (resp. $\mathcal{C}^-$) is the set of half-branches of $\mathcal{C}$ on which $x_1 > 0$ (resp. $x_1 < 0$).

Lemma 5.2. *Let $\mathbf{c} \in \mathcal{C}^+$. There exists $a_{\mathbf{c}} > 0$ such that the function $a \mapsto \deg_q \nabla f_{|x_1^{-1}(a)}$, where $\{q\} = \mathbf{c} \cap \{x_1 = a\}$, is constant on $]0, a_{\mathbf{c}}]$.*

Proof. It is enough to prove that there exists an interval $]0, a_{\mathbf{c}}]$ on which the function $a \mapsto \deg_q \nabla f_{|x_1^{-1}(a)}$ is locally constant. Let $d : \mathbb{R}^n \to \mathbb{R}$ be the distance function to $\mathbf{c}$. It is a continuous definable function and there exists an open definable neighbourhood $\mathcal{U}$ of $\mathbf{c}$ such that d is smooth on $\mathcal{U} \setminus \mathbf{c}$. Moreover we can assume that d is a (stratified) submersion on $\{f \leq 0\} \cap (\mathcal{U} \setminus \mathbf{c})$.

Let $\pi : \{f \leq 0\} \cap (\mathcal{U} \setminus \mathbf{c}) \to \mathbb{R}^2$ be the mapping defined by $\pi(p) = (x_1(p), d(p))$ and let $\Delta \subset \mathbb{R}^2$ be its (stratified) discriminant. It is a definable curve included in $\mathbb{R} \times \mathbb{R}^*$ and so $\bar{\Delta} \cap (\mathbb{R} \times \{0\})$ is a finite number of points. Let us choose $a_{\mathbf{c}} > 0$ such that

$$a_{\mathbf{c}} < \min\left\{x_1(u) \mid u \in \bar{\Delta} \cap (\mathbb{R}^* \times \{0\})\right\}.$$

If $0 < a \leq a_{\mathbf{c}}$, then there exists $t > 0$ and $\epsilon > 0$ such that $]a - t, a + t[\times]0, \epsilon[$ does not meet Δ. Hence the function

$$\begin{array}{ccc}]a-t, a+t[\times]0,\epsilon[& \to & \mathbb{R} \\ (a', \epsilon') & \mapsto & \chi\big(\{f \leq 0\} \cap \{x_1 = a'\} \cap \{d = \epsilon'\}\big) \end{array}$$

is constant. Therefore by Corollary 2.4, the function $a' \mapsto \deg_{q'} \nabla f_{|x_1^{-1}(a')}$ is constant on $]a - t, a + t[$. $\square$

Of course, a similar result is valid for $\mathbf{c} \in \mathcal{C}^-$. If $\mathbf{c} \in \mathcal{C}$, let us denote by $\tau(\mathbf{c})$ the value that the function $a \mapsto \deg_q \nabla f_{|x_1^{-1}(a)}$, $\{q\} = \mathbf{c} \cap \{x_1 = a\}$, takes close to the origin.

Definition 5.3. We set $\gamma^+ = \sum_{\mathbf{c} \in \mathcal{C}^+} \tau(\mathbf{c})$ and $\gamma^- = \sum_{\mathbf{c} \in \mathcal{C}^-} \tau(\mathbf{c})$.

In this setting, Theorem 4.12 admits the following formulation.

Theorem 5.4. *Assume that $e_1 \notin \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$. For $0 < \delta \ll \epsilon \ll 1$, we have*

$$\chi\big(f^{-1}(-\delta) \cap B_\epsilon\big) = 1 - \lambda^- - \gamma^- = 1 - \lambda^+ - \gamma^+,$$

$$\chi\big(f^{-1}(\delta) \cap B_\epsilon\big) = 1 - (-1)^n(\lambda^+ - \gamma^-) = 1 - (-1)^n(\lambda^- - \gamma^+).$$

Proof. Since $\Gamma_{x_1} \cap f^{-1}(0) = \emptyset$, the critical points of $f_{|\{x_1=\pm a\}}$ in $f^{-1}([-\alpha, \alpha])$, $0 < \alpha \ll \delta \ll \epsilon$, are exactly the points in $\Sigma_f \cap \{x_1 = \pm a\}$. An easy adaptation of the proof of the Khimshiashvili formula (Theorem 2.3) gives that

$$\gamma^{-,-} = \gamma^-,\ \gamma^{-,+} = \gamma^+,\ \gamma^{+,+} = (-1)^{n-1}\gamma^+ \text{ and } \gamma^{+,-} = (-1)^{n-1}\gamma^-.$$

$\square$

We remark that $\lambda^- + \gamma^- = \lambda^+ + \gamma^+$. Moreover, if n is even, we have

$$\chi\big(f^{-1}(\delta) \cap B_\epsilon\big) - \chi\big(f^{-1}(-\delta) \cap B_\epsilon\big) = \gamma^+ + \gamma^-,$$

$$\chi\big(f^{-1}(\delta) \cap B_\epsilon\big) + \chi\big(f^{-1}(-\delta) \cap B_\epsilon\big) = 2 - (\lambda^+ + \lambda^-),$$

and if n is odd, we have

$$\chi\big(f^{-1}(\delta) \cap B_\epsilon\big) - \chi\big(f^{-1}(-\delta) \cap B_\epsilon\big) = \lambda^+ + \lambda^-,$$

$$\chi\big(f^{-1}(\delta) \cap B_\epsilon\big) + \chi\big(f^{-1}(-\delta) \cap B_\epsilon\big) = 2 - (\gamma^+ + \gamma^-).$$

Therefore the two sums $\sum_{\mathbf{b}\in\mathcal{B}} \sigma(\mathbf{b})$ and $\sum_{\mathbf{c}\in\mathcal{C}} \tau(\mathbf{c})$ do not depend on the generic choice of linear function that we used to define them. Moreover, applying Lemma 2.5, we get that if n is even, $\chi(\mathrm{Lk}(\{f=0\})) = 2 - (\lambda^+ + \lambda^-)$ and if n is odd, $\chi(\mathrm{Lk}(\{f=0\})) = \gamma^+ + \gamma^-$.

Let us give an example. Let $f(x,y,z) = y^2 - zx^b$, $b > 1$ (see [26], Example 2.2). This polynomial is weighted-homogeneous but we cannot apply Corollary 2.7, for b may be arbitrary large. Then $\Sigma_f = \{(0,0,z) \mid z \in \mathbb{R}\}$.

Let $v = (1,1,1)$ so that $v^*(x,y,z) = x+y+z$. We have to check that v satisfies the conclusions of Lemma 4.1, Lemma 4.4 and Corollary 4.5, and Lemma 5.1. A straightforward computation shows that

$$\Gamma_v = \left\{ (x, -\frac{x^b}{2}, \frac{x}{b}) \mid x \neq 0 \right\}.$$

Since $\Gamma_v \cap \{v^* = 0\} = \emptyset$, we see that $\{v^* = 0\}$ intersects the stratum $f^{-1}(0) \setminus \Sigma_f$ transversally. Moreover, since v^* does not vanish on $\Sigma_f \setminus \{0\}$, $\{v^* = 0\}$ intersects the stratum $\Sigma_f \setminus \{0\}$ transversally and so v satisfies the conclusion of Lemma 4.1 (and of Lemma 5.1 as well). It is clear that Γ_v is a curve in the neighborhood of the origin. In order to check that the conclusion of Corollary 4.5 holds, thanks to the computations of Lemmas 4.6 and 4.7, it is enough to check that $\det[\nabla f_x, \nabla f_y, \nabla f_z]$ does not vanish on Γ_v. But

$$\det[\nabla f_x, \nabla f_y, \nabla f_z](x,y,z) = -2b^2 x^{2b-2},$$

and so the conclusion of Corollary 4.5 holds. Moreover, since

$$\frac{\partial f}{\partial v}(x,y,z) = -bx^{b-1}z + 2y - x^b,$$

we easily compute that $\lambda^+ = \lambda^- = -1$ if b is odd and that $\lambda^+ = 0$ and $\lambda^- = -2$ if b is even.

It remains to compute γ^+ and γ^-. But γ^+ is the local topological degree at $(0,0)$ of the function $f(x,y,a-x-y)$, $a > 0$, that is the local topological degree at $(0,0)$ of the function

$$(x,y) \mapsto y^2 - ax^b + x^{b+1} + yx^b.$$

Then it is not difficult to see that $\gamma^+ = -1$ if b is even and $\gamma^+ = 0$ if b is odd. Similarly $\gamma^- = 1$ if b is even and $\gamma^- = 0$ if b is odd. Therefore, applying Theorem 5.4 and Lemma 2.5, we obtain that

$$\chi\big(f^{-1}(-1)\big) = 2,\ \chi\big(f^{-1}(1)\big) = 0 \text{ and } \chi(\mathrm{Lk}(\{f=0\})) = 0.$$

In the rest of the section, we will apply Theorem 5.4 to establish a real version of the Lê-Iomdine formula. From now on, we assume that the structure is polynomially bounded.

Lemma 5.5. *There exists $n_0 \in \mathbb{N}$ such that*

$$|f_{x_1}(p)| > |x_1(p)|^{n_0} \text{ and } |f(p)| > |x_1(p)|^{n_0},$$

for $p \in \Gamma_{x_1}$ close to the origin.

Proof. For $u > 0$ small, we define $\beta(u)$ by

$$\beta(u) = \inf\big\{|f_{x_1}(p)| \mid p \in \Gamma_{x_1} \cap \{|x_1| = u\}\big\}.$$

It is well defined because $\Gamma_{x_1} \cap \{|x_1| = u\}$ is finite and $\beta(u) > 0$. The function β is definable and so is the function $\alpha(R) = \beta(\frac{1}{R})$, defined for $R > 0$ sufficiently big. Then there exists $n_0 \in \mathbb{N}$ such that $\frac{1}{\alpha(R)} < R^{n_0}$ for $R > 0$ sufficiently big. This implies that $\beta(\frac{1}{R}) > \frac{1}{R^{n_0}}$, i.e., $\beta(u) > u^{n_0}$ for $u > 0$ sufficiently small. Hence for $p \in \Gamma_{x_1}$ sufficiently close to the origin, we have

$$|f_{x_1}(p)| > |x_1(p)|^{n_0}.$$

A similar proof works for the second equality because f and x_1 do not vanish on Γ_{x_1}. □

Let us fix $k \in \mathbb{N}$ with $k > n_0 + 1$ and let us set $g(x) = f(x) + x_1^k$.

Lemma 5.6. *The function g has an isolated critical point at the origin.*

Proof. A point p belongs to $(\nabla g)^{-1}(0)$ if and only if

$$\frac{\partial f}{\partial x_1}(p) + kx_1^{k-1}(p) = 0 \text{ and } \frac{\partial f}{\partial x_i}(p) = 0 \text{ for } i \geq 2.$$

Let us suppose first that $p \in \Sigma_f \setminus \{0\}$. This implies that $x_1(p) = 0$. Since x_1 does not vanish on $\Sigma_f \setminus \{0\}$ close to the origin, this case is not possible. Let us suppose now that $p \notin \Sigma_f$. Then p belongs to Γ_{x_1} and so $x_1(p) \neq 0$ and $f_{x_1}(p) \neq 0$. By the previous lemma, $|f_{x_1}(p)| > |x_1(p)|^{n_0}$ which implies that $k|x_1(p)|^{k-1} > |x_1(p)|^{n_0}$, and so $|x_1(p)|^{k-n_0-1} > \frac{1}{k}$ in the neighborhood of the origin. This is impossible by the choice of k. The only possible case is when p is the origin. □

The previous lemma unables us to use the Khimshiashvili formula to compute the Euler characteristic of the Milnor fibre of g. We will relate $\deg_0 \nabla g$ to the indices λ^+, λ^-, γ^+ and γ^-. Before that we need some auxiliary results. Let

$$\Gamma_{x_1}(g) = \{x \in \mathbb{R}^n \setminus \Sigma_g \mid \operatorname{rank}(\nabla g(x), e_1) < 2\}.$$

Lemma 5.7. *We have $\Gamma_{x_1}(g) \cap \{g = 0\} = \emptyset$.*

Proof. If it is not the case this implies that the following set

$$\{g = 0\} \cap \{g_{x_2} = \ldots = g_{x_n} = 0\} \cap \{x_1 = 0\} \setminus \{0\}$$

is not empty in the neighbourhood of the origin. Therefore the set

$$\{f = 0\} \cap \{f_{x_2} = \ldots = f_{x_n} = 0\} \cap \{x_1 = 0\} \setminus \{0\}$$

is not empty in the neighbourhood of the origin. But this is not possible because

$$\{f = 0\} \cap \{f_{x_2} = \ldots = f_{x_n}\} = \Sigma_f$$

and $\{x_1 = 0\} \setminus \{0\} \cap \Sigma_f = \emptyset$. □

Lemma 5.8. *The set $\Gamma_{x_1}(g)$ admits the following decomposition:*

$$\Gamma_{x_1}(g) = \Gamma_{x_1} \sqcup (\Sigma_f \setminus \{0\}).$$

Proof. We see that $p \in \Gamma_{x_1}(g)$ if and only if $g_{x_2}(p) = \ldots = g_{x_n}(p) = 0$ and $g_{x_1}(p) \neq 0$. Since $g_{x_i}(p) = f_{x_i}(p)$, $i = 2, \ldots, n$, it is clear that $\Gamma_{x_1} \sqcup (\Sigma_f \setminus \{0\}) \subset \Gamma_{x_1}(g)$. If $p \in \Gamma_{x_1}(g)$ then $f_{x_2}(p) = \ldots = f_{x_n}(p) = 0$ and $f_{x_1}(p) + kx_1(p)^{k-1} \neq 0$. If $f_{x_1}(p) \neq 0$ then $p \in \Gamma_{x_1}$. If $f_{x_1}(p) = 0$ then $p \in \Sigma_f \setminus \{0\}$. □

Lemma 5.9. *If $p \in \Gamma_{x_1}$, then* $\det\big[\nabla g_{x_1}(p), \ldots, \nabla g_{x_n}(p)\big] \neq 0$ *and*

$$\text{sign} \det\big[\nabla g_{x_1}(p), \ldots, \nabla g_{x_n}(p)\big] = \text{sign} \det\big[\nabla f_{x_1}(p), \ldots, \nabla f_{x_n}(p)\big].$$

Proof. Let $p \in \Gamma_{x_1}$. We have $g_{x_1}(p) = f_{x_1}(p) + kx_1^{k-1}$. By the choice of k, this implies that sign $g_{x_1}(p) = $ sign $f_{x_1}(p)$. Using the computations of Lemmas 4.6 and 4.7, we see that

$$\text{sign } \det\big[\nabla f_{x_1}(p), \dots, \nabla f_{x_n}(p)\big] = \text{sign}\left(x_1(p) f_{x_1}(p) \frac{\partial(x_1, f_{x_2}, \dots, f_{x_n})}{\partial(x_1, \dots, x_n)}(p) \right).$$

But

$$\frac{\partial(x_1, f_{x_2}, \dots, f_{x_n})}{\partial(x_1, \dots, x_n)}(p) = \frac{\partial(x_1, g_{x_2} \dots, g_{x_n})}{\partial(x_1, \dots, x_n)}(p)$$

so $\frac{\partial(x_1, g_{x_2}, \dots, g_{x_n})}{\partial(x_1, \dots, x_n)}(p) \neq 0$. Since $x_1(p) g_{x_1}(p) \neq 0$, we obtain that

$$\det\big[\nabla g_{x_1}(p), \dots, \nabla g_{x_n}(p)\big] \neq 0$$

and since sign $g_{x_1}(p) = $ sign $f_{x_1}(p)$, we conclude that

$$\text{sign } \det\big[\nabla g_{x_1}(p), \dots, \nabla g_{x_n}(p)\big] = \text{sign } \det\big[\nabla f_{x_1}(p), \dots, \nabla f_{x_n}(p)\big].$$

□

Lemma 5.10. *Assume that k is even. If $q \in \Sigma_f \setminus \{0\}$ is close enough to the origin and $x_1(q) = a$, then $\deg_q \nabla f_{|\{x_1 = a\}}$ is equal to $\deg_q(\nabla g - \nabla g(q))$, where $\deg_q(\nabla g - \nabla g(q))$ is the topological degree of the mapping*

$$\frac{\nabla g - \nabla g(q)}{|\nabla g - \nabla g(q)|} : S_{\epsilon'}(q) \to S^{n-1}$$

with $0 < \epsilon' \ll 1$.

Proof. We have that $\deg_q \nabla f_{|\{x_1 = a\}}$ is equal to the topological degree of the mapping $\frac{W}{|W|} : S_{\epsilon'}(q) \to S^{n-1}$ where $W = (x_1 - a, f_{x_2}, \dots, f_{x_n})$. But

$$\nabla g - \nabla g(q) = (f_{x_1} + kx_1^{k-1} - ka^{k-1}, f_{x_2}, \dots, f_{x_n})$$

and so, since $f_{x_1}(q) = 0$ and $k - 1$ is odd, there exists a small neighborhood of q on which $f_{x_1} + kx_1^{k-1} - ka^{k-1}$ and $x_1 - a$ have the same sign. If ϵ' is small enough, then the mappings $\frac{\nabla g - \nabla g(q)}{|\nabla g - \nabla g(q)|}$ and $\frac{W}{|W|}$ are homotopic on $S_{\epsilon'}(q)$. Hence the two topological degrees are equal. □

Proposition 5.11. *If k is odd, then*

$$\deg_0 \nabla g = \lambda^- = \lambda^+ + \gamma^+ - \gamma^-.$$

If k is even, then

$$\deg_0 \nabla g = \lambda^- + \gamma^- = \lambda^+ + \gamma^+.$$

Proof. Let $\eta > 0$ be a small real number. The set $(\nabla g)^{-1}(-\eta, 0, \dots, 0)$ is finite because $\Gamma_{x_1}(g)$ is one-dimensional. Let us write

$$(\nabla g)^{-1}(-\eta, 0, \dots, 0) = \{p_1, \dots, p_s\} \cup \{q_1, \dots, q_r\},$$

where

$$\{p_1, \dots, p_s\} = (\nabla g)^{-1}(-\eta, 0, \dots, 0) \cap \Gamma_{x_1}$$

and

$$\{q_1, \dots, q_r\} = (\nabla g)^{-1}(-\eta, 0, \dots, 0) \cap (\Sigma_f \setminus \{0\}).$$

Therefore we have

$$\deg_0 \nabla g = \sum_{i=1}^{s} \deg_{p_i}(\nabla g - \nabla g(p_i)) + \sum_{j=1}^{r} \deg_{q_j}(\nabla g - \nabla g(q_j)).$$

If k is odd, $(\nabla g)^{-1}(-\eta, 0, \ldots, 0) \cap (\Sigma_f \setminus \{0\})$ is empty and, by the choice of k,

$$(\nabla g)^{-1}(-\eta, 0, \ldots, 0) \cap \Gamma_{x_1} = (\nabla g)^{-1}(-\eta, 0, \ldots, 0) \cap [\cup_{b \in \mathcal{B}^-} b] .$$

Using Lemma 5.9, we conclude that

$$\deg_0 \nabla g = \lambda^- = \lambda^+ + \gamma^+ - \gamma^- .$$

If k is even then

$$(\nabla g)^{-1}(-\eta, 0, \ldots, 0) \cap (\Sigma_f \setminus \{0\}) = (\nabla g)^{-1}(-\eta, 0, \ldots, 0) \cap [\cup_{c \in \mathcal{C}^-} c] .$$

Using Lemma 5.10, we conclude that $\deg_0 \nabla g = \lambda^- + \gamma^- = \lambda^+ + \gamma^+$. □

Now we are in position to formulate the real version of the Lê-Iomdine formula.

Theorem 5.12. *Assume that $e_1 \notin \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$ and that $k > n_0 + 1$. For $0 < \delta \ll \epsilon \ll 1$, we have*

- *if k is odd,*

$$\chi\big(g^{-1}(-\delta) \cap B_\epsilon\big) = \chi\big(f^{-1}(-\delta) \cap B_\epsilon\big) + \gamma^-,$$
$$\chi\big(g^{-1}(\delta) \cap B_\epsilon\big) = \chi\big(f^{-1}(\delta) \cap B_\epsilon\big) + (-1)^{n-1}\gamma^+,$$

- *if k is even,*

$$\chi\big(g^{-1}(-\delta) \cap B_\epsilon\big) = \chi\big(f^{-1}(-\delta) \cap B_\epsilon\big),$$
$$\chi\big(g^{-1}(\delta) \cap B_\epsilon\big) = \chi\big(f^{-1}(\delta) \cap B_\epsilon\big) + (-1)^{n-1}(\gamma^+ + \gamma^-).$$

Proof. We know that $\chi\big(g^{-1}(-\delta) \cap B_\epsilon\big) = 1 - \deg_0 \nabla g$. If k is odd, this gives

$$\chi\big(g^{-1}(-\delta) \cap B_\epsilon\big) = 1 - \lambda^- = \chi\big(f^{-1}(-\delta) \cap B_\epsilon\big) + \gamma^- .$$

If k is even, this gives

$$\chi\big(g^{-1}(-\delta) \cap B_\epsilon\big) = 1 - \lambda^- - \gamma^- = \chi\big(f^{-1}(-\delta) \cap B_\epsilon\big).$$

We know that $\chi\big(g^{-1}(\delta) \cap B_\epsilon\big) = 1 - (-1)^n \deg_0 \nabla g$. So if k is odd, then

$$\chi\big(g^{-1}(\delta) \cap B_\epsilon\big) = 1 - (-1)^n \lambda^- = \chi\big(f^{-1}(\delta) \cap B_\epsilon\big) - (-1)^n \gamma^+ .$$

If k is even, we get that

$$\begin{aligned} \chi\big(g^{-1}(\delta) \cap B_\epsilon\big) &= 1 - (-1)^n(\lambda^- + \gamma^-) \\ &= 1 - (-1)^n(\lambda^- - \gamma^+) - (-1)^n(\gamma^+ + \gamma^-) \\ &= \chi\big(f^{-1}(\delta) \cap B_\epsilon\big) - (-1)^n(\gamma^+ + \gamma^-). \end{aligned}$$

□

References

[1] V.I. Arnol'd, *Index of a singular point of a vector field, the Petrovski-Oleinik inequality, and mixed Hodge structures*, Funct. Anal. and its Appli. **12** (1978), 1–14. DOI: 10.1007/bf01077558

[2] K. Bekka, *Regular stratification of subanalytic sets*, Bull. London Math. Soc. **25**, no. 1 (1993), 7–16. DOI: 10.1112/blms/25.1.7

[3] J. Bochnak, M. Coste, M.F. Roy, *Géométrie algébrique réelle*, Ergebnisse der Mathematik **12**, Springer-Verlag 1987.

[4] J.W. Bruce, *Euler characteristics of real varieties*, Bull. London Math. Soc. **22** (1990), 547–552.

[5] G. Comte, M. Merle, *Equisingularité réelle II : invariants locaux et conditions de régularité*, Ann. Sci. Éc. Norm. Supér. (4) **41** (2008), no. 2, 221–269. DOI: 10.24033/asens.2067

[6] M. Coste, M. Reguiat, *Trivialités en famille*, Real algebraic geometry (Rennes, 1991), 193–204, Lecture Notes in Math., **1524**, Springer, Berlin, 1992. DOI: 10.1007/bfb0084620

[7] A. Dimca, L. Păunescu, *Real singularities and dihedral representations*, Workshop on Real and Complex Singularities (São Carlos, 1996). Mat. Contemp. **12** (1997), 67–82.

[8] A. Durfee, *Neighborhoods of algebraic sets*, Trans. Amer. Math. Soc. **276** (1983), no. 2, 517–530. DOI: 10.1090/s0002-9947-1983-0688959-3

[9] N. Dutertre, *On the topology of semi-algebraic functions on closed semi-algebraic sets*, Manuscripta Mathematica **139**, no. 3-4 (2012), 415–441. DOI: 10.1007/s00229-011-0523-0

[10] N. Dutertre, *Euler characteristic and Lipschitz-Killing curvatures of closed semi-algebraic sets*, Geom. Dedicata **158**, no.1 (2012), 167–189. DOI: 10.1007/s10711-011-9627-7

[11] N. Dutertre, *Stratified critical points on the real Milnor fibre and integral-geometric formulas*, Journal of Singularities **13** (2015), 87–126. DOI: 10.5427/jsing.2015.13e

[12] N. Dutertre, *Euler obstruction and Lipschitz-Killing curvatures*, Israël Journal of Mathematics **213** (2016), 109–137. DOI: 10.1007/s11856-016-1322-9

[13] N. Dutertre, R. Araújo dos Santos, *Topology of real Milnor fibrations for non-isolated singularities*, International Mathematics Research Notices **2016**, no. 16 (2016), 4849–4866. DOI: 10.1093/imrn/rnv286

[14] D. Eisenbud, H.I. Levine, *An algebraic formula for the degree of a C^∞ map-germ*, Annals of Mathematics **106** (1977), 19–44.

[15] T. Fukui, *An algebraic formula for a topological invariant of bifurcation of 1-parameter family of function-germs*, Stratifications, singularities, and differential equations, II (Marseille, 1990; Honolulu, HI, 1990), Travaux en cours **55** (1997), 45–54.

[16] G.M. Khimshiashvili, *On the local degree of a smooth map*, Soobshch. Akad. Nauk Gruz. SSR **85** (1977), 309–312.

[17] K. Kurdyka, *On gradients of functions definable in o-minimal structures*, Ann. Inst. Fourier **48** no. 3 (1998), 769–784. DOI: 10.5802/aif.1638

[18] I. N. Iomdin, *Complex surfaces with a one-dimensional set of singularities*, Sibirsk. Mat. Z. **15** (1974), 1061–1082.

[19] Lê D.T., *Calcul du nombre de cycles évanouissants d'une hypersurface complexe*, Ann. Inst. Fourier **23** (1973), no. 4, 261–270. DOI: 10.5802/aif.491

[20] Lê D. T., *Ensembles analytiques complexes avec lieu singulier de dimension un (d'après I.N. Iomdin)*, Séminaire sur les Singularités (Paris, 1976-1977), Publ. Math. Univ. Paris VII, (1980), 87–95.

[21] T. L. Loi, *Thom stratifications for functions definable in o-minimal structures on* $(\mathbb{R}, +, \cdot)$, C. R. Acad. Sci. Paris Sér. I Math. **324** (1997), no. 12, 1391–1394. DOI: 10.1016/s0764-4442(97)83581-1

[22] T. L. Loi, *Verdier and strict Thom stratifications in o-minimal structures*, Illinois J. Math. **42** (1998), no. 2, 347–356. DOI: 10.1215/ijm/1256045049

[23] J. Milnor, *Singular points of complex hypersurfaces*, Ann. Math. Stud. **61**, Princeton University Press (1968).

[24] D. B. Massey, *The Lê varieties. I.*, Invent. Math. **99** (1990), no. 2, 357–376. DOI: 10.1007/BF01234423

[25] D. B. Massey, *The Lê varieties. II.*, Invent. Math. **104** (1991), no. 1, 113–148. DOI: 10.1007/BF01245068

[26] D. B. Massey, *Lê cycles and hypersurface singularities*, Lecture Notes in Mathematics, **1615**, Springer-Verlag, Berlin, 1995. DOI: 10.1007/BFb0094409

[27] D. B. Massey, *Prepolar deformations and a new Lê-Iomdine formula*, Pacific J. Math. **174** (1996), no. 2, 459–469. DOI: 10.2140/pjm.1996.174.459

[28] D.B. Massey, D. Siersma, *Deformation of polar methods*, Ann. Inst. Fourier **42** (1992), no. 4, 737–778. DOI: 10.5802/aif.1308

[29] Nguyen, X. V. N., Trivedi, S., Trotman, D., *A geometric proof of the existence of definable Whitney stratifications*, Illinois J. Math. **58** (2014), no. 2, 381–389. DOI: 10.1215/ijm/1436275489

[30] D. Siersma, *The monodromy of a series of hypersurface singularities*, Comment. Math. Helv. **65** (1990), no. 2, 181–197. DOI: 10.1007/bf02566602

[31] Z. Szafraniec, *On the Euler characteristic of analytic and algebraic sets*, Topology **25** (1986), no. 4, 411–414. DOI: 10.1016/0040-9383(86)90020-0

[32] Z. Szafraniec, *Topological invariants of weighted homogeneous polynomial*, Glasgow Math. Journal **33** (1991), 241–245. DOI: 10.1017/s0017089500008302

[33] Z. Szafraniec, *A formula for the Euler characteristic of a real algebraic manifold*, Manuscripta Mathematica **85** (1994), 345-360. DOI: 10.1007/bf02568203

[34] B. Teissier, *Introduction to equisingularity problems*, Proc. Sympos. Pure Math. **29** (1975), 593–632. DOI:

[35] B. Teissier, *Variétés polaires. I. Invariants polaires des singularités d'hypersurfaces*, Invent. Math. **40** (1977), no. 3, 267–292. DOI: 10.1007/bf01425742

[36] D. van Straten, T. Warmt, *Gorenstein-duality for one-dimensional almost complete intersections–with an application to non-isolated real singularities*, Math. Proc. Cambridge Philos. Soc. **158** (2015), no. 2, 249–268. DOI: 10.1017/s0305004114000504

[37] C.T.C. Wall, *Topological invariance of the Milnor number mod 2*, Topology **22** (1983), 345–350. DOI: 10.1016/0040-9383(83)90019-8

Nicolas Dutertre, Laboratoire angevin de recherche en mathématiques, LAREMA, UMR6093, CNRS, UNIV. Angers, SFR MathStic, 2 Bd Lavoisier 49045 Angers Cedex 01, France.

Email address: `nicolas.dutertre@univ-angers.fr`

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A GEOMETRIC DESCRIPTION OF THE MONODROMY OF BRIESKORN-PHAM POLYNOMIALS

AURÉLIO MENEGON

Abstract. We give an explicit construction of Lê's vanishing polyhedra for a Brieskorn-Pham polynomial f. Then we use it to give a geometric description of the monodromy associated to f. It allows us to write the matrix that determines the induced algebraic monodromy. In particular, this provides another proof for the Brieskorn-Pham theorem, which says that the characteristic polynomial associated to the monodromy of f is given by $\Delta(t) = \Pi(t - \omega_1\omega_2 \dots \omega_n)$, where each ω_j ranges over all a_j-th roots of unity other than 1.

1. Introduction

Let $f : \mathbb{C}^n \to \mathbb{C}$ be the polynomial map given by

$$f(z_1, \dots, z_n) = z_1^{a_1} + \cdots + z_n^{a_n} \,,$$

with $a_j \in \mathbb{N}$ and $a_j \geq 2$, for $j = 1, \dots, n$.

Pham [8] constructed a polyhedron $\mathcal{P}$ in the Milnor fiber F_f of f which is a deformation retract of F_f. Moreover, he showed that $\mathcal{P}$ (and hence F_f) has the homotopy type of a wedge of $\mu(f)$-many spheres $\mathbb{S}^{n-1}$, with

$$\mu(f) = (a_1 - 1)(a_2 - 1) \dots (a_n - 1) \,.$$

Afterwards, Brieskorn [2] studied the topology of the complex variety $f^{-1}(0)$, so now the polynomials above are known as *Brieskorn-Pham polynomials*.

They also studied the algebraic monodromy

$$h^* : H_{n-1}(F_f; \mathbb{C}) \to H_{n-1}(F_f; \mathbb{C})$$

associated to the Milnor fibration of f. They showed that the characteristic roots of the linear transformation h^* are the products $\omega_1\omega_2 \dots \omega_n$, where each ω_j ranges over all the a_j-th roots of unity other than 1. So the characteristic polynomial of h^* is given by

$$\Delta(t) = \Pi(t - \omega_1\omega_2 \dots \omega_n) \,.$$

Later, many other mathematicians have studied the monodromy associated to singularities. See [3] for a survey on this subject.

In this paper, we use Lê's construction ([4] and [5]) of the vanishing polyhedron $\mathcal{P}$ in F_f to give a geometric description of the induced monodromy $h : \mathcal{P} \to \mathcal{P}$. It allows us to explicitly construct the matrix defined by the induced geometric monodromy h^* with respect to a given basis for $H_{n-1}(\mathcal{P})$ (compare to [7], page 75). In particular, it provides another proof for the Brieskorn-Pham theorem.

The approach suggested by this paper could be useful to study the monodromy associated to real analytic map-germs with an isolated critical point.

On the other hand, the explicit construction of a Lê's vanishing polyhedron for this family of complex functions is a quite interesting example illustrating Lê's construction in a concrete case.

There is another way of describing the geometric monodromy of certain classes of singularities, which have recently been developed by A'Campo. In the last section of his very interesting preprint [1] he explains the so-called *tête-à-tête monodromy* for Brieskorn-Pham polynomials in three variables.

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2. Lê's vanishing polyhedron

In [4] D.T. Lê sketched a proof of the following theorem, whose complete proof was given later in [5] by the author and himself.

Theorem 2.1. *Let $X \subset \mathbb{C}^N$ be a reduced equidimensional complex analytic space and let $\mathcal{S} = (S_\alpha)_{\alpha \in A}$ be a Whitney stratification of X. Let $f : (X, x) \to (\mathbb{C}, 0)$ be a germ of complex analytic function at a point $x \in X$. If f has an isolated singularity at x relatively to $\mathcal{S}$ and if ϵ and η are sufficiently small positive real numbers as above, then for each $t \in \mathbb{D}^*_\eta$ there exist:*

(*i*) *a polyhedron P_t of real dimension $\dim_{\mathbb{C}} X_t$ in the Milnor fiber X_t, compatible with the Whitney stratification $\mathcal{S}$, and a continuous simplicial map:*

$$\tilde{\xi}_t : \partial X_t \to P_t$$

compatible with $\mathcal{S}$, such that X_t is homeomorphic to the mapping cylinder of $\tilde{\xi}_t$;

(*ii*) *a continuous map $\psi_t : X_t \to X_0$ that sends P_t to $\{0\}$ and that restricts to a homeomorphism $X_t \backslash P_t \to X_0 \backslash \{0\}$.*

In this section, we review the general lines of Lê's construction of such a vanishing polyhedron in the case of a complex function-germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ with $n \geq 2$ and with isolated critical point.

Let $\ell : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be the germ of a linear form and consider the map-germ

$$\phi_\ell : (\mathbb{C}^n, 0) \to (\mathbb{C}^2, 0)$$

defined by $\phi_\ell(z) := (\ell(z), f(z))$.

For a generic choice of ℓ the critical set of ϕ_ℓ is either empty or a smooth reduced complex curve, whose closure Γ has image by ϕ_ℓ a complex curve Δ in $\mathbb{C}^2$ (Lemma 21 of [5]). We say that Γ is the *polar curve of f relatively to ℓ* and that Δ *is the polar discriminant of f relatively to ℓ*.

Then the map ϕ_ℓ induces a locally trivial fibration

$$\phi_| : \phi_\ell^{-1}(\mathbb{D}_{\eta_1} \times \mathbb{D}_{\eta_2} \setminus \Delta) \cap \mathbb{B}_\epsilon \to \mathbb{D}_{\eta_1} \times \mathbb{D}_{\eta_2} \setminus \Delta\,,$$

where η_1 and η_2 are small enough real numbers, with $0 < \eta_2 \ll \eta_1 \ll \epsilon$ (Proposition 22 of [5]). The Milnor fiber $f^{-1}(t) \cap \mathbb{B}_\epsilon$ of f is homeomorphic to the set $F_t := \phi_\ell^{-1}(D_t) \cap \mathbb{B}_\epsilon$ (see Theorem 2.3.1 of [6]) for $t \in \mathbb{D}_{\eta_2} \setminus \{0\}$, where

$$D_t := \mathbb{D}_{\eta_1} \times \{t\}\,.$$

Notice that for each $t \in \mathbb{D}_{\eta_2} \setminus \{0\}$ fixed, the restriction of ϕ_ℓ induces a locally trivial fibration

$$\ell_t : (F_t \setminus \{y_1(t), \ldots, y_k(t)\}) \cap \mathbb{B}_\epsilon \to D_t \setminus \{y_1(t), \ldots, y_k(t)\}\,,$$

where

$$\{y_1(t), \ldots, y_k(t)\} := \Delta \cap D_t\,.$$

We can suppose that $\lambda_t := (0,t)$ is in $D_t \setminus \{y_1(t),\ldots,y_k(t)\}$. For each $j = 1,\ldots,k$, let $\delta(y_j(t))$ be a simple path in D_t starting at λ_t and ending at $y_j(t)$. We can choose λ_t in such a way that these paths are disjoint away from λ_t. Finally, set

$$Q_t := \bigcup_{j=1}^{k} \delta(y_j(t))\,.$$

With this notation, we can now construct the Lê's vanishing polyhedron. This is done by induction on n.

For $n = 2$ we just set

$$P_t := \ell_t^{-1}(Q_t)$$

and the lifting of a suitable vector field on D_t that deformation retracts it onto Q_t gives a deformation retraction of F_t onto P_t (see Lemma 25 and Proposition 27 of [5]).

Actually, the constructions above can be made simultaneously for every t in a simple path γ in $\mathbb{D}_{\eta_2}$ joining 0 and some $t_0 \in \partial\mathbb{D}_{\eta_2}$. The resulting polyhedron P_γ is called a *collapsing cone along* γ.

Now suppose $n > 2$. By the induction hypothesis we have a vanishing polyhedron P'_t in the local Milnor fiber F'_t of the hyperplane section

$$f' : \mathbb{C}^n \cap \{\ell = 0\} \to \mathbb{C}\,.$$

For each point $y_j(t) \in \Delta \cap D_t$ let $x_j(t)$ be a point in the intersection of the polar curve Γ with $\ell_t^{-1}\big(y_j(t)\big)$. Without losing generality, we can assume that $x_j(t)$ is the only point in such intersection. Also by the induction hypothesis, there is a collapsing cone P_j for the restriction of the map ℓ_t to a small neighborhood of $x_j(t)$. The "basis" of a such cone is the polyhedron $P_j(a_j) := P_j \cap \ell_t^{-1}(a_j)$, where a_j is a point in $\delta(y_j(t))\backslash y_j(t)$ close to $y_j(t)$.

Since ℓ_t is a locally trivial fiber bundle over $\delta(y_j(t))\backslash y_j(t)$, we can "extend" the cone P_j until it reaches the "central" polyhedron P'_t. This gives a polyhedron C_j. The union of all the polyhedra C_j together with P'_t gives our vanishing polyhedron P_t.

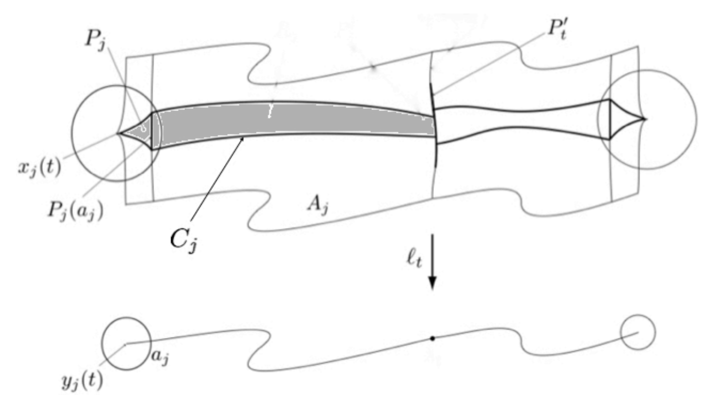

Figure 1.

3. Vanishing polyhedron for Brieskorn-Pham polynomials

In this section, we will follow the steps pointed in Section 2 above to construct a Lê's vanishing polyhedron for a Brieskorn-Pham polynomial

$$f(z_1,\ldots,z_n) = z_1^{a_1} + \cdots + z_n^{a_n}\,,$$

with $a_j \in \mathbb{N}$ and $a_j \geq 2$, for $j = 1, \ldots, n$.

3.1. **The two-dimensional case.** Since the construction of a Lê's vanishing polyhedron is made by induction on the dimension of the domain of the complex function f, we start with the two-dimensional case. That is, we consider a Brieskorn-Pham polynomial $f : \mathbb{C}^2 \to \mathbb{C}$ given by

$$f(x, y) = x^a + y^b \,,$$

with $a, b \in \mathbb{N}$ and $a, b \geq 2$.

Define the linear form $\ell(x, y) = x$ and consider $\phi : \mathbb{C}^2 \to \mathbb{C}^2$ given by $\phi := (\ell, f)$, that is

$$\phi(x, y) = (x, x^a + y^b) \,.$$

Its critical set is the curve $\Gamma = \{y = 0\}$, which we call the polar curve of f relatively to the form ℓ. We say that its image $\Delta = f(\Gamma)$ is the polar discriminant of f relatively to ℓ. It is the complex curve in $\mathbb{C}^2$ given by

$$\Delta = \{(u, v) \in \mathbb{C}^2;\ u^a - v = 0\} \,.$$

One can consider small real numbers $0 < \eta_2 \ll \eta_1 \ll \epsilon \ll 1$ such that the restriction

$$\phi_| : \phi^{-1}\big((\mathbb{D}_{\eta_1} \times \mathbb{D}_{\eta_2}) \setminus \Delta\big) \cap \mathbb{B}_\epsilon \to (\mathbb{D}_{\eta_1} \times \mathbb{D}_{\eta_2}) \setminus \Delta$$

is a topological locally trivial fibration (see Proposition 22 of [5]).

For any $t \in \mathbb{D}_{\eta_2}$ set

$$D_t := \mathbb{D}_\eta \times \{t\} \,.$$

If $t \neq 0$, the local Milnor fiber $f^{-1}(t) \cap \mathbb{B}_\epsilon$ of f at $0 \in \mathbb{C}^2$ is homeomorphic to

$$F_t := f^{-1}(t) \cap \ell^{-1}(\mathbb{D}_\eta) \cap \mathbb{B}_\epsilon$$

(see Theorem 2.3.1 of [6]).

Now, for any $t \in \mathbb{D}_{\eta_2}$ the map ϕ induces a map

$$\ell_t : F_t \to D_t$$

which is a locally trivial fibration over $D_t \setminus (\Delta \cap D_t)$.

Notice that

$$\Delta \cap D_t = \{(t^{\frac{1}{a}}\omega_a^\alpha, t) \in \mathbb{C}^2;\ 0 \leq \alpha \leq a - 1\} \,,$$

where $\omega_a := \exp(\frac{2\pi i}{a})$. Moreover, notice that for each $\alpha = 0, \ldots, a-1$ one has that

$$(\ell_t)^{-1}\big((t^{\frac{1}{a}}\omega_a^\alpha, t)\big) = \{(t^{\frac{1}{a}}\omega_a^\alpha, 0)\} \,.$$

Now, for each $\alpha = 0, \ldots, a-1$ fixed, consider the path $\delta_{t,\alpha}$ in D_t given by

$$\delta_{t,\alpha}(r) := (rt^{\frac{1}{a}}\omega_a^\alpha, t) \ ;\ 0 \leq r \leq 1$$

Notice that

$$(\ell_t)^{-1}\big((rt^{\frac{1}{a}}\omega_a^\alpha, t)\big) = \{(rt^{\frac{1}{a}}\omega_a^\alpha, (1 - r^a)^{\frac{1}{b}} t^{\frac{1}{b}} \omega_b^\beta) \in \mathbb{C}^2;\ 0 \leq \beta \leq b - 1\} \,.$$

Hence $(\ell_t)^{-1}(\delta_{t,\alpha})$ is the union of the b-many paths $p_{\alpha,\beta}$ in F_t given by

$$p_{\alpha,\beta}(r) := (rt^{\frac{1}{a}}\omega_a^\alpha, (1 - r^a)^{\frac{1}{b}} t^{\frac{1}{b}} \omega_b^\beta) \ ;\ 0 \leq r \leq 1$$

with $\beta = 0, \ldots, b-1$. Each path $p_{\alpha,\beta}$ start at the corresponding point $(0, t^{\frac{1}{b}}\omega_b^\beta) \in (\ell_t)^{-1}\big((0, t)\big)$. All the paths $p_{\alpha,\beta}$ end at the point $(t^{\frac{1}{a}}\omega_a^\alpha, 0) = (\ell_t)^{-1}\big((t^{\frac{1}{a}}\omega_a^\alpha, t)\big)$.

So the vanishing polyhedron P_t is given by

$$P_t := \bigcup_{\substack{0 \leq \alpha \leq a-1 \\ 0 \leq \beta \leq b-1}} tr(p_{\alpha,\beta})$$

where $tr(p_{\alpha,\beta})$ denotes the trace of the path $p_{\alpha,\beta}(r)$, with $0 \leq r \leq 1$.

Following [5] we have that P_t is a deformation retract of F_t. It is easy to see that P_t is homeomorphic to the join of $(\ell_t)^{-1}\big((0,t)\big)$ and $(\ell_t)^{-1}\big(\Delta \cap D_t\big)$. The first one is a set of b-many points and the second one is a set of a-many points. Hence the Milnor number of f is given by $\mu(f) = (a-1)(b-1)$.

3.2. **The general case.** Now, given $n > 2$, consider a Brieskorn-Pham polynomial

$$f(z_1, \ldots, z_n) = z_1^{a_1} + \cdots + z_n^{a_n}\,,$$

with $a_j \in \mathbb{N}$ and $a_j \geq 2$, for $j = 1, \ldots, n$.

Define the linear form $\ell(z_1, \ldots, z_n) = z_n$ and consider $\phi : \mathbb{C}^n \to \mathbb{C}^2$ given by $\phi := (\ell, f)$. Its critical set is the polar curve

$$\Gamma = \{z_1 = \cdots = z_{n-1} = 0\}\,,$$

and its image

$$\Delta = \{(u,v) \in \mathbb{C}^2;\ u^{a_n} - v = 0\}\,.$$

is the polar discriminant of f relatively to ℓ.

As before, one can consider small real numbers $0 < \eta_2 \ll \eta_1 \ll \epsilon \ll 1$ such that the restriction

$$\phi_| : \phi^{-1}\big((\mathbb{D}_{\eta_1} \times \mathbb{D}_{\eta_2}) \setminus \Delta\big) \cap \mathbb{B}_\epsilon \to (\mathbb{D}_{\eta_1} \times \mathbb{D}_{\eta_2}) \setminus \Delta$$

is a topological locally trivial fibration, so that for any $t \in \mathbb{D}_{\eta_2}$ the map ϕ induces a map

$$\ell_t : F_t \to D_t$$

which is a locally trivial fibration over $D_t \setminus (\Delta \cap D_t)$, where $D_t := \mathbb{D}_{\eta_1} \times \{t\}$ and

$$F_t := f^{-1}(t) \cap \ell^{-1}(\mathbb{D}_\eta) \cap \mathbb{B}_\epsilon$$

is homeomorphic to the local Milnor fiber of f at $0 \in \mathbb{C}^n$.

Notice that

$$\Delta \cap D_t = \{(t^{1/a_n}\omega_{a_n}^{\alpha_n}, t) \in \mathbb{C}^2;\ 0 \leq \alpha_n \leq a_n - 1\}\,,$$

where $\omega_{a_n} := \exp(\frac{2\pi i}{a_n})$.

Let f' be the restriction of f to $\ell^{-1}(0)$. That is

$$f'(z_1, \ldots, z_{n-1}, 0) := z_1^{a_1} + \cdots + z_{n-1}^{a_{n-1}}\,.$$

By induction on n, we have a Lê's polyhedron P'_t in $F'_t := F_t \cap \{z_n = 0\}$ such that

$$P'_t = \bigcup_{\substack{0 \leq \alpha_j \leq a_j - 1\\ 1 \leq j \leq n-1}} tr(p_{\alpha_1, \ldots, \alpha_{n-1}})$$

where each $p_{\alpha_1,\ldots,\alpha_{n-1}} : \big([0,1]\big)^{n-2} \to F'_t$ is a parametrized space.

Example 3.1. For $n = 3$ we have

$$p_{\alpha_1,\alpha_2}(r) = (rt^{\frac{1}{a_1}}\omega_{a_1}^{\alpha_1}, (1 - r^{a_1})^{\frac{1}{a_2}} t^{\frac{1}{a_2}}\omega_{a_2}^{\alpha_2})\ ;\ 0 \leq r \leq 1\,.$$

$\square$

Now, for each point $y_{\alpha_n} := (t^{1/a_n}\omega_{a_n}^{\alpha_n}, t)$ in $(\Delta \cap D_t)$, with $0 \leq \alpha_n \leq a_n - 1$, set

$$x_{\alpha_n} := (\ell_t)^{-1}(y_{\alpha_n}) \cap \Gamma = (0, \ldots, 0, t^{1/a_n}\omega_{a_n}^{\alpha_n})\,.$$

Then consider the map-germ

$$\tilde{\ell}_{\alpha_n} : (F_t, x_{\alpha_n}) \to (\mathbb{C}, y_{\alpha_n})$$

given by the restriction of ℓ to F_t. As in Section 2 above, we can use the induction hypothesis to construct a collapsing cone P_{α_n} of $\tilde{\ell}_{\alpha_n}$, for each $\alpha_n = 0, \ldots, a_n - 1$ fixed, so that:

(i) Each P_{α_n} is the union of parametrized spaces

$$q^{\alpha_n}_{\alpha_1,\ldots,\alpha_{n-1}} : \big([0,1]\big)^{n-1} \to F_t \text{ ; and}$$

(ii) Any two of them intersect exactly at P'_t.

So

$$P_t = \bigcup_{\substack{0\le\alpha_j\le a_j-1\\ 1\le j\le n}} tr(q^{\alpha_n}_{\alpha_1,\ldots,\alpha_{n-1}})\,.$$

Example 3.2. In the case $n=3$ we have the map-germ

$$\tilde\phi_{\alpha_3} : (F_t, x_{\alpha_3}) \to (\mathbb{C}^2, \tilde y_{\alpha_3})$$

given by $\tilde\phi_{\alpha_3}(z_1,z_2,z_3) := (z_1,z_3)$, where $\tilde y_{\alpha_3} := (0, t^{1/a_3}\omega^{\alpha_3}_{a_3})$. Its critical points are the points in F_t at which

$$\det\begin{pmatrix} \frac{\partial f}{\partial z_1} & \frac{\partial f}{\partial z_2} & \frac{\partial f}{\partial z_3} \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 0$$

Hence the relative polar curve of $\tilde\ell_{\alpha_3}$ is the curve

$$\tilde\Gamma_{\alpha_3} := F_t \cap \{z_2 = 0\}$$

and its polar discriminant is the curve

$$\tilde\Delta_{\alpha_3} = \{u^{a_1} + v^{a_3} = t\}\,.$$

Setting $D_\tau := \mathbb{D}_{\tilde\eta_1} \times \{\tau\}$ for $\tilde\eta_1$ sufficiently small, we have that

$$\tilde\Delta_{\alpha_3} \cap D_\tau = \{((t-\tau^{a_3})^{1/a_1}\omega^{\alpha_1}_{a_1}, \tau) \in \mathbb{C}^2;\ 0 \le \alpha_1 \le a_1 - 1\}\,.$$

So for each $\alpha_1 = 0,\ldots,a_1-1$ fixed, consider the path $\delta^{\alpha_3}_{\tau,\alpha_1}$ in D_τ given by

$$\delta^{\alpha_3}_{\tau,\alpha_1}(r) := (r(t-\tau^{\alpha_3})^{1/a_1}\omega^{\alpha_1}_{a_1}, \tau) \text{ ; } 0 \le r \le 1\,.$$

Then $(\tilde\phi_{\alpha_3})^{-1}\big(\delta^{\alpha_3}_{\tau,\alpha_1}(r)\big)$ is the set of points $(z_1,z_2,\tau) \in \mathbb{C}^3$ such that

$$z_1^{a_1} + z_2^{a_2} + \tau^{a_3} = t \quad\text{and}\quad z_1 = r(t-\tau^{\alpha_3})^{1/a_1}\omega^{\alpha_1}_{a_1}\,.$$

Since

$$r^{a_1}(t-\tau^{a_3}) + z_2^{a_2} + \tau^{a_3} = t \Leftrightarrow z_2 = (t-\tau^{a_3})^{\frac{1}{a_2}}(1-r^{a_1})^{\frac{1}{a_2}}\omega^{\alpha_2}_{a_2}\,,$$

with $\alpha_2 = 0,\ldots,a_2-1$, it follows that $(\tilde\phi_{\alpha_3})^{-1}\big(\delta^{\alpha_3}_{\tau,\alpha_1}\big)$ is the union of the a_2-many paths

$$q^{\alpha_3}_{\alpha_1,\alpha_2}(r) := \big(r(t-\tau^{a_3})^{\frac{1}{a_1}}\omega^{\alpha_1}_{a_1}, (1-r^{a_1})^{\frac{1}{a_2}}(t-\tau^{a_3})^{\frac{1}{a_2}}\omega^{\alpha_2}_{a_2}, \tau\big) \text{ ; } 0 \le r \le 1\,.$$

Now make τ move along the semi-line that passes through $t^{\frac{1}{a_3}}\omega^{\alpha_3}_{a_3}$, that is, consider:

$$\tau_{\alpha_3}(k) := (1-k)t^{\frac{1}{a_3}}\omega^{\alpha_3}_{a_3} \text{ ; } 0 \le k \le 1\,.$$

Then the collapsing cone P_{α_3} of $\tilde\ell_{\alpha_3}$, for each $\alpha_3 = 0,\ldots,a_3-1$, is given by

$$P_{\alpha_3} := \bigcup_{\substack{0\le\alpha_1\le a_1-1\\ 0\le\alpha_2\le a_2-1}} q^{\alpha_3}_{\alpha_1,\alpha_2}\big([0,1]\times[0,1]\big)\,,$$

where $q^{\alpha_3}_{\alpha_1,\alpha_2}$ is the parametrized surface in P_t given by

$$q^{\alpha_3}_{\alpha_1,\alpha_2}(r,k) = \big(rt^{\frac{1}{a_1}}(1-(1-k)^{a_3})^{\frac{1}{a_1}}\omega^{\alpha_1}_{a_1}, (1-r^{a_1})^{\frac{1}{a_2}}t^{\frac{1}{a_2}}(1-(1-k)^{a_3})^{\frac{1}{a_2}}\omega^{\alpha_2}_{a_2}, (1-k)t^{\frac{1}{a_3}}\omega^{\alpha_3}_{a_3}\big)\,.$$

Notice that $q^{\alpha_3}_{\alpha_1,\alpha_2}(r,1) = p_{\alpha_1,\alpha_2}(r)$, so any two collapsing cones of the type P_{α_3} as above intersect at P'_t.

So we finally have that

$$P_t = \bigcup_{0\le\alpha_3\le a_3-1} P_{\alpha_3}$$

and hence

$$P_t = \bigcup_{\substack{0\le\alpha_1\le a_1-1\\0\le\alpha_2\le a_2-1\\0\le\alpha_3\le a_3-1}} q^{\alpha_3}_{\alpha_1,\alpha_2}\big([0,1]\times[0,1]\big)$$

is the Lê's vanishing polyhedron for f.

Since P'_t has the homotopy type of a wedge of $(a_1-1)(a_2-1)$-many circles, it follows that P_t has the homotopy type of a wedge of $(a_1-1)(a_2-1)(a_3-1)$-many spheres $\mathbb{S}^2$. □

4. The monodromy of the Brieskorn-Pham polynomial

Consider the characteristic homeomorphism $h_t : F_t \to F_t$ given by

$$h_t(z_1,\dots,z_n) := (e^{2\pi i/a_1}z_1,\dots,e^{2\pi i/a_n}z_n)\,.$$

Identifying $a_i \sim 0$ for each $i=1,\dots,n$ one can check that the characteristic homeomorphism h_t takes each $q^{\alpha_n}_{\alpha_1,\dots,\alpha_{n-1}}$ onto $q^{\alpha_n+1}_{\alpha_1+1,\dots,\alpha_{n-1}+1}$. This gives a geometric view of the monodromy of f (see the examples below).

Notice that the homology group $H_{n-1}(P_t)$ is generated by $(n-1)$-cycles $\sigma(\alpha_1,\dots,\alpha_n)$, where each one of them is a sum (with signals) of 2^n-many parametrized spaces $q^{\alpha_n}_{\alpha_1,\dots,\alpha_{n-1}}$.

Moreover, one can check that

$$h_t\big(\sigma(\alpha_1,\dots,\alpha_n)\big) = \sigma(\alpha_1+1,\dots,\alpha_n+1)$$

if $0\le\alpha_i\le a_i-3$ for any $i=1,\dots,n$; and that $h_t\big(\sigma(\alpha_1,\dots,\alpha_n)\big)$ equals

$$(-1)^k\sum_{i_1=0}^{a_{i_1}-2+1}\cdots\sum_{i_k=0}^{a_{i_k}-2}\sigma(\alpha_1+1,\dots,i_1,\dots,i_2,\dots,i_k,\dots,\alpha_n+1)$$

if $\alpha_{i_j}=a_{i_j}-2$ for $j=1,\dots,k$ and $\alpha_i<a_i-2$ for $i\notin\{i_1,\dots,i_k\}$. This gives a homological view of the monodromy of f.

Next we consider the two and the three dimensional cases, so the reader can actually see this geometric description of the monodromy of a Brieskorn-Pham polynomial.

4.1. **Two-dimensional case.**

Consider $f(x,y)=x^a+y^b$ and let $h_t : F_t\to F_t$ be the characteristic homeomorphism, given by

$$h_t(x,y) := (e^{2\pi i/a}x, e^{2\pi i/b}y)\,.$$

Notice that

$$h_t\big(p_{\alpha,\beta}(r)\big) = h_t\Big(\big(rt^{\frac{1}{a}}\omega_a^{\alpha},(1-r^a)^{\frac{1}{b}}t^{\frac{1}{b}}\omega_b^{\beta}\big)\Big) = \big(rt^{\frac{1}{a}}\omega_a^{\alpha+1},(1-r^a)^{\frac{1}{b}}t^{\frac{1}{b}}\omega_b^{\beta+1}\big)$$

for any $0\le r\le 1$, $0\le\alpha\le a-1$ and $0\le\beta\le b-1$. So if we identify $a\sim 0$ and $b\sim 0$ we have that

$$h_t\big(p_{\alpha,\beta}(r)\big) = p_{\alpha+1,\beta+1}(r)\,.$$

In particular, $h_t(P_t)=P_t$.

Now observe that the homology group $H_1(P_t)$ is generated by the cycles

$$\sigma(\alpha,\beta) := p_{\alpha,\beta}-p_{\alpha,\beta+1}-p_{\alpha+1,\beta}+p_{\alpha+1,\beta+1}$$

with $0 \leq \alpha \leq a-2$ and $0 \leq \beta \leq b-2$. So we have that

$$h_t\big(\sigma(\alpha,\beta)\big) = p_{\alpha+1,\beta+1} - p_{\alpha+1,\beta+2} - p_{\alpha+2,\beta+1} + p_{\alpha+2,\beta+2}\,.$$

We have some cases:

(*i*) If $0 \leq \alpha \leq a-3$ and $0 \leq \beta \leq b-3$ then one clearly has

$$h_t\big(\sigma(\alpha,\beta)\big) = \sigma(\alpha+1,\beta+1)\,.$$

(*ii*) If $0 \leq \alpha \leq a-3$ and $\beta = b-2$ then

$$\begin{array}{rcl} h_t\big(\sigma(\alpha,b-2)\big) & = & -p_{\alpha+1,0} + p_{\alpha+1,b-1} + p_{\alpha+2,0} - p_{\alpha+2,b-1} \\ & = & -\sigma(\alpha+1,0) - \sigma(\alpha+1,1) - \cdots - \sigma(\alpha+1,b-1)\,. \end{array}$$

(*iii*) Analogously, if $\alpha = a-2$ and $0 \leq \beta \leq b-3$ we have that

$$h_t\big(\sigma(a-2,\beta)\big) = -\sigma(0,\beta+1) - \sigma(1,\beta+1) - \cdots - \sigma(a-1,\beta+1)\,.$$

(*iv*) If $\alpha = a-2$ and $\beta = b-2$ then

$$\begin{array}{rcl} h_t\big(\sigma(a-2,b-2)\big) & = & p_{0,0} - p_{0,b-1} - p_{a-1,0} + p_{a-1,b-1} \\ & = & \sum_{i=0}^{a-2}\sum_{j=0}^{b-2}\sigma(i,j)\,. \end{array}$$

So we have showed that

$$h_t\big(\sigma(\alpha,\beta)\big) = \begin{cases} \sigma(\alpha+1,\beta+1) & \text{if } 0 \leq \alpha \leq a-3 \text{ and } 0 \leq \beta \leq b-3 \\ -\sum_{j=0}^{b-2}\sigma(\alpha+1,j) & \text{if } 0 \leq \alpha \leq a-3 \text{ and } \beta = b-2 \\ -\sum_{i=0}^{a-2}\sigma(i,\beta+1) & \text{if } \alpha = a-2 \text{ and } 0 \leq \beta \leq b-3 \\ \sum_{i=0}^{a-2}\sum_{j=0}^{b-2}\sigma(i,j) & \text{if } \alpha = a-2 \text{ and } \beta = b-2 \end{cases}$$

Notice that since $H_1(P_t)$ has a finite basis, then h_t^* has finite order. So, by a theorem from Linear Algebra, we know that the minimal polynomial of h_t^* is a product of distinct cyclotomic polynomials. In particular, the roots of the characteristic polynomial of h_t are products of roots of the unity $\omega_a^k \omega_b^l$.

Example 4.1. Consider $f(x,y) = x^3 + y^3$. Then $a = b = 3$ and we have the following basis for $H_1(P_t)$:

$$B = \{\sigma(0,0), \sigma(0,1), \sigma(1,0), \sigma(1,1)\}\,.$$

So the matrix of the homomorphism $h_t^* : H_1(P_t) \to H_1(P_t)$ in the basis B is given by:

$$[h_t^*]_B^B = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & 1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

A simple calculation shows that the characteristic polynomial is

$$p(\lambda) = (\lambda - 1)(\lambda^3 + 1)\,.$$

Example 4.2. Consider $f(x,y) = x^3 + y^4$ and consider the following basis for $H_1(P_t)$:

$$B = \{\sigma(0,0), \sigma(0,1), \sigma(0,2), \sigma(1,0), \sigma(1,1), \sigma(1,2)\}\,.$$

So the matrix of the homomorphism $h_t^*: H_1(P_t) \to H_1(P_t)$ in the basis B is given by:

$$[h_t^*]_B^B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 & 0 & 1 \\ 1 & 0 & -1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 0 & -1 & 1 \end{pmatrix}.$$

4.2. **Three-dimensional case.** Consider

$$f(z_1, z_2, z_3) = z_1^{a_1} + z_2^{a_2} + z_3^{a_3}.$$

The characteristic homeomorphism $h_t: F_t \to F_t$ is given by

$$h_t(z_1, z_2, z_3) := (e^{2\pi i/a_1} z_1, e^{2\pi i/a_2} z_2, e^{2\pi i/a_3} z_3).$$

So if we identify $a_i \sim 0$, $i = 1, 2, 3$, we have that

$$h_t\big(q_{\alpha_1,\alpha_2}^{\alpha_3}(r,k)\big) = q_{\alpha_1+1,\alpha_2+1}^{\alpha_3+1}(r,k),$$

for any $(r,k) \in [0,1] \times [0,1]$. In particular, $h_t(P_t) = P_t$.

Now observe that the homology group $H_2(P_t)$ is generated by the 2-cycles given by

$$\begin{aligned}\sigma(\alpha_1, \alpha_2, \alpha_3) := q_{\alpha_1,\alpha_2}^{\alpha_3} - q_{\alpha_1+1,\alpha_2}^{\alpha_3} - q_{\alpha_1,\alpha_2+1}^{\alpha_3} - q_{\alpha_1,\alpha_2}^{\alpha_3+1} \\ + q_{\alpha_1+1,\alpha_2+1}^{\alpha_3} + q_{\alpha_1+1,\alpha_2}^{\alpha_3+1} + q_{\alpha_1,\alpha_2+1}^{\alpha_3+1} - q_{\alpha_1+1,\alpha_2+1}^{\alpha_3+1}\end{aligned}$$

with $0 \le \alpha_i \le a_i - 2$ for $i = 1, 2, 3$.

Then some calculations as before give that $h_t\big(\sigma(\alpha_1, \alpha_2, \alpha_3)\big)$ equals to:

$$\begin{cases} \sigma(\alpha_1+1, \alpha_2+1, \alpha_3+1) & \text{if } 0 \le \alpha_i \le a_i - 3, \text{for } i = 1,2,3 \\ -\sum_{i=0}^{a_1-2} \sigma(i, \alpha_2+1, \alpha_3+1) & \text{if } \alpha_1 = a_1 - 2,\ 0 \le \alpha_2 \le a_2 - 3 \text{ and } 0 \le \alpha_3 \le a_3 - 3 \\ -\sum_{j=0}^{a_2-2} \sigma(\alpha_1+1, j, \alpha_3+1) & \text{if } 0 \le \alpha_1 \le a_1 - 3,\ \alpha_2 = a_2 - 2 \text{ and } 0 \le \alpha_3 \le a_3 - 3 \\ -\sum_{k=0}^{a_3-2} \sigma(\alpha_1+1, \alpha_2+1, k) & \text{if } 0 \le \alpha_1 \le a_1 - 3,\ 0 \le \alpha_2 \le a_2 - 3 \text{ and } \alpha_3 = a_3 - 2 \\ \sum_{i=0}^{a_1-2} \sum_{j=0}^{a_2-2} \sigma(i, j, \alpha_3+1) & \text{if } \alpha_1 = a_1 - 2,\ \alpha_2 = a_2 - 2 \text{ and } 0 \le \alpha_3 \le a_3 - 3 \\ \sum_{i=0}^{a_1-2} \sum_{k=0}^{a_3-2} \sigma(i, \alpha_2+1, k) & \text{if } \alpha_1 = a_1 - 2,\ 0 \le \alpha_2 \le a_2 - 3 \text{ and } \alpha_3 = a_3 - 2 \\ \sum_{j=0}^{a_2-2} \sum_{k=0}^{a_3-2} \sigma(\alpha_1+1, j, k) & \text{if } 0 \le \alpha_1 \le a_1 - 3,\ \alpha_2 = a_2 - 2 \text{ and } \alpha_3 = a_3 - 2 \\ -\sum_{i=0}^{a_1-2} \sum_{j=0}^{a_2-2} \sum_{k=0}^{a_3-2} \sigma(i, j, k) & \text{if } \alpha_1 = a_1 - 2,\ \alpha_2 = a_2 - 2 \text{ and } \alpha_3 = a_3 - 2 \end{cases}$$

Example 4.3. Consider $f(x,y) = z_1^2 + z_2^3 + z_3^3$ and consider the following basis for $H_2(P_t)$:

$$B = \{\sigma(0,0,0), \sigma(0,0,1), \sigma(0,1,0), \sigma(0,1,1)\}.$$

So the matrix of the homomorphism $h_t^*: H_2(P_t) \to H_2(P_t)$ in the basis B is given by:

$$[h_t^*]_B^B = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 \\ -1 & 1 & 1 & -1 \end{pmatrix}.$$

References

[1] N. A'Campo, *Tête-à-tête twists and geometric monodromy*, Preprint.

[2] E. Brieskorn, *Beispiele zur Differentialtopologie von Singularitäten*, Invent. Math., 2: 1–14, 1966. DOI: 10.1007/bf01403388

[3] W. Ebeling, *Monodromy*, Singularities and Computer Algebra (Ch. Lossen, G. Prister, eds.), Conference on Occasion of Gert-Martin Greuel's 60th Birthday, London Math. Soc. Lecture Note Series 324, Cambridge University Press 2006, 129–155. DOI: 10.1017/CBO9780511526374.008

[4] D.T. Lê, *Polyèdres évanescents et effondrements*, A fête of topology, 293-329, Academic Press, Boston, MA, 1988. DOI: 10.1016/b978-0-12-480440-1.50018-6

[5] D.T. Lê and A. Menegon Neto, *Vanishing polyhedron and collapsing map*, Math. Zeitschrift, v. 286 (2017), p. 1003-1040. DOI: 10.1007/s00209-016-1793-8

[6] D.T. Lê and B. Teissier, *Cycles evanescents, sections planes et conditions de Whitney II*, in Singularities, Part 2., Proc. Sympos. Pure Math. 40, pp. 65–103. Amer. Math. Soc., Arcata (1981). DOI: 10.1090/pspum/040.2/713238

[7] J.W. Milnor, *Singular points of complex hypersurfaces*, Ann. of Math. Studies 61, Princeton, 1968.

[8] Frédéric Pham. Formules de Picard-Lefschetz généralisées et ramification des intégrales. *Bull. Soc. Math. France*, 93:333–367, 1965. DOI: 10.24033/bsmf.1628

Aurélio Menegon, Universidade Federal da Paraíba, Departamento de Matemática, CEP 58051-900, João Pessoa - PB, Brazil

Email address: aurelio@mat.ufpb.br

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BOUQUET DECOMPOSITION FOR DETERMINANTAL MILNOR FIBERS

MATTHIAS ZACH

Abstract. We provide a bouquet decomposition for the determinantal Milnor fiber of an Essentially Isolated Determinantal Singularity (EIDS) of arbitrary type. The building blocks of the decomposition are (suspensions of) hyperplane sections in general position off the origin of the generic determinantal varieties. For the special case of $2 \times n$-matrices we give a full description of the homotopy types of the determinantal Milnor fibers as a wedge of spheres.

1. Results

In this note we will apply a general Bouquet Decomposition Theorem by M. Tibăr [13] in the case of an Essentially Isolated Determinantal Singularity (EIDS, see [4]) to prove the following:

Theorem 1.1. *Let $(X_0, 0) = (A^{-1}(M^t_{m,n}), 0) \subset (\mathbb{C}^N, 0)$ be an EIDS of type (m, n, t) and dimension $d = \dim(X_0, 0) = N - (m - t + 1)(n - t + 1) > 0$ given by a holomorphic map germ*

$$A\colon (\mathbb{C}^N, 0) \to (\mathrm{Mat}(m, n; \mathbb{C}), 0).$$

Suppose A_u is a stabilization of A and $\overline{X}_u = A_u^{-1}(M^t_{m,n})$ the determinantal Milnor fiber. Define

$$s_0 := \min\{s \in \mathbb{N} : (m - s + 1)(n - s + 1) \leq N\}.$$

Then $\overline{X}_u$ is homotopy equivalent to the bouquet

$$L^{t,N}_{m,n} \vee \bigvee_{s_0 \leq s \leq t} \bigvee_{i=1}^{r(s)} S^{N-(m-s+1)(n-s+1)+1}(L^{t-s+1,(m-s+1)(n-s+1)-1}_{m-s+1,n-s+1}) \tag{1}$$

for some numbers $r(s)$ with $s_0 \leq s \leq t$.

The spaces $M^t_{m,n}$ and $L^{t,k}_{m,n}$ appearing in this theorem are defined as follows. For any triple (m, n, t) of non-negative integers we set

$$M^t_{m,n} := \{M \in \mathrm{Mat}(m, n; \mathbb{C}) : \mathrm{rank}\, M < t\},$$

the *generic determinantal variety*. We define the space $L^{t,k}_{m,n}$ to be the interior of the determinantal Milnor fiber of a *linear* EIDS of type (m, n, t), i.e. the singularity obtained from a generic linear map germ

$$\Phi\colon (\mathbb{C}^k, 0) \to (\mathrm{Mat}(m, n; \mathbb{C}), 0).$$

Note that for the particular case $k = m \cdot n - 1$ the space $L^{t,k}_{m,n}$ is the *complex link* of $M^t_{m,n}$.

In Formula (1) we denote by $S^r(X)$ the r-fold repeated suspension of a topological space X. We use the same convention as in [13] and set $S^1(\emptyset) = S^0$, the sphere of dimension 0, and $S^0(X) = X$ for any X.

Theorem 1.1 is a major reduction step in the understanding of the vanishing topology of essentially isolated determinantal singularities. In particular it implies the known results for the Milnor fiber of an isolated complete intersection singularity $(X_0, 0) = (f^{-1}(\{0\}), 0) \subset (\mathbb{C}^N, 0)$ given by a holomorphic map germ

$$f\colon (\mathbb{C}^N, 0) \to (\mathbb{C}^d, 0)$$

which can naturally be regarded as EIDS of type $(d, 1, 1)$. In this particular case one has $L^{t,N}_{m,n} \cong_{ht} \{pt\}$, $s_0 = t = 1$, and $r(s) = \mu$ is the classical Milnor number of $(X_0, 0)$. Formula (1) therefore reads

$$\begin{aligned} \overline{X}_u &\cong_{ht} \{pt\} \vee \bigvee_{i=1}^{r(1)} S^{N-d+1}(\emptyset) \\ &\cong_{ht} \bigvee_{i=1}^{\mu} S^{N-d} \end{aligned}$$

In fact, it has already been shown in [13, Corollary 4.2] how to apply the Handlebody Theorem to reprove the known results [8] on isolated complete intersection singularities and we will follow the ideas presented there to obtain our generalization for EIDS.

While for ICIS the Milnor fiber is always homotopy equivalent to a bouquet of spheres of the same dimension, this is no longer the case for determinantal Milnor fibers of EIDS, see e.g. [3], [5], and Section 4. Several groups have studied the vanishing Euler characteristic for EIDS, see e.g. [4], [6], and [12]. One approach is to study the behavior of a generic hyperplane equation h in a determinantal deformation of a given EIDS $(X_0, 0)$. The determinantal Milnor fiber $\overline{X}_u$ is then obtained from its hyperplane section $\overline{X}_u \cap \{h = 0\}$ by attaching cells, or, more generally in the context of stratified Morse theory, so-called "thimbles[1]", at Morse critical points of h on $\overline{X}_u$. This way, one obtains nice formulas for the vanishing Euler characteristic in terms of the polar multiplicities of the singularity $(X_0, 0)$. However, it is hardly possible to describe the loci in the hyperplane section $\overline{X}_u \cap \{h = 0\}$ at which the attachments take place. This fact destroys any hope to arrive at a precise description of the homotopy type of $\overline{X}_u$.

It is the Carrousel by Lê which sits at the heart of the proof of the Handlebody Theorem (stated as Theorem 2.4 below) from [13] and which allows us to understand the attachments of the thimbles. As we will see, however, the setup for the application of the Handlebody Theorem is quite different from the viewpoint of EIDS. We will describe the transformation of any EIDS $(X_0, 0) = (A^{-1}(M^t_{m,n}) \subset (\mathbb{C}^N, 0)$ to an isolated *relative* complete intersection singularity (IRCIS, see Definition 3.2)

$$(X_0, 0) = (\{f_{1,1} = \cdots = f_{m,n} = 0\}, 0) \subset (Z, 0)$$

on a controlled Whitney stratified ambient space

$$(Z, 0) \cong (\mathbb{C}^N, 0) \times (M^t_{m,n}, 0)$$

in Section 3.1. Then, rather than doing an induction argument by cutting down with generic hyperplanes, we proceed by an inductive argument where we always trade one equation $f_{i,j}$ defining $(X_0, 0)$ in $(Z, 0)$ for a generic hyperplane equation and eventually end up with the space $L^{t,N}_{m,n}$ – a generic linear section of $M^t_{m,n}$ off the origin. During this process, the Handlebody Theorem allows us to really keep track of the involved attachment processes.

The homotopy type of the spaces $L^{t,k}_{m,n}$ has been studied in a few particular cases, see e.g. [5]. The Euler obstructions of the generic determinantal varieties $M^t_{m,n}$, which are closely related to their hyperplane sections $L^{t,m\cdot n-1}_{m,n}$, can be found in [6] and the Chern-Schwartz-MacPherson classes of their projectivizations $\mathbb{P}(M^t_{m,n})$ have been studied in [16]. However, there is – at least to the knowledge of the author – no complete understanding of the homotopy and homology groups of $L^{t,k}_{m,n}$ for arbitrary values of m, n, t, and k.

[1]By a thimble we mean the pair of topological spaces given by the product of the tangential and the normal Morse data at a given critical point. This might differ from the cell $(D^\lambda, \partial D^\lambda)$ occurring in classical Morse theory, see [7].

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2. Preliminaries

2.1. **Notations and Background.** In this article we will make use of the common terms of stratified Morse theory. The reader may consult the standard textbook reference [7]. Suppose we are given a manifold N and a closed subspace $Z \subset N$ with a Whitney stratification $\Sigma = (S_\alpha)_{\alpha \in A}$. For any point $p \in Z$ we will write

$$T_p Z := T_p S_\alpha$$

for the tangent space of the stratum S_α containing p. Furthermore, we say that a smooth map

$$f\colon M \to N \supset Z$$

from a manifold M to N is transverse to Z if f is transverse to all the strata.

Consider the set $X = f^{-1}(Z)$. It naturally decomposes into the sets $\Sigma_\alpha = f^{-1}(S_\alpha)$ given by the preimages of the strata of Z. Whenever $f\colon M \to N \supset Z$ is transverse to Z in M, the Σ_α form a Whitney stratification for X and we also say that X *inherits* the stratification of Z. In particular, this applies to the case of a closed embedding such as for example the fiber of a stratified submersion on Z induced from a map on N.

Throughout this article we usually consider *closed* Milnor balls B for singularities. This convention always assures that one automatically keeps track of the boundary behavior in deformations which can be a particularly tricky task in the setting of non-isolated singularities. Moreover, the resulting Milnor fibers are always *compact* stratified spaces which simplifies their treatment by Morse theory.

Since this note is merely an application of methods which had been developed before, we will restrict ourselves to the description of how the techniques can be used on determinantal singularities. To this end, we will review the cornerstones of the proofs of e.g. the Handlebody Theorem by Tibăr and other ideas behind it. However, the reader who is unfamiliar with the mathematical rigor on singularity theory on Whitney stratified spaces is strongly encouraged to consult the articles [13], [11], the references given there, and the standard textbook on stratified Morse theory [7].

2.2. **Essentially Isolated Determinantal Singularities.** Let $(M_{m,n}^t, 0) \subset (\mathrm{Mat}(m,n;\mathbb{C}), 0)$ be the generic determinantal variety of type (m,n,t):

$$M_{m,n}^t = \{M \in \mathrm{Mat}(m,n;\mathbb{C}) : \operatorname{rank} M < t\}.$$

The canonical rank stratification by

$$S_{m,n}^s = M_{m,n}^s \setminus M_{m,n}^{s-1}$$

for $0 < s \le \min\{m,n\}+1$ is a Whitney stratification of $\mathrm{Mat}(m,n;\mathbb{C})$ and $M_{m,n}^t$. This can easily be deduced by induction from the observation that at any point $p \in S_{m,n}^s$ one has a product

$$(M_{m,n}^t, p) \cong (M_{m-s+1,n-s+1}^{t-s+1}, 0) \times (\mathbb{C}^{(m+n)\cdot(s-1)-(s-1)^2}, 0) \tag{2}$$

of analytic spaces. Consequently, the complex link of $M_{m,n}^t$ along the stratum $S_{m,n}^s$ is

$$L_{m-s+1,n-s+1}^{t-s+1,(m-s+1)(n-s+1)-1}.$$

The complex links play a central role in the stratified Morse theory on complex analytic varieties because they determine the *normal Morse data*, see [7]. In the case of the generic determinantal variety $M^t_{m,n}$ we find from (2) that the normal Morse data along the stratum $S^s_{m,n}$ for $s \leq t$ is given by the pair of spaces

$$\left(C(L^{t-s+1,(m-s+1)(n-s+1)-1}_{m-s+1,n-s+1}), L^{t-s+1,(m-s+1)(n-s+1)-1}_{m-s+1,n-s+1} \right), \tag{3}$$

where $C(X)$ denotes the real cone over a given topological space X. We adopt the convention that $C(\emptyset) = \{pt\}$ is just one point.

Definition 2.1 ([4])**.** A determinantal singularity of type (m, n, t) is given by a holomorphic map germ

$$A\colon (\mathbb{C}^N, 0) \to (\mathrm{Mat}(m, n; \mathbb{C}), 0)$$

such that the space

$$(X_0, 0) := (A^{-1}(M^t_{m,n}), 0) \subset (\mathbb{C}^N, 0)$$

has *expected codimension* $\operatorname{codim}(X_0, 0) = \operatorname{codim} M^t_{m,n} = (m - t + 1)(n - t + 1)$.

A determinantal singularity $(X_0, 0)$ given by a matrix A is called *essentially isolated*, if the map A is transverse to the rank stratification of $\mathrm{Mat}(m, n; \mathbb{C})$ in a punctured neighborhood of the origin.

It follows directly from this definition that, away from the origin, X_0 inherits a canonical stratification by the strata

$$\Sigma^s := A^{-1}(S^s_{m,n}).$$

Counting dimensions yields that these strata are nonempty if and only if

$$\min\{r \in \mathbb{N} : (m - r + 1)(n - r + 1) < N\} \leq s \leq t. \tag{4}$$

and that

$$\dim \Sigma^s = N - (m - s + 1)(n - s + 1) > 0.$$

We supplement this stratification with the one-point stratum $\{0\} \subset X_0$ at the origin.

An *essential smoothing* of $(X_0, 0)$ is a family

$$\begin{tikzcd} X_0 \arrow[r, hook] \arrow[d] & X \arrow[d, "u"] \\ \{0\} \arrow[r] & \mathbb{C} \end{tikzcd}$$

coming from a stabilization

$$\mathbf{A} : (\mathbb{C}^N, 0) \times (\mathbb{C}, 0) \to (\mathrm{Mat}(m, n; \mathbb{C}), 0) \times (\mathbb{C}, 0)$$

of the map A. That is $\mathbf{A} = \mathbf{A}(x, u) = (A_u(x), u)$ with $A_0 = A$ and A_u transversal to $M^t_{m,n}$ for all $u \neq 0$ sufficiently small. Then, the total space of the family above appears as $X = \mathbf{A}^{-1}(M^t_{m,n} \times \mathbb{C})$ and u is the map given by the deformation parameter.

From a stabilization we can construct the *determinantal Milnor fiber* as follows. Choose a representative

$$\mathbf{A} : W \times U \to \mathrm{Mat}(m, n; \mathbb{C}) \times U$$

of the stabilization $\mathbf{A}$ for some open sets $W \subset \mathbb{C}^N$ and $U \subset \mathbb{C}$ and let $B \subset \mathbb{C}^N$ be a Milnor ball for $(X_0, 0)$ in W. By this we mean a closed ball around the origin such that $\overline{X}_0 := X_0 \cap B$ is closed, the boundary ∂B intersects X_0 transversally, and

$$\overline{X}_0 \cong C(\partial \overline{X}_0)$$

is homeomorphic to the real cone over its boundary $\partial\overline{X}_0 = \partial B \cap X_0$. We can then consider the family $u : X \cap (B \times U) \to U$. It may be deduced from Thom's first Isotopy Lemma that u is a trivial topological fibration along the boundary $X \cap (\partial B \times U)$ over U and that

$$u : (X \cap (B \times U)) \setminus \overline{X}_0 \to U \setminus \{0\}$$

is a topological fiber bundle for U small enough.

Definition 2.2. It is the fiber of this bundle

$$\overline{X}_u \cong A_u^{-1}(M_{m,n}^t) \cap B$$

that we call the determinantal Milnor fiber.

Using the theory of versal unfoldings, one can show that in fact for any given EIDS $(X_0, 0)$ the determinantal Milnor fiber is unique up to homeomorphism, see [2] or [15].

Example 2.3. Consider the EIDS $(X_0, 0) \subset (\mathbb{C}^5, 0)$ of type $(2, 3, 2)$ given by the matrix

$$A = \begin{pmatrix} x & y & z \\ v & w & x \end{pmatrix}$$

together with the essential smoothing induced by the perturbation with

$$\begin{pmatrix} u & 0 & 0 \\ 0 & 0 & -u \end{pmatrix}.$$

It is easily seen that indeed the total space $(X, 0) \subset (\mathbb{C}^{5+1}, 0)$ is isomorphic to the generic determinantal variety $M_{2,3}^2 \subset \mathrm{Mat}(2, 3; \mathbb{C}) \cong \mathbb{C}^6$ and the map u is a generic linear form on it. Hence, the determinantal Milnor fiber of $(X_0, 0)$ is nothing but the (closure of the) complex link $L_{2,3}^{2,5}$ of $(M_{2,3}^2, 0)$. It is known that $L_{2,3}^{2,5}$ is homotopy equivalent to the 2-sphere S^2, see [5].

2.3. **The Handlebody Theorem.** In [13], M. Tibăr proofs the following theorem for the Milnor fiber F of an isolated hypersurface singularity

$$f : (Z, 0) \to (\mathbb{C}, 0)$$

on a complex analytic, Whitney stratified space $(Z, 0)$ of dimension $\dim(Z, 0) \geq 2$ and the complex link L of $(Z, 0)$:

Theorem 2.4 ([13], Handlebody Theorem)**.** *The Milnor fiber F is obtained from the complex link L to which one attaches cones over local Milnor fibers of stratified Morse singularities. The image of each such attaching map retracts within L to a point.*

We give a rough outline of the idea of the proof. We may assume $(Z, 0) \subset (\mathbb{C}^N, 0)$ to be embedded in some smooth ambient space. Let h be the linear equation on $\mathbb{C}^N$ defining the link L of $(Z, 0)$ and consider

$$\Phi = (h, f) : B \cap Z \cap \Phi^{-1}(D \times D') \to D \times D' \tag{5}$$

for a sufficiently small, closed ball B and discs $D, D' \subset \mathbb{C}$ around the origin. In [11], Lê has shown the following. There exists a Zariski open set $\Omega \subset \left(\mathbb{C}^N\right)^\vee$ of linear forms on the ambient space such that for $h \in \Omega$ the polar variety

$$\Gamma(h, f) := \overline{\{z \in Z \setminus f^{-1}(\{0\}) : \exists a \in \mathbb{C} : \mathrm{d}h(z)|_{T_z Z} = a \cdot \mathrm{d}f(z)|_{T_z Z}\}},$$

i.e. the critical locus of h on Z relative to f, is a curve which is branched over its image

$$\Delta = \Delta(h, f) = \Phi(\Gamma(h, f)) \subset D \times D',$$

the so-called Cerf-diagram. The proof for the set Ω of admissible hyperplane equations to be Zariski open can be found in [9]. Moreover, one can choose D' small enough such that the intersection $\Delta \cap (\partial D \times D')$ is empty. Then Φ is a topological fibration away from Δ and one has homeomorphisms

$$F \cong \Phi^{-1}(D \times \{\delta\})$$

and

$$L \cong \Phi^{-1}(\{\eta\} \times D')$$

for $0 \neq \delta$, resp. $0 \neq \eta$, sufficiently small. It is also shown in [9] that Ω can be chosen such that the restriction of $h \in \Omega$ to any fixed fiber $\Phi^{-1}(D \times \{\delta\})$ has only Morse singularities over the intersection points $\Delta \cap D \times \{\delta\}$ for $0 \neq \delta \in D'$.

At this point the so-called "carrousel" is furnished by the geometric monodromy of F along the boundary of D', i.e. by the variation of the value δ of f. But contrary to the classical viewpoint on monodromy one does not only construct a lifting of the unit tangent vector field along $\partial D'$ to $\Phi^{-1}(D \times \partial D')$, but one also keeps track of the monodromy induced on the disc $D \times \{\delta\}$, the intersection points $C = \Delta(h, f) \cap D \times \{\delta\}$, and the corresponding critical points of h on the Milnor fiber $\Phi^{-1}(D \times \{\delta\})$ over them.

Let $F' = \Phi^{-1}(\{(\eta, \delta)\})$. Then up to homotopy the Milnor fiber F is obtained from F' by attaching thimbles along suitably chosen paths in $D \times \{\delta\}$ from (η, δ) to the critical values of the stratified Morse points of h on F. The topology of each of these attachments is governed by the Morse data. In the situations we will encounter in the context of EIDS, the Morse data will always be of the following form:

Proposition 2.5. *Let $(X, p) \cong (M^s_{m,n}, 0) \times (\mathbb{C}^k, 0)$ and $h : (X, p) \to (\mathbb{C}, 0)$ a holomorphic map germ with a stratified Morse singularity at p. Then the thimble corresponding to this critical point is*

$$(C(S^k(L^{s,m\cdot n-1}_{m,n})), S^k(L^{s,m\cdot n-1}_{m,n})),$$

i.e. one attaches the real cone $C(S^k(L^{s,m\cdot n-1}_{m,n}))$ along its boundary $S^k(L^{s,m\cdot n-1}_{m,n})$.

The key observation from the Carrousel is that keeping track of the relative critical points of the hyperplane equation h on F allows one to determine exactly at which loci on F' these attachments take place.

As a final step, one constructs another homeomorphism $L \cong \Phi^{-1}(W) \subset F$ on a certain subspace $\Phi^{-1}(W)$ of F by "sliding along Δ". The space W is chosen such that $F' \subset \Phi^{-1}(W)$ and one can use the carrousel monodromy to show that for each thimble e one has to attach to $\Phi^{-1}(W)$ to complete it – up to homotopy – to F, there is already one thimble e' that had been attached to F' in the same spot as e to complete it to $\Phi^{-1}(W)$. This explains, why each attaching map in the statement of the Handlebody Theorem 2.4 retracts within L to a point.

3. Proof of the Main Theorem

3.1. **The Graph Transformation.** Let $(X_0, 0) \subset (\mathbb{C}^N, 0)$ be a determinantal singularity of type (m, n, t) given by a matrix A. In this section we will explain how to transform $(X_0, 0)$ into a *relative* complete intersection singularity on a canonical ambient space

$$(Z, 0) \cong (\mathbb{C}^N, 0) \times (M^t_{m,n}, 0),$$

see Definition 3.2.

Let $Y = \mathrm{Mat}(m, n; \mathbb{C}) \cong \mathbb{C}^{m\cdot n}$, $\mathbb{C}[\underline{y}] = \mathbb{C}[y_{i,j} | 1 \leq i \leq m, 1 \leq j \leq n]$ the associated coordinate ring and $\mathcal{O}_{m\cdot n} = \mathbb{C}\{\underline{y}\}$ the local ring of $(Y, 0)$. By abuse of notation, we will also write y for the tautological matrix $y \in \mathrm{Mat}(m,n;\mathbb{C})$ with entries $y_{i,j}$:

$$y = \begin{pmatrix} y_{1,1} & \cdots & y_{1,n} \\ \vdots & & \vdots \\ y_{m,1} & \cdots & y_{m,n} \end{pmatrix}.$$

Choose a representative $A : U \to Y$ of the matrix A defining $(X_0, 0)$ and let

$$\Gamma_A = \{(x,y) : y = A(x)\} \subset U \times Y$$

be the graph of A. Set $Z := U \times M^t_{m,n}$. Then, by construction, $X_0 \cong \Gamma_A \cap Z$.

We define two maps

$$\begin{aligned} p\colon U \times Y &\to Y, \qquad && (x,y) \mapsto y - A(x), \\ q\colon U \times Y &\to U, \qquad && (x,y) \mapsto x \end{aligned}$$

and form the commutative diagram

$$\begin{tikzcd} X_y \arrow[r, hook] \arrow[d] & Z \arrow[r, "q"] \arrow[d, "p"'] & U \\ \{y\} \arrow[r, hook] & Y. & \end{tikzcd} \tag{6}$$

While q is the projection to the first factor, the map p can be considered as the "projection to Y along the graph Γ_A". Clearly, for every point $y \in Y$ the space X_y is the determinantal variety

$$X_y = (A - y)^{-1}\,(M^t_{m,n}) = q\left(p^{-1}(\{y\})\right)$$

defined by the perturbation of A by the constant matrix y and we can consider X_y as a determinantal deformation of the EIDS $(X_0, 0)$.

Note that $(Z, 0)$ enjoys a canonical Whitney stratification by the strata

$$(\tilde{S}^s_{m,n}, 0) = (S^s_{m,n}, 0) \times (\mathbb{C}^N, 0)$$

inherited from the rank stratification on $M^t_{m,n}$. Whenever A is defining an EIDS, i.e. A is transverse to the rank stratification in a punctured neighborhood of the origin in $\mathbb{C}^N$, the above construction turns $(X_0, 0)$ into the fiber of a map $p|(Z, 0)$ which is a stratified submersion along $X_0 \subset Z$ on a punctured neighborhood of the origin in $\mathbb{C}^N \times \mathbb{C}^{m \cdot n}$:

Lemma 3.1. *Let $(x, A(x))$ be a point in the graph Γ_A of A. The restriction $p|Z$ is a stratified submersion on Z at $(x, A(x))$ if and only if the map $A : U \to Y$ is transverse to the rank stratification at $x \in U$.*

Proof. Let $(v_1, \dots, v_d)$ be local coordinates at $y = A(x)$ of the stratum $S^s_{m,n}$ containing y. Together with the standard coordinates of U, they form a coordinate system (x, v) of the stratum $\tilde{S}^s_{m,n}$ of Z at $(x, A(x))$. Now note that on the one hand the jacobian matrix of $p|Z$ at this point is of block form

$$\begin{pmatrix} \frac{\partial p(x,v)}{\partial x} & \frac{\partial p(x,v)}{\partial v} \end{pmatrix} = \begin{pmatrix} -\frac{\partial A(x)}{\partial x} & \frac{\partial y(v)}{\partial v} \end{pmatrix}$$

and p is a stratified submersion at $(x, A(x))$ if and only if this matrix has full rank $m \cdot n$. On the other hand, the map A is transverse to the rank stratification of Y at x, if and only if the tangent space $T_y Y$ of the ambient space Y at $y = A(x)$ can be generated by both the image of the differential of A – i.e. the span of the columns of the matrix $\frac{\partial A}{\partial x}$ – and the tangent space $T_y S^s_{m,n}$ of the stratum $S^s_{m,n}$. Since $T_y S^s_{m,n}$ is by definition the span of the second block $\frac{\partial y}{\partial v}$ in the jacobian matrix of p, the claim follows. $\square$

The components of the map p define the graph Γ_A via

$$p_{i,j}(x,y) = y_{i,j} - a_{i,j}(x) = 0$$

and clearly, Γ_A is a complete intersection in $U \times Y$. The determinantal singularity $X_0 \cong Z \cap \Gamma_A$ appears as the intersection of Γ_A with $Z = \mathbb{C}^N \times M^t_{m,n}$. While $M^t_{m,n}$ is not a complete intersection in general, it is nevertheless always a Cohen-Macaulay space, see [10]. Since $(X_0, 0)$ has expected dimension and

$$\mathcal{O}_{X_0,(x,y)} \cong \mathcal{O}_{Z,(x,y)}/\langle p_{1,1}, \dots, p_{m,n}\rangle,$$

the components $p_{i,j}(x,y)$ of p must also form a regular sequence on $\mathcal{O}_Z$, the structure sheaf of Z; cf. [1, Theorem 2.1.2 c)]. We give a general definition of the object we just encountered.

Definition 3.2. Let $(Z,0) \subset (\mathbb{C}^r, 0)$ be a germ of a complex analytic space and

$$f\colon (\mathbb{C}^r, 0) \to (\mathbb{C}^c, 0)$$

a holomorphic map.

We say that the restriction $f|(Z,0)$ is a complete intersection morphism, if the components $f_1, \dots, f_c$ form a regular sequence on $\mathcal{O}_{Z,0}$.

If, moreover, $(Z,0)$ is endowed with a Whitney stratification, we say that $f|(Z,0)$ has an isolated *relative* complete intersection singularity (IRCIS) on $(Z,0)$ whenever there exists a punctured neighborhood U of 0 in $\mathbb{C}^r$ such that at every point $z \in U \cap Z \cap f^{-1}(\{0\})$ in the central fiber, the restriction $f|(Z,0)$ is a stratified submersion at z.

We have just verified:

Proposition 3.3. *The restriction $p|(Z,0)$ is a complete intersection morphism which realizes $(X_0,0) = p^{-1}(\{0\}) \cap (Z,0)$ as an IRCIS of p on $(Z,0)$.*

We will refer to the above construction as the *graph transformation* of the EIDS $(X_0,0) = (A^{-1}(M^t_{m,n}), 0)$. This transformation allows us to study $(X_0,0)$ with the classical methods for complete intersections. To this end, we will fix some notation. Let

$$\mathbf{W} = (\{0\} = W_0 \subsetneq W_1 \subsetneq W_2 \subsetneq \cdots \subsetneq W_{m\cdot n-1} \subsetneq W_{m\cdot n} = \mathbb{C}^{m\cdot n})$$

be a maximal ascending flag in $Y = \mathrm{Mat}(m,n;\mathbb{C})$ and

$$\mathbf{V} = \left(\mathbb{C}^N \times \mathrm{Mat}(m,n;\mathbb{C}) = V_0 \supsetneq V_1 \supsetneq \cdots \supsetneq V_{m\cdot n-1} \supsetneq V_{m\cdot n}\right)$$

a descending flag in $\mathbb{C}^N \times \mathrm{Mat}(m,n;\mathbb{C})$ with $\dim V_i/V_{i+1} = 1$ for each i.

For each $k > 0$ we set

$$Z_k := Z \cap p^{-1}(W_k) \cap V_{k-1}. \tag{7}$$

The two projections p and q induce natural maps

(8)
$$\begin{tikzcd} & Z_k \arrow[dl, "f_k"'] \arrow[dr, "h_k"] & \\ W_k/W_{k-1} & & V_{k-1}/V_k. \end{tikzcd}$$

Proposition 3.4. *If the flags* $\mathbf{W}$ *and* $\mathbf{V}$ *are in general position, then the following holds.*

(1) Each of the spaces Z_k inherits the canonical Whitney stratification from $(Z,0)$ outside the origin.

(2) Each f_k defines an isolated hypersurface singularity on $(Z_k, 0)$ relative to the given stratification.

(3) *The function h_k is a linear equation on $(Z_k, 0)$, which can be used to define the complex link and the carrousel.*

Proof. We do induction on k. Let $k = 1$. By definition $V_{k-1} = V_0 = \mathbb{C}^N \times \mathrm{Mat}(m, n; \mathbb{C})$. Consider the map

$$\mathbb{P}p : Z \setminus X_0 \to \mathbb{P}^{m \cdot n - 1}, \quad (x, y) \mapsto (p_{1,1}(x, y) : \cdots : p_{m,n}(x, y))$$

and let $[W_1] \in \mathbb{P}^{m \cdot n-1}$ be a regular value of this map. Choose a splitting $\mathbb{C}^{m \cdot n} \cong (\mathbb{C}^{m \cdot n}/W_1) \oplus W_1$ and write $p = (\tilde{p}, f_1)$ with

$$\tilde{p} \colon z \mapsto p(z) + W_1 \in \mathbb{C}^{m \cdot n}/W_1 \cong \mathbb{C}^{m \cdot n - 1}.$$

Then $Z_1 = p^{-1}(W_1) = \tilde{p}^{-1}(\{0\})$ does not have critical points of $\tilde{p}$ outside $X_0 = \{f_1 = 0\} \subset Z_1$. Suppose $(x, y) \in X_0$, $x \neq 0$ was a critical point of $\tilde{p}$ on Z_1 in X_0 and S the stratum of Z containing it. Then the differential $\mathrm{d}(\tilde{p}|S)(x, y)$ does not have full rank and, hence, also $\mathrm{d}(p|S)(x, y)$ can not have full rank – a contradiction to X_0 being an IRCIS. We conclude that $\tilde{p}$ is a stratified submersion on Z at all points of Z_1 except the origin. Therefore, Z_1 inherits the Whitney stratification from $(Z, 0)$ and $f_1 : (Z_1, 0) \to \mathbb{C}$ defines an IRCIS on $(Z_1, 0)$.

For a given isolated singularity $f_1 \colon (Z_1, 0) \to (\mathbb{C}, 0)$ the condition on a linear equation h_1 to be sufficiently general to define the carrousel is Zariski open; cf. [13]. We may choose h_1 accordingly and set $V_1 = \{h_1 = 0\}$.

For the induction step we start by projectivizing the map $\tilde{p}$:

$$\mathbb{P}\tilde{p} \colon Z \cap V_k \setminus p^{-1}(W_{k-1}) \to \mathbb{P}(\mathbb{C}^{m \cdot n}/W_{k-1}), \quad (x, y) \mapsto [p(x, y) + W_{k-1}].$$

Choose a subspace $W_k \subset \mathbb{C}^{m \cdot n}$ such that $[W_k/W_{k-1}]$ is a regular value of this map. The rest of the induction step is merely a repetition of the above said and left to the reader. $\square$

In what follows, we will from now on assume that the flags $\mathbf{V}$ and $\mathbf{W}$ have been chosen to fulfill Proposition 3.4. For any $k > 0$ let

$$F_k = f_k^{-1}(\{\delta\}) \cap Z_k \cap B \tag{9}$$

be the Milnor fiber of f_k on Z_k for a suitable choice of a Milnor ball B and $\delta \in \mathbb{C} \setminus \{0\}$ small enough. We denote the complex link of Z_k by

$$L_k = h_k^{-1}(\{\eta\}) \cap Z_k \cap B, \tag{10}$$

$\eta \in \mathbb{C} \setminus \{0\}$ small enough.

3.2. **The induction argument.**

We can apply the Handlebody Theorem of Tibăr at each step k in the setup of the previous section to obtain our Main Theorem. The key lemma for this induction can already be extracted from [13, Corollary 4.2]:

Lemma 3.5. *In the final setup of the standard transformation we have for each $0 < k < m \cdot n$ a (non-canonical) homeomorphism*

$$L_k \cong F_{k+1}. \tag{11}$$

Proof. One has homeomorphisms

$$\begin{array}{rcl} F_{k+1} & = & Z_{k+1} \cap f_{k+1}^{-1}(\{\delta\}) \cap B \\ & = & Z \cap V_k \cap p^{-1}(W_{k+1}) \cap f_{k+1}^{-1}(\{\delta\}) \cap B \\ & = & Z \cap V_{k-1} \cap p^{-1}(W_{k+1}) \cap h_k^{-1}(\{0\}) \cap f_{k+1}^{-1}(\{\delta\}) \cap B \\ & \cong & Z \cap V_{k-1} \cap p^{-1}(W_{k+1}) \cap h_k^{-1}(\{\eta\}) \cap f_{k+1}^{-1}(\{\delta\}) \cap B \\ & \cong & Z \cap V_{k-1} \cap p^{-1}(W_{k+1}) \cap h_k^{-1}(\{\eta\}) \cap f_{k+1}^{-1}(\{0\}) \cap B \\ & \cong & Z \cap V_{k-1} \cap p^{-1}(W_k) \cap h_k^{-1}(\{\eta\}) \cap B \\ & \cong & L_k \end{array}$$

for a Milnor ball B and sufficiently small values for δ and η. The homeomorphisms are induced from the parallel transport in the fibration given by

$$\Phi = (h_k, f_{k+1}) \colon Z \cap V_{k-1} \cap p^{-1}(W_{k+1}) \cap B \to \mathbb{C} \times \mathbb{C}$$

as in (5) over suitably chosen paths connecting $(0, \delta)$, (η, δ), and $(\eta, 0)$. □

Proof. (of Theorem 1.1) After applying the graph transformation we obtain for $k = 1$:

$$\overline{X}_u = f_1^{-1}(\{\delta\}) \cap Z_1 \cap B = F_1,$$

because W_1 was in general position. This space is naturally stratified by the strata Σ^s of dimension

$$\dim \Sigma^s = N - (m - s + 1)(n - s + 1)$$

for $s_0 \leq s \leq t$ with $s_0 = \min\{r \in \mathbb{N}_0 : (m - r + 1)(n - r + 1) \leq N\}$ and the complex link along Σ^s is $L_{m-s+1,n-s+1}^{t-s+1,(m-s+1)(n-s+1)-1}$. We may apply Proposition 2.5 to determine the thimbles associated to Morse critical points on the strata. It is the pair of spaces consisting of

$$S^{N-(m-s+1)(n-s+1)}(L_{m-s+1,n-s+1}^{t-s+1,(m-s+1)(n-s+1)-1})$$

and the cone over it. According to the Handlebody Theorem [13], the space F_1 then has a bouquet decomposition

$$F_1 \cong_{ht} L_1 \vee \bigvee_{s_0 \leq s \leq t} \bigvee_{i=1}^{r_1(s)} S^{N-(m-s+1)(n-s+1)+1}(L_{m-s+1,n-s+1}^{t-s+1,(m-s+1)(n-s+1)-1}).$$

Note that, since the image of the attaching maps in L_1 retract to a point, we obtain one more suspension compared to the formula for the thimble.

We may now proceed inductively and replace L_k by F_{k+1} in this formula according to Lemma 3.5. At each step we attach a certain number $r_k(s)$ of thimbles and we may add them up to $r(s) = \sum_{k=1}^{m \cdot n-1} r_k(s)$. This finishes the proof. □

Corollary 3.6. *If the singularity $(X_0, 0)$ in the setting of Theorem 1.1 is smoothable (i.e. if $N < (m - t + 2)(n - t + 2)$), then*

$$\overline{X}_u \cong_{ht} L_{m,n}^{t,N} \vee \bigvee_{i=1}^{r} S^d, \tag{12}$$

with $d = N - (m - t + 1)(n - t + 1) = \dim(X_0, 0)$.

4. EIDS of type $(2,n,2)$

In this section we will be concerned with arbitrary EIDS $(X_0,0) \subset (\mathbb{C}^N,0)$ of type $(2,n,2)$. The requirement on $(X_0,0)$ to have expected dimension relates N and the dimension $d = \dim(X_0,0)$ via

$$d = N - (2-2+1)(n-2+1) = N - n + 1.$$

In particular, we always have $n-1 \leq N$. Note that Theorem 1.1 is only applicable if $n \leq N$.

If we require $(X_0,0)$ to be smoothable, we also obtain an upper bound on N given by

$$N < (2-2+2)(n-2+2) = 2n.$$

4.1. **The homotopy type of $L^{2,N}_{2,n}$.** We shall first determine the homotopy type of all the spaces $L^{2,N}_{2,n}$, see (13), (15), (17), and (18).

Whenever $N \geq 2n$, any generic linear map

$$\Phi\colon (\mathbb{C}^N,0) \to (\mathrm{Mat}(2,n;\mathbb{C}),0)$$

is a submersion and in particular stable. The interior of the determinantal Milnor fiber of Φ is therefore given by

$$L^{2,N}_{2,n} \cong \mathbb{C}^{N-2n} \times M^2_{2,n} \cong_{ht} \{pt\}. \tag{13}$$

Suppose $N < 2n$. Let $M = M^2_{2,n}$ be the generic determinantal variety and

$$\begin{tikzcd} W \arrow[r, hook] \arrow[d, "\pi"'] & \mathrm{Mat}(2,n;\mathbb{C}) \times \mathbb{P}^1 \arrow[r] \arrow[d] & \mathbb{P}^1 \\ M \arrow[r, hook] & \mathrm{Mat}(2,n;\mathbb{C}) & \end{tikzcd}$$

its Tjurina transform (see e.g. [14], or [15]) resulting from the blowup of the rational map

$$\Psi : M \dashrightarrow \mathbb{P}^1, \quad y \mapsto [\ker(y)].$$

If we let $y_{i,j}$ be the canonical coordinates of $\mathrm{Mat}(2,n;\mathbb{C})$ and $(s_1 : s_2)$ the homogeneous coordinates of $\mathbb{P}^1$ then the equations for W are

$$s_1 \cdot y_{2,j} - s_2 \cdot y_{1,j} = 0 \text{ for } j = 1,\dots,n. \tag{14}$$

We may consider $y_{1,j}$ and $y_{2,j}$ as linear fiber coordinates in local trivializations of the tautological bundle $\mathcal{O}_{\mathbb{P}^1}(-1)$ for every j. Thus, W is a smooth complex manifold isomorphic to the total space of the vector bundle $(\mathcal{O}_{\mathbb{P}^1}(-1))^n$.

Instead of describing an embedding

$$\Phi\colon \mathbb{C}^N \to \mathrm{Mat}(2,n;\mathbb{C})$$

of a linear subspace defining an EIDS $(X_0,0) = (\Phi^{-1}(M^2_{2,n}),0)$, we may also choose a linear form

$$l = (l^1,\dots,l^{2n-N}) \in \mathrm{Hom}_{\mathbb{C}}(\mathrm{Mat}(2,n;\mathbb{C}),\mathbb{C}^{2n-N})$$

such that $\Phi(\mathbb{C}^N) = \ker(l)$. Since all equations involved in this process are either linear or homogeneous, we may neglect the choice of Milnor balls. We obtain an extension of the above diagram to the left:

$$\begin{tikzcd}
\tilde{X}_0 \arrow[r, hook] \arrow[d] & W \arrow[r, hook] \arrow[d, "\pi"'] & \mathrm{Mat}(2,n;\mathbb{C}) \times \mathbb{P}^1 \arrow[r] \arrow[d] & \mathbb{P}^1 \\
X_0 \arrow[r, hook] \arrow[d] & M \arrow[r, hook] \arrow[d, "l"'] & \mathrm{Mat}(2,n;\mathbb{C}) & \\
\{0\} \arrow[r, hook] & \mathbb{C}^{2n-N}, & &
\end{tikzcd}$$

with $\tilde{X}_0 := \pi^{-1}(X_0)$. The interior of the determinantal Milnor fiber $L^{2,N}_{2,n}$ of Φ is then given by

$$L^{2,N}_{2,n} = M \cap l^{-1}(\{u\})$$

for some regular value u of l on M.

Utilizing the trace pairing (see e.g. [4])

$$\mathrm{Mat}(2,n;\mathbb{C}) \times \mathrm{Mat}(2,n;\mathbb{C}) \to \mathbb{C}, \quad (A,B) \mapsto \mathrm{trace}(A^T \cdot B),$$

we may write the components of l in the form

$$l^k = \begin{pmatrix} l^k_{1,1} & l^k_{1,2} & \cdots & l^k_{1,n} \\ l^k_{2,1} & l^k_{2,2} & \cdots & l^k_{2,n} \end{pmatrix}$$

for constant entries $l^k_{i,j} \in \mathbb{C}$. We leave it to the reader to verify that in the range $n \leq N < 2n$, a sufficiently general choice for l is given by choosing the $2n-N$ components l^k from the following n matrices:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \end{pmatrix}, \quad \cdots$$

$$\cdots, \begin{pmatrix} 0 & \cdots & 0 & 1 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \cdots & 0 & 0 & 1 & 0 \\ 0 & \cdots & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Fix one value $n \leq N < 2n$ and the linear form $l \colon \mathrm{Mat}(2,n;\mathbb{C}) \to \mathbb{C}^{2n-N}$ as above and consider the algebraic sets

$$W \supset \pi^{-1}\left(\{l=0\}\right) = \tilde{X}_0 \xrightarrow{\pi} X_0 = M \cap l^{-1}(\{0\}).$$

Using the above equations (14) for W and $\pi^* l^k, k = 1, \dots, 2n-N$ we see that $\tilde{X}_0$ is a local complete intersection in $\mathrm{Mat}(2,n;\mathbb{C}) \times \mathbb{P}^1$.

Moreover, whenever $N > n$ – i.e. whenever $d = \dim(X_0, 0) > 1$ – $\tilde{X}_0$ is isomorphic to the total space of the vector bundle

$$\mathcal{O}_{\mathbb{P}^1}(-(2n-N+1)) \oplus \left(\mathcal{O}_{\mathbb{P}^1}(-1)\right)^{N-n-1}$$

and in particular smooth of dimension $d = N-n+1$. Passing from $l = 0$ to a regular value $l = u$ therefore results in a flat deformation of $\tilde{X}_0$ which is topologically trivial due to Ehresmann's theorem. Since the set $X_u = M \cap \{l = u\}$ does not meet the locus $M^1_{2,n} = \{0\}$ where Ψ is not defined, the projection $\pi \colon \tilde{X}_u \to X_u$ is an isomorphism and we obtain homotopy equivalences

$$S^2 \cong \mathbb{P}^1 \cong_{ht} \tilde{X}_0 \cong_{ht} \tilde{X}_u \cong_{ht} X_u \cong_{ht} L^{2,N}_{2,n} \text{ for } n < N < 2n. \tag{15}$$

In the particular case where $N = n$ – i.e. when X_0 is a curve and the components of l comprise all of the above listed linear forms – we find the following system of equations for $\tilde{X}_0$ in the chart

$\{s_1 \neq 0\}$:

$$y_{2,j} = \frac{s_2}{s_1} y_{1,j}, \qquad j = 1, \dots, n,$$
$$y_{1,j} = \left(-\frac{s_2}{s_1}\right) y_{1,j+1}, \qquad j = 1, \dots, n-1,$$
$$y_{1,n} + y_{2,1} = 0.$$

We may eliminate the variables $y_{2,j}$ for all j and express all $y_{1,j}$ in terms of $y_{1,n}$ for $j < n$. Substituting this into the last equation yields

$$y_{1,n}\left(1 - \left(-\frac{s_2}{s_1}\right)^n\right) = 0.$$

Thus,

$$\tilde{X}_0 = \tilde{L}_1 \cup \tilde{L}_2 \cup \cdots \cup \tilde{L}_n \cup E \quad \xrightarrow{\pi} \quad L_1 \cup L_2 \cup \cdots \cup L_n = X_0 \subset \mathbb{C}^N \tag{16}$$

consists of exactly n lines $\tilde{L}_1, \dots, \tilde{L}_n$ meeting the exceptional set $E = \{0\} \times \mathbb{P}^1$ of π transversally in the points $(s_1 : s_2) = (1 : -\zeta_n^k)$, $\quad k = 0, \dots, n$ with ζ_n a primitive n-th root of unity. Since the projection π is an isomorphism outside E, the $\tilde{L}_i$ are taken to a set of lines $L_i \subset \mathbb{C}^N$, which meet pairwise at the origin.

The situation is depicted in Figure 1 for the case $n = 3$. Note that X_0 is drawn as three cones touching each other at their vertices. This is intrinsically homeomorphic to three complex lines meeting at the origin, but drawn as embedded in real 3-space. In fact, all the pictures really capture the described objects up to homeomorphism.

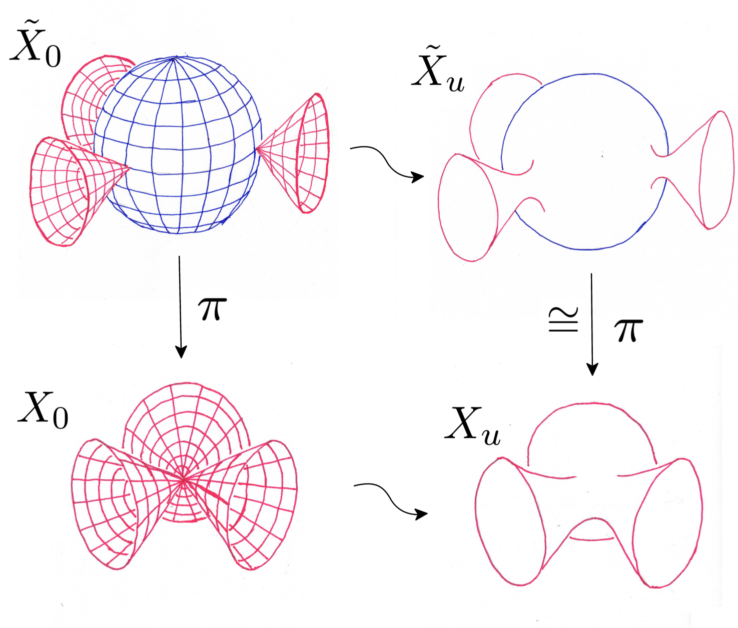

FIGURE 1. Deformation of a space curve and its Tjurina transform for $n = 3$

It is not clear a priori how the topology of X_u changes compared to X_0 when passing to a regular value u of l. For $\tilde{X}_0$, however, the induced deformation must be a smoothing of the n

distinct singularities of $\tilde{X}$ at the points $(1 : \zeta_n^k)$, because again $\pi\colon \tilde{X}_u \to X_u$ is an isomorphism and X_u is smooth. Locally and up to homotopy, the smoothing replaces a neighborhood $D_k \cup \tilde{L}_k$ of the line $\tilde{L}_k$ in X_0 by a punctured disc D_k^* at every such point. Thus $\tilde{X}_u$ has the homotopy type of a punctured 2-sphere with n points missing:

$$\tilde{X}_u \cong_{ht} L_{2,n}^{2,n} \cong_{ht} S^2 \setminus \{n \text{ points}\} \cong_{ht} \bigvee_{i=1}^{n-1} S^1. \tag{17}$$

For the last admissible value $N = n-1$ of N observe that the space $L_{2,n}^{2,n-1}$ is given by the intersection of

$$X_0 = L_1 \cup L_2 \cup \cdots \cup L_n$$

in (16) from the previous considerations with a further codimension one hyperplane in general position off the origin. Clearly, this intersection consists of precisely n points and therefore

$$L_{2,n}^{2,n-1} = \{n \text{ points}\}. \tag{18}$$

4.2. **Arbitrary EIDS of type** $(2, n, 2)$**.** Suppose that

$$A\colon (\mathbb{C}^N, 0) \to (\mathrm{Mat}(2, n; \mathbb{C}), 0)$$

defines an arbitrary EIDS $(X_0, 0) = (A^{-1}(M_{2,n}^2), 0) \subset (\mathbb{C}^N, 0)$ of type $(2, n, 2)$. We will describe the homotopy type of its determinantal Milnor fiber in all cases (19), (20), (21), and (22).

Whenever $N = n-1$, i.e. if $\dim(X_0, 0) = 0$ and $(X_0, 0)$ is a fat point, the determinantal Milnor fiber will consist of a finite number of distinct, regular points

$$\overline{X}_u = \{k \text{ points}\} \quad \text{if } N = n-1. \tag{19}$$

Since $(X_0, 0)$ is Cohen-Macaulay, we may use the principle of conservation of number and compute this number k directly from the local algebra:

$$k = \dim_{\mathbb{C}} \mathcal{O}_{X_0,0}.$$

Now let $(X_0, 0) = (A^{-1}(M_{2,n}^2), 0)$ be a curve, i.e. $d = 1 \Leftrightarrow N = n$. Theorem 1.1 is applicable and we have $s_0 = t = 2$. Hence, there is only one number $r = r(2)$ which is relevant in the bouquet decomposition (1). The homotopy type of the determinantal Milnor fiber $\overline{X}_u$ is

$$\overline{X}_u \cong_{ht} \left(\bigvee_{i=1}^{n-1} S^1 \right) \vee \left(\bigvee_{i=1}^{r} S^1 \right) \quad \text{if } N = n. \tag{20}$$

Suppose $d = \dim(X_0, 0) > 1$ and $(X_0, 0)$ is smoothable. This allows a range $n < N < 2n$ for N and according to the computations in the previous section we find

$$\overline{X}_u \cong_{ht} S^2 \vee \bigvee_{i=1}^{r} S^{N-n+1} \quad \text{if } n < N < 2n. \tag{21}$$

Note that whenever $d \geq 3$, there is still a 2-sphere in the decomposition! This is a striking difference to any behavior which can be observed for ICIS.

Finally, for values $N \geq 2n$, a determinantal singularity $(X_0, 0)$ of type $(2, n, 2)$ does not admit a determinantal smoothing. Nevertheless, the determinantal Milnor fiber $\tilde{X}_u$ is defined up to homeomorphism. In this case we find $s_0 = 1 \leq s \leq t = 2$ and we have different contributions in the bouquet decomposition. The complex link $L_{2,n}^{2,N}$ is homotopically trivial. But the thimble which is being attached to $L^{2,N}_{2,n}$ at a Morse critical point for $s = 1$ has a nontrivial normal Morse datum

$$\left(C(L^{2,2n-1}_{2,n}), L^{2,2n-1}_{2,n}\right).$$

Thus, according to (15) we find

$$\overline{X}_u \cong_{ht} \{pt\} \vee \left(\bigvee_{i=1}^{r(1)} S^3\right) \vee \left(\bigvee_{i=1}^{r(2)} S^{N-n+1}\right) \quad \text{for } N \geq 2n. \tag{22}$$

Remark 4.1. The decomposition (1) in Theorem 1.1 reduces the question about the homotopy type of a determinantal Milnor fiber to the question about the topology of the spaces $L^{t,k}_{m,n}$ appearing in the formula. In those cases, where all these $L^{t,k}_{m,n}$ themselves are homotopy equivalent to a bouquet of spheres, the same holds for the determinantal Milnor fiber.

Moreover, the generalized Milnor numbers $r(s)$ measuring the contributions from critical points on the different strata are invariants of the singularity. Using computer algebra systems like Singular, one can compute these numbers for any given singularity from the Cerf-diagrams Δ in the carrousel [13, Section 1.4] at each induction step in the proof of Theorem 1.1. However, these computations involve random choices of linear equations and it would be appealing to have a concise formula relating the numbers $r(s)$ to analytic invariants of the singularity itself.

References

[1] W. Bruns and J. Herzog. *Cohen-Macaulay rings*. Cambridge University Press, 1993.

[2] M. da Silva Pereira. Variedades determinantais e singularidades de matrizes, 2010. Thesis, Universidade de São Paulo, São Carlos, Brazil. DOI: 10.11606/t.55.2010.tde-22062010-133339

[3] J. Damon and B. Pike. Solvable groups, free divisors and nonisolated matrix singularities II: Vanishing topology. *Geom. Topol.*, 18:911–962, 2014. DOI: 10.2140/gt.2014.18.911

[4] W. Ebeling and S. Gusein-Zade. On indices of 1-forms on determinantal singularities. *Proc. Steklov Inst. Math*, 267:113–124, 2009. DOI: 10.1134/s0081543809040099

[5] A. Frühbis-Krüger and M. Zach. On the Vanishing Topology of Isolated Cohen-Macaulay Codimension 2 Singularities. To appear in *Geometry & Topology*. arχiv: 1501.01915

[6] T. Gaffney, Nivaldo G.Grulha Jr., and M.A.S. Ruas. The local Euler obstruction and topology of stabilization of associated determinantal varieties. *Math. Zeit.*, 291:905–930, 2019. DOI: 10.1007/s00209-018-2141-y

[7] M. Goresky and R. MacPherson. *Stratified Morse Theory*. Springer Verlag, 1988.

[8] H. Hamm. Lokale topologische Eigenschaften komplexer Räume. *Math. Ann.*, 191:235–252, 1972. DOI: 10.1007/bf01578709

[9] H. Hamm and Dũng Tráng Lê. Un théorème de Zariski du type de Lefschetz. *Ann. Sci. École Norm. Sup. (4)*, 6:317–355, 1973. DOI: 10.24033/asens.1250

[10] M. Hochster and J.A. Eagon. Cohen-Macaulay rings, invariant theory, and the generic perfection of determinantal loci. *American Journal of Mathematics*, 93:1020–1058, 1971. DOI: 10.2307/2373744

[11] Lê Dũng Tráng. Some remarks on relative monodromy. *Proc. Ninth Nordic Summer School/NAVF Sympos. Math., Oslo 1976*, pages 397–403, 1977. DOI: 10.1007/978-94-010-1289-8_11

[12] J.J. Nuño Ballesteros, B. Oréfice-Okamoto, and J.N. Tomazella. The vanishing Euler characteristic of an isolated determinantal singularity. *Israel J. Math.*, 197:475–495, 2013. DOI: 10.1007/s11856-012-0188-8

[13] M. Tibăr. Bouquet decomposition of the Milnor fibre. *Topology*, 35(1):227 – 241, 1995. DOI: 10.1016/0040-9383(95)00003-8

[14] G. N. Tjurina. Absolute isolatedness of rational singularities and triple rational points. *Func. Anal. Appl.*, 2:324–333, 1968. DOI: 10.1007/bf01075685

[15] M. Zach. Topology of isolated determinantal singularities, 2017. Thesis, Leibniz Universität Hannover, Germany.

[16] X. Zhang. Chern classes and characteristic cycles of determinantal varieties. 2016. arχiv: 1605.05380

Matthias Zach, Institut für Mathematik, FB 08 - Physik, Mathematik und Informatik, Johannes Gutenberg-Universität, Staudingerweg 9, 4. OG, 55128 Mainz, Germany

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ON A SINGULARITY APPEARING IN THE MULTIPLICATION OF POLYNOMIALS

SANTIAGO LÓPEZ DE MEDRANO AND ENRIQUE VEGA CASTILLO

To Cidinha, on her 70th birthday

Abstract. The multiplication of monic polynomials of degrees n and m defines a mapping $R^{n+m} \to R^{n+m}$. Singularities of this mapping at a point corresponding to two polynomials (P, Q) appear when the two polynomials have a common root. In [Ch-LdM] it was shown that, when every such common root is simple in one of the polynomials, the singularity type can be described using swallowtail singularities whose geometry is well understood. In this paper we consider the case where there are common double roots. We start with the minimal possible situation where both polynomials are of degree 2, and give a normal form for the singularity that allows us to describe its geometry quite thoroughly. This normal form is then extended to other polynomial pairs with only one common multiple root which is a double root in one of them. Finally we give a general statement for pairs whose greater common divisor has only single or double roots.

Introduction.

Let $\mathrm{MP}(\mathbf{K}, n)$ be the space of monic polynomials of degree n with coefficients in a field $\mathbf{K}$. We will consider only cases where $\mathbf{K}$ is either the real or the complex field. A polynomial in $\mathrm{MP}(\mathbf{K}, n)$ is given by n coefficients, so the space $\mathrm{MP}(\mathbf{K}, n)$ can be identified with $\mathbf{K}^n$.

Multiplication of polynomials gives a mapping:

$$\mathrm{Mult} : \mathrm{MP}(\mathbf{K}, n) \times \mathrm{MP}(\mathbf{K}, m) \to \mathrm{MP}(\mathbf{K}, n + m),$$

which can then be identified as a mapping from $\mathbf{K}^{n+m}$ to itself.

We are interested in understanding the properties of this differentiable map: at which pairs (P, Q) of polynomials is it a local diffeomorphism? When it is not so, can we describe the type of singularities that may appear, starting with the most simple situations?

In [Ch-LdM] these questions were given some first answers (which were then applied to the theory of deformations of linear operators):

(i) *The points (P, Q) where the mapping* Mult *is a local diffeomorphism are characterized as those where the two polynomials are relatively prime.*

(ii) *The singularity type is given at the pairs where the greatest common divisor of them has only simple roots (see Theorem 1 below).*

(iii) *A general normal form for every type of singularity appearing in* Mult*.*

It is the purpose of this article to study the singularity type of Mult when P and Q have a common double root. First, we give a new normal form for the simplest case that allows us to describe the geometry of its singularity type in the real and complex cases. Surprisingly, in the real case the critical set is not equivalent to, but still related to a well-known swallowtail singularity, typical of the cases where the greatest common divisor has only simple roots.

In the interesting paper [L-W] the Thom-Boardman symbol of the singularities of the mapping Mult at all points (P, Q) is computed. Our approach, following [Ch-LdM], is different: we search for a simple normal form and a complete topological description of the singularity. This objective looks difficult to achieve except in the simplest cases.

1. Known results.

The main result of [Ch-LdM] is the following:

Theorem 1. *For* $(P_0, Q_0) \in \mathrm{MP}(\mathbf{K}, n) \times \mathrm{MP}(\mathbf{K}, m)$*:*

(i) *The corank of the differential* $D\,\mathrm{Mult}(P_0, Q_0)$ *is the degree of* $\gcd(P_0, Q_0)$.

(ii) *In particular,* Mult *is a local diffeomorphism at* (P_0, Q_0) *if and only if* $\gcd(P_0, Q_0) = 1$.

(iii) *The mapping* Mult *is a* $(k+1)$*–swallowtail at* (P_0, Q_0) *for some positive integer* k *if, and only if,* $\deg\gcd(P_0, Q_0) = 1$*, the integer* k *being the maximum of the multiplicities in* P_0 *and* Q_0 *of their common root.*

(iv) *If* $\mathbf{K} = \mathbf{R}$*, the mapping* Mult *is a complex* $(k+1)$*–swallowtail at* (P_0, Q_0) *for some positive integer* k *if, and only if,* $\gcd(P_0, Q_0)$ *is an irreducible polynomial of degree* 2*,* k *being the maximum of the multiplicities in* P_0 *and* Q_0 *of their complex conjugate common roots.*

The proof consists in giving a simple normal form for such mappings. All these mappings are well-known and so is the general description of their singular and critical sets. A reduction lemma shows that the singularity type of Mult at a point (P_0, Q_0) splits into a product of the singularity types of the factors of the polynomials corresponding to the different roots:

Lemma 1. *The singularity type of* Mult *at a pair of polynomials with several common roots is the set-theoretical product of the singularity types ot* Mult *at each of the pairs consisting of the factors of the polynomials involving only one of those roots.*

This is because the multiplication of factors involving different roots is locally invertible by (ii) and so the product can be factored, multiplied separately and then multiplied together again, all the complementary multiplications being local bijections.

Another argument given in [Ch-LdM] can be formulated in general as follows:

Lemma 2. *Assume* $P_0 \in \mathrm{MP}(\mathbf{R}, 2k)$ *has no real roots and let* $P_0 = P_{01}\bar{P}_{01}$ *be a decomposition of* P_0 *such that* P_{01} *and* $\bar{P}_{01}$ *have no common roots. Then the mapping* $P_1 \mapsto P_1\bar{P}_1$ *is a diffeomorphism between a neighborhood of* P_{01} *in* $\mathrm{MP}(\mathbf{C}, k)$ *and a neighborhood of* P_0 *in* $\mathrm{MP}(\mathbf{R}, 2k)$.

This is because in a neighborhood of P_0 in $\mathrm{MP}(\mathbf{C}, 2k)$ every polynomial P can be written in a unique way as P_1P_2 with P_1, P_2 in neighborhoods of P_{01} and $\bar{P}_{01}$, respectively. When $P \in \mathrm{MP}(2k, \mathbf{R})$ then $P = \bar{P} = \bar{P}_1\bar{P}_2$. The uniqueness of the decomposition implies that $P_2 = \bar{P}_1$ and $P = P_1\bar{P}_1$ so the mapping $P \mapsto P_1$ is a local inverse of $P_1 \mapsto P_1\bar{P}_1$.

Also, in [Ch-LdM], Proposition 2, there are normal forms for all possible singularity types of Mult at pairs with only one root which is common. We still do not know how to use these normal forms to obtain a geometric description of the singularity types, so we looked for new normal forms in the cases we study.

2. Polynomials with common double roots.

We will start by describing the minimal case: two polynomials of degree 2 with one single root which is common and double in both of them. We will give a new normal form of the mapping Mult in the neighborhood of such a pair and a detailed description of its singularity type in the case $K = \mathbf{R}$. Section 2.4 treats the case of two real polynomials with a double common complex root.

In section 2.5 we give a new normal form for the case of two polynomials with only one root which is common, double in one of them and of multiplicity $k \geq 2$ in the other one.

In section 2.6 we will combine all the cases known to give a statement about pairs of polynomials whose greater common divisor has only simple and double roots.

2.1. **The minimal case for general K.** We consider now the case where both polynomials are of degree 2 with a common double root α which is in $\mathbf{K}$.

A change of variable $x = y + \alpha$ in those polynomials is an automorphism of $MP(\mathbf{K}, 2)$ that preserves the multiplication and gives us two polynomials in y whose common double root is zero. So we can assume that both $P_0(x)$ and $Q_0(x)$ are equal to x^2 and $\mathrm{Mult}(P_0, Q_0) = x^4$. A variation of the pair (P_0, Q_0) is given by the pair (P, Q) where $P(x) = x^2 + sx + t$ and $Q(x) = x^2 + ux + v$. Their product is then

$$P(x)Q(x) = x^4 + ux^3 + sx^3 + vx^2 + sux^2 + tx^2 + svx + tux + tv.$$

In terms of the parameters s, t, u, v the mapping is

$$F(s, t, u, v) = (u + s, v + su + t, sv + tu, tv).$$

This is a simple mapping of degree 2, but this fact does not give us an idea of its geometry. In several steps we will simplify this map through invertible changes of variables, obtaining a map of degree 4 that can be much better understood.

We begin by taking the first two components of F as new independent variables, through changes of coordinates:

$$s = s_1 - u, \quad t = t_1 - s_1 u - v + u^2$$

to obtain the equivalent map

$$F_1(s_1, t_1, u, v)\left(s_1, t_1, vs_1 - 2vu + ut_1 - s_1 u^2 + u^3, (t_1 - s_1 u - v + u^2)v\right).$$

To simplify the third component we use the changes of coordinates:

$$v = \frac{t_1}{2} - \frac{s_1 u}{2} + \frac{u^2}{2} - \frac{v_1}{2}, \quad u = u_1 + \frac{s_1}{2}$$

giving the new equivalent function

$$F_2(s_1, t_1, u_1, v_1) = \left(s_1, t_1, \frac{-s_1^3 + (4u_1^2 + 4t_1)s_1}{8} + u_1 v_1, \frac{(s_1^2 - 4u_1^2 - 4t_1 - 4v_1)^2}{64}\right).$$

Now we operate on the target space by substracting two functions of the first two components: $-\frac{s_1^3}{8} + \frac{s_1 t_1}{2}$ from the third component and $(-\frac{s_1^2}{8} + \frac{t_1}{2})^2$ from the fourth one.

Another change of variables finishes the simplification of the third coordinate:

$$v_1 = v_2 - \frac{s_1 u_1}{2},$$

$$F_3(s_1, t_1, u_1, v_2) = \left(s_1, t_1, u_1 v_2, \frac{u_1^4 + s_1 u_1 v_2 - v_2^2}{4} + \frac{(-3s_1^2 + 8t_1)u_1^2}{16}\right).$$

Now it is time to simplify the fourth coordinate through the substitutions

$$t_1 = \frac{t_2}{2} + \frac{3s_1^2}{8}, \quad s_1 = 4s_2,$$

$$F_4(s_2, t_2, u_1, v_2) = \left(4s_2, \frac{t_2}{2} + 6s_2^2, u_1 v_2, \frac{1}{4}u_1^4 + s_2 u_1 v_2 + \frac{1}{4}t_2 u_1^2 - \frac{1}{4}v_2^2\right).$$

We have messed with the first two components, but we can fix them back easily by acting on the target: divide the first component by 4 and then substract from the second one the function $6s_2^2$ of the first one. Then multiply the second component by 2 to make it again equal to t_2.

Finally, one can substract the product of the first and third components from the last one to obtain a remarkable simplification of the original mapping[1]:

$$F_5(s_2, t_2, u_1, v_2) = \left(s_2, t_2, u_1 v_2, \frac{1}{4}(u_1^4 + t_2 u_1^2 - v_2^2)\right).$$

Seen as an unfolding, we observe that the coordinate s_2 plays no role in the deformation of the mapping, so we can omit it from both sides and need only study the one-parameter unfolding, which in new coordinates can be written as:

$$f(a, x, y) = \left(a, xy, x^4 + ax^2 - y^2\right).$$

So Mult at (P_0, P_0) is equivalent to the *suspension* of f.

f is an unfolding of the mapping

$$f_0(x, y) = (xy, x^4 - y^2),$$

which for $\mathbf{K} = \mathbf{R}$ reminds us of the *square of a complex variable* mapping $(x, y) \mapsto (x^2 - y^2, 2xy)$ and, actually, the two mappings are topologically equivalent (see section 2.3).

The unfolding $f(a, x, y)$ is based on the deformation

$$f_a(x, y) = \left(xy, x^4 + ax^2 - y^2\right).$$

To obtain the singular points of f we compute its Jacobian matrix:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & y & x \\ x^2 & 4x^3 + 2ax & -2y \end{pmatrix}$$

so the singular set is given by:

$$J = -4x^4 - 2ax^2 - 2y^2 = 0,$$

which gives also the singular set of f_a for each fixed a.

2.2. **The minimal case for $\mathbf{K} = \mathbf{C}$.** When $\mathbf{K} = \mathbf{C}$ it turns out that for all $a \neq 0$, the deformations f_a are equivalent: the substitutions $x = \sqrt{a}X, y = aY$, followed by multiplication of the components by adequate constants, gives $\left(XY, X^4 + X^2 - Y^2\right)$, which is the case $a = 1$.

However, f_0 is not equivalent to f_a for $a \neq 0$. The jacobian determinant of f_a is in general $-4x^4 - 2ax^2 - 2y^2$, so the origin is always a zero and a singular point of J. Under those circumstances, equivalent maps must have jacobians with equivalent 2-jets, but for $a = 0$ the 2-jet of the jacobian determinant is degenerate, which is not the case for $a \neq 0$. Also, the singular sets are not equivalent.

In this case f_a is, for all a, surjective and, generically, four-to-one, since the corresponding equations have always a solution and generically four different ones (cf. the computations in the next section).

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[1]For the record, it will be useful for section 2.5 to take note now of the global substitution suffered by the coordinates s, t:

$$s = 2s_2 - u_1, \quad t = s_2^2 - s_2 u_1 + \frac{1}{2}u_1^2 + \frac{1}{4}t_2 + \frac{1}{2}v_2.$$

2.3. **The minimal case for $\mathbf{K} = \mathbf{R}$.** In the case $\mathbf{K} = \mathbf{R}$ at polynomials of degree 2 with two common double real roots, the computation in section 2.1 gives again that Mult is equivalent to the suspension of the mapping

$$f(a,x,y) = \left(a, xy, x^4 + ax^2 - y^2\right),$$

which is an unfolding of

$$f_0(x,y) = \left(xy, x^4 - y^2\right).$$

The mapping f_0 appears in Mather's classification of stable germs as being of the type $II_{2,4}$ with algebra $\mathbf{R}[[x,y]]/(xy, x^2 - y^4)$. See [M], p. 240. Its jacobian determinant is $-4x^4 - 2y^2$; so the origin is the only critical point of f_0.

Consider now f_0 as a (non-holomorphic) function of the complex variable $z = x + iy$. Since this function takes the same values for z and $-z$, it can be written as a function of z^2; so we can express f_0 as a composition

$$f_0(x,y) = g_0(x^2 - y^2, 2xy),$$

where g_0 is a differentiable function outside the origin. It follows from the computations below that g_0 is a homeomorphism of R^2 which is a diffeomorphism outside the origin.

As for f_a, its differentiable type now depends on the sign of a: For $a > 0$, the substitutions in the previous section show that f_a is equivalent to f_1. For $a < 0$ we have to use instead the substitutions $x = \sqrt{-a}X, y = aY$ to obtain in the same way that f_a is equivalent to f_{-1}.

By the same argument as in the case $\mathbf{K} = \mathbf{C}$ we obtain that f_0 is not equivalent to f_a for any $a \neq 0$.

We shall prove now that f_a is 2 to 1 outside the origin for $a \geq 0$ and surjective for all a:

If $a \geq 0$, take a point (x,y) and another point (x_1, y_1) with the same image:

$$f_a(x,y) = f_a(x_1, y_1);$$

so

$$xy = x_1 y_1, \quad x^4 + ax^2 - y^2 = x_1^4 + ax_1^2 - y_1^2.$$

If $x = 0$ then one of x_1, y_1 is zero.

If $x_1 = 0$ then $y_1 = \pm y$ and there is only one more point with the same image as (x,y).

If $x = 0$ and $y_1 = 0$ then the second equation gives

$$-y^2 = x_1^4 + ax_1^2,$$

which is only possible for $y = x_1 = 0$ and there is no other point with the same image as (x,y).

If $x \neq 0$ we can solve for y in the first equation and substitute its value in the second one. After multiplying by x^2 and factoring the resulting polynomial we get

$$(x - x_1)(x + x_1)(x^4 + x_1^2 x^2 + ax^2 + y_1^2) = 0.$$

The third factor must be positive since $x \neq 0$ and $a \geq 0$ so we must have $x_1 = \pm x$ and therefore $y_1 = \pm y$, with the same sign. So there is only one more point with the same image as (x,y). So f_a is 2-to-1 outside the origin.

To see that f_a is surjective for every a, we need to solve the equations

$$xy = \chi, \quad x^4 + ax^2 - y^2 = \eta$$

for a given $(\chi, \eta) \in \mathbf{R}^2$.

If $\chi = 0$ there is always a solution: $x = 0, y = \sqrt{-\eta}$ for $\eta \leq 0$; $y = 0$ and x a solution $x^4 + ax^2 = \eta$ for $\eta > 0$.

If $\chi \neq 0$ then x and y are non-zero. Then we can proceed as before: solve for y in the first equation, substitute its value in the second one and multiply by x^2. We obtain:

$$x^6 + ax^4 - \eta x^2 - \chi^2 = 0.$$

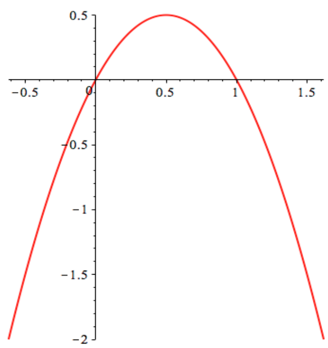
For $x = 0$ this polynomial is negative, while it tends to $+\infty$ when x tends to $+\infty$. Therefore there is a positive solution of this equation (and a negative one, too).

So we have shown that f_a is surjective for all a.

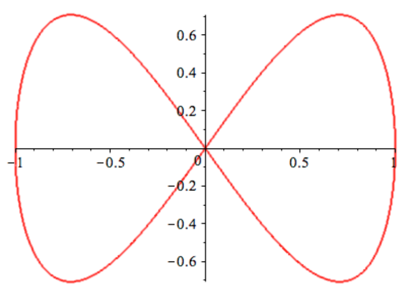
For $a > 0$ the jacobian determinant is again 0 only at the origin.

For $a < 0$ we can see the singular set as follows: Substituting $X = x^2$ and $Y = y^2$ in the jacobian determinant we obtain a parabola:

$$4X^2 - 2aX - 2Y = 0$$



The singular set is then the pre-image of the part of this parabola in the first quadrant under the mapping $(x, y) \mapsto (x^2, y^2)$ so it is the lemniscate:



This lemniscate is actually, up to linear changes of coordinates, the variant known as *Geromo's lemniscate*:

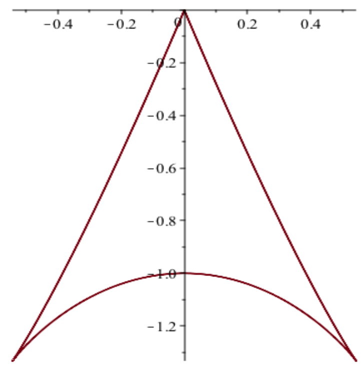
$$x^4 - x^2 + y^2 = 0.$$

A parametrization of this lemniscate is known (see [Wik]), which adapted to ours becomes

$$\gamma(\phi) = (\sqrt{-a/2}\,\cos(\phi), a\,\sin(\phi)\cos(\phi)/\sqrt{2})$$

as can be directly verified. We will use this parametrization to obtain the image of the singular set:

$$\sigma(\phi) = (a\sqrt{-a}\,\cos(\phi)^2\sin(\phi)/2, a^2\cos(\phi)^4/4 - a^2\cos(\phi)^2/2 - a^2\sin(\phi)^2\cos(\phi)^2/2)$$



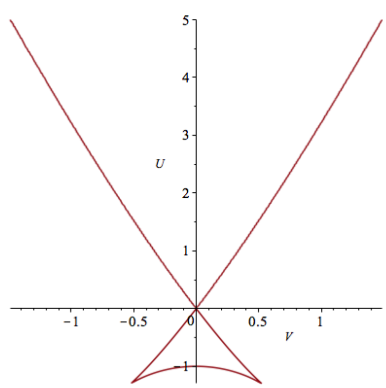
This figure has three singular points, two simple cusps (as can easily verified) at the lower level and a strange angle at the origin.

Let us call U, V the coordinates in the target plane containing this critical set. One can find the equations satisfied by the critical set by using the parameters $X = x^2, Y = y^2$ as before and eliminating the variables X, Y from the components of the mapping and the equation of the singular set. Alternatively, one can parametrize algebraically the intersections of the lemniscate with the four quadrants to carry out this elimination.

In any case, it can be verified directly that the points in the critical set satisfy the following equation:

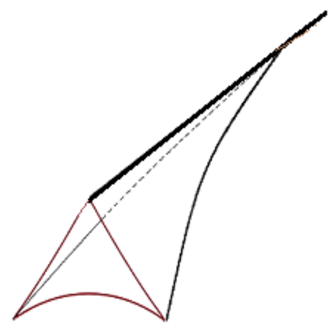
$$108\,a^3U^2 - 729\,U^4 + 486\,a\,U^2V + 27\,a^2V^2 + 108\,V^3 = 0$$

Drawing the zero set of this polynomial for a negative value of a, one obtains the following figure:

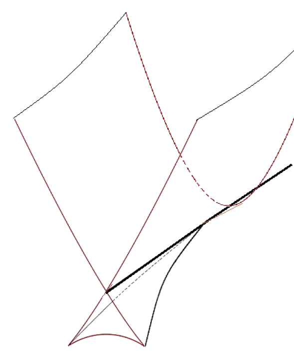


So the critical set of our mapping is just a semi-algebraic subset of this well-known swallowtail curve! (And this explains the angle).

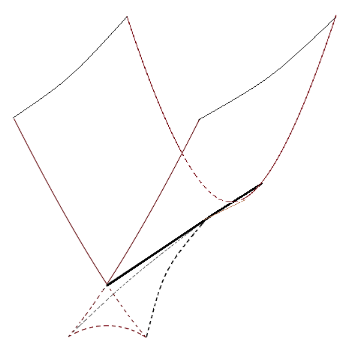
We can also draw the unfolding of the critical set by considering all values of a: for negative values of a it is the previous figure, where the triangular lower part shrinks to a single point when a approaches 0 and continues to be a single point when a is positive (we have highlighted the a axis):



Again, this is only the lower part of the swallowtail unfolding:



It is a curious fact that the complementary upper part of the swallowtail:



appears also as the singularity of a minimax solution of a Hamilton-Jacobi partial differential equation. See [Ch2] section 2.5 for the theory and [Ch1], appendix, for a specific example (the explicit figure appears in page 431).

2.4. **The minimal case of two real polynomials with a double complex root.** In this case we will have actually two conjugate double roots $\alpha, \bar{\alpha}$.

Here we apply Lemma 2 of section 1 to obtain that at such point Mult is equivalent to the suspension of the **complex** mapping

$$f : \mathbf{C}^2 \to \mathbf{C}^2,$$

$$f(a, x, y) = \left(a, xy, x^4 + ax^2 - y^2\right).$$

This is an unfolding of the mapping, in real variables:

$$f(x_1, x_2, y_1, y_2) = (x_1y_1 - x_2y_2, x_1y_2 + x_2y_1, x_1^4 - 6x_1^2x_2^2 + x_2^4 - y_1^2 + y_2^2, 4x_1^3x_2 - 4x_1x_2^3 - 2y_1y_2).$$

2.5. **The case $P_0(x) = (x-a)^2$, $Q_0(x) = (x-a)^k$.** We give now a formula for the general case of two polynomials with a single root which is common, double in one of them and of degree $k \geq 2$ in the other one. So we can assume as before that $P_0(x) = x^2, Q_0(x) = x^k$. The method consists in applying the same changes of variables as in the minimal case $k = 2$ and is valid for any field $\mathbf{K}$. This gives a reasonable closed normal form, while other methods we have tried do not seem to produce one.

First, we illustrate it with small values of k. For $k = 3$ the mapping is given by the coefficients of $(x^2+sx+t)(x^3+ux^2+vx+w)$. After applying the sequence of changes of variable of section 2.1, adjusting factors and renaming the variables, one obtains the following normal form:

$$(a,b,x,y,w) \mapsto \big(a,b,xy+w,x^4+bx^2-y^2+(2a-x)w,(4a^2-4ax+2x^2+b+2y)w\big)\,.$$

One could also linearize the third component by using the coordinate $w_1 = xy + w$, thus obtaining a normal form which would be an unfolding of f_0. This would, however, increase the complexity of the expressions of the following components (without much hope of simplification).

Observe that here both parameters a, b appear in the formula, so there are no mute parameters. Also, that the new coordinate w appears only with degree 1 multiplied by factors of degrees 0 to 2 and increasing complexity. It does not seem easy to simplify them with new changes of coordinates.

The good news is that for greater values of k the coefficients of the new coordinates not only do not increase in complexity, but are actually exactly the same as for $k = 3$. It will be therefore convenient to use a short notation for them:

$$\sigma(a,x) = 2a - x, \quad \tau(a,b,x,y) = 4a^2 - 4ax + 2x^2 + b + 2y.$$

Then, for $k = 4$ we get by the same method the following map:

$$(a,b,x,y,w_3,w_4) \mapsto (a,b,xy+w_3,x^4+bx^2-y^2+\sigma w_3+w_4,\tau w_3+\sigma w_4,\tau w_4).$$

For $w = 0$ we obtain essentially the normal form for $k = 2$. This shows that this mapping is a deformation of the mapping f_0 we studied before, and is the basis of the proof by induction of the general normal form for every k:

Let $P(x) = x^2+sx+t$ and $Q_k(x) = x^k+ux^{k-1}+vx^{k-2}+\Sigma_{i=3}^k w_i x^{k-i}$ and $F_k(x) = P(x)Q_k(x)$. Then, clearly

$$F_{k+1}(x) = xF_k(x) + P(x)w_{k+1}.$$

In terms of the coordinates $(s,t,u,v,w_3,\ldots,w_k,w_{k+1})$, this is expressed as

$$F_{k+1}(s,t,u,v,w_3,\ldots,w_k,w_{k+1}) = (F_k(s,t,u,v,w_3,\ldots,w_k),0) + (0,\ldots,0,w_{k+1},w_{k+1}s,w_{k+1}t).$$

Passing to the coordinates (s_2,t_2,u_1,v_2) as in section 2.1, we obtain

$$\begin{gathered}F_{k+1}(s_2,t_2,u_1,v_2,w_3,\ldots,w_k,w_{k+1}) = \\ \big(F_k(s_2,t_2,u_1,v_2,w_3,\ldots,w_k),0\big) + \\ \Big(0,\ldots,0,w_{k+1},(2s_2-u_1)w_{k+1},\Big(s_2^2-s_2u_1+\tfrac{1}{2}u_1^2+\tfrac{1}{4}t_2+\tfrac{1}{2}v_2\Big)w_{k+1}\Big),\end{gathered}$$

since the coefficients of w_{k+1} are precisely the results of applying the coordinate changes of section 2.1 to the variables s, t (cf. footnote 1).

Starting with $k = 2$ this gives the inductive proof that the mapping Mult at P, P_k is equivalent to the mapping (in new coordinates):

$$\begin{gathered}G_k(a,b,x,y,w_3,\ldots,w_k) = \\ (a,b,xy,x^4+bx^2-y^2,0,\ldots,0) + \\ (0,0,\tau(a,b,x,y)w_1+\sigma(a,x)w_2+w_3,\ldots,\tau(a,b,x,y)w_k+\sigma(a,x)w_{k+1}+w_{k+2}),\end{gathered}$$

where σ and τ are as above and it is understood that $w_1 = w_2 = w_{k+1} = w_{k+2} = 0$.

As before, the components $\tau(a, b, x, y)w_i + \sigma(a, x)w_{i+1} + w_{i+2}$ can, in principle, be linearized for $i = 3$ to $k - 2$ to present G as an unfolding of $f_0(x, y) = (xy, x^4 - y^2)$ with k parameters.

2.6. **The general result.** Putting together the previous results we can conclude that:

If $P_0 \in \mathrm{MP}(\mathbf{K}, n)$ and $Q_0 \in \mathrm{MP}(\mathbf{K}, m)$ are two polynomials such that their greatest common divisor has only simple and double roots then:

1)*If $\mathbf{K} = \mathbf{C}$ then at (P_0, Q_0),* Mult *is equivalent to the suspension of a product of complex swallowtails and complex mappings G_k.*

2)*If $\mathbf{K} = \mathbf{R}$ then at (P_0, Q_0),* Mult *is equivalent to the suspension of a product of real complex swallowtails and real and complex mappings G_k.*

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Bibliography.

[Ch1] M. Chaperon, *Un témoignage*, in Michèle Porte, *Passion des formes, à René Thom*, Editions Fontenay-Saint Cloud (1994), pp. 419-438.

[Ch2] M. Chaperon, *Singularities in contact geometry*, in *Geometry and topology of caustics-Caustics '02*, Banach Center Publications 62 (2004), pp. 39-55. DOI: 10.4064/bc62-0-3

[Ch-LdM] M. Chaperon and S. López de Medrano, *Some regularities and singularities appearing in the study of polynomials and operators*, Astérisque 323 (2009), pp. 123-160, Société Mathématique de France. numdam: AST_2009__323__123_0

[L-W] J. Lin and J. Wethington, *On the Thom-Boardman Symbols for Polynomial Multiplication Maps*, Asian J. Math. Volume 16, Number 3 (2012), pp. 367-386. See also J. Wethington, *On computing the Thom-Boardman Symbols for Polynomial Multiplication Maps*, Ph.D. Thesis, University of Athens, Georgia, 2002. DOI: 10.4310/ajm.2012.v16.n3.a1

[M] J. Mather, *Stability of C^∞ mappings: VI The nice dimensions.* Proceedings of Liverpool Singularities Symposium I, LNM, volume 192, pp. 207-253 DOI: 10.1007/BFb0066824

[Wik] *Lemniscate of Geromo*, Wikipedia.

Santiago López de Medrano, Instituto de Matemáticas, UNAM
Email address: santiago@matem.unam.mx

Enrique Vega Castillo, Facultad de Ciencias, UNAM
Email address: enriquevegac@ciencias.unam.mx

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REAL AND COMPLEX INTEGRAL CLOSURE, LIPSCHITZ EQUISINGULARITY AND APPLICATIONS ON SQUARE MATRICES

THIAGO F. DA SILVA, NIVALDO G. GRULHA JR., AND MIRIAM S. PEREIRA

Dedicated to Terence Gaffney and Maria Ruas, on the occasion of their 70th birthday, and to Marcelo Saia, on the occasion of his 60th birthday.

Abstract. Recently the authors investigated the Lipschitz triviality of simple germs of matrices. In this work, we improve some previous results and we present an extension of an integral closure result for the real setting. These tools are applied to investigate classes of square matrices singularities classified by Bruce and Tari.

Introduction

The study of Lipschitz equisingularity has risen from works of Zariski [25], Pham [22] and Teissier [23] and further developed by Parusiński ([18, 19]), Gaffney ([15, 12, 13]), Fernandes, Ruas ([11]) and others.

In [17], Mostowski introduced a new technique for the study of this subject from the existence of Lipschitz vector fields. In general, these vector fields are not canonical from the varieties. Nevertheless, Gaffney [12] presented conditions to find a canonical Lipschitz vector field in the context of a family of irreducible curves using the *double* structure, defined for ideals in [13] and generalized for modules in [15].

Families of square matrices were first studied by Arnold in [2], where the parametrised invertible matrices act by conjugation. Recently, many authors have presented a series of interesting results about determinacy and classification using parametrised families or smooth changes of coordinates in the source of the germ ([3], [4], [9], [10] and [21]).

More recently, Gaffney's result was extended in [8], where the authors presented conditions which ensure the canonical vector field is Lipschitz in the context of 1-unfoldings of singularities of matrices, following the approach of Pereira and Ruas [24].

In this work we prove a real version of the result proved in [8] in order to investigate the Lipschitz triviality in the real case. Finally, we study some deformations of simple singularities classified by Bruce and Tari [3, 4] in the real and complex cases, using a similar approach as that in [8].

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Key words and phrases. Bi-Lipschitz Equisingularity, Real and Complex integral closure, The double structure, Finite Determinacy, Canonical vector fields.

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1. Notation and Background

We start with some notation. Let $\mathbb{K}$ be a field which is $\mathbb{R}$ or $\mathbb{C}$ and let $\mathcal{R}$ be the group of diffeomorphisms $\mathbb{K}^r, 0 \to \mathbb{K}^r, 0$. Let $\mathcal{H}$ denote the set of germs of smooth mappings

$$\mathbb{K}^r, 0 \to GL_n(V) \times GL_p(W),$$

and M the set of germs $F : \mathbb{K}^r, 0 \to \mathrm{Hom}(V; W)$. The set $\mathcal{H}$ can be endowed with a group structure inherited from the product group in the target.

We define a notion of bi-Lipschitz equivalence between two matrices as in [20].

Definition 1.1. *Let $\mathcal{G} = \mathcal{R} \ltimes \mathcal{H}$ be the semi-direct product of $\mathcal{R}$ and $\mathcal{H}$. We say that two germs*

$$F_1, \quad F_2 : \mathbb{K}^r, 0 \to Hom(V; W)$$

are $\mathcal{G}$-Lipschitz equivalent if there exist a germ $\phi : (\mathbb{K}^r, 0) \to (\mathbb{K}^r, 0)$ of a bi-Lipschitz homeomorphism and germs of continuous mappings $X : (\mathbb{K}^r, 0) \to GL_n(V)$, $Y : (\mathbb{K}^r, 0) \to GL_p(W)$ such that $F_1 = X^{-1}(F_2 \circ \phi^{-1})Y$.

An element of M can also be considered as a map $\mathbb{K}^r, 0 \to \mathbb{K}^N$, where we identify $\mathrm{Hom}(V; W)$ with the $n \times p$ matrices, and $N = np$.

It is not difficult to see that $\mathcal{G}$ is one of Damon's geometric subgroups of $\mathcal{K}$. As a consequence of Damon's result we can use the techniques of singularity theory, for instance, those concerning finite determinacy (see [6], [21] and [4]).

It is possible to determine the tangent space to the orbit for the action of the group $\mathcal{G}$ on M. Given a matrix F, we write $F_{x(i)}$ for the matrix $\dfrac{\partial F}{\partial x_i}$ and we denote $\mathcal{E}_r$ for the ring of smooth functions $\mathbb{K}^r, 0 \to \mathbb{K}$. So the tangent space could be viewed as an $\mathcal{E}_r$-submodule of $\mathcal{E}_N$ spanned by the set of matrices R_{il} (respectively C_{jm}) with l^{th} row (respectively m^{th} th column) the i^{th} row of F (respectively j^{th} column) and with zeros elsewhere, for $1 \leq i$, $l \leq n$ and $1 \leq j$, $m \leq p$ (see [6], [21] and [4]).

2. Real integral closure and Lipschitz Equisingularity

For the complex case, in [8] the authors obtained conditions so that the canonical vector field defined in a family of simple germs of matrices is Lipschitz, depending of a specific inclusion of ideals, involving the integral closure and the double of an ideal.

A new comprehension of the integral closure in the real case plays a key role in the proof of Theorem 2.4. Let us recall this notion.

Let $(\mathcal{A}_n, m_n)$ be the local ring of real analytic functions germs at the origin in $\mathbb{R}^n$, and let $\mathcal{A}_n^p$ be the $\mathcal{A}_n$-free module of rank p. For a germ of a real analytic set (X, x), denote by $\mathcal{A}_{X,x}$ the local ring of real analytic function germs at (X, x).

Definition 2.1. *Let I be an ideal of $\mathcal{A}_{X,x}$. An element $h \in \mathcal{A}_{X,x}$ is in the* ***real integral closure*** *of I, denoted $\overline{I}$, if $h \circ \phi \in \phi^*(I)\mathcal{A}_1$, for all real analytic path $\phi : (\mathbb{R}, 0) \to (X, x)$.*

For an algebraic definition of the real integral closure of an ideal one can see [5].

The key step to obtain the main results of [8] for the real case is the fact that the definition of the real integral closure of an ideal is equivalent to the following formulation using analytic inequalities.

Theorem 2.2 ([14])**.** *Let I be an ideal of $\mathcal{A}_{X,x}$ and $h \in \mathcal{A}_{X,x}$. Then: $h \in \overline{I}$ if and only if for each choice of generators $\{f_i\}$ there exist a positive constant C and a neighborhood U of x such that $\| h(z) \| \leq C \max_i \| f_i(z) \|$ for all $z \in U$.*

Let us recall some definitions and fix some notations.

Here we work with one parameter deformations and unfoldings. The parameter space is denoted by $Y = \mathbb{R} \equiv \mathbb{R} \times 0$.

Definition 2.3. *Let $h \in \mathcal{A}_N$. The* **double of** h *is the element denoted by $h_D \in \mathcal{A}_{2N}$ defined by the equation $h_D(z, z') := h(z) - h(z')$.*

If $h = (h_1, ..., h_r)$ is a map, with $h_i \in \mathcal{A}_N$, for all i, then we define $I_D(h)$ as the the ideal of $\mathcal{A}_{2N}$ generated by $\{(h_1)_D, ..., (h_r)_D\}$.

We obtain a relation between the real integral closure of the double and the canonical vector field induced by a one parameter unfolding to be Lipschitz.

Let $\tilde{F} : \mathbb{R} \times \mathbb{R}^q \longrightarrow \mathbb{R} \times \mathbb{R}^n$ be an analytic map, which is a homeomorphism onto its image, and such that we can write $\tilde{F}(y, x) = (y, \tilde{f}(y, x))$, with $\tilde{f}(y, x) = (\tilde{f}_1(y, x), ..., \tilde{f}_n(y, x))$. Let us denote by

$$\frac{\partial}{\partial y} + \sum_{j=1}^{n} \frac{\partial \widetilde{f}_j}{\partial y} \cdot \frac{\partial}{\partial z_j}$$

the vector field $v : \tilde{F}(\mathbb{R} \times \mathbb{R}^q) \longrightarrow \mathbb{R} \times \mathbb{R}^n$ given by

$$v(y, z) = \left(1, \frac{\partial \tilde{f}_1}{\partial y}(\tilde{F}^{-1}(y, z)), ..., \frac{\partial \tilde{f}_n}{\partial y}(\tilde{F}^{-1}(y, z))\right).$$

Theorem 2.4. *The vector field $\frac{\partial}{\partial y} + \sum_{j=1}^{n} \frac{\partial \widetilde{f}}{\partial y} \cdot \frac{\partial}{\partial z_j}$ is Lipschitz if and only if*

$$I_D\Big(\frac{\partial \tilde{F}}{\partial y}\Big) \subseteq \overline{I_D(\tilde{F})}.$$

Proof. Since we are working in a finite dimensional $\mathbb{R}$-vector space then all the norms are equivalent. To simplify the argument, we use the notation $\|.\|$ for the *maximum norm* on $\mathbb{R} \times \mathbb{R}^q$ and $\mathbb{R} \times \mathbb{R}^n$, i.e., $\|(x_1, ..., x_{n+1})\| = \max_{i=1}^{n+1}\{\|x_i\|\}$.

Suppose the canonical vector field is Lipschitz. By hypothesis there exists a constant $c > 0$ such that $\| v(y, z) - v(y', z') \| \leq c \| (y, z) - (y', z') \|$ for all $(y, z), (y', z') \in U$, where U is an open subset of $\tilde{F}(\mathbb{R} \times \mathbb{R}^q)$.

Thus, given $(y, x), (y', x') \in \tilde{F}^{-1}(U)$, and applying the above inequality on these points, we get

$$\left\|(\frac{\partial \tilde{f}_j}{\partial y})_D(y, x, y', x')\right\| \leq c \| \tilde{F}(y, x) - \tilde{F}(y', x') \|$$

for all $j = 1, ...n$. By the previous theorem, each generator of $I_D(\frac{\partial \tilde{F}}{\partial y})$ belongs to $\overline{I_D(\tilde{F})}$.

Now suppose that $I_D(\frac{\partial \tilde{F}}{\partial y}) \subset \overline{I_D(\tilde{F})}$. Using the hypothesis and Theorem 2.2, for each $j \in \{1, ...n\}$ there exists a constant $c_j > 0$ and an open subset $U_j \subset \mathbb{R} \times \mathbb{R}^q$ such that

$$\left\|(\frac{\partial \tilde{f}_j}{\partial y})_D(y, x, y', x')\right\| \leq c_j \| \tilde{F}(y, x) - \tilde{F}(y', x')\right\|$$

for all $(y, x), (y', x') \in U_j$. Take $U := \bigcap_{j=1}^{n} U_j$, $c := \max\{c_j\}_{j=1}^{n}$ and $V := \tilde{F}(U)$, which is an open subset of $\tilde{F}(\mathbb{R} \times \mathbb{R}^q)$, since $\tilde{F}$ is a homeomorphism onto its image. Hence,

$$\| v(y, z) - v(y', z') \| \leq c \| (y, z) - (y', z') \|$$

for all $(y,z),(y',z')\in V$.

Therefore, the vector field $\frac{\partial}{\partial y}+\sum\limits_{j=1}^{n}\frac{\partial \tilde{f}_j}{\partial y}\cdot\frac{\partial}{\partial z_j}$ is Lipschitz. □

Corollary 2.5. *Suppose that $\tilde{F}:\mathbb{R}\times\mathbb{R}^q\longrightarrow\mathbb{R}\times Hom(\mathbb{R}^m,\mathbb{R}^n)$ is an analytic map and a homeomorphism onto its image, and suppose we can write $\tilde{F}(y,x)=(y,F(x)+y\theta(x))$.*

a) *The vector field $\frac{\partial}{\partial y}+\sum\limits_{j=1}^{n}\frac{\partial \widetilde{f}}{\partial y}\cdot\frac{\partial}{\partial z_j}$ is Lipschitz if, and only if, $I_D(\theta)\subseteq\overline{I_D(\tilde{F})}$.*

b) *If θ is constant then the vector field $\frac{\partial}{\partial y}+\sum\limits_{j=1}^{n}\frac{\partial \widetilde{f}}{\partial y}\cdot\frac{\partial}{\partial z_j}$ is Lipschitz.*

3. Applications in some classes of square matrices

In this section we study if the Lipschitz condition is satisfied on the canonical vector field naturally associated to the 1-unfolding of a $\mathcal{G}$-simple square matrices singularities classified in [3, 4]. Our goal is to obtain a better understanding of its behaviour. In [8] we consider versal deformation of determinantal singularities of codimension 2 and we showed this behaviour depends on the type of the normal form.

The next result presents a part of the classification of $\mathcal{G}$-simple symmetric matrices obtained by Bruce on Theorem 1.1 of [3].

Proposition 3.1. *The $\mathcal{G}$-simple germs $F:\mathbb{C}^2\to Sym_2$ of rank 0 at the origin are given in the following table.*

	Normal Form		Discriminant
1.	$\begin{pmatrix} y^k & x \\ x & y^\ell \end{pmatrix}$	$k\geq 1,\ell\geq 2$	$\mathcal{A}_{k+\ell+1}$
2.	$\begin{pmatrix} x & 0 \\ 0 & y^2+x^k \end{pmatrix}$	$k\geq 2$	D_{k+2}
3.	$\begin{pmatrix} x & 0 \\ 0 & xy+y^k \end{pmatrix}$	$k\geq 2$	D_{2k}
4.	$\begin{pmatrix} x & y^k \\ y^k & xy \end{pmatrix}$	$k\geq 2$	D_{2k+1}
5.	$\begin{pmatrix} x & y^2 \\ y^2 & x^2 \end{pmatrix}$		E_6
6.	$\begin{pmatrix} x & 0 \\ 0 & x^2+y^3 \end{pmatrix}$		E_7

In the following result we establish conditions for the Lipschitz triviality of the canonical vector field associated to the normal forms introduced in the above proposition. Differently from the cases exhibited on [8], here we present examples with a different nature. Taking the versal deformation of a normal formal we can find directions that produce Lipschitz trivial deformations, Lipschitz deformations off the origin or non-Lipschitz.

Proposition 3.2. *Following the table of normal forms of $\mathcal{G}$-simple germs $F : \mathbb{C}^2 \to Sym_2$ of rank 0 at the origin, the canonical vector field associated to the 1-parameter deformation $\tilde{F}$ induced by $\theta \in \dfrac{Sym_2}{T\mathcal{G}_eF}$ is Lipschitz in the following conditions:*

1. *For the normal form 1 of the table, if the canonical vector field associated to $\tilde{F}$ is Lipschitz then θ can be written in the form*
$$\theta = \begin{pmatrix} a_0 + \sum\limits_{i=r}^{k-1} a_i y^i & 0 \\ 0 & b_0 + \sum\limits_{j=r}^{\ell-2} b_j y^j \end{pmatrix}$$
with $a_i, b_j \in \mathbb{C}$ and $r = \min\{k, \ell\}$.
2. *For the normal form 2 of the table, the canonical vector field associated to $\tilde{F}$ is Lipschitz if and only of θ can be written in the form*
$$\theta = \begin{pmatrix} a & b \\ b & \sum\limits_{i=0}^{k-2} d_i x^i \end{pmatrix},$$
with $a, b, d_i \in \mathbb{C}$.
3. *For the normal form 3 of the table, the canonical vector field associated to $\tilde{F}$ is Lipschitz if and only of θ is constant.*
4. *For the normal form 4 of the table,the canonical vector field associated to $\tilde{F}$ is Lipschitz if and only of $\frac{\partial \tilde{F}}{\partial y} = \frac{\partial F}{\partial y}$, i.e., θ can be written in the form*
$$\theta = \begin{pmatrix} a & b \\ b & \sum\limits_{j=0}^{k-1} b_j x^j \end{pmatrix},$$
with $a, b, b_j \in \mathbb{C}$.
5. *For the normal form 5 of the table, the canonical vector field associated to the 1-parameter deformation $\tilde{F}$ induced by $\theta \in \frac{Sym_2}{T\mathcal{G}_eF}$ is Lipschitz if and only if the 1-jet type of $\tilde{F}$ and F agree.*
6. *For the normal form 6 of the table, the canonical vector field associated to $\tilde{F}$ is Lipschitz if and only of θ is constant.*

The proof follows from the following lemmas.

Lemma 3.3. *Let $F : (\mathbb{C}^2, 0) \to Sym_2$ be a $\mathcal{G}$-simple germ of rank 0 at the origin whose discriminant of type $\mathcal{A}_{k+\ell-1}$. Let $\tilde{F}$ be a deformation induced by $\theta \in \frac{Sym_2}{T\mathcal{G}_eF}$. If the canonical vector field associated to $\tilde{F}$ is Lipschitz then θ can be written in the form*
$$\theta = \begin{pmatrix} a_0 + \sum\limits_{i=r}^{k-1} a_i y^i & 0 \\ 0 & b_0 + \sum\limits_{j=r}^{\ell-2} b_j y^j \end{pmatrix},$$
with $a_i, b_j \in \mathbb{C}$ and $r = \min\{k, \ell\}$.

In particular, in the case $\ell = k$, the canonical vector field associated to $\tilde{F}$ is Lipschitz if and only if θ is constant.

Proof. The normal form of F is
$$\begin{pmatrix} y^k & x \\ x & y^\ell \end{pmatrix}.$$

Then, the normal space $\dfrac{Sym_2}{T\mathcal{G}_eF}$ is generated by
$$\left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} y & 0 \\ 0 & 0 \end{pmatrix}, \dots, \begin{pmatrix} y^{k-1} & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & y \end{pmatrix}, \dots, \begin{pmatrix} 0 & 0 \\ 0 & y^{\ell-2} \end{pmatrix}\right\}.$$

If $\theta \in \dfrac{Sym_2}{T\mathcal{G}_eF}$ then θ is a $\mathbb{C}$-linear combination of the above elements, i.e., there exist $a_i, b_j \in \mathbb{C}$ such that
$$\theta = \begin{pmatrix} \sum\limits_{i=0}^{k-1} a_i y^i & 0 \\ 0 & \sum\limits_{j=0}^{\ell-2} b_j y^j \end{pmatrix}.$$

Thus,
$$\tilde{F} = \begin{pmatrix} y^k + t\sum\limits_{i=0}^{k-1} a_i y^i & x \\ x & y^k + t\sum\limits_{j=0}^{k-2} b_j y^j \end{pmatrix}.$$

Notice that $I_D(\tilde{F})$ is generated by
$$\{x - x', y^k - y'^k + t\sum_{i=1}^{k-1} a_i(y^i - y'^i), y^\ell - y'^\ell + t\sum_{j=1}^{\ell-2} b_j(y^j - y'^j)\}$$
and $I_D(\theta)$ is generated by $\left\{\sum\limits_{i=1}^{k-1} a_i(y^i - y'^i), \sum\limits_{j=1}^{\ell-2} b_j(y^j - y'^j)\right\}$.

Consider the curve $\phi(s) = (s^{k+\ell}, 2s^{k+\ell}, 2s, s^{k+\ell}, s^{k+\ell}, s)$. Thus,
$$\phi^*(I_D(\tilde{F})) = \langle s^{k+\ell}, (2^k - 1)s^k + s^{k+\ell}\sum_{i=1}^{k-1} a_i(2^i - 1)s^i, (2^\ell - 1)s^\ell + s^{k+\ell}\sum_{j=1}^{\ell-2} b_j(2^j - 1)s^j\rangle,$$
which is contained in $\langle s^r\rangle$. Since $I_D(\theta) \subseteq \overline{I_D(\tilde{F})}$, then
$$\left\langle \sum_{i=1}^{k-1} a_i(2^i - 1)s^i, \sum_{j=1}^{\ell-2} b_j(2^j - 1)s^j \right\rangle \subseteq \langle s^r\rangle,$$
which finishes the proof. $\square$

Lemma 3.4. *Let $F : (\mathbb{C}^2, 0) \to Sym_2$ be a $\mathcal{G}$-simple germ of rank 0 at the origin whose discriminant of type $\mathcal{D}_{k+2}$, $k \geq 2$. Let $\tilde{F}$ be a deformation induced by $\theta \in \frac{Sym_2}{T\mathcal{G}_eF}$. Then the canonical vector field associated to $\tilde{F}$ is Lipschitz if and only of θ can be written in the form*
$$\theta = \begin{pmatrix} a & b \\ b & \sum\limits_{i=0}^{k-2} d_i x^i \end{pmatrix},$$
with $a, b, d_i \in \mathbb{C}$.

Proof. The normal form of F is

$$\begin{pmatrix} x & 0 \\ 0 & y^2+x^k \end{pmatrix}.$$

Then, the normal space $\dfrac{Sym_2}{T\mathcal{G}_eF}$ is generated by

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & y \\ y & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix}, \dots, \begin{pmatrix} 0 & 0 \\ 0 & x^{k-2} \end{pmatrix} \right\}.$$

Thus, we can write

$$\theta = \begin{pmatrix} a & b+cy \\ b+cy & \sum\limits_{i=0}^{k-2} d_i x^i \end{pmatrix},$$

with $a, b, c, d_i \in \mathbb{C}$,

$$I_D(\theta) = \langle c(y-y'), \sum_{i=1}^{k-2} d_i(x^i - x'^i)\rangle$$

and

$$I_D(\tilde{F}) = \langle x - x', tc(y-y'), y^2 - y'^2 + x^k - x'^k + t\sum_{i=1}^{k-2}(x^i - x'^i)\rangle.$$

Consider the curve $\phi(s) = (s, 2s^2, 2s, s, s^2, s)$. Notice that

$$\phi^*(I_D(\tilde{F})) = \langle s^2, cs^2, 3s^2 + (2^k-1)s^{2k} + s\sum_{i=1}^{k-2} d_i(2^i-1)s^i\rangle \subseteq \langle s^2\rangle.$$

Suppose the canonical vector field is Lipschitz, i.e., $I_D(\theta) \subseteq \overline{I_D(\tilde{F})}$. Then,

$$cs = \phi^*(c(y-y')) \in \langle s^2\rangle$$

and so $c = 0$.

Conversely, if $c = 0$ then $I_D(\theta) = \langle \sum\limits_{i=1}^{k-2} d_i(x^i - x'^i)\rangle \subseteq \langle x - x'\rangle \subseteq I_D(\tilde{F})$. $\square$

Lemma 3.5. *Let $F : (\mathbb{C}^2, 0) \to Sym_2$ be a $\mathcal{G}$-simple germ of rank 0 at the origin whose discriminant of type $\mathcal{D}_{2k}$, $k \geq 2$. Let $\tilde{F}$ be a deformation induced by $\theta \in \frac{Sym_2}{T\mathcal{G}_eF}$. Then the canonical vector field associated to $\tilde{F}$ is Lipschitz if and only of θ is constant.*

Proof. The normal form of F is

$$\begin{pmatrix} x & 0 \\ 0 & xy+y^k \end{pmatrix}.$$

Then, the normal space $\dfrac{Sym_2}{T\mathcal{G}_eF}$ is generated by

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} y & 0 \\ 0 & 0 \end{pmatrix}, \dots, \begin{pmatrix} y^{k-2} & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & y \end{pmatrix}, \dots, \begin{pmatrix} 0 & 0 \\ 0 & y^{k-1} \end{pmatrix} \right\}.$$

So we can write

$$\theta = \begin{pmatrix} \sum\limits_{i=0}^{k-2} a_i y^i & a \\ a & \sum\limits_{j=0}^{k-1} b_j y^j \end{pmatrix},$$

for some $a, a_i, b_j \in \mathbb{C}$, $I_D(\theta) = \langle \sum_{i=1}^{k-2} a_i(y^i - y'^i), \sum_{j=1}^{k-1} b_j(y^j - y'^j)\rangle$ and

$$I_D(\tilde{F}) = \langle x - x' + t\sum_{i=1}^{k-2} a_i(y^i - y'^i), xy - x'y' + y^k - y'^k + t\sum_{j=1}^{k-1} b_j(y^j - y'^j)\rangle.$$

Consider the curve $\phi(s) = (s^k, 2s^k, 2s, s^k, s^k, s)$. Then

$$\phi^*(I_D(\tilde{F})) = \langle s^k + s^k\sum_{i=1}^{k-2} a_i(2^i - 1)s^i, 3s^{k+1} + (2^k - 1)s^k + s^k\sum_{j=1}^{k-1} b_j(2^j - 1)s^j\rangle \subseteq \langle s^k\rangle.$$

If the canonical vector field is Lipschitz then $\sum_{i=1}^{k-2} a_i(2^i - 1)s^i$ and $\sum_{j=1}^{k-1} b_j(2^j - 1)s^j$ belong to $\langle s^k\rangle$. Hence, $a_i = 0$ and $b_j = 0$ for all i and j. Therefore, θ is constant.

□

Lemma 3.6. *Let $F : (\mathbb{C}^2, 0) \to Sym_2$ be a $\mathcal{G}$-simple germ of rank 0 at the origin whose discriminant of type $\mathcal{D}_{2k+1}$, $k \geq 2$. Let $\tilde{F}$ be a deformation induced by $\theta \in \dfrac{Sym_2}{T\mathcal{G}_eF}$. Then the canonical vector field associated to $\tilde{F}$ is Lipschitz if and only of $\dfrac{\partial\tilde{F}}{\partial y} = \dfrac{\partial F}{\partial y}$, i.e., θ can be written in the form*

$$\theta = \begin{pmatrix} a & b \\ b & \sum_{j=0}^{k-1} b_j x^j \end{pmatrix},$$

with $a, b, b_j \in \mathbb{C}$.

Proof. The normal form of F is

$$\begin{pmatrix} x & y^k \\ y^k & xy \end{pmatrix}.$$

Then, the normal space $\dfrac{Sym_2}{T\mathcal{G}_eF}$ is generated by

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} y & 0 \\ 0 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} y^{k-1} & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix}, \ldots, \begin{pmatrix} 0 & 0 \\ 0 & x^{k-1} \end{pmatrix} \right\}.$$

Thus, we can write

$$\theta = \begin{pmatrix} a + \sum_{i=1}^{k-1} a_i y^i & b \\ b & \sum_{j=0}^{k-1} b_j x^j \end{pmatrix},$$

with $a, a_i, b, b_j \in \mathbb{C}$,

$$I_D(\theta) = \langle \sum_{i=1}^{k-1} a_i(y^i - y'^i), \sum_{j=1}^{k-1} b_j(x^j - x'^j)\rangle$$

and

$$I_D(\tilde{F}) = \langle x - x' + t\sum_{i=1}^{k-1} a_i(y^i - y'^i), y^k - y'^k, xy - x'y' + t\sum_{j=1}^{k-1} b_j(x^j - x'^j)\rangle.$$

Consider the curve $\phi(s) = (s^k, 2s^k, 2s, s^k, s^k, s)$. Then

$$\phi^*(I_D(\tilde{F})) = \langle s^k + s^k \sum_{i=1}^{k-1} a_i(2^i - 1)s^i, (2^k - 1)s^k, 3s^{k+1} + s^k \sum_{j=1}^{k-1} b_j(2^j - 1)s^{kj} \rangle \subseteq \langle s^k \rangle.$$

If $I_D(\theta) \subseteq \overline{I_D(\tilde{F})}$ then $\sum_{i=1}^{k-1} a_i(2^i - 1)s^i \in \langle s^k \rangle$, hence $a_i = 0$ for all $i \in \{1, ..., k-1\}$. Conversely, if $a_i = 0$, for all $i \in \{1, ..., k-1\}$ then $I_D(\theta) = \langle \sum_{j=1}^{k-1} b_j(x^j - x'^j) \rangle \subseteq \langle x - x' \rangle \subseteq I_D(\tilde{F})$. □

Lemma 3.7. *Let $F : (\mathbb{C}^2, 0) \to Sym_2$ be a $\mathcal{G}$-simple germ of rank 0 at the origin with discriminant of type E_6. Then the canonical vector field associated to the 1-parameter deformation $\tilde{F}$ induced by $\theta \in \dfrac{Sym_2}{T\mathcal{G}_e F}$ is Lipschitz if and only if the 1-jet type of $\tilde{F}$ and F agree.*

Proof. The normal form of F is

$$\begin{pmatrix} x & y^2 \\ y^2 & x^2 \end{pmatrix}.$$

Then, the normal space $\dfrac{Sym_2}{T\mathcal{G}_e F}$ is generated by

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} y & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & y \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & y^2 \end{pmatrix} \right\}.$$

If $\theta \in \dfrac{Sym_2}{T\mathcal{G}_e F}$ induces a non-trivial deformation $\tilde{F}$ then we can write

$$\theta(x, y) = \begin{pmatrix} a_1 + a_3 y + a_4 y^2 & 0 \\ 0 & a_2 + a_5 y + a_6 y^2 \end{pmatrix}.$$

Thus

$$\tilde{F} = \begin{pmatrix} x + t(a_1 + a_3 y + a_4 y^2) & y^2 \\ y^2 & x^2 + t(a_2 + a_5 y + a_6 y^2) \end{pmatrix}.$$

Notice that $I_D(\theta) = \langle a_3(y - y') + a_4(y^2 - y'^2), a_5(y - y') + a_6(y^2 - y'^2) \rangle$.

Suppose the 1-jet type of $\tilde{F}$ and F agree. Then $a_3 = a_5 = 0$ and in this case

$$I_D(\theta) = \langle a_4(y^2 - y'^2), a_6(y^2 - y'^2) \rangle.$$

Since $y^2 - y'^2 \in I_D(\tilde{F})$ then $I_D(\theta) \subseteq I_D(\tilde{F})$ and the canonical vector field is Lipschitz.

Conversely, if the canonical vector field is Lipschitz then $a_3 = a_5 = 0$. In fact, we are assuming that $I_D(\theta) \subseteq \overline{I_D(\tilde{F})}$.

We have $I_D(\tilde{F})$ is generated by

$$\{y^2 - y'^2, x - x' + t(a_3(y - y') + a_4(y^2 - y'^2)), x^2 - x'^2 + t(a_5(y - y') + a_6(y^2 - y'^2))\}.$$

Consider the curve $\phi(s) = (s, 2s^3, 2s^2, s, s^3, s^2)$. Then we have that

$$\phi^*(I_D(\tilde{F})) = \langle 3s^4, s^3 + s(a_3 s^2 + 3a_4 s^4), 3s^6 + s(a_5 s^2 + 3a_6 s^4) \rangle \subseteq \langle s^3 \rangle.$$

Since $\phi^*(I_D(\theta)) \subseteq \phi^*(I_D(\tilde{F})) \subseteq \langle s^3 \rangle$ then $\phi^*(a_3(y - y') + a_4(y^2 - y'^2)) = a_3 s^2 + 3a_4 s^4 \in \langle s^3 \rangle$ which implies that $a_3 s^2 \in \langle s^3 \rangle$, hence $a_3 = 0$. Analogously, using the same curve, we prove that $a_5 = 0$. □

Lemma 3.8. *Let $F : (\mathbb{C}^2, 0) \to Sym_2$ be a $\mathcal{G}$-simple germ of rank 0 at the origin whose discriminant of type E_7. Let $\tilde{F}$ be a deformation induced by $\theta \in \dfrac{Sym_2}{T\mathcal{G}_e F}$. Then the canonical vector field associated to $\tilde{F}$ is Lipschitz if and only of θ is constant.*

Proof. The normal form of F is
$$\begin{pmatrix} x & 0 \\ 0 & x^2+y^3 \end{pmatrix}.$$

Then, the normal space $\dfrac{Sym_2}{T\mathcal{G}_e F}$ is generated by
$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & y \end{pmatrix}, \begin{pmatrix} y & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & y \\ y & 0 \end{pmatrix}, \begin{pmatrix} 0 & y^2 \\ y^2 & 0 \end{pmatrix} \right\}.$$
So we can write
$$\theta = \begin{pmatrix} a_1 + a_5 y & a_3 + a_6 y + a_7 y^2 \\ a_3 + a_6 y + a_7 y^2 & a_2 + a_4 y \end{pmatrix},$$
for some $a_i \in \mathbb{C}$,
$$I_D(\theta) = \langle a_5(y-y'), a_4(y-y'), a_6(y-y') + a_7(y^2 - y'^2) \rangle$$
and
$$I_D(\tilde{F}) = \langle x - x' + t a_5(y-y'), t(a_6(y-y') + a_7(y^2-y'^2)), x^2 - x'^2 + y^3 - y'^3 + t a_4(y-y') \rangle.$$
Consider the curve $\phi(s) = (s^2, 2s^3, 2s, s^2, s^3, s)$. Thus,
$$\phi^*(I_D(\tilde{F})) = \langle s^3 + a_5 s^3, a_6 s^3 + 3a_7 s^4, 3s^6 + 7s^3 + a_4 s^3 \rangle \subseteq \langle s^3 \rangle.$$
If the canonical vector field is Lipschitz then $a_5 s, a_4 s, a_6 s + 3a_7 s^2 \in \langle s^3 \rangle$ which implies that $a_4 = a_5 = a_6 = a_7 = 0$. Therefore, θ is constant. □

As in [8], the canonical vector field associated to the 1-parameter deformation $\tilde{F}$ of the normal forms presented in [3] induced by $\theta \in \frac{Sym_3}{T\mathcal{G}_e F}$ is Lipschitz if and only if the 1-jet type of $\tilde{F}$ and F agree. The proof of the next result is analogous to the proof of the main result of [8].

Proposition 3.9. *For all $\mathcal{G}$-simple germs $F : \mathbb{C}^r \to Sym_3$ of rank 0 at the origin we have that the canonical vector field associated to the 1-parameter deformation $\tilde{F}$ induced by $\theta \in \dfrac{Sym_3}{T\mathcal{G}_e F}$ is Lipschitz.*

Proof. Suppose that F is of 1-jet-type of the form in the tables in items (5) and (6) of Theorem 1.1 from [5]. Since $\theta \in \dfrac{Mat_{(3)}(\mathcal{O}_r)}{T\mathcal{G}F}$, the r order 1 entries of the matrix F stay unperturbed, thus the differences of the monomial generators of the maximal ideal are in $I_D(\tilde{F})$. In particular the ideal I_Δ from the diagonal satisfies the inclusion $I_\Delta \subseteq I_D(\tilde{F})$. Let θ_i, $i \in \{1, ..., 6\}$ be the components of θ. Notice that every $(\theta_i)_D$ vanishes on the diagonal Δ which implies that all the generators of $I_D(\theta)$ belong to I_Δ. Therefore, $I_D(\theta) \subseteq I_\Delta \subseteq I_D(\tilde{F})$ and Proposition 3.4 of [8] ensures the canonical vector field is Lipschitz.

□

Remark 3.10 ([3], Remark 1.2.). *In the cases when $r = 2$ and $n = 2, 3$ the $\mathcal{G}$-codimension of the germs and the Milnor number of the discriminant coincide.*

The next result is an application of the results of the previous section for the real case. The proof follows the same steps of Theorem 2.8 of [8].

Theorem 3.11. *Consider the $\mathcal{G}$-simple germs $F : \mathbb{R}^r \to Hom(\mathbb{R}^n, \mathbb{R}^n)$ of rank 0 at the origin, classified in Theorem 1.1 of* [4]*, and consider the semi-universal unfolding*

$$\tilde{F} : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \times Hom(\mathbb{R}^n, \mathbb{R}^n),$$

where $\theta \in \dfrac{Mat_n(\mathcal{A}_r)}{T\mathcal{G}_e F}$.

If the ideal of 1-minors of F defines a reduced point then the canonical vector field is Lipschitz.

Proof. Since the ideal of 1-minors of F defines a reduced point and $\theta \in \dfrac{Mat_n(\mathcal{A}_r)}{T\mathcal{G}_e F}$, then the r order 1 entries of F stay unperturbed, thus the differences of the monomial generators of the maximal ideal are in $I_D(\tilde{F})$. Consequentely, $I_\Delta \subseteq I_D(\tilde{F})$. Let θ_{ij} be the components of θ, $i, j \in \{1, \ldots, n\}$. Clearly all $(\theta_{ij})_D$ vanish on Δ. Hence, $I_D(\theta) \subseteq I_\Delta$ and the proof is done by Corollary 2.5. □

Remark 3.12. *In* [20]*, the author obtained sufficient conditions for topological triviality of 1-parameter deformations of weighted homogeneous matrix M (see Proposition 6.1 and Proposition 6.2). Considering the action defined in the Definition 1.1, the triviality condition is related to the tangent space to the $\mathcal{G}$-orbit of M. These conditions ensure that the canonical vector field is integrable.*

At this point, one way to continue our study is to show that the homeomorphism obtained by integration of the canonical Lipschitz vector fields gives the bi-Lipschitz equivalence of the members of the respective family of square matrix map-germs according to Definition 1.1.

References

[1] C. Andradas, L. Bröcker, J. M. Ruiz, *Constructible Sets in Real Geometry*, V. 33, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge - A Series of Modern Surveys in Mathematics, Springer Science & Business Media, 2012. DOI: 10.1007/978-3-540-77270-5_19

[2] V. I. Arnold, *Matrices depending on parameters*. Russian Math. Surveys 26, no. 2, 29-43 (1971).

[3] J. W. Bruce, *Families of symmetric matrices*, *Moscow Math. J.*, **3**, no 2, 335-360 (2003).

[4] J. W. Bruce and F. Tari, *Families of square matrices*, Proc. London Math. Soc. (3) 89, 738-762 (2004). DOI: 10.1112/s0024611504014911

[5] G. W. Brumfiel, *Real valuation rings and ideals*, Géométrie algébrique réelle et formes quadratiques, Rennes 1981, Springer Lecture Notes 959, 55-97, (1982). DOI: 10.1007/bfb0062250

[6] J. Damon, *The unfolding and determinancy theorems for subgoups of $\mathcal{A}$ and $\mathcal{K}$, Memoirs of the American Mathematical Society*, Providence RI, (1984).

[7] J. Damon and B. Pike *Solvable groups, free divisors and nonisolated matrix singularities* II: Vanishing topology. Geom. Topol. 18, no. 2, 911-962 (2014). DOI: 10.2140/gt.2014.18.911

[8] T. da Silva, N. G. Grulha Jr. and M. Pereira, *The Bi-Lipschitz Equisingularity of Essentially Isolated Determinantal Singularities*, Bull Braz Math Soc, New Series 49, 637-645 (2018). DOI: 10.1007/s00574-017-0067-3

[9] A. Frühbis-Krüger, *Classification of Simple Space Curves Singularities, Communications in Algebra*, 27 (8), 3993-4013, (1999). DOI: 10.1080/00927879908826678

[10] A. Frühbis-Krüger and A. Neumer, *Simple Cohen-Macaulay Codimension 2 Singularities*, Communications in Algebra, 38:2, 454-495 (2010). DOI: 10.1080/00927870802606018

[11] A. Fernandes and M. A. S. Ruas, *Bilipschitz determinacy of quasihomogeneous germs*, Glasgow Mathematical Journal, 46 (1), 77-82 (2004). DOI: 10.1017/s001708950300154x

[12] T. Gaffney, *Bi-Lipschitz equivalence, integral closure and invariants,* Proceedings of the 10th International Workshop on Real and Complex Singularities. Edited by: M. Manoel, Universidade de São Paulo, M. C. Romero Fuster, Universitat de Valencia, Spain, C. T. C. Wall, University of Liverpool, London Mathema DOI: tical Society Lecture Note Series (No.380) November (2010). DOI: 10.1017/cbo9780511731983.010

[13] T. Gaffney, *The genericity of the infinitesimal Lipschitz condition for hypersurfaces*, J. Singul. 10, 108-123 (2014). DOI: 10.5427/jsing.2014.10g

[14] T. Gaffney, *Integral Closure of Modules and Whitney equisingularity,* Invent. Math. 107, 301-322 (1992). DOI: 10.1007/bf01231892

[15] T. Gaffney and T. da Silva, *Infinitesimal Lipschitz conditions on a family of analytic varieties*, (2019). arχiv: 1902.03194

[16] M. Lejeune-Jalabert and B. Teissier, *Clôture intégrale des idéaux et equisingularité*, Séminaire Lejeune-Teissier, Centre de Mathématiques École Polytechnique, (1974) Publ. Inst. Fourier St. Martin d'Heres, F-38402 (1975). DOI: 10.5802/afst.1203

[17] T. Mostowski, *A criterion for Lipschitz equisingularity.* Bull. Polish Acad. Sci. Math. 37 (1989), no. 1-6, 109-116 (1990).

[18] A. Parusiński, *Lipschitz stratification of real analytic sets.* Singularities (Warsaw, 1985), 323-333, Banach Center Publ., 20, PWN, Warsaw, (1988). DOI: 10.4064/-20-1-323-333

[19] A. Parusiński, *Lipschitz properties of semi-analytic sets.* Ann. Inst. Fourier (Grenoble) 38, no. 4, 189-213 (1988). DOI: 10.5802/aif.1154

[20] M. S. Pereira, *Properties of G-Equivalence of Matrices*, (2017). arχiv: 1711.02156

[21] M. S. Pereira, *Variedades Determinantais e Singularidades de Matrizes*, Tese de Doutorado, *ICMC-USP*, (2010). DOI: 10.11606/t.55.2010.tde-22062010-133339

[22] F. Pham, *Fractions lipschitziennes et saturation de Zariski des algèbres analytiques complexes.* Exposé d'un travail fait avec Bernard Teissier. Fractions lipschitziennes d'une algèbre analytique complexe et saturation de Zariski, Centre Math. lÉcole Polytech., Paris, 1969. Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 2, pp. 649-654. Gauthier-Villars, Paris (1971). DOI: 10.1515/9783110879872-toc

[23] F. Pham and B. Teissier, *Fractions lipschitziennes d'une algèbre analytique complexe et saturation de Zariski*, Centre Math. l'École Polytech., Paris (1969).

[24] M. A. S. Ruas and M. S. Pereira, *Codimension two determinantal varieties with isolated singularities.* Math. Scand. 115, no. 2, 161-172 (2014). DOI: 10.7146/math.scand.a-19220

[25] O. Zariski, *General theory of saturation and of saturated local rings. II. Saturated local rings of dimension 1.* Amer. J. Math. 93, 872-964 (1971). DOI: 10.2307/2373741

Thiago Filipe da Silva, Departamento de Matemática, Universidade Federal do Espírito Santo, Av. Fernando Ferrari, 514 - Goiabeiras, 29075-910 - Vitória - ES, Brazil
Email address: thiago.silva@ufes.br

Nivaldo de Góes Grulha Júnior, Instituto de Ciências Matemáticas e de Computaćão - USP, Av. Trabalhador São Carlense, 400 - Centro, 13566-590 - São Carlos - SP, Brazil
Email address: njunior@icmc.usp.br

Miriam da Silva Pereira, Departamento de Matemática, Universidade Federal da Paraíba, 58.051-900 João Pessoa, Brazil
Email address: miriam@mat.ufpb.br

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CHAOS IN PERIODICALLY FORCED REVERSIBLE VECTOR FIELDS

ISABEL S. LABOURIAU AND ELISA SOVRANO

Abstract. We discuss the appearance of chaos in time-periodic perturbations of reversible vector fields in the plane. We use the normal forms of codimension 1 reversible vector fields and discuss the ways a time-dependent periodic forcing term of pulse form may be added to them to yield topological chaotic behaviour. Chaos here means that the resulting dynamics is semiconjugate to a shift in a finite alphabet. The results rely on the classification of reversible vector fields and on the theory of topological horseshoes. This work is part of a project of studying periodic forcing of symmetric vector fields.

1. Introduction

A standard classification of continuous dynamical systems defined by a set of first order ordinary differential equations distinguishes between conservative systems and dissipative ones [9]. On the one hand, conservative systems can be described by a Hamiltonian function. By varying the initial conditions, these systems can exhibit regions of regular motions surrounded by a sea of chaotic ones. Instead, dealing with dissipative systems, conserved quantities are no longer guaranteed, and chaotic regions could coexist with stable equilibria, limit cycles, and strange attractors.

In between conservative and dissipative systems, there are systems with reversing symmetries. By reversible dynamical systems we mean those admitting an involution in phase space which reverses the direction of time (see [1, 4, 10, 13]). It is shown that these systems despite having similar features to Hamiltonian ones (e.g., at an elliptic equilibrium can possess the same structure), yet they are different because they can also have attractors and repellers. The additional structure given by reversing symmetries allows exhibiting complex behaviors for codimension one bifurcations, and so, it can be responsible for chaotic dynamics.

The goal of this paper is to find chaos for a class of planar periodically perturbed reversible systems whose normal form analysis is studied in [13]. We take into account the local bifurcations of low codimension by arguing what dynamical behaviors we can expect. Our main result is the following.

Theorem 1.1. *Let $X_\lambda(x,y)$ be a fixed type of normal form for a one-parameter family of codimension 1 reversible vector fields, of either saddle type or of cusp type. Let λ_1 and λ_2 be two real distinct values. Suppose that the dynamical system $\dot{X} = X(x,y)$ switches in a T-periodic manner between*

$$\dot{X} = X_{\lambda_1}(x,y) \text{ for } t \in [0,\tau_1) \quad \text{and} \quad \dot{X} = X_{\lambda_2}(x,y) \text{ for } t \in [\tau_1, \tau_1+\tau_2)$$

with $\tau_1 + \tau_2 = T$. Then for open sets of the parameters (λ_1,λ_2) and for τ_1 and τ_2 in open intervals there exist infinitely many T-periodic solutions as well as chaotic-like dynamics for the problem $\dot{X} = X(x,y)$.

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The paper is organized as follows. In Section 2 we discuss the classification of plane reversible vector fields of codimension 0 and 1. In Section 3 we give a review of the concept of symbolic dynamics and topological horseshoes. We collect preliminary topological results in the phase-plane that can produce chaotic dynamics. In Section 4 we prove Theorem 1.1 for the two of the four normal forms of codimension 1 reversible vector fields: *i*) saddle type and *ii*) cusp type. We conjecture that the other two possible normal forms, namely *iii*) nodal type and *iv*) focal type, may also be amenable to the same treatment.

2. Planar reversible systems

In [13], M. A. Teixeira has provided a local classification of 2D reversible systems of codimension less than or equal to two. A dynamical system $\dot{X} = V(X)$ is called *reversible* if there is a phase space *involution* h (i.e., $h^2 = \mathrm{Id}$) such that $Dh(p)V(p) = -V(h(p))$ for $p \in \mathbb{R}^2$. We deal with reversible planar systems where the involution is $h(x, y) = (x, -y)$. Hence, we consider a dynamical system of the following form

$$\begin{cases} \dot{x} = yf(x, y^2), \\ \dot{y} = g(x, y^2), \end{cases} \tag{2.1}$$

where the functions f and g are smooth. We consider the behaviour of (2.1) near the origin, often making the assumption that it has an equilibrium at the origin. In the half-plane $y > 0$, by using the transformation $u = x$ and $v = y^2$, we can write system (2.1) equivalently as follows

$$\begin{cases} \dot{u} = \sqrt{v}\, f(u, v), \\ \dot{v} = 2\sqrt{v}\, g(u, v). \end{cases}$$

Through the symmetry properties of the vector field $X(x, y)$ associated with (2.1), the behavior of X near $(0, 0)$ may be described by the analysis in the half-plane $\{(u, v) \in \mathbb{R}^2 \colon v \geq 0\}$ of the vector field $Y(u, v) = (f(u, v), g(u, v))$.

2.1. **Normal forms.** Following the work in [13], the generic equilibria of reversible ODEs near the origin are either centers and saddles on the line of symmetry or a couple of repellers and attractors, as in Figure 1.

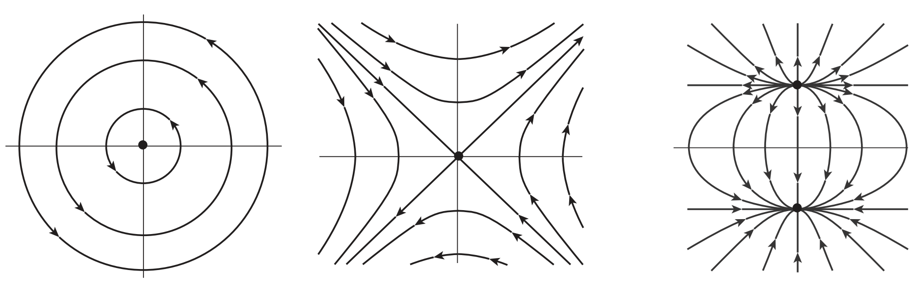


Figure 1. Phase-portraits of equilibria occurring in generic 2D reversible fields. The local geometry may be of a center (left), a saddle (middle), or a pair of attractor and repeller (right).

Let S be the line $\{(x, 0) \colon x \in \mathbb{R}\}$, the set of fixed points for h. An equilibrium point of V that lies on S is called a *symmetric equilibrium*.

Theorem 2.1 ([13])**.** *The normal forms around a symmetric equilibrium at* $(0,0)$ *of a structurally stable reversible vector field* X *are:*

- $X(x,y) = (y,x)$,
- $X(x,y) = (y,-x)$.

In the first case the origin is a center, and in the second one it is a saddle. The next result classifies one parameter families X_λ of reversible vector fields such that X_0 has a symmetric equilibrium at the origin.

Theorem 2.2 ([13])**.** *The normal forms of one-parameter families of structurally stable reversible vector fields* X_λ *near a symmetric equilibrium at* $(0,0)$ *are:*

i) saddle type: $X_\lambda(x,y) = (xy, x - y^2 + \lambda)$,
ii) cusp type: $X_\lambda(x,y) = (y, x^2 + \lambda)$,
iii) nodal type: $X_\lambda(x,y) = (xy, x + 2y^2 + \lambda)$ *or* $X_\lambda(x,y) = (-xy, x - 2y^2 + \lambda)$,
iv) focal type: $X_\lambda(x,y) = (xy + y^3, -x + y^2 + \lambda)$.

Depending on λ, the phase-portraits of the above normal forms can be described as follows.

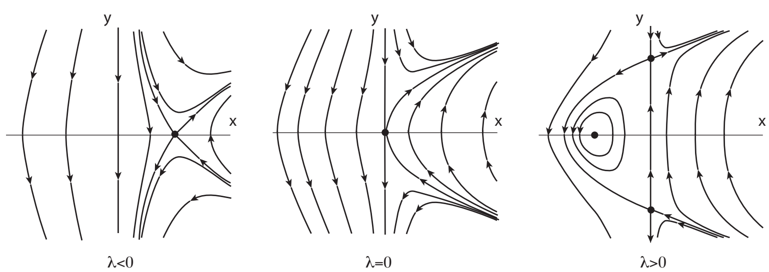


FIGURE 2. Phase-portraits reversible vector fields of saddle type.

Figure 2 shows the phase portraits of the *saddle type*. When $\lambda \le 0$ there is an equilibrium at $(-\lambda, 0)$ which is a saddle. When $\lambda > 0$ there are three equilibria: a center and two saddles at $(-\lambda, 0)$, $(0, -\sqrt{\lambda})$ and $(0, \sqrt{\lambda})$, respectively. The saddle points are connected through heteroclinic trajectories which surround periodic orbits.

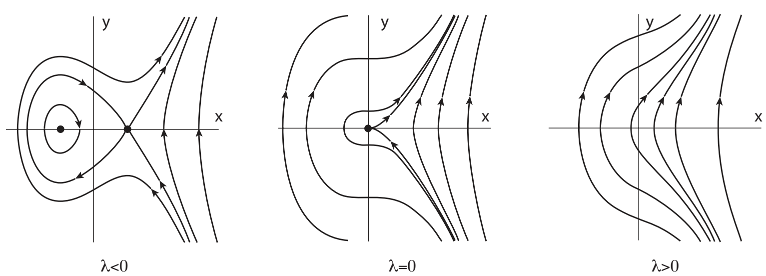


FIGURE 3. Phase-portraits reversible vector fields of cusp type.

Concerning the *cusp type* when $\lambda < 0$ there are two equilibria: a center and a saddle which are at $(-\sqrt{-\lambda}, 0)$ and $(\sqrt{-\lambda}, 0)$, respectively. Due to the reversibility, the only periodic orbits are the ones that meet the points $(x,0)$ with $-2\sqrt{-\lambda} < x < \sqrt{-\lambda}$, as in Figure 3. Moreover, these orbits are located inside the homoclinic trajectory that passes through $(-2\sqrt{-\lambda}, 0)$. When

$\lambda = 0$ there is only an equilibrium which is a degenerate saddle at $(0, 0)$ and all the orbits are unbounded. When $\lambda > 0$ there are no equilibria.

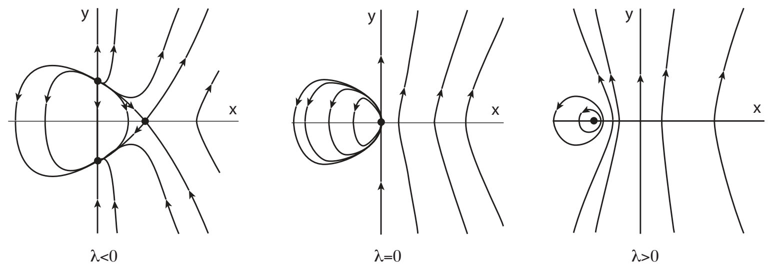


FIGURE 4. Phase-portraits reversible vector fields of nodal type (first case).

For the *nodal type* (first case, shown in Figure 4) when $\lambda < 0$ there are three equilibria: an attractor, a repeller and a saddle, located respectively at $(0, -\sqrt{-\lambda/2})$, $(0, \sqrt{-\lambda/2})$ and $(-\lambda, 0)$. When $\lambda = 0$ there is only an equilibrium at $(0, 0)$. When $\lambda > 0$ there is only an equilibrium at $(-\lambda, 0)$ which is a center and in the half-plane $x < 0$ all the orbits are periodic. In the second case there is always an equilibrium at $(-\lambda, 0)$ and for $\lambda > 0$ there is also a pair of equilibria at $(0, \pm\sqrt{\lambda/2})$.

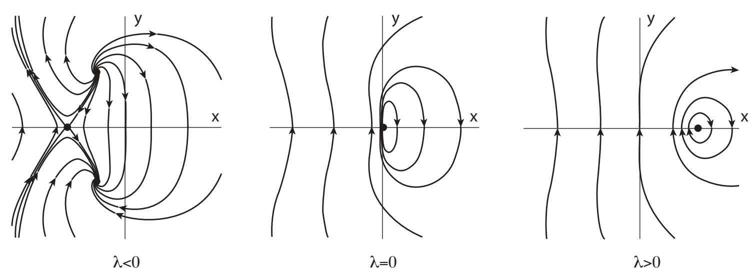


FIGURE 5. Phase-portraits reversible vector fields of focal type.

For the *focal type* when $\lambda < 0$ there are three equilibria: a saddle and two foci at $(\lambda, 0)$, $(\lambda/2, -\sqrt{-\lambda/2})$ and $(\lambda/2, \sqrt{-\lambda/2})$, respectively. When $\lambda \geq 0$ there is only an equilibrium at $(\lambda, 0)$ which is a center and all the orbits are periodic as in Figure 5.

3. Background on chaotic dynamics and preliminary results

3.1. **Symbolic dynamics and chaos.** To review the topological approach exploited throughout the paper, we start by introducing some notation and definitions of symbolic dynamics. General information on the subject may be found in the book by Guckenheimer and Holmes [2], with examples in Chapter 2 and a more general case in Chapter 5. A more detailed treatment is given by Wiggins and Ottino [14]. The point of view used here is similar to that of Kennedy and Yorke in [3] of Margheri *et al* in [5] and of Medio *et al* in [6].

Let $\Sigma_m := \{0, \ldots, m-1\}^{\mathbb{Z}}$ be the set of all two-sided sequences $S = (s_i)_{i\in\mathbb{Z}}$ with $s_i \in \{0, \ldots, m-1\}$ for each $i \in \mathbb{Z}$ endowed with a standard metric that makes Σ_m a compact space with the product topology. We define the shift map $\sigma\colon \Sigma_m \to \Sigma_m$ by $\sigma(S) = S' = (s'_i)_{i\in\mathbb{Z}}$ with $s'_i = s_{i+1}$ for all $i \in \mathbb{Z}$. We say that a map h on a metric space is semiconjugate (respectively, conjugate) to the shift map on m symbols if there exists a compact invariant set Λ and a continuous and surjective (respectively, bijective) map $\Pi\colon \Lambda \to \Sigma_m$ such that $\Pi \circ h(w) = \sigma \circ \Pi(w)$, for all $w \in \Lambda$.

The deterministic chaos is usually associated with the possibility to reproduce all the possible outcomes of a coin-tossing experiment, by varying the initial conditions within the dynamical system. We can express this concept using the symbolic dynamics of the shift map on the sets of two-sided sequences of 2 symbols. However, by considering a finite alphabet made by m symbols the possible dynamics can be more complex. Hence, in the sequel we adopt the following definition of chaos (cf., [5, 6]).

Definition 3.1 (Symbolic dynamics)**.** *Let $h\colon \operatorname{dom} h \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ be a map and let $\mathcal{D} \subseteq \operatorname{dom} h$ be a nonempty set. We say that h* induces chaotic dynamics on $m \geq 2$ symbols on a set $\mathcal{D}$ *if there exist m nonempty pairwise disjoint compact sets $\mathcal{K}_0, \dots, \mathcal{K}_{m-1} \subseteq \mathcal{D}$ such that for each two-sided sequence $(s_i)_{i\in\mathbb{Z}} \in \Sigma_m$ there exists a corresponding sequence $(w_i)_{i\in\mathbb{Z}} \in \mathcal{D}^{\mathbb{Z}}$ such that*

$$w_i \in \mathcal{K}_{s_i} \text{ and } w_{i+1} = h(w_i) \text{ for all } i \in \mathbb{Z}, \tag{3.1}$$

and, whenever $(s_i)_{i\in\mathbb{Z}} \in \Sigma_m$ is a k-periodic sequence for some $k \geq 1$ there exists a k-periodic sequence $(w_i)_{i\in\mathbb{Z}} \in \mathcal{D}^{\mathbb{Z}}$ satisfying (3.1).

For a one-to-one map h, Definition 3.1 ensures the existence of a nonempty compact invariant set $\Lambda \subseteq \cup_{i=0}^{m-1}\mathcal{K}_i \subseteq \mathcal{D}$ and a continuous surjection Π such that $h_{|\Lambda}$ is semiconjugate to the Bernoulli shift map on $m \geq 2$ symbols. Moreover, it guarantees that the set of the periodic points of h is dense in Λ and, for all two-sided periodic sequences $S \in \Sigma_m$, the preimage $\Pi^{-1}(S)$ contains a periodic point of h with the same period (cf. [6, Th. 2.2]). In this respect Definition 3.1 is related, by means of [6, Th. 2.3], to the concept of topological horseshoe introduced in [3]. This is a weaker notion of chaos than the Smale's horseshoe (see [2, ch. 5]) because the latter requires the full conjugacy between $h_{|\Lambda}$ and the shift map on m symbols.

We introduce the notion of an oriented topological rectangle and the stretching along the path property by borrowing the notations and definitions from [5, 7]. The pair $\widehat{\mathcal{R}} := (\mathcal{R}, \mathcal{R}^-)$ is called *oriented topological rectangle* if $\mathcal{R} \subseteq \mathbb{R}^2$ is a set homeomorphic to $[0,1] \times [0,1]$, and $\mathcal{R}^- = \mathcal{R}^-_l \cup \mathcal{R}^-_r$, where $\mathcal{R}^-_l$ and $\mathcal{R}^-_r$ are two disjoint compact arcs contained in $\partial\mathcal{R}$.

Definition 3.2 (SAP property)**.** *Given two topological oriented rectangles $\widehat{\mathcal{R}}_1 := (\mathcal{R}_1, \mathcal{R}^-_1)$, $\widehat{\mathcal{R}}_2 := (\mathcal{R}_2, \mathcal{R}^-_2)$ and a continuous map $h : \operatorname{dom} h \subseteq \mathbb{R}^2 \to \mathbb{R}^2$, we say that h* stretches $\widehat{\mathcal{R}}_1$ to $\widehat{\mathcal{R}}_2$ along the paths *if there exists a compact subset $\mathcal{K}$ of $\mathcal{R}_1 \cap \operatorname{dom} h$ and for each path $\gamma\colon [0,1] \to \mathcal{R}_1$ such that $\gamma(0) \in \mathcal{R}^-_{1,l}$ and $\gamma(1) \in \mathcal{R}^-_{1,r}$ (or vice-versa), there exists $[t_0, t_1] \subseteq [0,1]$ such that*

- *$\gamma(t) \in \mathcal{K}$ for all $t \in [t_0, t_1]$,*
- *$h(\gamma(t)) \in \mathcal{R}_2$ for all $t \in [t_0, t_1]$,*
- *$h(\gamma(t_0))$ and $h(\gamma(t_1))$ belong to different components of $\mathcal{R}^-_2$.*

In this case, we write

$$(\mathcal{K}, h)\colon \widehat{\mathcal{R}}_1 \Leftrightarrow\!\!\!\rightarrow \widehat{\mathcal{R}}_2.$$

Given a positive integer m, we say that h *stretches $\widehat{\mathcal{R}}_1$ to $\widehat{\mathcal{R}}_2$ along the paths with crossing number m* and we write

$$h\colon \widehat{\mathcal{R}}_1 \Leftrightarrow\!\!\!\rightarrow^m \widehat{\mathcal{R}}_2$$

if there exist m pairwise disjoint compact sets

$$\mathcal{K}_0, \dots, \mathcal{K}_{m-1} \subseteq \mathcal{R}_1 \cap \operatorname{dom} h$$

such that $(\mathcal{K}_i, h)\colon \widehat{\mathcal{R}}_1 \Leftrightarrow\!\!\!\rightarrow \widehat{\mathcal{R}}_2$ for each $i \in \{0, \dots, m-1\}$.

Finally, in order to detect chaos, a useful topological tool is the Stretching Along the Paths (SAP) method introduced in [6]. In our framework, it can be stated as follows (cf., [5, Th. 2.1]).

Theorem 3.1 (SAP method)**.** *Let* $h_1\colon \operatorname{dom}\nu \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ *and* $h_2\colon \operatorname{dom}\eta \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ *be continuous maps. Let* $\widehat{\mathcal{R}}_1 = (\mathcal{R}_1, \mathcal{R}_1^-)$ *and* $\widehat{\mathcal{R}}_2 = (\mathcal{R}_2, \mathcal{R}_2^-)$ *be two oriented rectangles in* $\mathbb{R}^2$*. Suppose that*

- *there exist* $n \geq 1$ *pairwise disjoint compact subsets of* $\mathcal{R}_1 \cap \operatorname{dom}\nu$*,* $\mathcal{Q}_0, \ldots, \mathcal{Q}_{n-1}$*, such that* $(\mathcal{Q}_i, h_1)\colon \widehat{\mathcal{R}}_1 \Leftrightarrow\!\!\!\rightarrow \widehat{\mathcal{R}}_2$ *for* $i = 0, \ldots, n-1$*,*
- *there exist* $m \geq 1$ *pairwise disjoint compact subsets of* $\mathcal{R}_2 \cap \operatorname{dom}\eta$*,* $\mathcal{K}_0, \ldots, \mathcal{K}_{m-1}$*, such that* $(\mathcal{K}_i, h_2)\colon \widehat{\mathcal{R}}_2 \Leftrightarrow\!\!\!\rightarrow \widehat{\mathcal{R}}_1$ *for* $i = 0, \ldots, m-1$*.*

If at least one between n *and* m *is greater than or equal to* 2*, then the map* $h = h_2 \circ h_1$ *induces chaotic dynamics on* $n \times m$ *symbols on*

$$\mathcal{Q}^* = \bigcup_{\substack{i=0,\ldots,n-1 \\ j=0,\ldots,m-1}} \mathcal{Q}_i \cap \nu^{-1}(\mathcal{K}_j).$$

For the proof of Theorem 3.1 we refer to [5, Th. 2.1].

3.2. **Topological tools in the phase-plane.** The geometry associated to the phase-portrait of (2.1) exhibits unbounded solutions and periodic trajectories. These configurations guarantee the existence of two types of invariant regions: topological strips and topological annuli confined between unbounded and bounded solutions, respectively. In this section we will give some preliminary topological results on the phase-plane (x, y) needed to establish the dynamics induced by (2.1).

By a *topological strip* $\mathcal{S}$ we mean the image of a straight strip of finite width

$$\mathbf{S} := \{(x, y) \in \mathbb{R}^2 \colon x_1 < x < x_2,\ -1 \leq y \leq 1\}$$

through a locally defined homeomorphism

$$h_{\mathbf{S}}\colon (x_1, x_2) \times [-1, 1] \to \mathcal{S}.$$

Let a *bridge* in $\mathcal{S}$ be the image by $h_{\mathbf{S}}$ of any simple continuous curve $\gamma\colon [a, b] \to \mathbf{S}$ such that $\gamma(a) = (\hat{x}, -1)$ and $\gamma(b) = (\check{x}, 1)$ for some $\hat{x}, \check{x} \in (x_1, x_2)$ or, viceversa, $\gamma(a) = (\check{x}, 1)$ and $\gamma(b) = (\hat{x}, -1)$.

A *topological annulus* $\mathcal{A}$ is defined as the image of a rectangular region

$$\mathbf{A} := \{(x, y) \in \mathbb{R}^2 \colon 1 \leq x \leq 2,\ -1 \leq y \leq 1\}$$

through a continuous map

$$h_{\mathbf{A}}\colon [1, 2] \times [-1, 1] \to \mathcal{A},$$

such that the restriction of $h_{\mathbf{A}}$ to $(1, 2) \times [-1, 1]$ is a homeomorphism and $h_{\mathbf{A}}(1, y) = h_{\mathbf{A}}(2, y)$. We notice that the restriction to $(1, 2) \times [-1, 1]$ yields a strip. Moreover, the boundary of the topological annulus $\partial\mathcal{A}$ is the union of two Jordan curves $\partial^i\mathcal{A} := h_{\mathbf{A}}(x, -1)$ and $\partial^e\mathcal{A} := h_{\mathbf{A}}(x, 1)$. We denote the portion of the plane outside a generic Jordan curve Γ by $out(\Gamma)$ and the one inside by $in(\Gamma)$. For identification purposes, let $\partial^i\mathcal{A} \subset in(\partial^e\mathcal{A})$. In this manner, we can identify two connected sets, one bounded and another one unbounded given by $in(\partial^i\mathcal{A})$ and $out(\partial^e\mathcal{A})$, respectively. Let a *ray* in $\mathcal{A}$ be any simple continuous curve $\gamma\colon [a, b] \to \mathcal{A}$ such that $\gamma(a) \in \partial^i\mathcal{A}$ and $\gamma(b) \in \partial^e\mathcal{A}$ or, viceversa, $\gamma(a) \in \partial^e\mathcal{A}$ and $\gamma(b) \in \partial^i\mathcal{A}$.

We are interested in crossing configurations between either an annulus and a strip or two annuli. In particular we are looking for similarities with the geometry of the linked-twist maps (see [8, 14]). Hence, we introduce the following definition and in Figure 6 we provide a visual representation of the linkage condition between an annulus and a strip.

Definition 3.3 (Linkage condition)**.** *Let $\mathcal{A}$ be a topological annulus and $\mathcal{S}$ be a topological strip. We say that $\mathcal{A}$ is linked with $\mathcal{S}$ if there exist a bridge γ_1 in $\mathcal{S}$, a ray γ_2 in $\mathcal{A}$, and a topological ball B containing $\mathcal{A}$ such that:*

- $\gamma_1 \subset in(\partial^i \mathcal{A})$;
- $\gamma_2 \cap \mathcal{S} = \emptyset$;
- $(\mathcal{S} \setminus \gamma_1) \cap \partial B$ *consists of exactly two disjoint bridges.*

From Definition 3.3 we observe that when $\mathcal{A}$ is linked with $\mathcal{S}$, then the topological ball B is cut into two connected components B^+ and B^-.

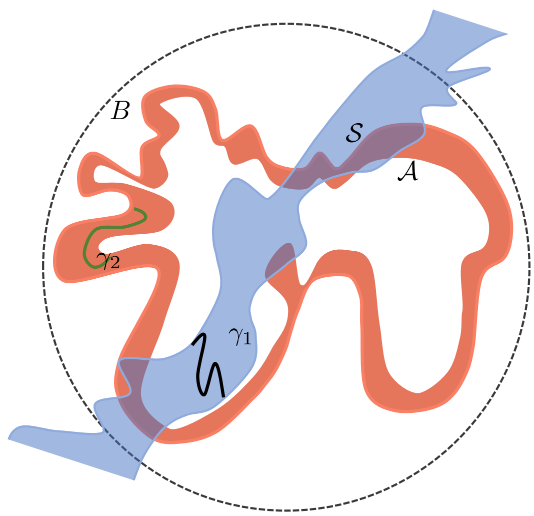


FIGURE 6. Linkage condition. The figure represents an example of a topological annulus (red) linked with a topological strip (blue) through the existence of a bridge (black) and a ray (green).

Notice that Definition 3.3 involves only the geometry inside a topological ball B. Therefore it could include the case when the strip $\mathcal{S}$ is the intersection of an annulus $\mathcal{A}_2$ with the ball B. In this manner we are generalizing the definition of the linkage between two annuli $\mathcal{A}_1$, $\mathcal{A}_2$ given in [7, Definition 3.2]. In the following proposition we also recover some of the properties collected in [7, Proposition 3.1] for the linkage of two annuli.

From the third requirement of Definition 3.3 it follows that the set $B \setminus \mathcal{S}$ has two connected components that will be denoted B^+ and B^-.

Proposition 3.2. *If the topological strip $\mathcal{S}$ is linked with the topological annulus $\mathcal{A}$, then there exists a topological ball B containing $\mathcal{A}$, a bridge γ_3 in $\mathcal{S}$ and a ray γ_4 in $\mathcal{A}$ such that $\gamma_3 \subset B \setminus in(\partial^e \mathcal{A})$, and denoting by B^+ the component of $B \setminus \mathcal{S}$ that contains $\gamma_2 \subset B^+$, then $\gamma_4 \subset B^-$.*

Proof. First of all we observe that the existence of a bridge $\gamma_3 \subset B \setminus in(\partial^e \mathcal{A})$ follows immediately from Definition 3.3. Indeed, we can choose γ_3 between one of the two components of $(\mathcal{S} \setminus \gamma_1) \cap \partial B$ and one of the bridges in $(\mathcal{S} \setminus \gamma_1) \cap \partial B$.

The proof of the existence of the ray γ_4 is entirely analogous to that of [7, Proposition 3.1] and is omitted. $\square$

In the sequel, we deal with the study of the dynamics in a strip $\mathcal{S}$ and in an annulus $\mathcal{A}$. If they are linked, then there exist two disjoint topological rectangular regions $\mathcal{R}_1 \subset \mathcal{A} \cap \mathcal{S} \cap B$ and $\mathcal{R}_2 \subset \mathcal{A} \cap \mathcal{S} \cap B$.

Firstly, we consider the following continuous map

$$\phi_{\mathcal{S}} \colon \mathcal{S} \to \mathcal{S}. \tag{3.2}$$

Without loss of generality, we can assume that $\mathcal{R}_1$, $\mathcal{R}_2$ are homeomorphic to

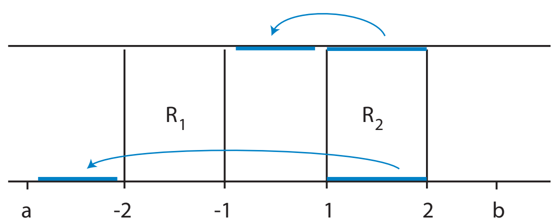
$$R_1 = [-2,-1] \times [-1,1] \quad \text{and} \quad R_2 = [1,2] \times [-1,1],$$

respectively. We suppose that the map $\phi_{\mathcal{S}}$ in (3.2) admits a lift $\widetilde{\phi}_{\mathcal{S}}$ to the covering space $[a,b] \times [-1,1]$, with $a < -2$ and $b > 2$, defined as

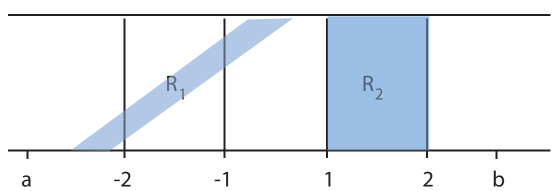
$$\widetilde{\phi}_{\mathcal{S}} \colon (x,y) \mapsto (x + \Xi(x,y), \zeta(x,y))$$

where ζ, Ξ are continuous functions.

Definition 3.4 (Strip boundary invariance condition)**.** *The condition holds for the map $\phi_{\mathcal{S}}$ if the second coordinate of its lift $\widetilde{\phi}_{\mathcal{S}}$ satisfies $\zeta(x,-1) \equiv -1$ and $\zeta(x,1) \equiv 1$.*



(A) Image of $[1,2] \times \{-1\}$ and $[1,2] \times \{1\}$ under a twist condition with respect to the rectangle R_1.



(B) Image of the rectangle R_2 under a twist condition with respect to the rectangle R_1.

FIGURE 7. Example of strip twist condition.

Definition 3.5 (Strip twist condition)**.** *The condition holds with respect to R_1 for $x \in [1,2]$ if either*

$$\Xi(x,-1) \le -4 \quad \textit{and} \quad \Xi(x,1) \ge -2,$$

or

$$\Xi(x,-1) \ge -2 \quad \textit{and} \quad \Xi(x,1) \le -4.$$

The condition holds with respect to R_2 for $x \in [-2,-1]$ if either

$$\Xi(x,-1) \le 2 \quad \textit{and} \quad \Xi(x,1) \ge 4,$$

or

$$\Xi(x,-1) \geq 4 \quad \textit{and} \quad \Xi(x,1) \leq 2.$$

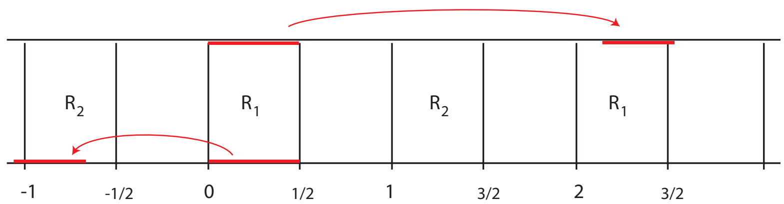
Secondly, we consider the following continuous map

$$\phi_{\mathcal{A}} \colon \mathcal{A} \to \mathcal{A}. \tag{3.3}$$

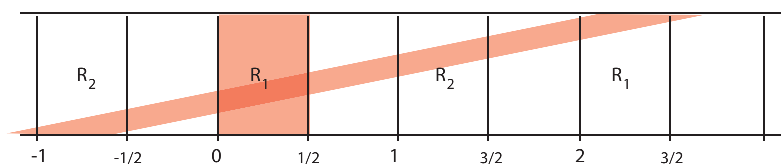
We suppose that the map $\phi_{\mathcal{A}}$ in (3.3) admits a lift $\widetilde{\phi}_{\mathcal{A}}$ to the covering space $\mathbb{R} \times [-1,1]$ defined as

$$\widetilde{\phi}_{\mathcal{A}} \colon (\theta,\rho) \mapsto (\theta + \Theta(\theta,\rho), \omega(\theta,\rho)),$$

where θ, ρ are generalized polar coordinates, and Θ, ω are continuous functions 1-periodic in the θ-variable. Without loss of generality, we can assume that $\mathcal{R}_1$ and $\mathcal{R}_2$ are represented in the covering by $R_1 = [2k, 2k+\frac{1}{2}] \times [-1,1]$ and $R_2 = [2k+1, 2k+\frac{3}{2}] \times [-1,1]$, respectively.



(A) Image of $[1,1/2] \times \{-1\}$ and $[1,1/2] \times \{1\}$ under a twist condition with respect to the rectangle R_1.



(B) Image of the rectangle R_1 under a twist condition it goes across a copy of R_2. Here $j_{-1} = j_1 = 0$.

FIGURE 8. Example of an annular twist condition.

Definition 3.6 (Annular boundary invariance condition)**.** *The condition holds for the map $\phi_{\mathcal{A}}$ if the second coordinate of its lift $\widetilde{\phi}_{\mathcal{A}}$ satisfies $\omega(\theta,-1) \equiv -1$ and $\omega(\theta,1) \equiv 1$.*

Definition 3.7 (Annular twist condition)**.** *There exist integers j_{-1} and j_1 such that the condition holds with respect to R_1 for $\theta \in [0,1/2]$ if either*

$$\Theta(\theta,-1) \leq 2j_{-1} + \tfrac{1}{2} \quad \textit{and} \quad \Theta(\theta,1) \geq 2j_1 + \tfrac{3}{2}, \ \textit{with } j_1 + 1 - j_{-1} > 0$$

or

$$\Theta(\theta,-1) \geq 2j_{-1} + \tfrac{3}{2} \quad \textit{and} \quad \Theta(\theta,1) \leq 2j_1 + \tfrac{1}{2}, \ \textit{with } j_{-1} + 1 - j_1 > 0$$

hold.

We notice that when the annular twist condition holds with respect to R_1 then the rectangle R_1 is stretched across R_2 a number of times which is given by $|j_{-1} - j_1| + 1$.

Theorem 3.3. *Let $\mathcal{A}$ be a topological annulus linked with a topological strip $\mathcal{S}$. Let $\mathcal{R}_i$ for $i = 1, 2$ be two disjoint oriented topological rectangles given through the linkage. Let $\phi_{\mathcal{A}} \colon \mathcal{A} \to \mathcal{A}$ and $\phi_{\mathcal{S}} \colon \mathcal{S} \to \mathcal{S}$, be two continuous maps that satisfy the boundary invariance conditions, and the twist conditions. Then,*

$$\phi_{\mathcal{A}} \circ \phi_{\mathcal{S}} \colon \widehat{\mathcal{R}}_j \Bumpeq\!\!\!\!\!\!\!\longrightarrow^{m-1} \widehat{\mathcal{R}}_j \quad \textit{and} \quad \phi_{\mathcal{S}} \circ \phi_{\mathcal{A}} \colon \widehat{\mathcal{R}}_{j+1} \Bumpeq\!\!\!\!\!\!\!\longrightarrow^{m-1} \widehat{\mathcal{R}}_{j+1}$$

for some $j \pmod 2$ with $m = |j_{-1} - j_1| + 1$.

We notice that [7, Theorem 3.1] becomes a corollary of Theorem 3.3. For the proof we use the following lemma.

Lemma 3.4. *Consider*

$$K_\ell = \tilde{\phi}_{\mathcal{A}} \left([2\ell + 1, 2\ell + 3/2] \times [-1, 1] \right) \cap R_{1,0}, \quad \ell \in \mathbb{Z}$$

where $R_{1,0} = [0, 1/2] \times [-1, 1]$. If $\phi_{\mathcal{A}}$ satisfies the annular twist condition then at least $m - 1$ of the K_ℓ are non empty with $m = |j_1 - j_{-1}| + 1$.

Proof. We will prove the lemma in the case of the first annular strip condition, the proof for the second condition being similar.

Let $\theta_0 \in [0, 1/2]$ be fixed. The vertical segment (θ_0, ρ), $\rho \in [-1, 1]$ is mapped by $\tilde{\phi}_{\mathcal{A}}$ in to a curve. Its end points satisfy

$$\begin{aligned} \tilde{\phi}_{\mathcal{A}}(\theta_0, -1) &= (\theta_{-1}, -1) \text{ where } \theta_{-1} \le \theta_0 + 2j_{-1} + \tfrac{1}{2}, \\ \tilde{\phi}_{\mathcal{A}}(\theta_0, 1) &= (\theta_1, 1) \text{ where } \theta_1 \ge \theta_0 + 2j_{-1} + \tfrac{1}{2} + 2m - 1. \end{aligned}$$

Hence, $|\theta_{-1} - \theta_1| \ge 2m - 1|$ and $K_\ell \neq \emptyset$ for $\ell = j_{-1}, \ldots, j_{-1} + m - 1$. $\square$

Proof of Theorem 3.3. First of all without loss of generality we assume that $\phi_{\mathcal{S}}$ maps $\mathcal{R}_2$ across $\mathcal{R}_1$ thanks to the strip twist condition. Hence we prove that $\phi_{\mathcal{S}} \circ \phi_{\mathcal{A}} \colon \widehat{\mathcal{R}}_1 \Bumpeq\!\!\!\!\!\!\!\longrightarrow^{m} \widehat{\mathcal{R}}_1$. The other situations are just an adaptation of this proof.

We want to find disjoint compact subsets $\mathcal{K}_1, \ldots, \mathcal{K}_{m-1} \subset \mathcal{R}_1$ such that for any continuous path γ across $\mathcal{R}_1$ with $\gamma(0)$, $\gamma(1)$ in different components of $\partial \mathcal{R}_1$, the restriction $\phi_{\mathcal{A}}(\gamma(t))|_{\mathcal{K}_\ell}$ goes across $\mathcal{R}_2$. In order to do this we work on the covering space, where the $\mathcal{K}_\ell$ will be represented by the K_ℓ of Lemma 3.4. The $\mathcal{K}_\ell$ are pairwise disjoint because the K_ℓ lie in a single representative $R_{1,0}$ of $\mathcal{R}_1$.

The arguments used in the proof of Lemma 3.4 ensure that the curve $\tilde{\gamma}(t)$ in the covering, satisfying $\tilde{\gamma}(0) = (\theta_0, -1)$, and $\tilde{\gamma}(1) = (\theta_1, 1)$ with θ_0, $\theta_1 \in [0, 1/2]$ goes across all the K_ℓ, and that the restriction of $\tilde{\gamma}$ to each K_ℓ goes across some copy, $[2\ell + 1, 2\ell + 3/2] \times [-1, 1]$, of R_2. $\square$

4. Application to codimension 1 reversible vector fields

To detect chaotic dynamics, we apply the topological results of the previous section to some periodically forced reversible ODEs. In particular, we consider a T-periodic step-wise forcing term $p(t)$ that switches between two different values as follows

$$p(t) := \begin{cases} \lambda_1 & \text{for } t \in [0, \tau_1), \\ \lambda_2 & \text{for } t \in [\tau_1, \tau_1 + \tau_2), \end{cases}$$

where $\lambda_1 \neq \lambda_2$ and $0 < \tau_1 < \tau_2 < T$ with $\tau_1 + \tau_2 = T$. We investigate the T-periodic problem associated with the system

$$\begin{cases} \dot{x} = y f(x, y^2), \\ \dot{y} = g(x, y^2) + p(t), \end{cases} \tag{4.1}$$

where f and g are smooth functions that identify the normal forms of codimension 1 reversible systems introduced in [13].

Our goal is to prove the existence of chaotic dynamics for system (4.1). First, we look at the flow of the vector field $X(x,y)$ associated with (4.1) which is given by the unique solution $(x(t),y(t)) = \varphi(t,x_0,y_0)$ of $\dot{X} = X(x,y)$ satisfying $x(0) = x_0$ and $y(0) = y_0$. We study the Poincaré map $\Phi\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\Phi(x_0,y_0) = \varphi(T,x_0,y_0)$ for every point $(x_0,y_0) \in \mathbb{R}^2$. Second, we notice that the full dynamics of the problem can be broken into two sub-systems

$$\begin{cases} \dot{x} = yf(x,y^2), \\ \dot{y} = g(x,y^2) + \lambda_1, \end{cases} \tag{4.2}$$

and

$$\begin{cases} \dot{x} = yf(x,y^2), \\ \dot{y} = g(x,y^2) + \lambda_2. \end{cases} \tag{4.3}$$

Hence, we have that the Poincaré map Φ may be decomposed as $\Phi = \Phi_{\lambda_2} \circ \Phi_{\lambda_1}$, where, for any $(x_0,y_0) \in \mathbb{R}^2$, $\Phi_{\lambda_1}(x_0,y_0) = \varphi_{\lambda_1}(\tau_1,x_0,y_0)$ and $\Phi_{\lambda_2}(x_0,y_0) = \varphi_{\lambda_2}(\tau_2,x_0,y_0)$ are the Poincaré maps associated with (4.2) and (4.3), respectively. We outline here the structure of the proof for the saddle case, done by applying Theorem 3.3.

1) Locate a flow invariant line $\Gamma_{1,*}$ for, say λ_1 and a closed flow invariant line $\Gamma_{2,*}$ for λ_2, making sure they intersect in at least two points. Then $\Gamma_{2,*}$ is going to be $\partial^e\mathcal{A}$ and $\Gamma_{1,*}$ will be of one component of $\partial\mathcal{S}$.
2) Take τ_1 to be the time it takes for φ_{λ_1} to move one intersection point to the next one.
3) Look at a curve γ_1 ending at the first intersection point as a candidate for a bridge and make sure Φ_{λ_1} maps it to *in* $(\Gamma_{2,*})$. Take P to be the other end point of γ_1.
4) Take the φ_{λ_1} trajectory through P to be the other component of $\partial\mathcal{S}$ and take the (closed) φ_{λ_2} trajectory through P to be $\partial^i\mathcal{A}$. This ensures that the strip twist condition (Definition 3.5) holds.
5) Obtain the time τ_2 for the annular-strip condition (Definition 3.7).

In this way we can prove that the dynamics of (4.1) is semiconjugate to a shift in a finite alphabet.

4.1. **Saddle case.** We assume that system (4.1) has a saddle structure by considering

$$\begin{cases} \dot{x} = xy, \\ \dot{y} = x - y^2 + p(t). \end{cases} \tag{4.4}$$

Depending on $p(t)$, the phase-portrait of system (4.4) switches between different configurations as described in Section 2.

Theorem 4.1. *Let Φ be the Poincaré map associated with system (4.4). Then for each $\lambda_1 > 0$ and each λ_2 with $\lambda_1 > \lambda_2$ and for an open set of values of τ_1 and τ_2 the map Φ induces chaotic dynamics on m symbols, for some $m \geq 2$.*

Proof. First of all we notice that the following two cases can occur: $\lambda_1 > \lambda_2 > 0$ or $\lambda_1 > 0 \geq \lambda_2$.

Let us suppose that λ_1 and λ_2 are two fixed positive values satisfying the first case. Then for both systems (4.2) and (4.3) there exist three equilibria. In particular, there exists a heteroclinic cycle around the center $(-\lambda_i, 0)$ which joins the two saddles $(0, -\sqrt{\lambda_i})$ and $(0, \sqrt{\lambda_i})$, for $i = 1, 2$.

Let $(x^*, 0)$ be the point where the heteroclinic cycle of system (4.3) crosses the negative part of the x-axis. Then two configurations are possible: $-\lambda_1 < x^* < -\lambda_2$ or $x^* < -\lambda_1$. It will be not restrictive to consider the first configuration since the other situation can be treated similarly.

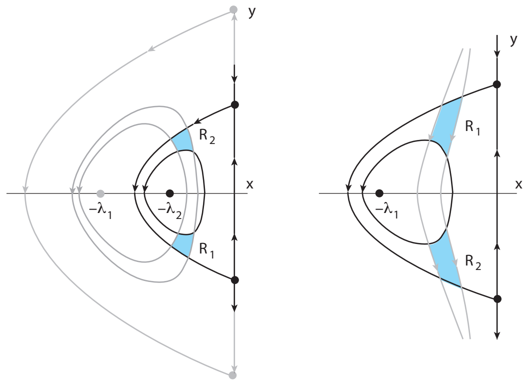


FIGURE 9. Construction of the annulus $\mathcal{A}$ and the strip $\mathcal{S}$ in the saddle case. Left: $0 < \lambda_2 < \lambda_1$; right: $\lambda_2 \leq 0 < \lambda_1$.

We proceed with the construction of an annulus $\mathcal{A}$ and a strip $\mathcal{S}$ which satisfy the topological conditions required to apply Theorem 3.3.

For any $(x, y) \in \mathbb{R}^2$, we call $\Gamma_1(x, y)$ and $\Gamma_2(x, y)$ the trajectories through the point (x, y) of system (4.2) and (4.3), respectively. Let $\Gamma_2(x^*, 0)$ be the heteroclinic trajectory through $(x^*, 0)$, then we define the outer component of $\partial\mathcal{A}$ as

$$\partial^e\mathcal{A} := \Gamma_2(x^*, 0) \cup \{(0, -\sqrt{\lambda_2})\} \cup \Gamma_2(0, 0) \cup \{(0, \sqrt{\lambda_2})\}.$$

Let $\alpha < 0$ with $-\lambda_2 < \alpha$ be any number so the trajectory $\Gamma_1(\alpha, 0)$ through $(\alpha, 0)$ will cross the heteroclinic connection $\Gamma_2(x^*, 0)$. We take $\Gamma_1(\alpha, 0) \cap \{x^* \leq x \leq 0\}$ to be one of the components of $\partial\mathcal{S}$, and we construct the other two boundary pieces of the annulus and the strip so as to satisfy the linkage condition and the twist conditions.

Let τ_1 be the minimum positive time such that, if $r(t)$ is a solution of (4.2) through $(\alpha, 0)$ with $r(0) \in \Gamma_2(x^*, 0) \cap \{y < 0\}$, then $r(\tau_1) \in \Gamma_2(x^*, 0) \cap \{y > 0\}$. For any point

$$(x, y) \in \Gamma_2(x^*, 0) \cap \{y < 0\}$$

close to $r(0)$ the points $\varphi_{\lambda_1}(\tau_1, x, y)$ form a curve through $r(\tau_1)$. Generically this curve goes across $\Gamma_2(x^*, 0)$ (otherwise, make a small change in α). Suppose that the curve is below $\Gamma_2(x^*, 0)$ to the left of $r(\tau_1)$ (otherwise the arguments are similar). Take $\beta < 0$ with $-\lambda_2 < \beta < \alpha < 0$ such that the points in the trajectory $\Gamma_1(\beta, 0)$ of system (4.2) through $(\beta, 0)$ satisfy the condition on the curve. Then we take the other component of $\partial\mathcal{S}$ as $\Gamma_1(\beta, 0) \cap \{x^* \leq x \leq 0\}$. It remains to obtain the inner component of $\partial\mathcal{A}$.

Let $\Pi\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the projection on the second component, namely $\Pi(x, y) = y$. For any $(x, y) \in \mathbb{R}^2$ let $\psi(x, y) = \Pi(\varphi_{\lambda_1}(\tau_1, x, y))$ and let $\overline{\psi}(x, y) = \psi(x, y) + \Pi(x, y)$, so $\overline{\psi}(x, y)$ compares the height of $\varphi_{\lambda_1}(\tau_1, x, y)$ to that of the symmetric point of (x, y).

Let $q(t)$ be the solution of (4.2) through $(\beta, 0)$ with $q(0) \in \Gamma_2(x^*, 0) \cap \{y < 0\}$. Then $\overline{\psi}(q(0)) < 0$. Also there exists a $\sigma > 0$ such that $q(\sigma) \in \Gamma_2(x^*, 0) \cap \{y > 0\}$. By construction, $\overline{\psi}(q(\sigma)) > 0$. Therefore, there exists $\widehat{\sigma} \in (0, \sigma)$ such that $\overline{\psi}(q(\widehat{\sigma})) = 0$. This means that $\varphi_{\lambda_1}(\tau_1, q(\tau_1))$ is symmetric to $q(\tau_1)$. The trajectory $\Gamma_2(q(\widehat{\sigma}))$ will go through both $q(\widehat{\sigma})$ and $\varphi_{\lambda_1}(\tau_1, q(\widehat{\sigma}))$. We define the inner component of $\partial\mathcal{A}$ as $\partial^i\mathcal{A} := \Gamma_2(q(\widehat{\sigma}))$.

In this manner, the topological annulus $\mathcal{A}$ and the topological strip $\mathcal{S}$ are linked by construction (see Figure 9). The linkage condition gives two symmetric topological rectangles $\mathcal{R}_1$ and $\mathcal{R}_2$ (in the lower and upper half-plane, respectively) that satisfy the twist conditions. Indeed, a strip-twist condition holds for $\Phi_{\lambda_1}\colon \mathcal{S} \to \mathcal{S}$ because the rectangle $\mathcal{R}_1 \subset \mathcal{A} \cap \mathcal{S} \cap \{y < 0\}$ is stretched across $\mathcal{R}_2 \subset \mathcal{A} \cap \mathcal{S} \cap \{y > 0\}$. Since $\Gamma_2(x^*, 0)$ is a heteroclinic connection then for every $m \geq 2$ there exists τ_2 large enough such that an annulus-twist condition also holds for $\Phi_{\lambda_2} \colon \mathcal{A} \to \mathcal{A}$ because $\mathcal{R}_2$ is stretched across $\mathcal{R}_1$ m-times (depending on τ_2). The result follows by an application of Theorem 3.3 to the Poincaré map $\Phi = \Phi_{\lambda_2} \circ \Phi_{\lambda_1}$. This concludes the first case.

The proof above holds for a fixed value of τ_1 and for sufficiently large τ_2. However, we may obtain the result for τ_1 in an open interval by taking different values of α.

The arguments above yield a proof for the case $\lambda_1 > 0 \geq \lambda_2$, we just indicate where it needs to be adapted. The outer component of $\partial\mathcal{A}$ may be taken as

$$\partial^e \mathcal{A} := \Gamma_1(x^*, 0) \cup \{(0, -\sqrt{\lambda_1})\} \cup \Gamma_1(0,0) \cup \{(0, \sqrt{\lambda_1})\},$$

where $\Gamma_1(x^*, 0)$ is the heteroclinic trajectory of φ_{λ_1} going through $(x^*, 0)$. One of the components of $\partial\mathcal{S}$ will be $\Gamma_2(\alpha, 0)$ with $-\lambda_1 < \alpha < 0$.

Then take τ_2 to be the least positive time to go from $\Gamma_2(\alpha, 0) \cap \Gamma_1(x^*, 0) \cap \{y > 0\}$ to $\Gamma_1(x^*, 0) \cap \{y < 0\}$. Apply the arguments above to obtain the other component of $\partial\mathcal{S}$ as a φ_{λ_2} trajectory that starting at $\Gamma_1(x^*, 0) \cap \{y > 0\}$ arrives above $\Gamma_1(x^*, 0) \cap \{y < 0\}$ in time τ_2. Then find a point q in this trajectory and in the upper half-plane, such that Φ_{λ_2} maps q to its symmetric $h(q)$. Take $\partial^i \mathcal{A} := \Gamma_2(q)$ to complete the construction. □

In the case when both λ_1 and λ_2 are negative there are no annular invariant regions, so the results cannot be applied. Moreover, in this case there are no non-trivial periodic orbits, so we do not expect periodic forcing to yield chaos. The same holds for the cusp case below, when both λ_1 and λ_2 are positive.

4.2. **Cusp case.** When system (4.1) has the following form

$$\begin{cases} \dot{x} = y, \\ \dot{y} = x^2 + p(t). \end{cases} \tag{4.5}$$

then its phase-portrait is of cusp type. We notice that system (4.5) has also a Hamiltonian structure, and at this juncture, when $\lambda_1 < 0$ and $\lambda_2 \leq 0$ the geometry is similar to the one investigated in [11, 12]. Hence, we expect that chaotic dynamics occurs for τ_1 and τ_2 large enough. For Theorem 4.1 we have used a heteroclinic connection to obtain an annulus twist condition. Here the existing homoclinic connection may be used for the same purpose and, by applying the procedure exploited for Theorem 4.1, we can prove what follows.

Theorem 4.2. *Let Φ be the Poincaré map associated with system (4.5). Then for each $\lambda_1 \leq 0$ and each λ_2 with $\lambda_1 < \lambda_2$ and for an open set of values of τ_1 and τ_2 the map Φ induces chaotic dynamics on $m \geq 2$ symbols.*

The case $\lambda_1 < \lambda_2 < 0$ of Theorem 4.2 may also be obtained as a corollary to [5, Theorem 4.1]. Our methods provide an alternative proof and extend the result to the case $\lambda_1 < 0$, $\lambda_2 > 0$. In the latter case there is no invariant annulus for $\lambda_2 > 0$ and for the proof we need to use a strip condition.

References

[1] R. L. Devaney, Reversible diffeomorphisms and flows, Trans. Amer. Math. Soc. 218 (1976) 89–113. DOI: 10.1090/s0002-9947-1976-0402815-3

[2] J. Guckenheimer, P. Holmes, Nonlinear oscillations, dynamical systems and bifurcations of vector fields, vol 42 of Applied Mathematical Sciences, Springer-Verlag, Berlin, 1983. DOI: 10.1007/978-1-4612-1140-2_6

[3] J. Kennedy, J. A. Yorke, Topological horseshoes, Trans. Amer. Math. Soc. 353 (6) (2001) 2513–2530. DOI: 10.1090/s0002-9947-01-02586-7

[4] J. S. W. Lamb, J. A. G. Roberts, Time-reversal symmetry in dynamical systems: a survey, Phys. D 112 (1-2) (1998) 1–39.
[5] A. Margheri, C. Rebelo, F. Zanolin, Chaos in periodically perturbed planar Hamiltonian systems using linked twist maps, J. Differential Equations 249 (12) (2010) 3233–3257. DOI: 10.1016/j.jde.2010.08.021
[6] A. Medio, M. Pireddu, F. Zanolin, Chaotic dynamics for maps in one and two dimensions: a geometrical method and applications to economics, Internat. J. Bifur. Chaos Appl. Sci. Engrg. 19 (10) (2009) 3283–3309. DOI: 10.1142/s0218127409024761
[7] D. Papini, G. Villari, F. Zanolin, Chaotic dynamics in a periodically perturbed Liénard system, Differential Integral Equations 32 (11-12) (2019) 595–614.
[8] A. Pascoletti, M. Pireddu, F. Zanolin, Multiple periodic solutions and complex dynamics for second order ODEs via linked twist maps, in: The 8th Colloquium on the Qualitative Theory of Differential Equations, vol. 8 of Proc. Colloq. Qual. Theory Differ. Equ., Electron. J. Qual. Theory Differ. Equ., Szeged, 2008, pp. No. 14, 32. DOI:
[9] D. Ruelle, Differentiable dynamical systems and the problem of turbulence, Bull. Amer. Math. Soc. (N.S.) 5 (1) (1981) 29–42. DOI: 10.14232/ejqtde.2007.7.14
[10] M. B. Sevryuk, Reversible systems, vol. 1211 of Lecture Notes in Mathematics, Springer-Verlag, Berlin, 1986.
[11] E. Sovrano, How to construct complex dynamics? A note on a topological approach, Internat. J. Bifur. Chaos Appl. Sci. Engrg. 30 (2) (2020) 2050034, 7 pp. DOI: 10.1090/s0273-0979-1981-14917-x
[12] E. Sovrano, F. Zanolin, The Ambrosetti-Prodi periodic problem: Different routes to complex dynamics, Dynam. Systems Appl. 26 (2017) 589–626.
[13] M. A. Teixeira, Singularities of reversible vector fields, Phys. D 100 (1-2) (1997) 101–118.
[14] S. Wiggins, J. M. Ottino, Foundations of chaotic mixing, Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 362 (1818) (2004) 937–970.

Isabel S. Labouriau, Centro de Matemática da Universidade do Porto, Rua do Campo Alegre 687, 4169-007 Porto, Portugal
Email address: islabour@fc.up.pt

Elisa Sovrano, École des Hautes Études en Sciences Sociales, Centro de Matemática da Universidade do Porto
Present address: Centre d'Analyse et de Mathématique Sociales (CAMS), CNRS, 54 Boulevard Raspail, 75006, Paris, France
Email address: elisa.sovrano@ehess.fr

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CONLEY THEORY FOR GUTIERREZ-SOTOMAYOR FIELDS

H. MONTÚFAR AND K. A. DE REZENDE

Abstract. In [6], a characterization and genericity theorem for C^1-structurally stable vector fields tangent to a 2-dimensional compact subset M of $\mathbb{R}^k$ are established. Also in [6], new types of structurally stable singularities and periodic orbits are presented. In this work we study the continuous flows associated to these vector fields, which we refer to as the Gutierrez-Sotomayor flows on manifolds M with simple singularities, GS flows, by using Conley Index Theory. The Conley indices of all simple singularities are computed and an Euler characteristic formula is obtained. By considering a stratification of M which decomposes it into a union of its regular and singular strata, certain Euler type formulas which relate the topology of M and the dynamics on the strata are obtained. The existence of a Lyapunov function for GS flows without periodic orbits and singular cycles is established. Using long exact sequence analysis of index pairs we determine necessary and sufficient conditions for a GS flow to be defined on an isolating block. We organize this information combinatorially with the aid of Lyapunov graphs and using a Poincaré-Hopf equality we construct isolating blocks for all simple singularities.

1. Introduction

In [6], C. Gutierrez and J. Sotomayor generalize characterization and genericity theorems obtained by M. Peixoto [8] for structurally stable vector fields tangent to smooth compact two-manifolds. The following definitions 1.1, 1.2 and 1.3 were introduced in [6] and the reader is referred to the original paper for more details.

Definition 1.1. *A subset $M \subset \mathbb{R}^l$ is called a* two-dimensional manifold with simple singularities *if for every point $p \in M$ there is a neighborhood V_p of p in M and a C^∞-diffeomorphism $\Psi : V_p \to \mathcal{G}$ such that $\Psi(p) = 0$, where $\mathcal{G}$ is one of the following subsets of $\mathbb{R}^3$:*

$\mathcal{R} = \{(x, y, z); z = 0\}$, *plane;*

$\mathcal{C} = \{(x, y, z); z^2 - y^2 - x^2 = 0\}$, *cone;*

$\mathcal{D} = \{(x, y, z); xy = 0\}$, *double crossing;*

$\mathcal{W} = \{(x, y, z); zx^2 - y^2 = 0\}$, *Whitney's umbrella;*

$\mathcal{T} = \{(x, y, z); xyz = 0\}$, *triple crossing.*

Ψ *is called a local chart at p.*

These local charts are essential in order to define the stratified set M in the sense of Thom [10], endowed with the partition $\{M(\mathcal{G}), \mathcal{G}\}$ where $\mathcal{G} = \mathcal{R}$, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{W}$ or $\mathcal{T}$ and $M(\mathcal{G})$ is the set of points $p \in M$ such that $\Psi(p) = 0$ for $\Psi : V_p \to \mathcal{G}$. Note that $M(\mathcal{R})$ is a smooth two-dimensional manifold called the regular part of M, $M(\mathcal{D})$ is a one-dimensional smooth manifold, while $M(\mathcal{C})$, $M(\mathcal{W})$ and $M(\mathcal{T})$ are discrete sets.

Definition 1.2. *A vector field X of class C^r on $\mathbb{R}^l$ is said to be* tangent *to a manifold $M \subset \mathbb{R}^l$ with simple singularities if it is tangent to the smooth submanifolds $M(\mathcal{G})$, for all $\mathcal{G}$. The space of such vector fields is denoted by $\mathfrak{X}^r(M)$.*

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In [6], C. Gutierrez and J. Sotomayor determine conditions of stability for fixed points, periodic orbits and singular cycles.

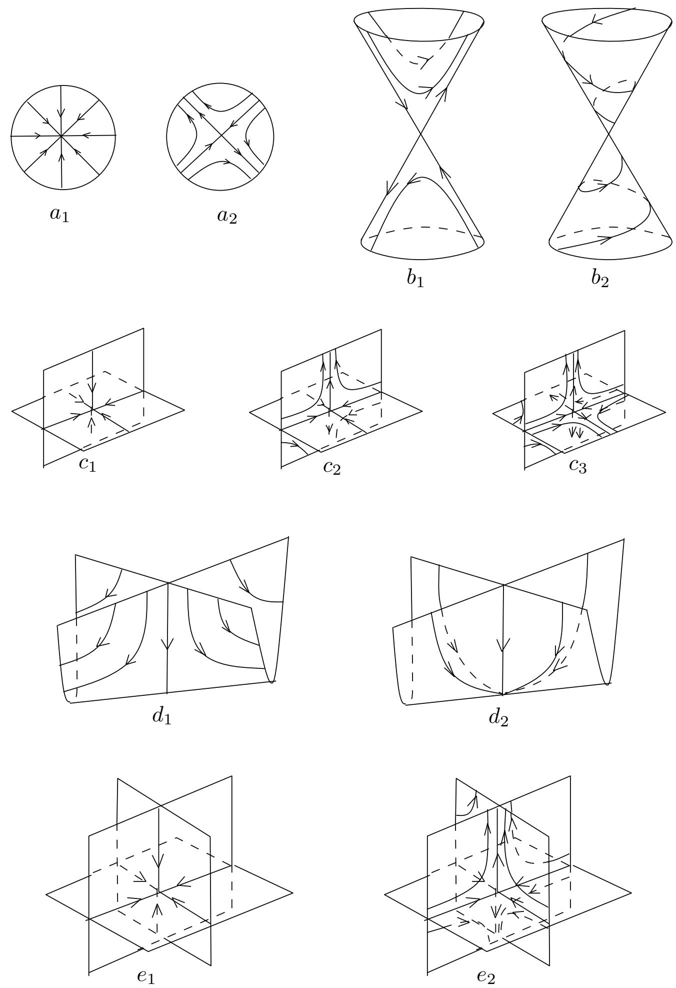

FIGURE 1. Local types of hyperbolic fixed points

Definition 1.3. *Let $M \subset \mathbb{R}^l$ be a two-manifold with simple singularities. We call $\Sigma^r(M)$ the set of vector fields $X \in \mathfrak{X}^r(M)$ such that:*

(1) *X has finitely many fixed points and periodic orbits, all hyperbolic.*
(2) *The singular limit cycles of X are simple and X has no saddle connection.*
(3) *The $\alpha-$limit set and $\omega-$limit set of every trajectory of X is either a fixed point, a periodic orbit or a singular cycle.*

In this work, we refer to the flow X_t associated with the field $X \in \Sigma^r(M)$ as the *Gutierrez-Sotomayor flow*.

In [6], C. Gutierrez and J. Sotomayor proved the following formidable theorem.

Theorem 1.4. *Under either of the following hypotheses on $\mathfrak{X}^r(M)$:*

- $r = 1$, *or*
- $r = 2, 3, \ldots, \infty$ *and each connected component of* $M(\mathcal{R})$ *is either an orientable two-manifold or an open subset of* $P^2 \cup K^2 \cup (T^2 \sharp P^2)$,

we have that:

(1) $\Sigma^r(M)$ *is open and dense in* $\mathfrak{X}^r(M)$, *and*
(2) $X \in \mathfrak{X}^r(M)$ *is structurally stable if and only if* $X \in \Sigma^r(M)$.

In this article, we study Gutierrez-Sotomayor flows from a topological perspective, using Conley index theory. In Section 2 we define a Lyapunov function and in this Gutierrez-Sotomayor context we show its existence for flows without periodic orbits and singular cycles. In proving the existence of Lyapunov functions, we also prove that there is a neighborhood, N of p, in M and a function f on N such that f is continuous and decreases along the orbits of X_t on $N - p$.

In Section 3, we develop the classical Conley theory. In Theorem 3.2, the homotopical index of singularities of a Gutierrez-Sotomayor flow, X_t, on M are obtained. Therefore, by calculating the ranks of the homology of the Conley index of a singularity $p \in M$, denoted by (h_0, h_1, h_2), we present several Euler characteristic type formulas in Section 3.2 which relate the topology of M to the dynamics of the flow X_t.

In Section 4, a more general handle theory is introduced in order to establish a procedure for constructing special isolating neighborhoods of simple singularities of a Gutierrez-Sotomayor flow. GS handles are defined. In Theorem 4.2, a Poincaré-Hopf equality is presented, which relates the first Betti number of the branched one-manifolds which makeup the boundary of the isolating block (N_1, N_0) of the singularity $p \in M$ with the number of boundary components in N_0 and the numerical Conley index (h_0, h_1, h_2) of p. This theorem will guide our constructions of isolating blocks.

In Section 5 we adopt a combinatorial approach, by associating a Lyapunov graph L to a GS flow X_t and a Lyapunov function f, by identifying to a point each connected component of a level set of f.

In Theorem 5.3, through a long exact homological sequence analysis of index pairs we determine properties that a Lyapunov graph must satisfy in order to be associated to a GS-flow. The main results herein generalize results of K. de Rezende and R. Franzosa [3] where Morse-Smale flows and more generally continuous flows are classified on smooth surfaces.

2. Lyapunov function

A Lyapunov function on M is a collection of Lyapunov functions on the strata of $M \subset \mathbf{R}^l$. Note, however, that we do not require the function to be smooth, only continuous.

Definition 2.1. Let M be a two-manifold with simple singularities. If X_t is a Gutierrez-Sotomayor flow on M then a function $f : M \to \mathbb{R}$ is called a *Lyapunov function* if:

(1) For each stratum $M(\mathcal{G})$ of M:
   (a) $f\,|_{M(\mathcal{G})}$ is a smooth function and f is continuous on M.
   (b) The critical points of $f\,|_{M(\mathcal{G})}$ are nondegenerate and coincide with the singularities of X_t.
   (c) $\dfrac{d}{dt}(f\,|_{M(\mathcal{G})}\,(X_t x)) < 0$, if x is not a singularity of X_t.
(2) If p and q are singularities of X_t, then $f(p) \neq f(q)$.

In Section 2.1, we will construct a Lyapunov function f locally on a neighborhood of a GS singularity. In Section 2.2 we extend this construction to isolating blocks and subsequently to GS two-manifolds.

2.1. **Local Construction.** Throughout this work, for simplicity, a two dimensional disk will be referred to as disk D and a one dimensional disk as a segment I.

Theorem 2.2. *Let M be a 2-dimensional manifold with simple singularities. If $p \in M$ is a singularity of a Gutierrez-Sotomayor flow X_t on M then there exists a neighborhood, N of p on M, sufficiently small, and a function f on N such that f is a Lyapunov function on N.*

Proof.

Case 1:: If $p \in M(\mathcal{R})$ then a neighborhood N of p on M is a disk. Without loss of generality, we can assume the disk N as in Figure 1 (a_1) and (a_2). If p is of type (a_1) then in local coordinates its dynamics in $\mathbb{R}^2$ is given by:

$$\begin{cases} \dot{x} = -2x \\ \dot{y} = -2y \end{cases}$$

Define a function f on N given by $f(x,y) = x^2 + y^2$. Since $\frac{df}{dt} = -4(x^2+y^2) < 0$ then f is a Lyapunov function on N. If p is as in (a_2) then in these local coordinates its dynamics are given by:

$$\begin{cases} \dot{x} = -y \\ \dot{y} = -x \end{cases}$$

Define a function f on N given by $f(x,y) = xy$. Since $\frac{df}{dt} = -(y^2+x^2) < 0$ then f is a Lyapunov function on N. If p is as in (a_1) with the reverse dynamics then consider $-f$.

In any case, we can summarize this by writing X in local coordinates as $\dot{x} = Ax + \phi(x)$ where $\phi(0) = d\phi(0) = 0$ and the eigenvalues of A have real part different from zero. This condition is equivalent to the existence of symmetric matrices Q and C with C positive definite and Q non-singular such that the Lyapunov equation $A^TQ + QA = -C$ holds, where the superscript T denotes the transpose of the matrix. Define a function f given by $f(x) = x^TQx$. Since

$$\frac{df}{dt} = \dot{x}^TQx + x^TQ\dot{x}$$

$$\frac{df}{dt} = (Ax + \phi(x))^TQx + x^TQ(Ax + \phi(x))$$

$$\frac{df}{dt} = x^T(A^TQ + QA)x + 2x^TQ\phi(x)$$

$$\frac{df}{dt} = -x^TCx + 2x^TQ\phi(x)$$

where $2x^TQ\phi(x)$ has higher order terms. For N sufficiently small, f is a Lyapunov function on N.

Case 2:: If $p \in M(\mathcal{C})$ then a neighborhood N of p in M is formed by two disks D_1 and D_2 identified at the singularity p, see Figure 1 (b_1) and (b_2). We can assume without loss of generality that the disks D_i, $i = 1, 2$, in $\mathbb{R}^2$ are as in Figure 2.

If the disks are as in (a) and (b) then we are in the previous case. If D_i is as in (c) then in local coordinates its dynamics are given by:

$$\begin{cases} \dot{x} = 0 \\ \dot{y} = -x^2 - y^2 \end{cases}$$

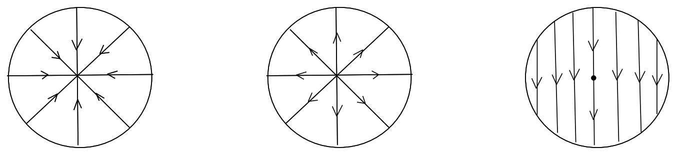

FIGURE 2. Disks D_i in N

Let f_i be a function on D_i given by $f_i(x,y) = y$. As $\frac{df_i}{dt} = -x^2 - y^2 < 0$ then f_i decreases along orbits of X_t on D_i. Define the function f on N:

$$f(x) = \begin{cases} f_1(x) \text{ if } x \in D_1 \\ f_2(x) \text{ if } x \in D_2 \end{cases}$$

Then $f|_{N\setminus\{p\}}$ is a Lyapunov function, hence f is a Lyapunov function on N.

Case 3:: If $p \in M(\mathcal{D})$ then a neighborhood N of p in M is formed by two disks D_i, $i = 1, 2$, that intersect transversally along diameters d_1 and d_2 on D_1 and D_2 respectively, see Figure 1 (c_1), (c_2) and (c_3). Let $d = D_1 \cap D_2$. On each disk D_i the dynamics are the same as defined for $p \in M(\mathcal{R})$, hence a Lyapunov function f_i is defined as in Case 1. By adding appropriate constants we can assume $f_1(p) = f_2(p)$.

Let γ be the orbit on d. By using a diffeomorphism $h : f_1(\gamma) \to f_2(\gamma)$, redefine $f_1 := h \circ f_1$ so that $f_1(x) = f_2(x)$ for $x \in d$. Thus, the transversal intersection of the disks D_i is attained via homeomorphisms on the orbit γ on d given by $x \to (f_2|_\gamma^{-1} \circ f_1|_\gamma)(x)$ we have that for $x \in D_1 \cap D_2$ then $f_1(x) = f_2(x)$. Hence, $f : N \to \mathbb{R}$ is given by

$$f(x) = \begin{cases} f_1(x) \text{ if } x \in D_1 \\ f_2(x) \text{ if } x \in D_2 \end{cases}$$

is a Lyapunov function on N. Indeed for each stratum $M(\mathcal{G}) \subset N$, with $\mathcal{G} = \mathcal{R}$ or $\mathcal{D}$, we have that $f\mid_{M(\mathcal{G})}$ is a Lyapunov function on $M(\mathcal{G})$.

Case 4:: If $p \in M(\mathcal{W})$ then a neighborhood N of p in M can be formed by identifying two distinct rays r_1 and r_2 on a disk D. See Figure 1 (d_1) and (d_2). On the disk D the dynamics are defined as in the case $p \in M(\mathcal{R})$, hence, a Lyapunov function f is defined. Define $\overline{f}$ on $N = D/\!\sim$ where $\sim$ is given by:

$$x \sim y \Leftrightarrow x = y \text{ or } f(x) = f(y) \text{ with } x \in r_1 \subset W^s(p),\ y \in r_2 \subset W^s(p).$$

Hence, $\overline{f} : N \to \mathbb{R}$ given by $\overline{f}(\overline{x}) = f(x)$ is a Lyapunov function on N. Indeed for each stratum $M(\mathcal{G}) \subset N$ with $\mathcal{G} = \mathcal{R}$ or $\mathcal{D}$, we have that $\overline{f}\mid_{M(\mathcal{G})}$ is a Lyapunov function on $M(\mathcal{G})$. Similarly, when considering the reverse flow the equivalence relation $\sim$ is taken in $W^u(p)$.

Case 5:: If $p \in M(\mathcal{T})$ then a neighborhood N of p in M is formed by three disks D_i, $i = 1, 2, 3$, that intersect transversally in pairwise distinct diameters that intersect at the point p. See Figure 1 (e_1) and (e_2). On the disks D_i the dynamics are as in $p \in M(\mathcal{R})$, hence a Lyapunov function f_i is defined on each disk D_i. If $\widetilde{N}$ is formed by disks D_i and D_j, intersecting transversally, with p a double crossing in $\widetilde{N}$ then define $\widetilde{f}$ on $\widetilde{N}$, decreasing along the orbits of X_t, as in $p \in M(\mathcal{D})$.

Denote by $d_{ki} \subset D_k$ and $d_{kj} \subset D_k$ the lines where $\widetilde{N}$ and D_k intersect transversally. By adding appropriate constants we assume $\widetilde{f}(p) = f_k(p)$ and by using a diffeomorphism $h : f_k(\gamma) \to \widetilde{f}(\gamma)$, redefine $f_k := h \circ f_k$ on the orbits γ of $d_{ki} \cup d_{kj}$ such that $\widetilde{f}(x) = f_k(x)$.

Thus, the transversal intersection of the disk D_k with $\widetilde{N}$ is obtained and via the homeomorphisms defined on the orbits γ on $d_{ki} \cup d_{kj}$

given by $x \to (\widetilde{f}|_{\gamma}^{-1} \circ f_k|_{\gamma})(x)$ we obtain:

$$\begin{cases} \text{if } x \in D_1 \cap D_2 \text{ then } f_1(x) = f_2(x) \\ \text{if } x \in D_1 \cap D_3 \text{ then } f_1(x) = f_3(x) \\ \text{if } x \in D_2 \cap D_3 \text{ then } f_2(x) = f_3(x) \end{cases}$$

since $\widetilde{f}|_{D_i} = f_i$ and $\widetilde{f}|_{D_j} = f_j$. Thus $f : N \to \mathbb{R}$ given by

$$f(x) = \begin{cases} f_1(x) \text{ if } x \in D_1 \\ f_2(x) \text{ if } x \in D_2 \\ f_3(x) \text{ if } x \in D_3 \end{cases}$$

is a Lyapunov function on N. Indeed, for each stratum $M(\mathcal{G}) \subset N$, with $\mathcal{G} = \mathcal{R}$ or $\mathcal{D}$, we have that $f\,|_{M(\mathcal{G})}$ is a Lyapunov function on $M(\mathcal{G})$.

□

We now prove the existence of a continuous real valued function on a neighborhood of a saddle cone singularity, see Proposition 2.3, as well as, in a neighborhood of a periodic orbit or cycle, see Theorem 2.4 that decreases along orbits of the local flow defined on that neighborhood.

Proposition 2.3. *Let M be a two-manifold with simple singularities. If $p \in M(\mathcal{C})$ is a saddle cone type singularity of a Gutierrez-Sotomayor flow X_t on M then there exists a sufficiently small neighborhood N of p, in M, and a function f on N such that f is continuous and decreases along orbits of X_t on $N - \{p\}$.*

Proof. If $p \in M(\mathcal{C})$ is a saddle cone type singularity in M then a neighborhood N of p in M is formed by a union of two discs D_1 and D_2 identified at the singularity p, $D_1 \vee_p D_2$, see Figure 1.1 (b_1). We can assume, without loss of generality, that via a homeomorphism, the discs D_i, $i = 1, 2$, are on the plane $\mathbb{R}^2$, see Figure 2 (c). In these local coordinates, the dynamics are given by:

$$\begin{cases} \dot{x} = 0 \\ \dot{y} = -x^2 - y^2 \end{cases}$$

Let f_i be the function on D_i given by $f_i(x, y) = y$. Since $\frac{df_i}{dt} = -x^2 - y^2 < 0$ then f_i decreases along the orbits of X_t on D_i. Now let the function f on N be such that:

$$f(x) = \begin{cases} f_1(x) \text{ se } x \in D_1 \\ f_2(x) \text{ se } x \in D_2 \end{cases}$$

Then $f\,|_{N \setminus \{p\}}$ is a continuous function that decreases along the orbits of X_t on $N - \{p\}$. □

Theorem 2.4. *Let M be a two manifold with simple singularities. If $\gamma \subset M(\mathcal{R})$ is a periodic orbit of a Gutierrez-Sotomayor flow X_t on M then there exists a neighborhood, sufficiently small, N of γ, on M, and a function f on N such that f decreases along orbits of X_t on $N \setminus \gamma$ and is constant on γ.*

Proof. If $\gamma \subset M(\mathcal{R})$ then a neighborhood N of γ in M is an annulus.

In local coordinates, the dynamics are given by:

$$\begin{cases} \dot{x} = x - y - x(x^2 + y^2) \\ \dot{y} = x + y - y(x^2 + y^2) \end{cases}$$

Define a function f on N by $f(x,y) = \frac{1}{4}\ln^2(x^2+y^2)$. Since

$$\begin{aligned}\frac{df}{dt} &= \frac{x}{x^2+y^2}(\ln(x^2+y^2))(x-y-x(x^2+y^2)) + \frac{y}{x^2+y^2}(\ln(x^2+y^2))(x+y-y(x^2+y^2))\\ &= (\ln(x^2+y^2))(1-(x^2+y^2)) < 0,\end{aligned}$$

we have that f decreases along orbits of X_t on $N\backslash\gamma$ and is constant on γ. □

2.2. **Lyapunov functions - global construction.**

In this section, we study Gutierrez-Sotomayor flows, X_t, with no periodic orbits and no singular cycles on a compact two-manifold with boundary ∂M (which maybe empty). We assume X_t has only GS simple singularities and is transversal to ∂M. Denote by ∂M^- the boundary on which the flow exits and $\partial M^+ = \partial M\backslash\partial M^-$ the boundary on which the flow enters. In general, ∂M is not connected, however there are some attractors as well as repellers defined on manifolds M with boundary where ∂M connected.

If a point p is on the stratum S of M then the tangent space T_pS is well defined. But if M is singular on S then there are possibly infinitely many "tangent spaces" on M at p and we denote them by generalized tangent spaces. Formally, a *generalized tangent space* at $p \in S$ is any plane Q_p of the form $Q_p = \lim_{p_i\to p} T_{p_i}S'$ where p_i is a sequence of points in a stratum S' whose limit is p. See [5] for more details. The *generalized tangent bundle* Q of M is the set of all pairs (x,v) such that $x \in M$ and $v \in Q_p$. Given a Riemannian metric on $\mathbb{R}^l$, for each $p \in S$, the inner product on the space Q_p splits it in a direct sum $Q_p = T_pS \oplus (T_pS)^\perp$ where $(T_pS)^\perp$ is the orthogonal complement of T_pS in Q_p. This means that, locally, the part of the generalized tangent bundle Q that projects on S splits in a tangent bundle TS and a *generalized normal bundle* $TS^\perp$.

Lemma 2.5. *Let M be a two-manifold with simple singularities. If X_t is a Gutierrez-Sotomayor flow on M then there exists a collection of disjoint branched one-submanifolds B_i of M, $i = 0, 1, \dots, m$, with the following properties:*

1. $B_0 = \partial M^-$, $B_m = \partial M^+$
2. *the flow X_t is transversal to each B_i*
3. *each B_k, $k \neq 0, m$, splits M in two regions whose closures are denoted by G_k and H_k with $G_k \supset G_{k-1}$, $H_k \supset H_{k+1}$ and G_k contains exactly k singularities. Define $G_0 = B_0$, $H_0 = M$, $G_m = M$ and $H_m = B_m$. Hence, for $i = 0,\dots,m$, $G_i \cap H_i = B_i$ and $G_i \cup H_i = M$.*
4. *B_k is the entering boundary of the flow X_t on G_k.*

Proof. By induction on k, let $B_0 = \partial M^-$ and assume we constructed B_{k-1} with

$$M = G_{k-1} \cup H_{k-1}, \quad G_{k-1} \cap H_{k-1} = B_{k-1},$$

G_{k-1} contains $k-1$ singularities and the entering boundary of the flow X_t in G_{k-1} is B_{k-1}. Now we will construct B_k.

Let $B_{k-1}\times[-1,1]$ be a product neighborhood[1] of B_{k-1} (in the case $k=1$ consider $B_{k-1}\times[0,1]$) with $B_{k-1} = B_{k-1}\times 0$, $B_{k-1}\times[0,1] \subset H_{k-1}$ and the flow X_t is transversal to $B_{k-1}\times t$ for each t.

(1) Let $p \in M(\mathcal{G})$, with $\mathcal{G} = \mathcal{R}$, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{W}$ or $\mathcal{T}$, be an attracting simple singularity of X_t.

By Theorem 2.2 we can choose a neighborhood N of p such that X_t is transversal to the boundary. Consider the disjoint union of N with G'_k to obtain G_k where G'_k is obtained by gluing to G_{k-1} the collar of B_{k-1} (contained in H_{k-1}), see Figure 3.

[1]Bicollar of B_{k-1} and a collar in the case $k=1$.

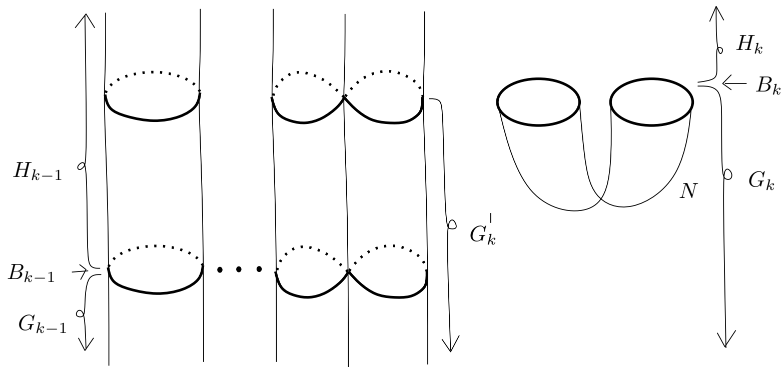

FIGURE 3. Construction of B_k

Hence, $B_k = \partial G_k$ is a disjoint union of branched one-manifolds with one more component than B_{k-1} if $p \in M(\mathcal{G})$ where $\mathcal{G} = \mathcal{R}$, $\mathcal{D}$, $\mathcal{W}$ or $\mathcal{T}$, and with two more components than B_{k-1} if $p \in M(\mathcal{C})$.

(2) Let $p \in M(\mathcal{G})$, with $\mathcal{G} = \mathcal{R}$, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{W}$ or $\mathcal{T}$, be a singularity of X_t which is not an attractor or repeller. We first construct S_ϵ for each singularity $p \in M(\mathcal{G})$ as the image of the exponential map $Exp : U \subset TM \to M$.

(a) If $p \in M(\mathcal{R}) \cup M(\mathcal{C})$ then by Theorem 2.2 we can choose a neighborhood N of p, a real valued function f on N and $\delta > 0$ such that the disk bounded by

$$f^{-1}(\delta) \cap W^s(p) = \widetilde{W}$$

is contained in N. Let E_ϵ be the normal bundle of $W^s(p)\backslash\{p\}$ in $M(\mathcal{R})$ restricted to $\widetilde{W}$ with vectors of magnitude $\leq \epsilon$. Denote by S_ϵ the image of E_ϵ under the exponential map.

(b) If $p \in M(\mathcal{W})$ then by Theorem 2.2 we can choose a neighborhood N of p, a real valued function f on N and $\delta > 0$ such that the disks bounded by

$$f^{-1}(\delta) \cap W^s(p) = \widetilde{W}$$

are in N. Let E_ϵ be the generalized normal bundle of $W^s(p)\backslash\{p\}$ in M restricted to $\widetilde{W}$ with vectors of magnitude $\leq \epsilon$. Denote by S_ϵ the image of E_ϵ under the exponential map restricted to each Q_p.

(c) If $p \in M(\mathcal{D})$ then by Theorem 2.2 we can choose a neighborhood N of p, a real valued function f on N and $\delta > 0$ such that the disks bounded by

$$f^{-1}(\delta) \cap W^s(p) \cap \overline{N\backslash W^s(p)} = \widetilde{W}$$

are in N. Let E_ϵ be the generalized normal bundle of $W^s(p) \cap \overline{N\backslash W^s(p)}$ in $\overline{N\backslash W^s(p)}$ restricted to $\widetilde{W}$ with vectors of magnitude $\leq \epsilon$. Denote by S_ϵ the image of E_ϵ under the exponential map restricted to each Q_p.

(d) if $p \in M(\mathcal{T})$ then by Theorem 2.2 we can choose a neighborhood N of p, a real valued function f on N and $\delta > 0$ such that the disks bounded by

$$f^{-1}(\delta) \cap W^s(p) \cap \overline{N\backslash W^s(p)} = \widetilde{W}$$

are in N. Let E_ϵ be the generalized normal bundle of $W^s(p) \cap \overline{N\backslash W^s(p)}$ in $\overline{N\backslash W^s(p)}$ restricted to $\widetilde{W}$ with vectors of magnitude $\leq \epsilon$. Denote by S_ϵ the image of E_ϵ under the exponential map restricted to each Q_p.

Choose ϵ sufficiently small such that S_ϵ is transversal to X. By the continuity of the flow X_t of X we can define $T : S_\epsilon \backslash \widetilde{W} \to \widetilde{V}$ which maps $x \in S_\epsilon \backslash \widetilde{W}$ to the point on the orbit of x which intersects $\widetilde{V}$.

Now, define a C^∞ embedding $F : \partial S_\epsilon \times [-1, 1] \to M$ by $F(x, -1) = x$, $F(x, 1) = T(x)$ and $F(x, t)$ is on the orbit that joins x to $T(x)$ and the distance from x to $F(x, t)$ is proportional to t. Extend F to a C^∞ embedding of $\partial S_\epsilon \times [-2, 2]$ that sends $x \times [-2, 2]$ to a regular orbit, for each x.Fix a Riemannian metric on $M(\mathcal{R})$ and let $v(p, t)$ be the unit normal vector field on the image of F with orientation given by (induced by) the vectors on ∂S_ϵ pointing outwards on $\widetilde{W}$. Let $\eta > 0$, be a small constant and $F_\eta(p, t)$ be the point at a distance ηt of $F(p, t)$ along the geodesic determined by $v(p, t)$, see Figure 4.

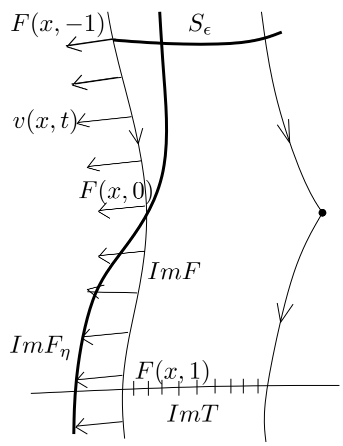

FIGURE 4. Construction of F_η

Choose small η such that the image of F_η, imF_η, is disjoint from the image of T, imT. Also, we have that X_t is transversal to imF_η, and $imF_\eta \cap S_\epsilon$, $imF_\eta \cap \widetilde{V}$ are diffeomorphic to $imF \cap S_\epsilon$, $imF \cap \widetilde{V}$, respectively.

In this way, we obtain a one-dimensional singular submanifold B'_k of M made up of:

- the part of S_ϵ bounded by $imF_\eta \cap S_\epsilon$;
- $\widetilde{V}$ except for regions bounded by $imF_\eta \cap \widetilde{V}$ that contains $W^u(p) \cap \widetilde{V}$;
- the part of imF_η bounded by $imF_\eta \cap S_\epsilon$ and $imF_\eta \cap \widetilde{V}$.

Thus, we have that X_t is transversal to B'_k. We verify that $M \backslash B'_k = G'_k \cup H'_k$ with G'_k containing G_{k-1} and the singular point p. Moreover, G'_k differs from G_k since $B'_k = \partial G'_k$ is not a differentiable submanifold, *i.e.,* differentiability fails along $imF_\eta \cap S_\epsilon$ and $imF_\eta \cap \widetilde{V}$. This can be smoothened easily in order to obtain the desired G_k and B_k. See Figure 5.

(3) Finally, if $p \in M(\mathcal{G})$, with $\mathcal{G} = \mathcal{R}$, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{W}$ or $\mathcal{T}$, is a repeller singularity of X_t then by Theorem 2.2 choose a neighborhood N of p whose boundary is transversal to X_t. Thus, $G_k = G_{k-1} \cup B_{k-1} \times [0, 1] \cup N$.

□

Lemma 2.6. *Let M be a two-manifold with simple singularities. If X_t is a Gutierrez-Sotomayor flow with only one singularity p then there is a Lyapunov function f on M such that f has value $c - \frac{1}{2}$ on ∂M^-, $c + \frac{1}{2}$ on ∂M^+ and $f(p) = c$.*

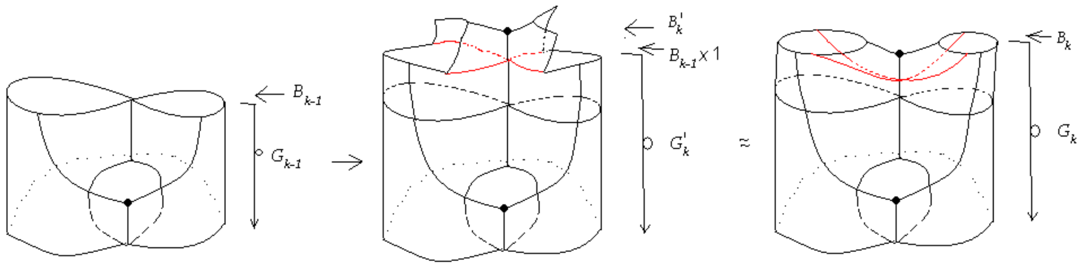

FIGURE 5. Smoothening B'_k to obtain B_k on a Whitney block

Proof. First define a function in a neighborhood of $W^s(p) \cup W^u(p)$. Let N be a neighborhood of p and f a function on N as in Theorem 2.2 and assume $f(p) = c$ by adding appropriate constants. Then let $f^{-1}(c+\delta) \cap N = R^+$, $f^{-1}(c-\delta) \cap N = R^-$, with δ chosen as in the previous lemma[2], $R^+_\epsilon = \{(u,v) \in R; \|v\| \leq \epsilon\}$ and $R^-_\epsilon = \{(u,v) \in R^-; \|u\| \leq \epsilon\}$.

Fix a Riemannian metric on $\mathbb{R}^l$ and take $\epsilon = \frac{1}{10}$. For $x \in R^+_\epsilon$ redefine f on $X_t(x)$, $t \leq 0$, such that $f(X_0(x)) = c + \delta$, $f(y) = c + \frac{1}{2}$ where y is the point of $X_t(x)$ that intersects ∂M^+. Define f proportional to the arclength of the points on the orbit that connect $X_0(x)$ and y. In this way, we obtain a function f in a neighborhood of $W^s(p)$ satisfying the required conditions on the boundary, although non-differentiable on $f^{-1}(c+\delta)$. We can smoothen f, see [7], so that it is C^∞ on $f^{-1}(c+\delta)$.

In a similar fashion, using R^-_ϵ, we obtain a real-valued function f defined in a neighborhood Q of $W^u(p)$ as well as in a neighborhood of $W^s(p)$, satisfying $f(Q \cap \partial M^-) = c - \frac{1}{2}$. Hence, we obtain the desired function f in an open neighborhood P of $W^s(p) \cup W^u(p)$. Without loss of generality, we can assume that if $x \in P$ then $X_t(x) \in P$, $\forall t$.

Now extend f to M. Choose $U \subset \partial M^- \cap P$ a compact neighborhood of $W^u(p) \cap \partial M^-$. Let λ be a C^∞ real valued function on ∂M^- satisfying $0 \leq \lambda \leq 1$ with $\lambda = 1$ on U and $\lambda = 0$ on $\partial M^- \backslash P \cap \partial M^-$. For $x \in M \backslash (W^s(p) \cup W^u(p))$ let $l(x)$ be the length of the orbit passing through x, $v(x)$ arclength of the orbit joining $\{X_t(x)\} \cap \partial M^-$ to x and $g(x) = c - \frac{1}{2} + \frac{v(x)}{l(x)}$. Hence, the function $\overline{\lambda} f + (1 - \overline{\lambda}) g$ on M is the desired function where $\overline{\lambda}(x) = \lambda(X_t(x) \cap \partial M^-)$ or equals one if $X_t(x)$ does not intersect ∂M^-. □

Theorem 2.7. *Let M be a compact two-manifold with simple singularities. If X_t is a Gutierrez-Sotomayor flow on M then there exists a Lyapunov function f on M.*

Proof. Consider $G_k - G_{k-1}$, $\forall k$, defined in Lemma 2.5. Let f_k be the function in Lemma 2.6 defined on the closure of $G_k - G_{k-1}$. Juxtaposing the f_k we obtain a function f well defined on M and smooth [3] in a neighborhood of $B_1, \ldots, B_{m-1}$. Therefore, the desired Lyapunov function is obtained. □

3. The Conley Index

In this section, we compute the Conley homotopy index and homology index of simple singularities of a Gutierrez-Sotomayor flow X_t on M. We also prove a result relating the singularities on the regular and singular parts of X_t with the homology of M.

[2]The disk bounded by $f^{-1}(c+\delta) \cap W^s(p)$ is in N.

[3]As in the proof of Lemma 2.6.

A compact set $N \subset M$ is an *isolating neighborhood* if

$$Inv(N) := \{x \in N; X_t(x) \subset N,\ \forall t\} \subset int(N),$$

where $int(N)$ denotes the interior of N. Λ is an *isolated invariant set* if $\Lambda = Inv(N)$ for some isolating neighborhood N.

If Λ is an isolated invariant set, a topological pair of spaces[4] (N, L) is an *index pair* for Λ if:

(1) $\Lambda = Inv(cl(N \backslash L))$ and $N \backslash L$ is an isolating neighborhood for Λ.

(2) L is *positively invariant* in N, *i.e.,* given $x \in L$ and $X_t(x) \subset N$ for $t \in [0, t_0]$ then $X_t(x) \subset L$ for $t \in [0, t_0]$.

(3) L is an *exit set* for N; *i.e.,* given $x \in N$ and $t_1 > 0$ such that $X_{t_1}(x) \notin N$ then there exists $t_0 \in [0, t_1]$ such that $X_t(x) \subset N$, for $t \in [0, t_0]$, and $X_{t_0}(x) \in L$.

In [2], Conley proves the existence of an index pair (N, L) for an isolated invariant set Λ. Furthermore, if (N, L) and (N', L') are index pairs for an isolated invariant set Λ then $(N/L, [L])$ has the same homotopy type as $(N'/L', [L'])$.

In what follows we define, the homotopy index as the homotopy type of the pointed space $(N/L, [L])$. Since homology is an invariant of homotopic spaces thus the homology index is well defined.

Definition 3.1. *We define:*

(1) *The Conley homotopic index of* Λ*,* $h(\Lambda)$*, is the homotopy type of the pointed space* $(N/L, [L])$ *where* (N, L) *is an index pair for* Λ*.*

(2) *The Conley homology index of* Λ *is defined by* $CH_*(\Lambda) := H_*(h(\Lambda))$ *where* H_* *denotes the homology on* $\mathbb{Z}$*.*

(3) *The numerical Conley indices of* Λ *are defined as the ranks of the Conley homology indices of* Λ*,* $h_* = rank CH_*(\Lambda)$*.*

In order to compute the Conley homology indices we make use of the isomorphism:

$$\widetilde{H}_n(X \vee Y) \approx \widetilde{H}_n(X) \oplus \widetilde{H}_n(Y)$$

if the base points of X and Y which are identified in $X \vee Y$ are deformation retracts of neighborhoods $U \subset X$ and $V \subset Y$.

If $p \in M$ is a singularity of a Gutierrez-Sotomayor flow X_t and N a sufficiently small neighborhood, as in the proof of Lemma 2.2. We say that p is of:

- type $\mathbf{a}$ if $p \in M(\mathcal{G})$, where $\mathcal{G} = \mathcal{R}$, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{W}$ or $\mathcal{T}$, is an attracting singularity.
- type $\mathbf{s}$ if $p \in M(\mathcal{R}) \cup M(\mathcal{C})$ is neither an attracting or repelling singularity.
- type $\mathbf{r}$ if $p \in M(\mathcal{G})$, where $\mathcal{G} = \mathcal{R}$, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{W}$ or $\mathcal{T}$, is a repelling singularity.
- type $\mathbf{s_u}$ if $p \in M(\mathcal{W})$ is a saddle singularity on a bidimensional disc with the unstable manifold identified to the fold.
- type $\mathbf{s_s}$ if $p \in M(\mathcal{W})$ is a saddle singularity on a bidimensional disc with the stable manifold identified to the fold.
- type $\mathbf{sa}$ if $p \in M(\mathcal{D})$ and N is formed by a sink and a saddle.
- type $\mathbf{sr}$ if $p \in M(\mathcal{D})$ and N is formed by a source and a saddle identified at the fold.
- type $\mathbf{ss_u}$ if $p \in M(\mathcal{D})$ and N is formed by two saddles with their unstable manifolds identified to the fold.
- type $\mathbf{ss_s}$ if $p \in M(\mathcal{D})$ and N is formed by two saddles with their stable manifolds identified to the fold.

[4]A *topological pair of spaces* is an ordered pair (N, L) of spaces such that L is a closed subspace of N.

- type **ssa** if $p \in M(\mathcal{T})$ and N is formed by a sink and two saddles.
- type **ssr** if $p \in M(\mathcal{T})$ and N is formed by a source and two saddles.

3.1. **Conley index of GS Singularities.** In the next theorem we compute the Conley homotopy index, as well as, the ranks of the homology indices.

Theorem 3.2. *Let M be a two-manifold with simple singularities and X_t a Gutierrez-Sotomayor flow on M. Let p be a singularity of X_t with type specified in the table below. Then, the numerical Conley index of each type of singularity is as given in the table.*

Type	$p \in M(\mathcal{R})$	$p \in M(\mathcal{C})$	$p \in M(\mathcal{W})$	$p \in M(\mathcal{D})$	$p \in M(\mathcal{T})$
a	S^0	S^0	S^0	S^0	S^0
	$(1,0,0)_{\mathcal{R}}$	$(1,0,0)_{\mathcal{C}}$	$(1,0,0)_{\mathcal{W}}$	$(1,0,0)_{\mathcal{D}}$	$(1,0,0)_{\mathcal{T}}$
s	S^1	S^1	—	—	—
	$(1,0,0)_{\mathcal{R}}$	$(0,1,0)_{\mathcal{C}}$	—	—	—
$\mathbf{s_u}$	—	—	$\bar{0}$	—	—
	—	—	$(0,0,0)_{\mathcal{W}}$	—	—
$\mathbf{s_s}$	—	—	S^1	—	—
	—	—	$(0,1,0)_{\mathcal{W}}$	—	—
sa	—	—	—	S^1	—
	—	—	—	$(0,1,0)_{\mathcal{D}}$	—
sr	—	—	—	S^2	—
	—	—	—	$(0,0,1)_{\mathcal{D}}$	—
$\mathbf{ss_u}$	—	—	—	S^1	—
	—	—	—	$(0,1,0)_{\mathcal{D}}$	—
$\mathbf{ss_s}$	—	—	—	$\vee_{i=1}^{3} S^1$	—
	—	—	—	$(0,3,0)_{\mathcal{D}}$	—
ssa	—	—	—	—	S^1
	—	—	—	—	$(0,1,0)_{\mathcal{T}}$
ssr	—	—	—	—	S^2
	—	—	—	—	$(0,0,1)_{\mathcal{T}}$
r	S^2	$S^2 \vee S^2 \vee S^1$	$S^2 \vee S^2$	$\vee_{i=1}^{3} S^2$	$\vee_{i=1}^{7} S^2$
	$(0,0,1)_{\mathcal{R}}$	$(0,1,2)_{\mathcal{C}}$	$(0,0,2)_{\mathcal{W}}$	$(0,0,3)_{\mathcal{D}}$	$(0,0,7)_{\mathcal{T}}$

Proof. If p is a singularity of X_t, we choose an index pair (N, L) for p in M and calculate the Conley homotopic index $h(p)$. The homology $CH_i(p)$ has a factor $\mathbb{Z}$ for each S^i of the homotopical index, thus the Conley numerical index (h_0, h_1, h_2) in each case of Theorem 3.2 is obtained. See Figures 6 through 22.

(1) If $p \in M(\mathcal{R})$, let N be a closed disk and $L = \partial N^-$ the exiting set of N. Thus, the Conley homotopy index of p is S^0 (S^1 or S^2) if p is an attractor (saddle or repeller) singularity.

(2) If $p \in M(\mathcal{C})$, a neighborhood N of p in M is formed by two disks D_1 and D_2 centered at p such that $D_1 \cap D_2 = \{p\}$.
   (a) If p is of type **a** then $L = \emptyset$ and thus is identified to a point. On the other hand, it is easy to see that the double cone, when retracted along the stable manifold of N, has the homotopy type of a point. Hence, $h(p) = S^0$. See Figure 6.
   (b) If p is of type **s** then $w^u(p) \cap \partial N = \{x_1, x_2\}$ where $x_i \in \partial D_i$, $i = 1, 2$. Let $C_i \subset \partial D_i$, $i = 1, 2$, be the two arcs from which the flow exits, then $x_i \in C_i$, $i = 1, 2$ and $L = C_1 \cup C_2$ is the exit set for N. Collapsing L to a point and retracting along

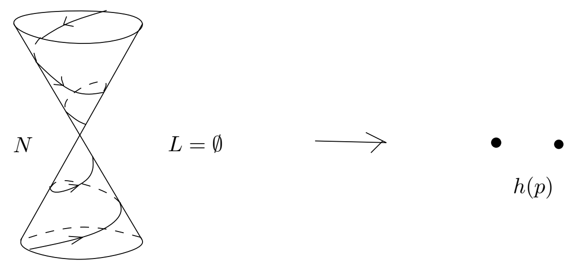

FIGURE 6. Conley index of a singularity of type **a** in $M(\mathcal{C})$

the stable manifold of N we conclude that N/L has the homotopy type of S^1, *i.e.*, $h(p) = S^1$. See Figure 7.

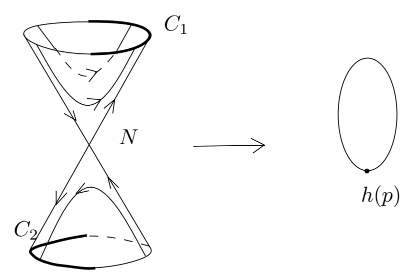

FIGURE 7. Conley index of a singularity of type **s** in $M(\mathcal{C})$

(c) If p is of type **r** then $L = \partial N = \partial D_1 \cup \partial D_2$. Collapsing L to a point we conclude that N/L has the homotopy type of $S^2 \vee S^2 \vee S^1$, *i.e.*, $h(p) = S^2 \vee S^2 \vee S^1$. See Figure 8.

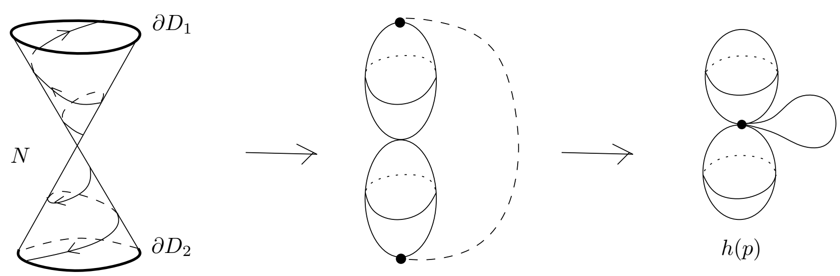

FIGURE 8. Conley index of a singularity of type **r** in $M(\mathcal{C})$

(3) If $p \in M(\mathcal{D})$, a neighborhood N of p in M is formed by two disks D_i, $i = 1, 2$, that intersect transversally along two diameters d_1 and d_2 in D_1 and D_2 respectively.
   (a) If p is of type **a** then $L = \emptyset$ and hence is identified to a point. On the other hand, it is easy to see that by retracting the stable manifold on N it has the homotopy type of a point, hence, $h(p) = S^0$. See Figure 9.
   (b) If p is of type **r** then $L = \partial N = \partial D_1 \cup \partial D_2$ where ∂D_1 and ∂D_2 intersect transversally at two points. Collapsing L to a point we conclude that N/L has the homotopy type of $S^2 \vee S^2 \vee S^2$, *i.e.*, $h(p) = S^2 \vee S^2 \vee S^2$. See Figure 10.

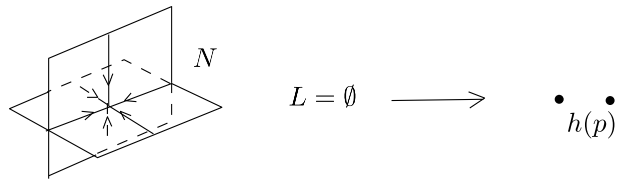

FIGURE 9. Conley index of a singularity of type **a** in $M(\mathcal{D})$

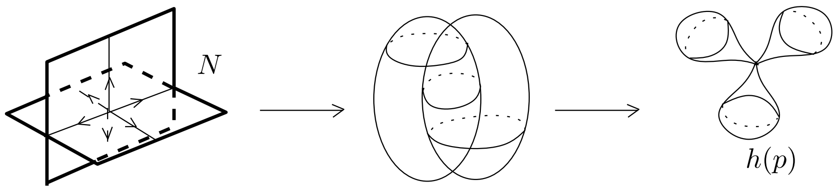

FIGURE 10. Conley index of a singularity of type **r** in $M(\mathcal{D})$

(c) If p is of type **sa** then $w^u(p) \cap \partial N = \{x_1, x_2\}$ where $x_1, x_2 \in \partial D_i$ and D_i is the disk that contains the saddle. Let $C_1, C_2 \subset \partial D_i$ be the two arcs from which the flow exits N hence $x_i \in C_i$, $i = 1, 2$ and $L = C_1 \cup C_2$ is the exit set for N. Collapsing L to a point and retracting along the stable manifold of N we conclude that N/L has the homotopy type of S^1, *i.e.*, $h(p) = S^1$. See Figure 11.

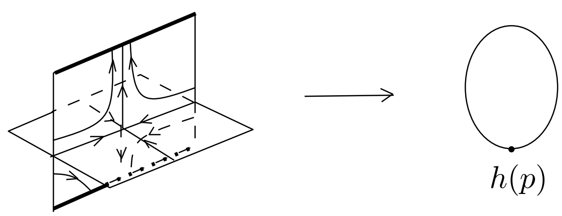

FIGURE 11. Conley index of a singularity of type **sa** in $M(\mathcal{D})$

(d) If p is of type **sr** then $w^u(p) \cap \partial N = \partial D_i$ where D_i is the disk that contains the repeller. Let $C_1, C_2 \subset \partial D_j$, $j \neq i$, be the two transversal arcs to ∂D_i from where the flow exits hence $L = \partial D_i \cup C_1 \cup C_2$ is the exit set for N. Collapsing L to a point and retracting along the stable manifold of N we conclude that N/L has the homotopy type of S^2, *i.e.*, $h(p) = S^2$. See Figure 12.

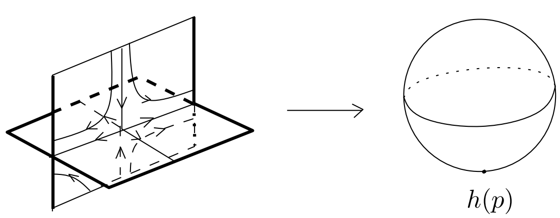

FIGURE 12. Conley index of a singularity of type **sr** in $M(\mathcal{D})$

(e) If p is of type $\mathbf{ss_u}$ then

$$w^u(p) \cap \partial N = \{x_1, x_2\},$$

where $x_1, x_2 \in \partial D_1$ and $x_1, x_2 \in \partial D_2$. Let $B_1, B_2 \subset \partial D_1$ and $C_1, C_2 \subset \partial D_2$ be the arcs from where the flow exits, $B_i \pitchfork C_i = \{x_i\}$ and $L = (B_1 \cup C_1) \sqcup (B_2 \cup C_2)$ is the exit set for N. Collapsing L to a point and retracting along the stable manifold of N we conclude that N/L has the homotopy type of S^1, *i.e.*, $h(p) = S^1$. See Figure 13.

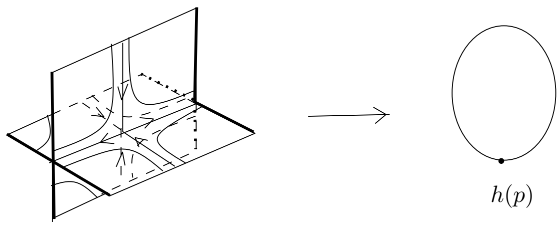

FIGURE 13. Conley index of a singularity of type $\mathbf{ss_u}$ in $M(\mathcal{D})$

(f) If p is of the type $\mathbf{ss_s}$ then $w^u(p) \cap \partial N = \{x_1, x_2, y_1, y_2\}$ where $x_1, x_2 \in \partial D_1$ and $y_1, y_2 \in \partial D_2$. Let $B_1, B_2 \subset \partial D_1$ and $C_1, C_2 \subset \partial D_2$ be the arcs from where the flow exits, $x_i \in B_i$, $i = 1, 2$, $y_i \in C_i$, $i = 1, 2$, and $L = B_1 \sqcup B_2 \sqcup C_1 \sqcup C_2$ is the exit set for N. Collapsing L to a point and retracting along the stable manifold of N we conclude that N/L has the homotopy type of $S^1 \vee S^1 \vee S^1$, *i.e.*, $h(p) = \bigvee_{i=1}^3 S^1$. See Figure 14.

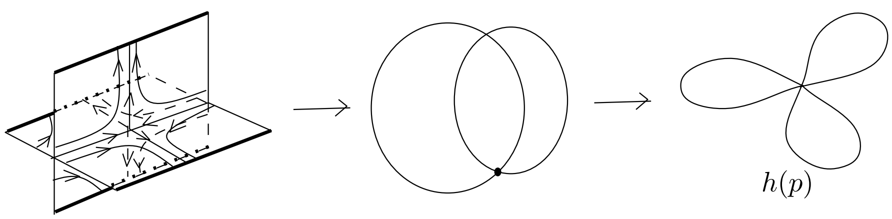

FIGURE 14. Conley index of a singularity of type $\mathbf{ss_s}$ in $M(\mathcal{D})$

(4) If $p \in M(\mathcal{W})$, a neighborhood N of p in M is a disk D with two distinct rays r_1 and r_2 identified.
  (a) If p is of type $\mathbf{a}$ then $L = \emptyset$ and hence is identified to a point. On the other hand, it is easy to see that by retracting the stable manifold of N, it has the homotopy type of a point, hence, $h(p) = S^0$.
  (b) If p is of type $\mathbf{r}$ then $L = \partial N$ is homeomorphic to a figure "eight". Collapsing L to a point we conclude that N/L has the homotopy type of $S^2 \vee S^2$, *i.e.*, $h(p) = S^2 \vee S^2$. See Figure 16.
  (c) If p is of type $\mathbf{s_u}$ then $w^u(p) \cap \partial N = \{x\}$. Let $C_1, C_2 \subset \partial N$ be the arcs from where the flow exits, hence, $C_1 \pitchfork C_2 = \{x\}$ and $L = C_1 \cup C_2$ is the exit set for N. Collapsing L to a point and retracting along the stable manifold of N we conclude that N/L has the homotopy type of a point, *i.e.*, $h(p) = \overline{0}$. See Figure 17.

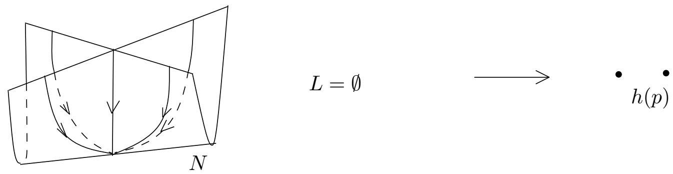

FIGURE 15. Conley index of a singularity of type $\mathbf{a}$ in $M(\mathcal{W})$

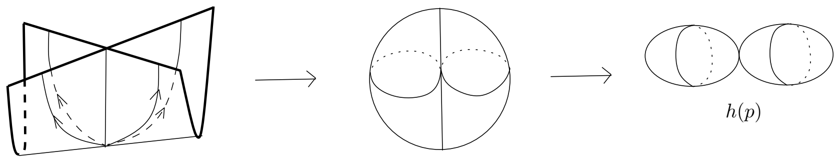

FIGURE 16. Conley index of a singularity of type $\mathbf{r}$ in $M(\mathcal{W})$

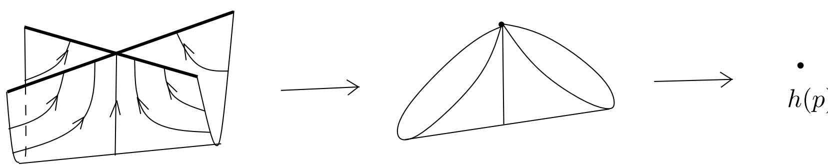

FIGURE 17. Conley index of a singularity of type $\mathbf{s_u}$ in $M(\mathcal{W})$

(d) If p is of type $\mathbf{s_s}$ then $w^u(p) \cap \partial N = \{x_1, x_2\}$. Let $C_i \subset \partial N$, $i = 1, 2$, be the arcs from where the flow exits, hence, $x_i \in C_i$, $i = 1, 2$ and $L = C_1 \cup C_2$ is the exit set for N. Collapsing L to a point and retracting along the stable manifold of N we conclude that N/L has the homotopy type of S^1, *i.e.,* $h(p) = S^1$. See Figure 18.

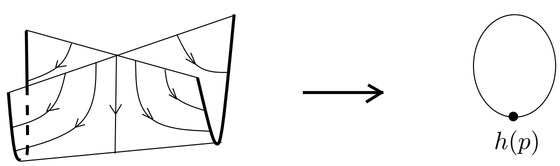

FIGURE 18. Conley index of a singularity of type $\mathbf{s_s}$ in $M(\mathcal{W})$

(5) If $p \in M(\mathcal{T})$, a neighborhood N of p in M is formed by three disks D_i, $i = 1, 2, 3$, that intersect transversally in pairwise disjoint diagonals that go through the point p.

(a) If p is of the type $\mathbf{a}$ then $L = \emptyset$ and thus is identified to a point. On the other hand,it is easy to see that by retracting the stable manifold of N, it has the homotopy type of a point, hence, $h(p) = S^0$.

(b) If p is of type $\mathbf{r}$ then $L = \partial N = \partial D_1 \cup \partial D_2 \cup \partial D_3$ where ∂D_1, ∂D_2 and ∂D_3 intersect transversally pairwise at two points. Collapsing L to a point we conclude that N/L has the homotopy type of $\vee_{i=1}^{7} S^2$, *i.e.,* $h(p) = \vee_{i=1}^{7} S^2$. See Figure 20.

(c) If p is of type $\mathbf{ssa}$ then $w^u(p) \cap \partial N = \{x_1, x_2\}$ where $x_1, x_2 \in \partial D_2$ and $x_1, x_2 \in \partial D_3$. Let $B_1, B_2 \subset \partial D_2$ and $C_1, C_2 \subset \partial D_3$ be the arcs from where the flow exits, hence, $B_i \pitchfork C_i = \{x_i\}$ and $L = (B_1 \cup C_1) \sqcup (B_2 \cup C_2)$ is the exit set for N. Collapsing L

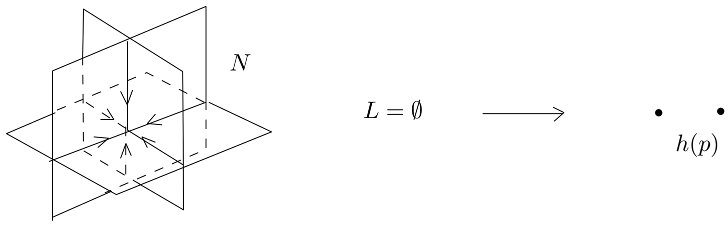

FIGURE 19. Conley index of a singularity of type **a** in $M(\mathcal{T})$

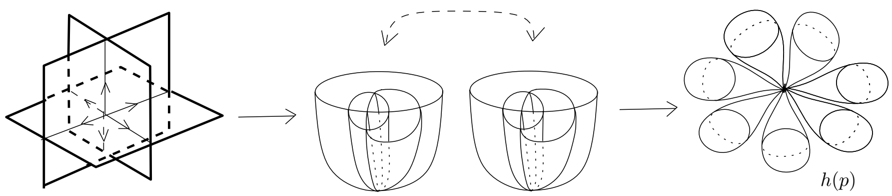

FIGURE 20. Conley index of a singularity of type **r** in $M(\mathcal{T})$

to a point and retracting along the stable manifold of N we conclude that N/L has the homotopy type of S^1, *i.e.,* $h(p) = S^1$. See Figure 21.

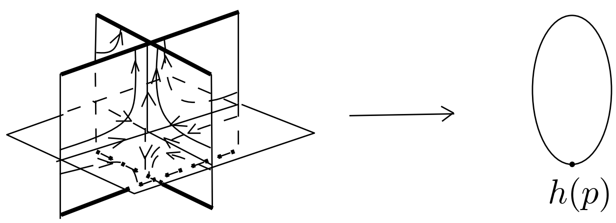

FIGURE 21. Conley index of a singularity of type **ssa** in $M(\mathcal{T})$

(d) If p is of type **ssr** then $w^u(p) \cap \partial N = \partial D_1$ where D_1 is the disk which contains the repeller. Let $B_1, B_2 \subset \partial D_2$ and $C_1, C_2 \subset \partial D_3$ transversal arcs to ∂D_1 from where the flow exits, hence, $L = \partial D_1 \cup B_1 \cup B_2 \cup C_1 \cup C_2$ is the exit set for N. Collapsing L to a point and retracting along the stable manifold of N we conclude that N/L has the homotopy type of S^2, *i.e.,* $h(p) = S^2$. See Figure 22.

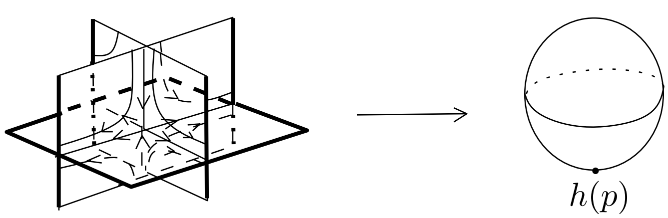

FIGURE 22. Conley index of a singularity of type **ssr** in $M(\mathcal{T})$

It is straightforward to compute the homology of the Conley indices $CH_*(\Lambda)$ and its ranks $h_* = \text{rank } CH_*(\Lambda)$ in each case of $\Lambda = \{p\}$ within this proof. Thus, this numerical Conley index appears in the table as

$$(h_0, h_1, h_2) = (\text{rank } CH_0(\Lambda),\ \text{rank } CH_1(\Lambda),\ \text{rank } CH_2(\Lambda)).$$

□

3.2. **Euler type Characteristic Formulas for GS manifolds.** Let $X = |K|$ be a topological space of dimension n. Define α_j as the number of j-simplices of K. The Poincaré Theorem asserts that the sum $\displaystyle\sum_{j=0}^{n}(-1)^j\alpha_j$ is independent of the simplicial complex K, such that $X = |K|$. This number is the *Euler-Poincaré Characteristic* and is denoted by $\chi(X)$. Also, Poincaré asserts the equality

$$\chi(X) = \sum_{j=0}^{n}(-1)^j\beta_j,$$

where β_j where the rank of $H_j(K)$ is the j-th Betti number of K.

For example, $\chi(S^2) = 2$, $\chi(\text{pinched sphere}) = 3$, $\chi(\text{pinched torus}) = 1$,

$$\chi(\text{sine torus or torus with a fold}) = 1$$

and $\chi(\text{crosscap}) = 2$. See Figure 24 and Figure 25.

We next present the Morse-Conley inequalities for manifolds with simple singularities. We make use of the ranks of the homology indices computed in Theorem 3.2.

Proposition 3.3. *Let M be a two-manifold with simple singularities and X_t a Gutierrez-Sotomayor flow on M with limit set $\mathfrak{L} = \bigcup_{i=1}^{m} L_i$. If (h_0^i, h_1^i, h_2^i) is the numerical Conley index of L_i then*

$$\sum_{i=1}^{m}(h_0^i - h_1^i + h_2^i) = \chi(M) \tag{1}$$

where $\chi(M)$ is the Euler characteristic of M.

Proof. Let f be a Lyapunov function associated to X_t and $G_k \subset M$ as in the proof of Theorem 2.7. Hence, $G_0 \subset G_1 \subset \cdots \subset G_m$ such that (G_i, G_{i-1}) is an index pair for L_i. Consider the long exact sequence of the pair (G_i, G_{i-1})

$$\cdots \stackrel{p_j}{\to} H_j(G_i, G_{i-1}) \stackrel{\partial_j}{\to} H_{j-1}(G_{i-1}) \stackrel{i_*}{\to} H_{j-1}(G_i) \stackrel{p_{j-1}}{\to} H_{j-1}(G_i, G_{i-1}) \stackrel{\partial_{j-1}}{\to} \cdots$$

By exactness,

$$\begin{aligned} \dim \text{ im } (p_j) &= \dim \ker (\partial_j) = \dim H_j(G_i, G_{i-1}) - \dim \text{ im } (\partial_j) \\ &= \dim H_j(G_i, G_{i-1}) - \dim \ker (i_*) \end{aligned}$$

$$\begin{aligned} \dim \text{ im } (p_{j-1}) &= -\dim \ker (p_{j-1}) + \dim H_{j-1}(G_i) \\ &= -\dim \text{ im } (i_*) + \dim H_{j-1}(G_i). \end{aligned}$$

Thus,

$$\begin{aligned} &\dim \text{ im } (p_j) + \dim \text{ im } (p_{j-1}) = \\ &\dim H_j(G_i, G_{i-1}) - \dim \ker (i_*) - \dim \text{ im } (i_*) + \dim H_{j-1}(G_i) = \\ &\dim H_j(G_i, G_{i-1}) - \dim H_{j-1}(G_{i-1}) + \dim H_{j-1}(G_i). \end{aligned}$$

Since $CH_*(L_i) \cong H_*(G_i, G_{i-1})$, then $h_j(L_i) = \dim H_j\,(G_i, G_{i-1})$. Thus,

$$\dim \text{ im } (p_j) + \dim \text{ im } (p_{j-1}) = h_j(L_i) - \beta_{j-1}(G_{i-1}) + \beta_{j-1}(G_i).$$

For fixed i, consider the alternated sum over j:

$$\sum_{j=0}^{2}(-1)^j h_j(L_i) + \sum_{j=0}^{3}(-1)^j(\beta_{j-1}(G_i) - \beta_{j-1}(G_{i-1})) = 0.$$

Now, consider the sum of the above expression for $i = 1, \ldots, m$

$$\sum_{i,j}(-1)^j h_j(L_i) + \sum_{j=0}^{3}(-1)^j(\beta_{j-1}(G_m)) = 0.$$

Since $G_m = M$, we obtain the desired result $\chi(M) = \sum_{i,j}(-1)^j h_j^i$, for $i = 1, \ldots, m$ and $j = 0, 1, 2$. $\square$

3.3. **Conley Index restricted to the Strata.** The calculations in the previous section were realized considering isolating neighborhoods of a simple singularity in M. However, one may also compute the Conley indices of the simple singularities of X_t with respect to subspaces of M. In particular, with respect to the singular part of a stratification of M.

A two-manifold with simple singularities M equipped with a partition $\{M(\mathcal{G}), \mathcal{G}\}$ is a stratified manifold. One can define a partition, by distinguishing the regular part from the singular part, as follows:

- $\mathcal{R}$ is the union of the strata of dimension 2.
- $\mathcal{S} = M \backslash \mathcal{R}$ is the union of the strata of dimension 0 and 1.

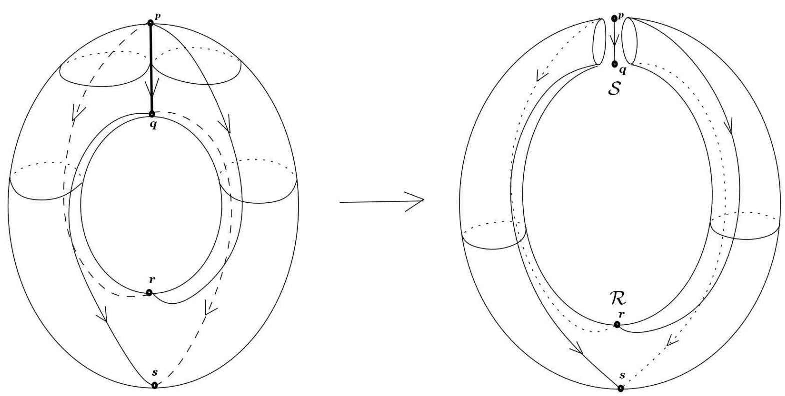

FIGURE 23. Stratification of the sine torus.

A stratification for $M = \mathcal{R} \sqcup \mathcal{S}$, where $\sqcup$ is a disjoint union. Hence, all points in $\mathcal{S}$ are singular points of the stratification. Observe that $p \in \mathcal{S}$ is not necessarily a singular point of the manifold nor of the flow. In the same way, a singular point of the manifold is not necessarily a singular point of the flow.

Consider the example in Figure 23. The points p, q, r, s are singularities of the flow. All points in $\mathcal{S}$ are singular points of the stratification as well as singular points of the manifold. In the example in Figure 24 (left), $\mathcal{S}$ is the figure "eight" and on it there are 5 singularities of the flow and on $\mathcal{R}$ there are an additional 4 singularities of the flow. All points on the figure "eight" are singular points of the stratification but only the cone point is a singular point of the manifold.

Consider the polar flow on S^2, one repeller and one attractor. Define the singular part, $\mathcal{S}$, to be a great circle C that contains these two singularities. The flow has two singularities, north and south pole. All points in C are singular points of the stratification and there are no singular points of the manifold.

Note that in this last example, a neighborhood U of $\mathcal{S}$, contains orbits of the flow that are both entering and exiting U. We will not consider this type of stratification. We will require that a neighborhood U of $\mathcal{S}$ is either an attracting or repelling basin.

Definition 3.4. *Let $\mathcal{E}$ be a stratification of M and $U_{\mathcal{S}}$ a tubular neighborhood of $\mathcal{S}$, the singular part of the stratification $\mathcal{E}$, of M. We define the distinguished class $\Sigma_{\mathcal{E}}$ of the stable vector fields, as*

$$\Sigma_{\mathcal{E}} = \{X \in \Sigma(M)\colon X \text{ either points inward i.e., } \partial U_{\mathcal{S}} \text{ is the incoming set, or points outward i.e., } \partial U_{\mathcal{S}} \text{ is the exit set, but not both}\}$$

The pair $(X, \mathcal{E})$ is called a distinguished field on M if $X \in \Sigma_{\mathcal{E}}$ and in the case of a flow $(X_t, \mathcal{E})$ is called a distinguished flow.

In what follows we will compute the Conley indices of a GS flow with respect to the stratification $\mathcal{E}$ of $M = \mathcal{R} \sqcup \mathcal{S}$, *i.e.,* if $p \in \mathcal{R}$ is a singularity of X_t the Conley index will be computed with respect to $\mathcal{R}$ and if $p \in \mathcal{S}$ it will be computed with respect to $\mathcal{S}$. In order to compute the Conley index relative to the singular strata, choose an index pair (N, L) in $\mathcal{S}$. Then the Conley numerical index

$$(s_0, s_1) = (\text{rank } H_0(N/L), \text{rank } H_1(N/L)),$$

of $p \in \mathcal{S}$.

We establish the following notation:

- $\mathcal{R}_0 = \sum_{p\in\mathcal{R}} h_0(p)$, $\mathcal{R}_1 = \sum_{p\in\mathcal{R}} h_1(p)$ and $\mathcal{R}_2 = \sum_{p\in\mathcal{R}} h_2(p)$, where $h_i(p)$ is the i-th Conley numerical index of p.
- $\mathcal{S}_0 = \sum_{p\in\mathcal{S}} s_0(p)$ and $\mathcal{S}_1 = \sum_{p\in\mathcal{S}} s_1(p)$.

Note that in Proposition 3.3, we did not take into account a stratification on M. Hence, if we do not take into account a stratification, the above notation implies that equation (1) in Proposition 3.3 can be rewritten as:

$$\mathcal{R}_2 - \mathcal{R}_1 + \mathcal{R}_0 = \chi(M). \tag{2}$$

Let us consider an example of this calculation restricted to the strata.

Example 3.5. *Consider the pinched torus in Figure 24, where the singular part is a circle. The two dimensional stratum is the complement of this circle, a disk and is the regular part. Although the circle itself is not singular, the cone singularity on that circle is a singular point of the manifold and of the flow. This cone singularity is a zero-dimensional stratum and its complement on the circle is the one-dimensional stratum. This flow has three singularities, a repeller in the regular part and two singularities in the singular part.*

Hence, the Conley index of this repeller is $h(p) = S^2$ and its homology index,

$$CH_i(p) = \begin{cases} \mathbb{Z} & \text{if } i = 2 \\ 0 & \text{otherwise.} \end{cases}$$

Hence, the Conley numerical index of the regular part is $(h_0, h_1, h_2) = (0, 0, 1)$.

The singularities of the singular part $\mathcal{S}$, are in S a repeller and an attractor. The repeller in $\mathcal{S}$ has Conley index $h(p) = S^1$ and homological index:

$$CH_i(p) = \begin{cases} \mathbb{Z} & \text{if } i = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Hence, the numerical Conley index is $(s_0, s_1) = (0, 1)$. *The attractor in* $\mathcal{S}$ *has Conley index* $h(p) = S^0$ *and homology index:*

$$CH_i(p) = \begin{cases} \mathbb{Z} & se \;\; i = 0 \\ 0 & otherwise. \end{cases}$$

Hence, the numerical Conley index is $(s_0, s_1) = (1, 0)$.

Theorem 3.7 relates the Euler characteristic of the regular part and the Euler characteristic of the singular part of Gutierrez-Sotomayor flows X_t on M, both expressed in terms of the numerical Conley indices to the Euler characteristic of M.

We first prove a lemma that shows that the numerical Conley indices of the singular part $\mathcal{S}$ of M is the same if computed with respect to M or with respect to $\mathcal{S}$.

Lemma 3.6. *Let M be a two-manifold with simple singularities and X_t the Gutierrez-Sotomayor flow on M. If M admits a stratification $\mathcal{E}$ such that $(X_t, \mathcal{E})$ is a distinguished flow then for the singularities $\{p_1, p_2, \dots, p_n\} \subset \mathcal{S}$ the following holds:*

$$\mathcal{R}_0 - \mathcal{R}_1 + \mathcal{R}_2 = \mathcal{S}_0 - \mathcal{S}_1 \tag{3}$$

Proof.

$$\mathcal{R}_2 - \mathcal{R}_1 + \mathcal{R}_0 = \chi(\overline{U_{\mathcal{S}}}) = \chi(\mathcal{S}) = -\mathcal{S}_1 + \mathcal{S}_0$$

The first equality follows from Proposition 3.3, the second equality follows from the fact that $\overline{U_{\mathcal{S}}}$ is a deformation retract of $\mathcal{S}$. Finally the third equality follows from Proposition 3.3 adjusted to the one dimensional setting. □

Theorem 3.7. *Let M be a two-manifold with simple singularities and X_t a Gutierrez-Sotomayor flow on M. If M admits a stratification $\mathcal{E}$ such that $(X_t, \mathcal{E})$ is a distinguished flow then*

$$(\mathcal{R}_2 - \mathcal{R}_1 + \mathcal{R}_0)|_{M \setminus \mathcal{S}} = \mathcal{S}_1 - \mathcal{S}_0 + \chi(M) \tag{4}$$

Proof. Consider a sufficiently small tubular neighborhood, $U_{\mathcal{S}}$ of the singular part $\mathcal{S}$ of M which contains no other singularities apart from the ones in $\mathcal{S}$. Suppose that on $\partial U_{\mathcal{S}}$, X points inward to $U_{\mathcal{S}}$ and denote by $\tilde{M} = M - U_{\mathcal{S}}$. Then by Proposition 3.3 we have that:

$$(\mathcal{R}_2 - \mathcal{R}_1 + \mathcal{R}_0)|_{\tilde{M}} = \chi(\tilde{M}, \partial \tilde{M}^-) \tag{5}$$

On the other hand, M is a CW-complex formed by the union of subcomplexes $\tilde{M}$ and $\overline{U_S}$ hence, $\chi(M) = \chi(\tilde{M}) + \chi(\overline{U_S}) - \chi(\partial \tilde{M}^-)$ since $\tilde{M} \cap \overline{U_S} = \partial \tilde{M} = \partial \tilde{M}^-$. Using the exact sequence of the pair $(\tilde{M}, \partial \tilde{M}^-)$ we have that $\chi(\tilde{M}, \partial \tilde{M}^-) = \chi(\tilde{M}) - \chi(\partial \tilde{M}^-)$. Thus,

$$\chi(\tilde{M}) + \chi(\overline{U_S}) - \chi(\partial \tilde{M}^-) = \chi(M)$$
$$\chi(\tilde{M}, \partial \tilde{M}^-) + \chi(\overline{U_S}) = \chi(M)$$
$$(\mathcal{R}_2 - \mathcal{R}_1 + \mathcal{R}_0)|_{\tilde{M}} + \chi(S) = \chi(M)$$

Since $X \in \Sigma_{\mathcal{E}}$ in $U_{\mathcal{S}} \setminus \mathcal{S}$ has no fixed points then $(\mathcal{R}_2 - \mathcal{R}_1 + \mathcal{R}_0)|_{U_{\mathcal{S}} \setminus \mathcal{S}} = 0$, thus from the above equality we have that:

$$(\mathcal{R}_2 - \mathcal{R}_1 + \mathcal{R}_0)|_{M \setminus \mathcal{S}} + \mathcal{S}_0 - \mathcal{S}_1 = \chi(M)$$
$$(\mathcal{R}_2 - \mathcal{R}_1 + \mathcal{R}_0)|_{M \setminus \mathcal{S}} = \mathcal{S}_1 - \mathcal{S}_0 + \chi(M)$$

□

Corollary 3.8. *Let M be a two-manifold with simple singularities and X_t a Gutierrez-Sotomayor flow on M. If M admits a stratification $\mathcal{E}$ such that $(X_t, \mathcal{E})$ is a distinguished flow then*

$$(\mathcal{R}_2 - \mathcal{R}_1 + \mathcal{R}_0)|_{M \setminus \mathcal{S}} = \chi(M) - \chi(S) \tag{6}$$

Proof. Follows directly from Theorem 3.7. □

3.3.1. *Examples.* Fix a stratification $\mathcal{E}$ on a a two-manifold with simple singularities of X_t, a Gutierrez-Sotomayor flow on M. Let $(X_t, \mathcal{E})$ be a distinguished flow.

Example 3.9. *Let M be a two manifold with cone singularities (e.g. a pinched sphere or a pinched torus) with stratification $\mathcal{E}$.*

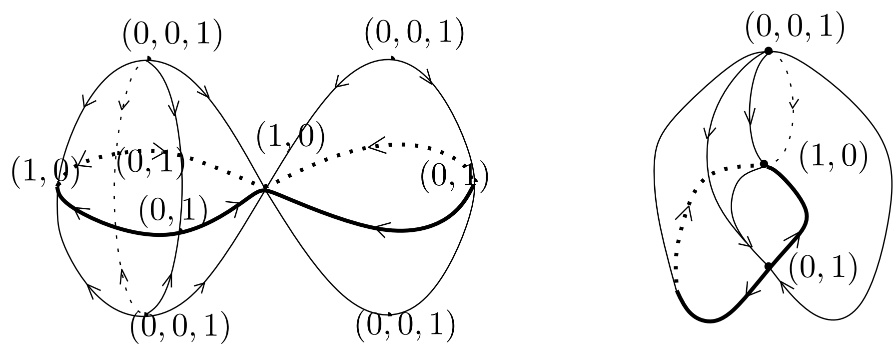

FIGURE 24. Flows on the pinched sphere and the pinched torus

(1) *Let X_t be Gutierrez-Sotomayor flow on the pinched sphere with 9 singularities: two attractors, three saddles and four repellers on M, see Figure 24(left).*

With respect to the stratification $\mathcal{E}$, R has four components homeomorphic to disks with one repelling singularity in the center of each disk. Hence, each singularity has numerical Conley indices equal to $(h_0, h_1, h_2) = (0, 0, 1)$ and thus,

$$R_0 = 0 + 0 + 0 + 0 = 0, \quad R_1 = 0 + 0 + 0 + 0 = 0 \quad e \quad R_2 = 1 + 1 + 1 + 1 = 4.$$

In the singular part of M one has 5 singularities of X_t two of which are attractors and three repellers. Hence, the numerical Conley indices are (s_0, s_1) equal to $(1, 0)$ and $(0, 1)$ respectively. Hence,

$$S_0 = 1 + 1 + 0 + 0 + 0 = 2 \quad and \quad S_1 = 0 + 0 + 1 + 1 + 1 = 3.$$

Substituting these values in equation (4): $4 - 0 + 0 = 3 - 2 + \chi(M)$. Thus, $\chi(M) = 3$.

(2) *Let X_t be Gutierrez-Sotomayor flow on the pinched torus with 3 singularities: one attractor, one saddle and one repeller on M, see Figure 24(right). In Example 3.5 we computed on the regular part*

$$R_0 = 0, \quad R_1 = 0 \quad and \quad R_2 = 1.$$

and on the singular part

$$S_0 = 1 + 0 = 1 \quad and \quad S_1 = 0 + 1 = 1.$$

Substituting these values in equation (4): $1 - 0 + 0 = 1 - 1 + \chi(M)$. Thus, $\chi(M) = 1$.

Example 3.10. *Let M be a manifold with Whitney umbrella singularity (e.g. a crosscap or a torus with a fold) with stratification $\mathcal{E}$.*

(1) *Let X_t be a Gutierrez-Sotomayor flow on a crosscap with 3 singularities: one attractor, one saddle and one repeller on M, see Figure 25(a).*

With respect to the stratification $\mathcal{E}$, R has one component homeomorphic to a disk with an attracting singularity at its center. Hence, $(h_0, h_1, h_2) = (1, 0, 0)$ and thus,

$$R_0 = 1, \quad R_1 = 0 \quad e \quad R_2 = 0.$$

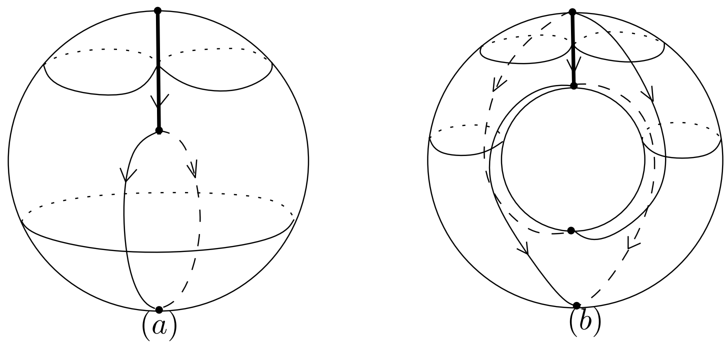

FIGURE 25. Flows on a crosscap and on a torus with a fold.

In the singular part of M there are two singularities of X_t one of which is an attractor and the other a repeller. Hence the numerical Conley indices (s_0, s_1) are equal to $(1,0)$ and $(0,0)$ respectively. Thus,

$$S_0 = 1 + 0 = 1 \quad \textit{and} \quad S_1 = 0 + 0 = 0.$$

Substituting these values in equation (4):

$$0 - 0 + 1 = 0 - 1 + \chi(M).$$

Thus, $\chi(M) = 2$.

(2) *Let X_t be a Gutierrez-Sotomayor flow on a torus with a fold with 4 singularities: an attractor, two saddles and one repeller on M, see Figure 25(b).*

With respect to the stratification $\mathcal{E}$, R has one component homeomorphic to a cylinder with two singularities in its interior a saddle and an attractor. The numerical Conley indices are $(h_0, h_1, h_2) = (0,1,0)$ for the saddle and $(h_0, h_1, h_2) = (1,0,0)$ for the attractor. Hence,

$$R_0 = 0 + 1 = 1, \quad R_1 = 1 + 0 = 1 \quad \textit{and} \quad R_2 = 0 + 0 = 0.$$

On the singular part of M there are two singularities of X_t one of which is an attractor and the other a repeller with numerical Conley indices equal to $(s_0, s_1) = (1,0)$ for the attractor and $(s_0, s_1) = (0,0)$ for the repeller. Hence,

$$S_0 = 0 + 1 = 1 \quad \textit{and} \quad S_1 = 0 + 0 = 0.$$

Substituting these values in equation (4):

$$0 - 1 + 1 = 0 - 1 + \chi(M).$$

Thus, $\chi(M) = 1$.

4. Isolating Blocks

In this section we will develop a theory of generalized handles to present a procedure of constructing special isolating neighborhoods for a simple singularity of a Gutierrez-Sotomayor flow. These isolating neighborhoods have the property that the flow is transversal to their boundary. Furthermore, we require that:

Definition 4.1. An *isolating block* is an isolating neighborhood N for an isolated invariant set Λ of the flow φ such that

$$N^- = \{x \in N | \varphi([0,T), x) \nsubseteq N, \forall T > 0\}$$

is closed.

A similar condition is required for the entering boundary N^+ for $T < 0$.

The existence of isolating blocks is an immediate consequence of the existence of Lyapunov functions f for Gutierrez-Sotomayor flows with simple singularities. If p is a singular point with $f(p) = c$ and $\epsilon > 0$ such that in $[c-\epsilon, c+\epsilon]$ there are no critical values then define an isolating block, N, for p as the connected component $f^{-1}([c-\epsilon, c+\epsilon])$ that contains p and $N^- = f^{-1}(c-\epsilon) \cap N$. Moreover, (N, N^-) is an index pair for $Inv(N) = \{p\}$.

4.1. **The Poincaré-Hopf Condition.** The following theorem establishes a relation between the first Betti number of the branched one-manifolds which make up the boundary N_0 of an isolating block N_1 for the singularity p, the number of boundary components of N_0 and the numerical Conley indices of p, (h_0, h_1, h_2).

Theorem 4.2. *Let (N_1, N_0) be an index pair where N_1 is an isolating block for a singularity p in a two dimensional manifold with simple singularities M. Let $X \in \Sigma^r(M)$ and (h_0, h_1, h_2) be the numerical Conley indices for p. Then*

$$(h_2 - h_1 + h_0) - (h_2 - h_1 + h_0)^* = e^+ - \mathcal{B}^+ - e^- + \mathcal{B}^- \tag{7}$$

where * indicates the index of the time-reversed flow, $e^+(e^-)$ is the number of entering (exiting) boundary components of N_1 and $\mathcal{B}^+ = \sum_{k=1}^{e^+} b_k^+$ ($\mathcal{B}^- = \sum_{k=1}^{e^-} b_k^-$) where $b_k^+(b_k^-)$ is the first Betti number of the kth entering (exiting) boundary components of N_1.

Proof. Proposition 3.3 asserts that $h_2 - h_1 + h_0 = \chi(N_1, N_0)$. By the long exact sequence of the pair (N_1, N_0) we have that $\chi(N_1, N_0) = \chi(N_1) - \chi(N_0)$. But $N_0 = \partial N_1^-$ hence,

$$h_2 - h_1 + h_0 + \chi(\partial N_1^-) = \chi(N_1)$$

Using the same arguments for the reverse flow, we obtain

$$(h_2 - h_1 + h_0)^* + \chi(\partial N_1^+) = \chi(N_1).$$

Subtracting these two equations, one concludes that

$$(h_2 - h_1 + h_0) - (h_2 - h_1 + h_0)^* = \chi(\partial N_1^+) - \chi(\partial N_1^-)$$

$$(h_2 - h_1 + h_0) - (h_2 - h_1 + h_0)^* = \sum_{k=1}^{e^+}(1 - b_k^+) - \sum_{k=1}^{e^-}(1 - b_k^-)$$

$$(h_2 - h_1 + h_0) - (h_2 - h_1 + h_0)^* = e^+ - \mathcal{B}^+ - e^- + \mathcal{B}^-$$

□

4.2. **The Gutierrez-Sotomayor Handle Theory.** In this section we will define a notion of generalized handles and specify their attaching regions. As in classical handle theory, the attaching regions produce different topological spaces depending on how the handle is glued.

Since the fixed points of $X \in \Sigma^r(M)$ are in $M(\mathcal{G})$, with $\mathcal{G} = \mathcal{R}$, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{W}$ or $\mathcal{T}$, one must consider different types of handles which we refer to as two dimensional Gutierrez-Sotomayor handles, GS handles for short.

A GS handle $\mathcal{H}_x^{\mathcal{G}}$ is a subspace of $\mathbb{R}^3$ with well defined dynamics where a fixed point is on $M(\mathcal{G})$, *i.e.,* it may be on the regular part, on the cone, on the Whitney fold, on double crossings or triple crossings. Hence, we will denote them by regular handles, cone handles, Whitney handles, double handles or triple handles respectively.

In order to specify the dynamics on the handles we consider the following vector fields defined on disks in $\mathbb{R}^2$:

(*a*) $X(x, y) = (-2x, -2y)$ (*b*) $X(x, y) = (x, -y)$

(c) $X(x, y) = (2x, 2y)$ (d) $X(x, y) = (0, -x^2 - y^2)$

Equivalently, one can consider the respective singularities on the handles and their local flow. We will now proceed to define the handles in each case above.

Definition 4.3. *A regular handle is formed by a disk D centered at p with a flow defined as in the cases below: See Figure 26.*

(1) *A regular handle $\mathcal{H}_a^{\mathcal{R}}$ has a flow defined by the vector field in (a). The attaching region of the handle is the empty set.*
(2) *A regular handle $\mathcal{H}_s^{\mathcal{R}}$ has a flow defined by the vector field in (b). The attaching region of the handle is homeomorphic to two disjoint segments from where the flow exits.*
(3) *A regular handle $\mathcal{H}_r^{\mathcal{R}}$ has a flow defined by the vector field in (c). The attaching region of the handle is homeomorphic to a circle from where the flow exits.*

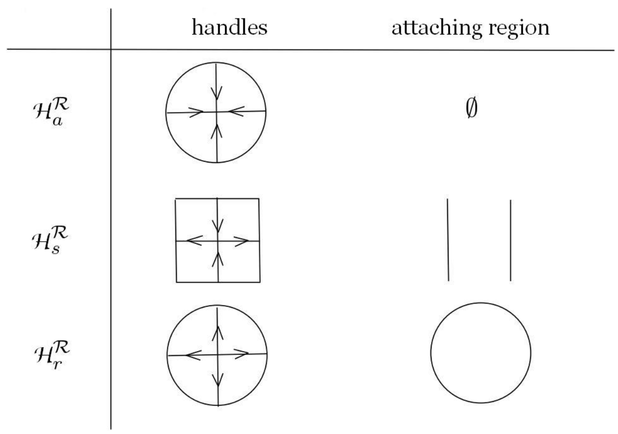

FIGURE 26. Regular handles $\mathcal{H}_a^{\mathcal{R}}$, $\mathcal{H}_s^{\mathcal{R}}$ and $\mathcal{H}_r^{\mathcal{R}}$.

Definition 4.4. *A cone handle is formed by two disks D_1 and D_2 centered at p such that $D_1 \cap D_2 = \{p\}$ with a flow defined as in the cases below. See Figure 27.*

(1) *A cone handle $\mathcal{H}_a^{\mathcal{C}}$ has a flow defined on both disks by the vector field in (a). The attaching region of the handle is the empty set.*
(2) *A cone handle $\mathcal{H}_s^{\mathcal{C}}$ has a flow defined on both disks by the vector field in (d). The attaching region of the handle is the disjoint union of two arcs in ∂D_1 and ∂D_2 respectively, from where the flow exits.*
(3) *A cone handle $\mathcal{H}_r^{\mathcal{C}}$ has a flow defined on both disks by the vector field in (c). The attaching region of the handle are the two circles that correspond to ∂D_i, from where the flow exits.*

Definition 4.5. *A Whitney handle is formed by a disk D with two regular orbits identified as in the cases below. See Figure 28.*

(1) *A Whitney handle $\mathcal{H}_a^{\mathcal{W}}$ has a flow defined on D by the vector field in (a), with two regular orbits identified to a ray of D. The attaching region of the handle is the empty set.*

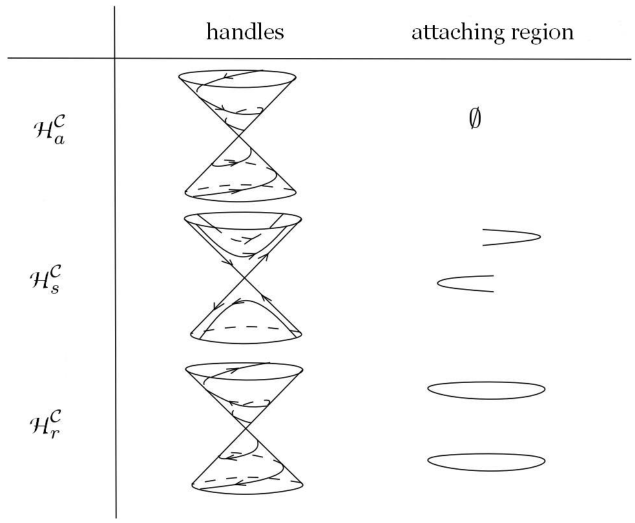

FIGURE 27. Cone handles $\mathcal{H}_a^{\mathcal{C}}$, $\mathcal{H}_s^{\mathcal{C}}$ and $\mathcal{H}_r^{\mathcal{C}}$.

(2) *A Whitney handle $\mathcal{H}_{s_s}^{\mathcal{W}}$ has a flow defined on D by the vector field in (b), with two regular orbits on the stable manifold identified to a ray of D. The attaching region of the handle is the disjoint union of two arcs in ∂D, from where the flow exits.*

(3) *A Whitney handle $\mathcal{H}_{s_u}^{\mathcal{W}}$ has a flow defined on D by the vector field in (b), with two regular orbits on the unstable manifold identified to a ray of D. The attaching region of the handle is a transversal intersection of two arcs from where the flow exits.*

(4) *A Whitney handle $\mathcal{H}_r^{\mathcal{W}}$ has a flow defined on D by the vector field in (c), with two regular orbits identified to a ray of D. The attaching region of the handle is the boundary ∂D which after the identification is homeomorphic to a figure "eight" from where the flow exits.*

Definition 4.6. *A double handle is formed by two disks D_1 and D_2 centered at p and intersecting transversally along diameters d_1 and d_2 of D_1 and D_2 respectively. These diameters are formed by a union of orbits as described below. See Figure 29.*

(1) *A double handle $\mathcal{H}_a^{\mathcal{D}}$ has a flow defined on D_1 and D_2 by the vector field in (a). The attaching region of the handle is the empty set.*

(2) *A double handle $\mathcal{H}_{sa}^{\mathcal{D}}$ has a flow defined on D_1 by the vector field in (a) and defined on D_2 by the vector field in (b) where d_2 is the stable manifold in D_2. The attaching region of the handle is homeomorphic to two disjoint segments from where the flow exits.*

(3) *A double handle $\mathcal{H}_{ss_u}^{\mathcal{D}}$ has a flow defined on D_1 and D_2 by the vector field in (b) where d_1 and d_2 are the unstable manifolds on the respective disks. The attaching region of the handle is homeomorphic to two copies of two segments that intersect transversally and from where the flow exits.*

(4) *A double handle $\mathcal{H}_{ss_s}^{\mathcal{D}}$ has a flow defined on D_1 and D_2 by the vector field in (b) where d_1 and d_2 are the stable manifolds on the respective disks. The attaching region of the handle is homeomorphic to two copies of two segments from where the flow exits.*

(5) *A double handle $\mathcal{H}_{sr}^{\mathcal{D}}$ has a flow defined on D_1 by the vector field in (c) and has a flow defined on D_2 by the vector field in (b) where d_2 is the unstable manifold in D_2. The*

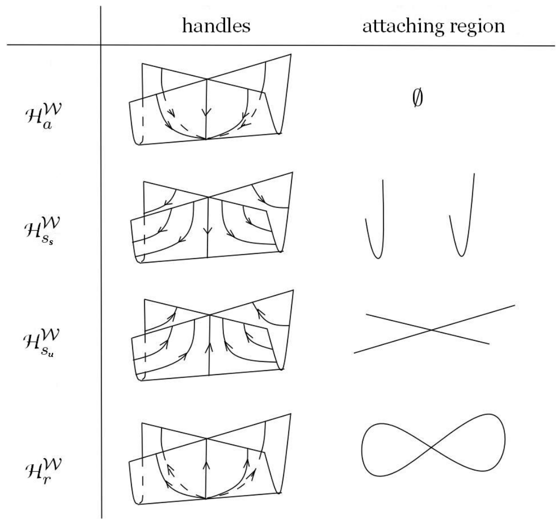

FIGURE 28. Whitney handles $\mathcal{H}_a^{\mathcal{W}}$, $\mathcal{H}_{s_s}^{\mathcal{W}}$, $\mathcal{H}_{s_u}^{\mathcal{W}}$ and $\mathcal{H}_r^{\mathcal{W}}$.

attaching region of the handle is homeomorphic to ∂D_2 on which two segments intersect transversally and from where the flow exits.

(6) *A double handle $\mathcal{H}_r^{\mathcal{D}}$ has a flow defined on D_1 and D_2 by the vector field in (c). The attaching region of the handle is homeomorphic to ∂D_1 and ∂D_2 intersecting transversally at two distinct points and from where the flow exits.*

Definition 4.7. *A triple handle is formed by three disks D_1, D_2 and D_3 centered at p with diameters $d_1 \subset D_1$, $d_2 \subset D_2$ and $d_3 \subset D_3$ intersecting transversally at p and pairwise disjoint. These diameters are formed by a union of orbits as described below. See Figure 30.*

(1) *A triple handle $\mathcal{H}_a^{\mathcal{T}}$ has a flow defined on D_1, D_2 and D_3 by the vector field in (a). The attaching region of the handle is the empty set.*

(2) *A triple handle $\mathcal{H}_{ssa}^{\mathcal{T}}$ has a flow defined on D_1 by the vector field in (a) and has a flow defined on D_2 and D_3 by the vector field in (b) where d_2 and d_3 are stable manifolds of D_2 and D_3 respectively. The attaching region of the handle is homeomorphic to two copies of two segments that intersect transversally from where the flow exits.*

(3) *A triple handle $\mathcal{H}_{ssr}^{\mathcal{T}}$ has a flow defined on D_1 by the vector field in (c) and has a flow defined on D_2 and D_3 by the vector field in (b) where d_2 and d_3 are unstable manifolds of D_2 and D_3 respectively. The attaching region of the handle is homeomorphic to ∂D_2 from where the flow exits with four segments intersecting ∂D_2 transversally and also from where the flow exits.*

(4) *A triple handle $\mathcal{H}_r^{\mathcal{T}}$ has a flow defined on D_1, D_2 and D_3 by the vector field in (c). The attaching region of the handle is homeomorphic to three circles, all from which the flow exits and that pairwise intersect transversally at two points.*

4.3. **Constructing Isolating Blocks.** In this section, we construct an isolating block by gluing a GS handle $\mathcal{H}_x^{\mathcal{G}}$ to a collar of a distinguished branched one manifold $N^- \times [0,1]$.

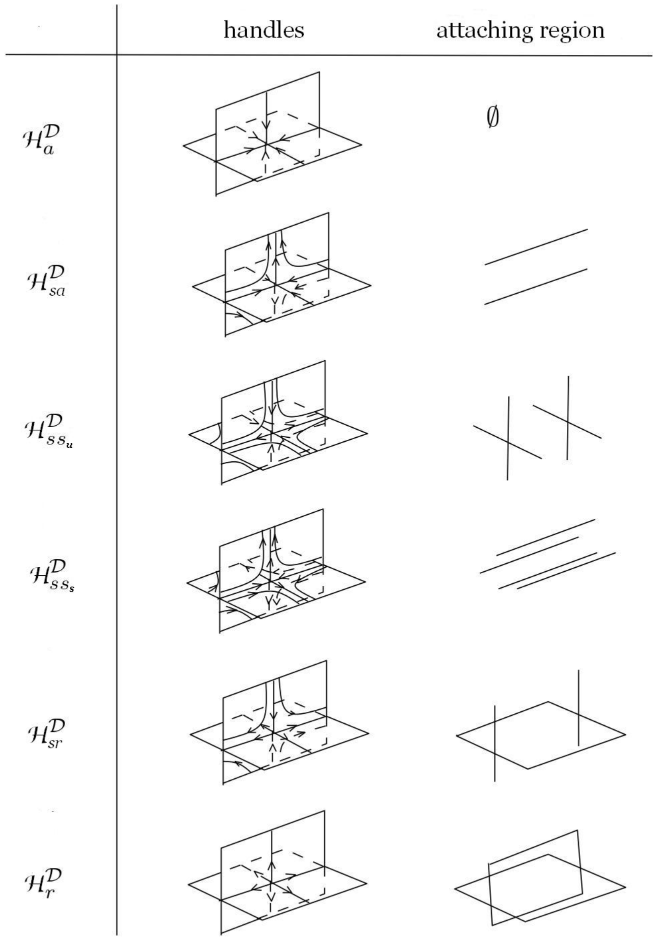

FIGURE 29. Double handles $\mathcal{H}_a^{\mathcal{D}}$, $\mathcal{H}_{sa}^{\mathcal{D}}$, $\mathcal{H}_{ss_u}^{\mathcal{D}}$, $\mathcal{H}_{ss_s}^{\mathcal{D}}$, $\mathcal{H}_{sr}^{\mathcal{D}}$ and $\mathcal{H}_r^{\mathcal{D}}$.

Definition 4.8. *A distinguished branched one manifold is a topological space, having at most four connected components, locally constructed from a finite number of branched charts. Each branched chart is the transversal intersection of two arcs in the plane.*

In Figure 31, we present examples of distinguished branched 1-manifolds.

It is interesting to note that the different attachments of a given GS handle $\mathcal{H}_x^{\mathcal{G}}$ produces non-homeomorphic isolating blocks (N, N^-). However, all isolating blocks have the same Conley index, *i.e.,* the homotopy type of N/N^- is the same and independent of the block.

Theorem 4.9. *Let p be a simple singularity of a Gutierrez-Sotomayor flow X_t on M. Suppose that p satisfies the Poincaré-Hopf condition for the positive numbers e^+ , e^-, $\{b_k^+\}_{k=1}^{e^+}$ and $\{b_k^-\}_{k=1}^{e^-}$. Then there exists an isolating block N for p with $\partial N = \partial N^+ \cup \partial N^-$ such that the following holds:*

(1) *e^+ (respectively e^-) is the number of connected components of ∂N^+ (respectively ∂N^-), corresponding to the entering (respectively exiting) boundary components of the flow. In*

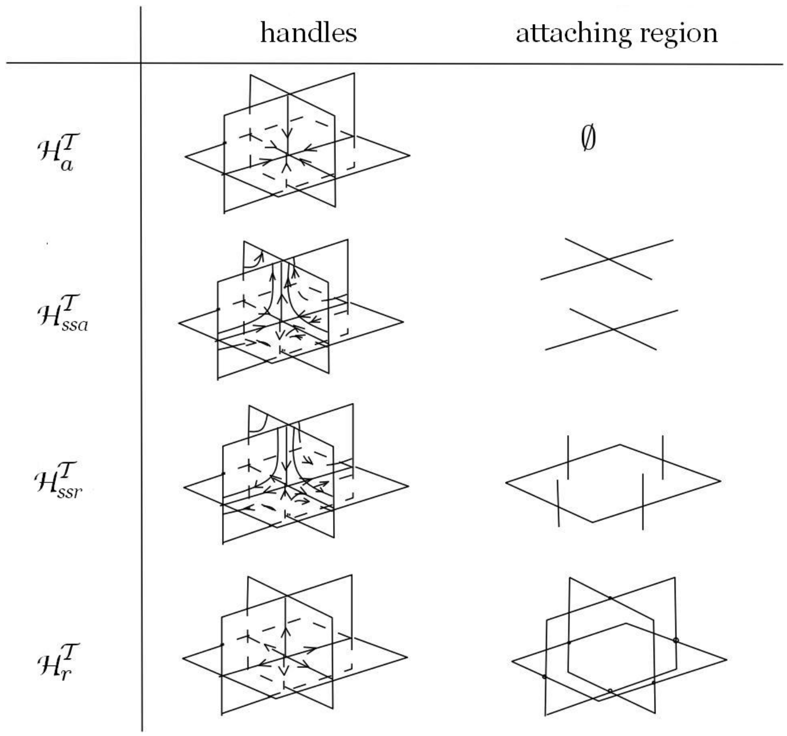

FIGURE 30. Triple handles $\mathcal{H}_a^{\mathcal{T}}$, $\mathcal{H}_{ssa}^{\mathcal{T}}$, $\mathcal{H}_{ssr}^{\mathcal{T}}$ e $\mathcal{H}_r^{\mathcal{T}}$.

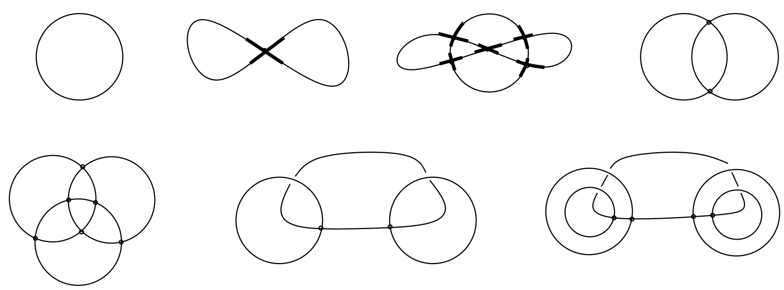

FIGURE 31. Distinguished branched one-manifolds.

other words, we have a disjoint union

$$\partial N^+ = \bigcup_{k=1}^{e^+} \partial N_k^+ \ \ \left(\text{respectively } \partial N^- = \bigcup_{k=1}^{e^-} \partial N_k^-\right).$$

(2) *the rank* $H_1(\partial N_k^+) = b_k^+$ *with* $k = 1, ..., e^+$ *and the rank*$H_1(\partial N_k^-) = b_k^-$ *with* $k = 1, ..., e^-$.

(3) *the rank* $H_*(N/\partial N^-) = h_*$ *where* (h_0, h_1, h_2) *is the numerical Conley index of* p.

Proof. For each attractor and repeller, the GS handle $\mathcal{H}_a^{\mathcal{G}}$ where $\mathcal{G} = \mathcal{R}$, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{W}$ or $\mathcal{T}$ is always an isolating block. For saddle handles there are different topological types of isolating blocks depending on the distinguished branched one manifolds and the attaching maps to their collars. Consider a distinguished branched one manifold $N^- = \bigcup_{k=1}^{e^-} N_k^-$ with e^- components and each N_k^- with b_k^- as its first Betti number. Let $\mathcal{H}_x^{\mathcal{G}}$ be a GS handle with attaching region A_k and the collar $\bigcup_{k=1}^{e^-}(N_k^- \times I)$ of N_k^-. Attach the handle to the distinguished branched one manifold via an embedding

$$f : A_k \to \bigcup_{k=1}^{e^-}(N_k^- \times 1).$$

$\square$

See Figures 32, 33, 34, 35, 36 and 37, where we present constructions for specific cases of saddle type isolating blocks for a simple singularity of a Gutierrez-Sotomayor flow X_t.

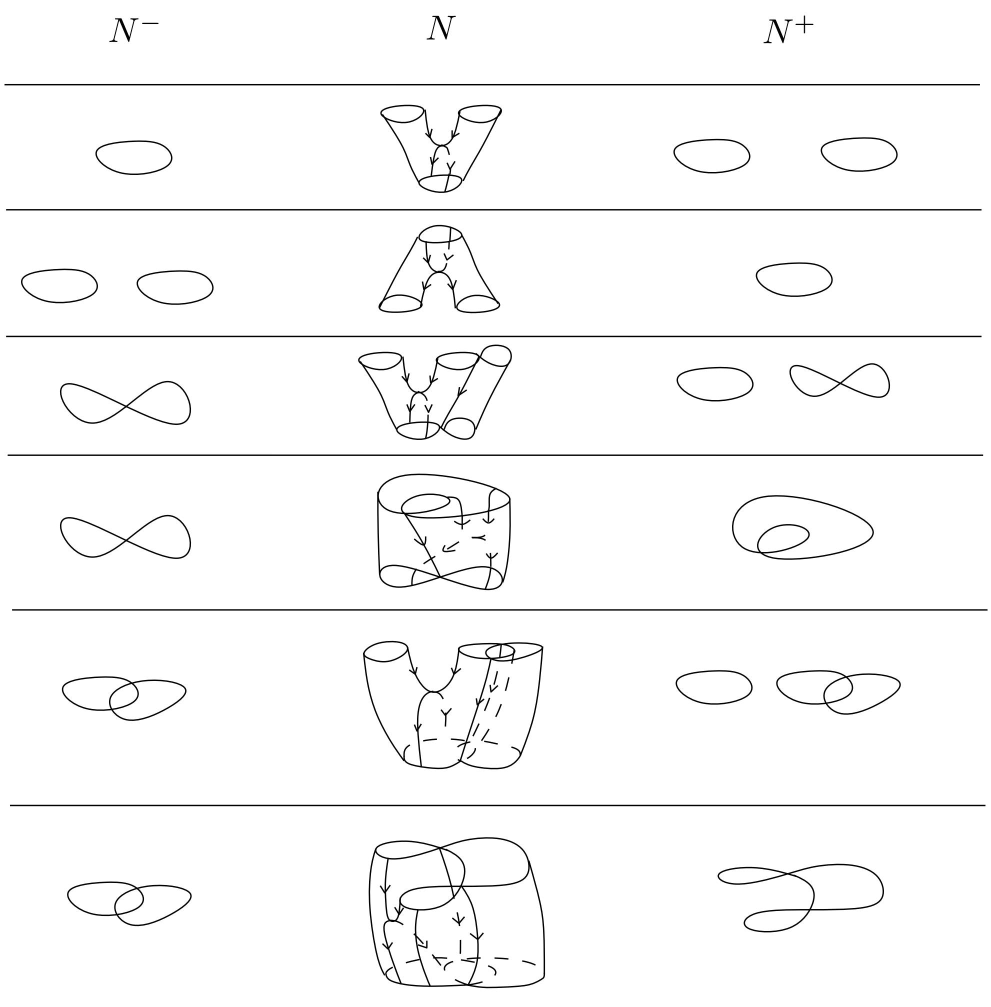

FIGURE 32. Isolating blocks containing a regular handle $\mathcal{H}_s^{\mathcal{R}}$.

Other blocks can be constructed from these by adding cylinders where the flow is trivial. See Figure 38.

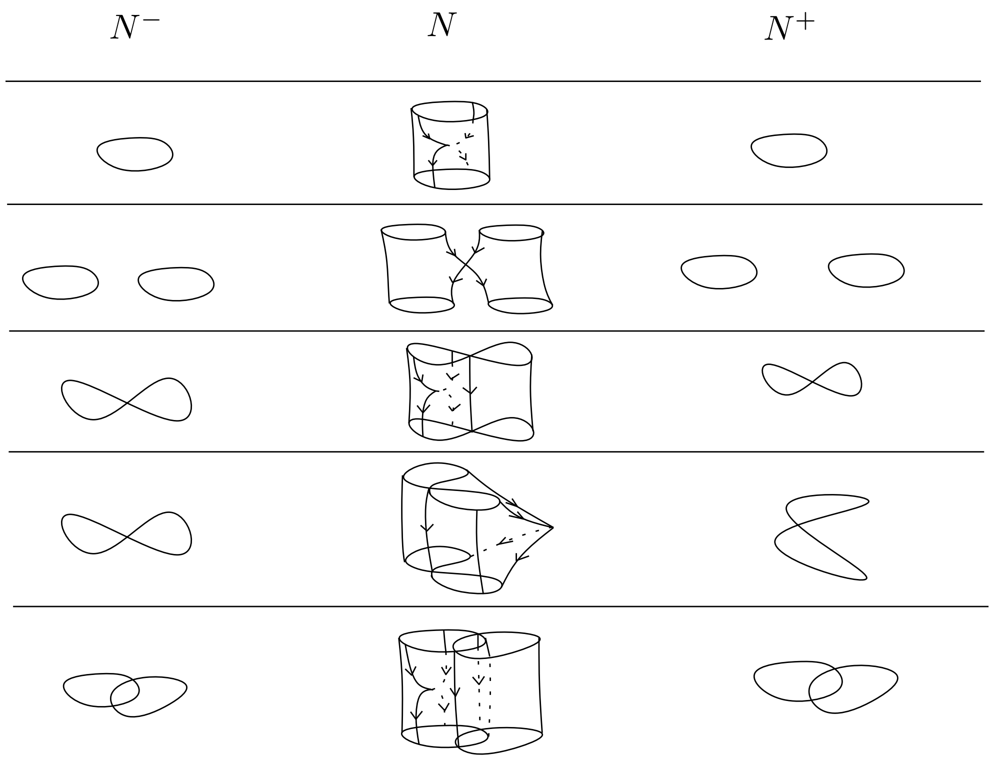

FIGURE 33. Isolating blocks containing a cone handle $\mathcal{H}_s^{\mathcal{C}}$.

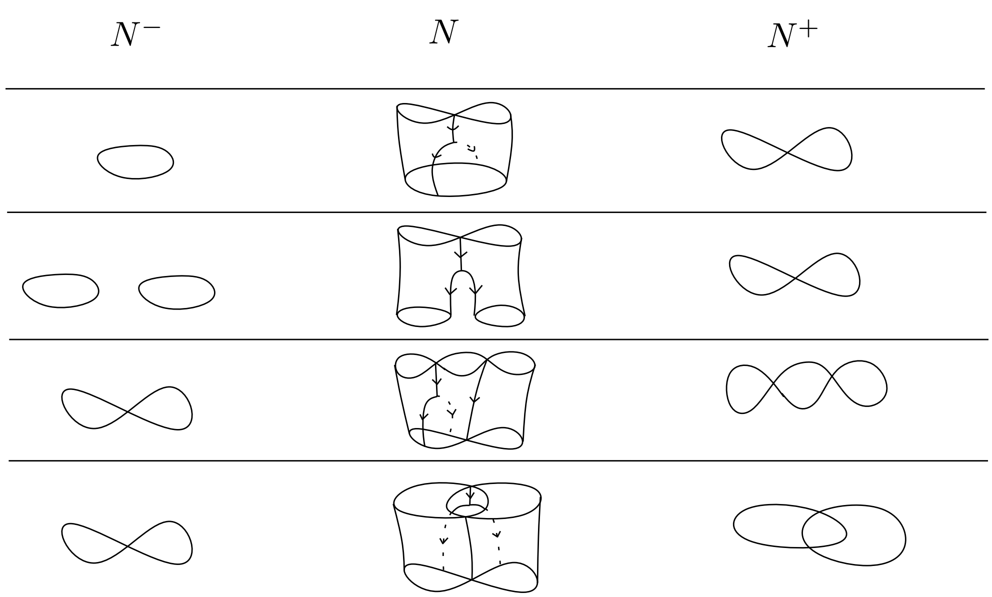

FIGURE 34. Isolating blocks containing a Whitney handle $\mathcal{H}_{s_s}^{\mathcal{W}}$.

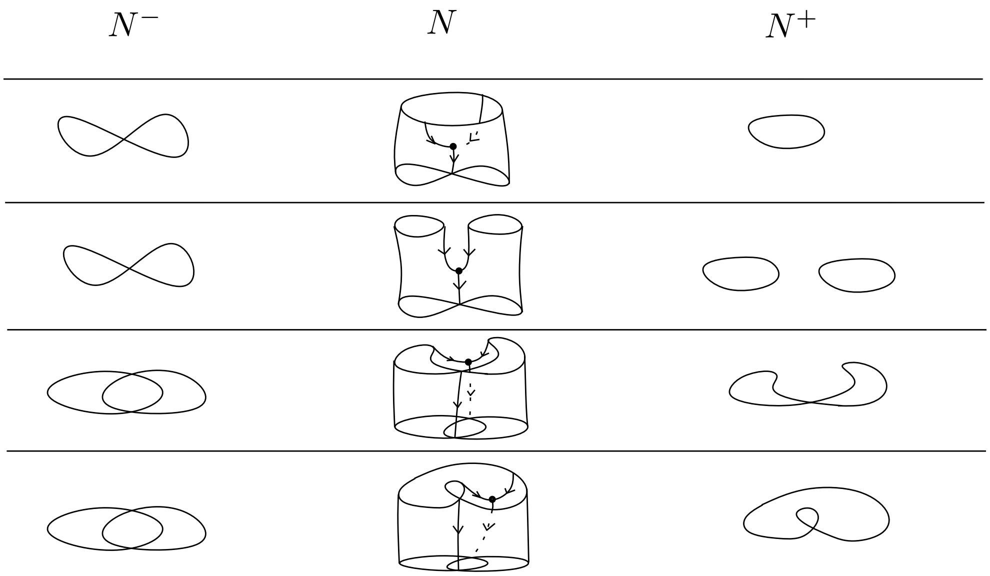

FIGURE 35. Isolating blocks containing a Whitney handle $\mathcal{H}^{\mathcal{W}}_{s_u}$.

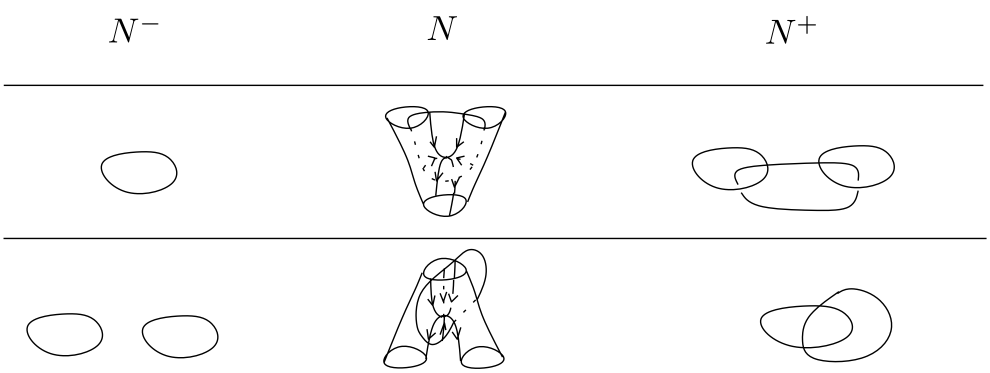

FIGURE 36. Isolating blocks containing a double handle $\mathcal{H}^{\mathcal{D}}_{sa}$.

5. Lyapunov Graphs

Let f be a Lyapunov function associated to the Gutierrez-Sotomayor flow X_t on the two-manifold M with simple singularities. We define the following equivalence relation on M: $x \sim_f y \Leftrightarrow x$ and y belong to the same connected component of a level set of f.

We call $M/\sim_f$ the Lyapunov graph associated to X_t and f.

On $M/\sim_f$ each connected component of a level set $f^{-1}(c)$ collapses to a point, thus $f^{-1}(c)/\sim_f$ is a finite set of distinct points on $M/\sim_f$. A point on $M/\sim_f$ is a vertex if by the equivalence relation it corresponds to a component of a level set containing a unique singularity. All other points are edge points. The vertices v of $M/\sim_f$ can be labelled with the type of singularity and we denote by e_v^+ the number of positively incident edges and e_v^- the number of negatively incident edges to v.

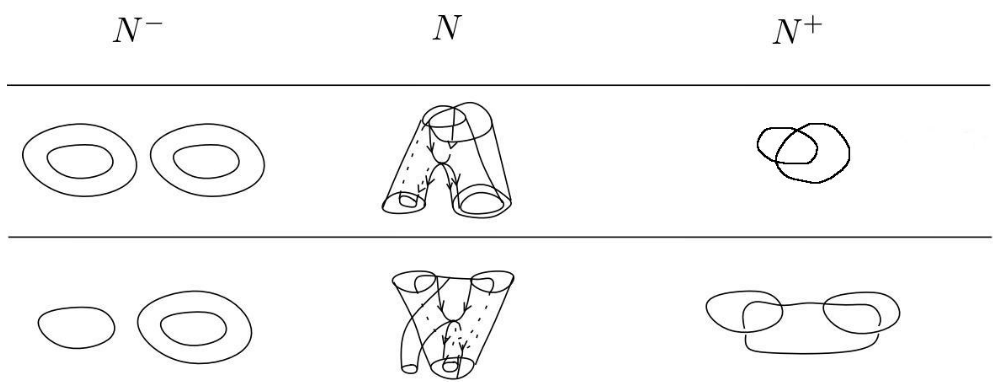

FIGURE 37. Isolating blocks containing a double handle $\mathcal{H}^{\mathcal{D}}_{ss_s}$.

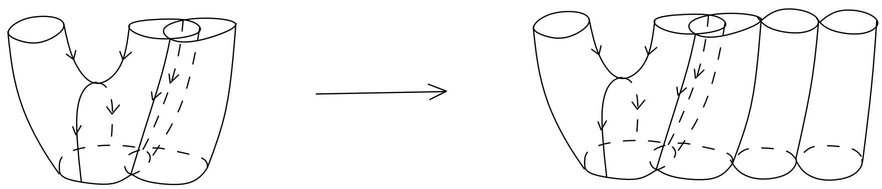

FIGURE 38. Isolating blocks containing a regular handle.

Theorem 5.1. *Supose that $X_t : M \to M$ is a Gutierrez-Sotomayor flow with Lyapunov function $f : M \to \boldsymbol{R}$. Let $L = M/\sim_f$, then L is a finite directed graph without oriented cycles.*

Proof. By the definition of a Lyapunov function we have that the critical points of f correspond to the singularities of X_t. Since X_t has a finite number of singularities then there exists a finite number of critical values of f, $c_1, c_2, \ldots, c_n$. Thus, $f^{-1}(c_i, c_{i+1})$ is diffeomorphic to $N \times (0, 1)$ where $N = f^{-1}(c)$ with $c \in (c_i, c_{i+1})$. Hence by Lemma 2.5, N is a branched one manifold with a finite number of components.

Also, $f^{-1}(c_i)$ has a finite number of components since if this were not the case $f^{-1}(c_i + \epsilon)$ would have infinite components for any $\epsilon > 0$. Only one of these components, denoted by X_i, contains the critical point of f since by definition a Lyapunov function f separates critical points.

Now if $N_0 \subset f^{-1}(c_i)$ does not contain critical points of f then the component of $f^{-1}(c_{i-1}, c_{i+1})$ that contains N_0 is diffeomorphic to $N_0 \times (0, 1)$. Indeed, $M - \bigcup_i X_i$ is diffeomorphic to the disjoint union of $N_j \times (0, 1)$ where each N_j is a connected compact branched one-manifold of M. Thus, if $P : M \to L$ is the quotient mapping that identifies each component of a level set of f to a point and $x_i = P(X_i)$ then it follows that $L - \{x_i\}$ is a finite set of open intervals. Hence, since L is compact, it is a graph.

Since f decreases along orbits of X_t then the Lyapunov graph L associated to X_t and f has no oriented cycle. $\square$

On the other hand, to construct a flow that satisfies a given dynamics, a great combinatorial tool is an abstract Lyapunov graph which can aggregate topological and dynamical information.

Definition 5.2. *An abstract Lyapunov graph is a finite connected oriented graph L which possesses no oriented cycles and with each vertex labelled with the numerical Conley indices. Each edge a that is incoming (resp. outgoing) i.e., positively incident to v (resp. negatively incident* *to v) will be labelled with a nonnegative integer* b_a^+ *(resp.* b_a^-*) where* $a \in \{1,\ldots,e^+\}$ *(resp.* $a \in \{1,\ldots,e^-\}$*), which we refer to as the weight on an edge.*

The question becomes: once necessary conditions on Lyapunov graphs are found, are they sufficient to realize an abstract graph as a GS flow on a manifold?

Theorem 5.3. *A Lyapunov graph L of a Gutierrez-Sotomayor flow X_t with simple singularities on M, satisfies the following conditions:*

(1) *If a vertex v is labelled with a repelling (attracting) singularity then:*
  (a) *If* $p \in M(\mathcal{R})$ *then* $e_v^- = 1$ *and* $b_1^- = 1$ ($e_v^+ = 1$ *and* $b_1^+ = 1$).
  (b) *If* $p \in M(\mathcal{C})$ *then* $e_v^- = 2$ *and* $b_1^- = b_2^- = 1$ ($e_v^+ = 2$ *and* $b_1^+ = b_2^+ = 1$).
  (c) *If* $p \in M(\mathcal{W})$ *then* $e_v^- = 1$ *and* $b_1^- = 2$ ($e_v^+ = 1$ *and* $b_1^+ = 2$).
  (d) *If* $p \in M(\mathcal{D})$ *then* $e_v^- = 1$ *and* $b_1^- = 3$ ($e_v^+ = 1$ *and* $b_1^+ = 3$).
  (e) *If* $p \in M(\mathcal{T})$ *then* $e_v^- = 1$ *and* $b_1^- = 7$ ($e_v^+ = 1$ *and* $b_1^+ = 7$).

(2) *If a vertex v is labelled with a saddle singularity p then:*
  (a) *If* $p \in M(\mathcal{R})$ *then* $1 \le e_v^- \le 2$ *and* $1 \le e_v^+ \le 2$.
  (b) *If* $p \in M(\mathcal{C})$ *then* $1 \le e_v^- \le 2$ *and* $1 \le e_v^+ \le 2$.
  (c) *If* $p \in M(\mathcal{W})$ *then*
    (i) *If p is of type* **si** *then* $e_v^- = 1$ *and* $1 \le e_v^+ \le 2$.
    (ii) *If p is of type* **se** *then* $1 \le e_v^- \le 2$ *and* $e_v^+ = 1$.
  (d) *If* $p \in M(\mathcal{D})$ *then*
    (i) *If p is of type* **as** *then* $1 \le e_v^- \le 2$ *and* $e_v^+ = 1$.
    (ii) *If p is of type* **rs** *then* $e_v^- = 1$ *and* $1 \le e_v^+ \le 2$.
    (iii) *If p is of type* **si** *then* $1 \le e_v^- \le 2$ *and* $1 \le e_v^+ \le 4$.
    (iv) *If p is of type* **se** *then* $1 \le e_v^- \le 4$ *and* $1 \le e_v^+ \le 2$.
  (e) *If* $p \in M(\mathcal{T})$ *then*
    (i) *If p is of type* **ssa** *then* $1 \le e_v^- \le 2$ *and* $e_v^+ = 1$.
    (ii) *If p is of type* **ssr** *then* $e_v^- = 1$ *and* $1 \le e_v^+ \le 2$.

All weights on the entering and exiting edges of v must satisfy the table.

$M(\mathcal{G})$	type	e_v^-	e_v^+	weights
$p \in M(\mathcal{R})$	**a**	0	1	$b_1^+ = 1$
	s	1	1	$b_1^- = b_1^+$
	s	1	2	$b_1^- = b_1^+ + b_2^+ - 1$
	s	2	1	$b_1^+ = b_1^- + b_2^- - 1$
	r	1	0	$b_1^- = 1$
$p \in M(\mathcal{C})$	**a**	0	2	$b_1^+ = b_2^+ = 1$
	s	1	1	$b_1^- = b_1^+$
	s	2	2	$b_1^- + b_2^- = b_1^+ + b_2^+$
	r	2	0	$b_1^- = b_2^- = 1$

$p \in M(\mathcal{W})$	**a**	0	1	$b_1^+ = 2$
	$\mathbf{s_u}$	1	1	$b_1^- = b_1^+ + 1$
	$\mathbf{s_u}$	1	2	$b_1^- = b_1^+ + b_2^+$
	$\mathbf{s_s}$	1	1	$b_1^+ = b_1^- + 1$
	$\mathbf{s_s}$	2	1	$b_1^+ = b_1^- + b_2^-$
	r	1	0	$b_1^- = 2$
$p \in M(\mathcal{D})$	**a**	0	1	$b_1^+ = 3$
	sa	1	1	$b_1^+ = b_1^- + 2$
	sa	2	1	$b_1^+ = b_1^- + b_2^- + 1$
	sr	1	1	$b_1^- = b_1^+ + 2$
	sr	1	2	$b_1^- = b_1^+ + b_2^+ + 1$
	$\mathbf{ss_u}$	1	1	$b_1^- = b_1^+ + 2$
	$\mathbf{ss_u}$	1	2	$b_1^- = b_1^+ + b_2^+ + 1$
	$\mathbf{ss_u}$	1	3	$b_1^- = b_1^+ + b_2^+ + b_3^+$
	$\mathbf{ss_u}$	1	4	$b_1^- = b_1^+ + b_2^+ + b_3^+ + b_4^+ - 1$
	$\mathbf{ss_u}$	2	1	$b_1^+ = b_1^- + b_2^- - 3$
	$\mathbf{ss_u}$	2	2	$b_1^- + b_2^- = b_1^+ + b_2^+ + 2$
	$\mathbf{ss_u}$	2	3	$b_1^- + b_2^- = b_1^+ + b_2^+ + b_3^+ + 1$
	$\mathbf{ss_u}$	2	4	$b_1^- + b_2^- = b_1^+ + b_2^+ + b_3^+ + b_4^+$
	$\mathbf{ss_s}$	1	1	$b_1^+ = b_1^- + 2$
	$\mathbf{ss_s}$	1	2	$b_1^- = b_1^+ + b_2^+ - 3$
	$\mathbf{ss_s}$	2	1	$b_1^+ = b_1^- + b_2^- + 1$
	$\mathbf{ss_s}$	2	2	$b_1^+ + b_2^+ = b_1^- + b_2^- + 2$
	$\mathbf{ss_s}$	3	1	$b_1^+ = b_1^- + b_2^- + b_3^-$
	$\mathbf{ss_s}$	3	2	$b_1^+ + b_2^+ = b_1^- + b_2^- + b_3^- + 1$
	$\mathbf{ss_s}$	4	1	$b_1^+ = b_1^- + b_2^- + b_3^- + b_4^- - 1$
	$\mathbf{ss_s}$	4	2	$b_1^+ + b_2^+ = b_1^- + b_2^- + b_3^- + b_4^-$
	r	1	0	$b_1^- = 3$
$p \in M(\mathcal{T})$	**a**	0	1	$b_1^+ = 7$
	ssa	1	1	$b_1^+ = b_1^- + 2$
	ssa	2	1	$b_1^+ = b_1^- + b_2^- + 1$
	ssr	1	1	$b_1^- = b_1^+ + 2$
	ssr	1	2	$b_1^- = b_1^+ + b_2^+ + 1$
	r	1	0	$b_1^- = 7$

Proof. First, we prove the inequalities on the degree of the vertices v in L.

Let L be a Lyapunov graph associated to a Gutierrez-Sotomayor flow X_t and f a Lyapunov function on a two-manifold with simple singularities M. If p is a singularity such that $f(p) = c$, denote by N_1 the component of $f^{-1}([c-\epsilon, c+\epsilon])$, with $\epsilon > 0$ sufficiently small so that it contains only one singular point p. Let $N_0 = N_1 \cap f^{-1}(c-\epsilon)$. Then (N_1, N_0) is an index pair for p.

Since p is a singularity then $\partial N_1 \neq \emptyset$, thus, $H_2(N_1) = 0$. Also, N_1 is connected, thus $\widetilde{H}_0(N_1) = 0$. Let v be the vertex of L labelled with p then dim $H_0(N_0) = e_v^-$ and if $N_0 \neq \emptyset$ then dim $\widetilde{H}_0(N_0) = e_v^- - 1$.

Hence, for N_0 we have the following long exact sequence:

$$0 \longrightarrow CH_2(p) \xrightarrow{\partial_2} H_1(N_0) \xrightarrow{i_1} H_1(N_1) \xrightarrow{p_1} CH_1(p) \xrightarrow{\partial_1} \widetilde{H}_0(N_0) \longrightarrow 0.$$

Secondly, we prove the conditions on the weights of the edges incident to v.

The Theorem 4.2 relates the first Betti number of the boundary components that are entering sets and exiting sets for the flow, ∂N_1^+ and ∂N_1^-, the isolating block (N_1, N_0) of a singularity, $p \in M$, of X_t with the number of boundary components of N_1 and the numerical Conley indices of $p \in M$. Since the fixed point $p \in M$ corresponds to a vertex v on the Lyapunov graph, ∂N_1^+ (∂N_1^-) corresponds to edges positively (negatively) incident to v then Theorem 4.2 relates the degree (of the entering and exiting edges) of v, to the weights on the edges (entering and exiting) incidents to v and the numerical Conley index with which v was labelled.

$$(h_2 - h_1 + h_0) - (h_2 - h_1 + h_0)^* = e_v^+ - \mathcal{B}^+ - e_v^- + \mathcal{B}^- \tag{8}$$

where $\mathcal{B}^+ = \sum_{k=1}^{e_v^+} b_k^+$ and $\mathcal{B}^- = \sum_{k=1}^{e_v^-} b_k^-$.

Considering all the possibilities for e^+, e^- in the inequalities involving the degree of v and using the above equations, we obtain the weights on the table and the result follows.

□

Example 5.4. *Gluing the isolating blocks to obtain a Gutierrez-Sotomayor flow.*

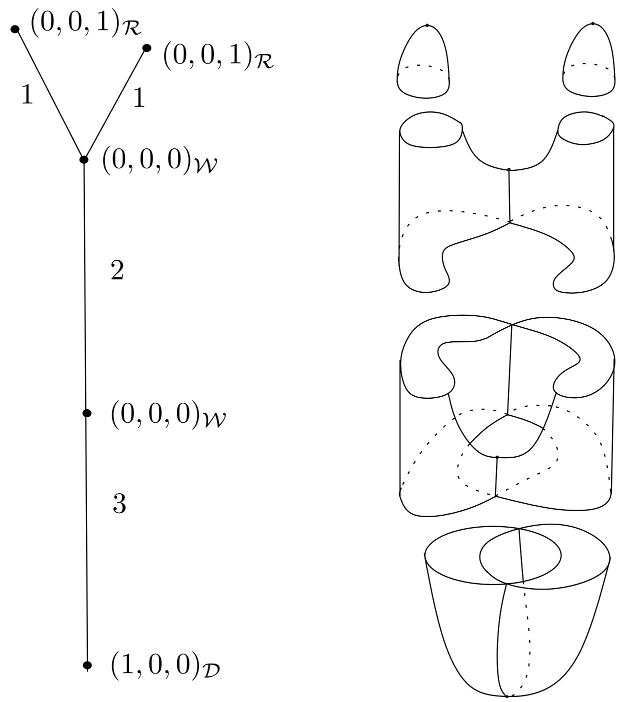

FIGURE 39. An abstract Lyapunov graph and its realization as a GS flow.

We conclude this paper with a couple of remarks. Example 5.4 suggests that one may be able to find sufficient conditions on abstract Lyapunov graphs in order to check their realizability. See Figure 39. This has not yet been done and remains an open question.

Also, one would like to include in a study similar to this one, the inclusion of periodic orbits and singular cycles.

References

[1] M. A. Bertolim, M. P. Mello and K. A. de Rezende. *Lyapunov graph continuation*. Ergodic Theory Dyn. Systems, 23, no. 1, pp. 1- 58, 2003. DOI: 10.1017/s014338570200086x

[2] C. Conley. *Isolated Invariant sets and the Morse index*. CBMS Regional Conf. Ser. in Math., 38. A.M.S. 1978.

[3] K. de Rezende and R. Franzosa. *Lyapunov graphs and flows on surfaces*. Trans. Amer. Math. Soc., no. 340, pp. 767-784, 1993. DOI: 10.1090/s0002-9947-1993-1127155-9

[4] J. Franks. *Nonsingular Smale flows on* S^3. Topology, 24 no. 3, pp. 265-282, 1985. DOI: 10.1016/0040-9383(85)90002-3

[5] M. Goresky and R. MacPherson. *Stratified Morse Theory*. Springer, 1988.

[6] C. Gutierrez and J. Sotomayor. *Stable Vector Fields on Manifolds with Simple Singularities*. Proc. London Math. Soc. (3) 45, no. 1, pp. 97-112, 1982. DOI: 10.1112/plms/s3-45.1.97

[7] J. Milnor *Collected Papers of John Milnor, III, Differential Topology*. AMS Collected Works, vol. 19, Stony Brook University, Stony Brook, NY, pp. 65-88, 2007.

[8] M. Peixoto *Structural stability on two-dimensional manifolds*. Topology, no. 1, pp. 101-120, 1962. DOI: 10.1016/0040-9383(65)90018-2

[9] S. Smale. *On Gradient Dynamical Systems*. Annals of Mathematics, vol. 74, pp.199-206, 1961.

[10] R. Thom *Ensembles et morphismes stratifis*. Bull. Amer. Math. Soc., 75, pp. 240-284, 1969. DOI: 10.1090/s0002-9904-1969-12138-5

H. Montúfar, Faculty of Mathematics, Federal University of Uberlândia, Uberlândia - Brazil
Email address: montufar@ufu.br

K. A. de Rezende, Institute of Mathematics and Scientific Computing, University of Campinas, Campinas - Brazil
Email address: ketty@unicamp.br

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MORIN SINGULARITIES OF COLLECTIONS OF ONE-FORMS AND VECTOR FIELDS

CAMILA M. RUIZ

Abstract. Inspired by the properties of a collection of n gradient vector fields $\nabla f_1, \dots, \nabla f_n$ from a Morin map $f = (f_1, \dots, f_n) : M \to \mathbb{R}^n$, with $\dim M \geq n$, we introduce the notion of Morin singularities in the context of collections of one-forms and collections of vector fields. We also study the singularities of generic one-forms which are related to specific collections (Morin collections) and we generalize a result of T. Fukuda on Euler characteristic ([5, Theorem 1]) for the case of collections of one-forms and vector fields.

1. Introduction

Morin maps are those which admit only Morin singularities. It is well known that these singularities are stable, and conversely, that corank one stable map-germs are Morin singularities. Thereby, Morin singularities are fundamental and frequently arise as singularities of maps from one manifold to another, as observed by K. Saji in [15]. These singularities have been studied by many authors in different contexts as [9, 1, 5, 12, 13], and more recently [7, 18, 21, 6, 3, 8, 2, 15, 16, 14, 11]. In particular, J.M. Èliašberg [4], J.R. Quine [10], T. Fukuda [5], O. Saeki [12] and N. Dutertre and T. Fukui [3] investigate relations between the topology of a manifold and the topology of the critical locus of maps with Morin singularities.

In this work, we introduce the notion of Morin singularities in the context of collections of one-forms that are not necessarily differential (Definition 2.26) and collections of vector fields that are not necessarily gradient (Definition 2.28). Our main result (Theorem 4.13) is a generalization of Fukuda's Theorem on Euler characteristic [5, Theorem 1] for the case of Morin collections of smooth one-forms: we show that if $\omega = \{\omega_i\}_{1 \leq i \leq n}$ is a Morin collection (Definition 2.26) defined on an m-dimensional compact manifold M then

$$\chi(M) \equiv \sum_{k=1}^{n} \chi(\overline{A_k(\omega)}) \mod 2,$$

where $\chi(M)$ denotes the Euler characteristic of M and $A_k(\omega)$ is the set given by the A_k-type singular points of ω.

Our original inspiration was provided by the following properties of a collection $\{\nabla f_1, \dots, \nabla f_n\}$ of n gradient vector fields from a Morin map $f = (f_1, \dots, f_n)$.

Let $f : M^m \to \mathbb{R}^n$ be a smooth Morin map defined on an m-dimensional Riemannian manifold M, with $m \geq n$. The singular points of $f = (f_1, \dots, f_n)$ are the points $x \in M$ where the rank of the derivative $df(x)$ is equal to $n-1$. By taking the gradient of each coordinate function $f_1, \dots, f_n$, we obtain a "singular collection" of n vector fields $\{\nabla f_1, \dots, \nabla f_n\}$ defined on M whose singular locus Σ is given by

$$\Sigma = \{x \in M \mid \operatorname{rank}(\nabla f_1(x), \dots, \nabla f_n(x)) = n-1\}.$$

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For any $k = 1, \dots, n$, it is known that the sets $A_k(f)$ and $\overline{A_k(f)}$, given by the A_k-type singular points of f and its topological closure, respectively, are $(n-k)$-dimensional smooth submanifolds of M satisfying

(i) $\Sigma = \overline{A_1(f)}$;

(ii) $\overline{A_k(f)} = \bigcup_{i=k}^{n} A_i(f)$;

(iii) For each $x \in \Sigma$,

$$\operatorname{rank} df|_{\overline{A_k(f)}}(x) = \begin{cases} n-k, & \text{if } x \in A_k(f), \\ n-k-1, & \text{if } x \in \overline{A_{k+1}(f)}; \end{cases}$$

(see [5], [9], [12] for Morin singularities). By item (iii), the intersection of the vector space spanned by $\nabla f_1(x), \dots, \nabla f_n(x)$ and the normal vector space to $\overline{A_k(f)}$ at x is a vector subspace whose dimension is given by

$$\dim(\langle \nabla f_1(x), \dots, \nabla f_n(x)\rangle \cap N_x\overline{A_k(f)}) = \begin{cases} k-1, & \text{if } x \in A_k(f), \\ k, & \text{if } x \in \overline{A_{k+1}(f)}. \end{cases}$$

Then, $\langle \nabla f_1(x), \dots, \nabla f_n(x)\rangle$ and $N_x\overline{A_k(f)}$ intersect transversally at x if and only if $x \in A_k(f)$. Otherwise, if $x \in \overline{A_{k+1}(f)}$ and $\{z_1(x), \dots, z_{n-k-1}(x)\}$ is a basis of a vector subspace complementary to $\langle \nabla f_1(x), \dots, \nabla f_n(x)\rangle \cap N_x\overline{A_k(f)}$ in $\langle \nabla f_1(x), \dots, \nabla f_n(x)\rangle$ then

$$\dim(\langle z_1(x), \dots, z_{n-k-1}(x)\rangle \cap N_x\overline{A_{k+1}(f)}) = \begin{cases} 0, & \text{if } x \in A_{k+1}(f), \\ 1, & \text{if } x \in \overline{A_{k+2}(f)}. \end{cases}$$

Therefore $\langle z_1(x), \dots, z_{n-k-1}(x)\rangle$ and $N_x\overline{A_{k+1}(f)}$ intersect transversally at x if and only if $x \in A_{k+1}(f)$, and A_{k+1}-type singular points of f can be distinguished from $\overline{A_{k+2}(f)}$ by this transversality or, equivalently, by the dimension of such intersection. We will follow this idea to define Morin singularities of collections.

This paper is organized as follows. In Section 2, we consider a non-degenerate collection of smooth one-forms $\omega = \{\omega_i\}_{1\le i\le n}$ (Definition 2.2) defined on a smooth m-dimensional manifold M, with $m \ge n$. Then, we define the A_k-type singularities of ω, for $k = 1, \dots, n$, in order to decompose the singular set $\Sigma^1(\omega)$ of ω into disjoint submanifolds according to the type of each singular point. To do that, we give an inductive definition of the singular subsets $\Sigma^k(\omega)$ and $A_k(\omega)$, in which we take successive transversality conditions (Definitions 2.3, 2.9, 2.10, 2.11, 2.18, 2.19, 2.25 and Remark 2.14). In particular, if the required transversality conditions hold, we show that the singular subsets $A_k(\omega)$ and $\Sigma^k(\omega) = \overline{A_k(\omega)}$ are $(n-k)$-dimensional smooth submanifolds of M (Lemmas 2.4, 2.12, 2.20 and Theorem 2.22) such that $\overline{A_k(\omega)} = \cup_{i\ge k} A_i(f)$ (Remark 2.24). Furthermore, in Proposition 2.23 (a) and Lemma 4.5 we provide equations that define the singular sets $\Sigma^k(\omega)$ locally.

We will say that $\omega = \{\omega_i\}_{1\le i\le n}$ is a Morin collection of one-forms (Definition 2.26) if it admits only Morin A_k-type singular points, for $k = 1, \dots, n$ (see Remark 2.27).

The definition of Morin singularities for collections of n one-forms can be analogously adapted to collections of n vector fields as follows. When considering a smooth manifold M, differential one-forms are naturally dual to vector fields, more specifically, if we fix a Riemannian metric on M then there exists an isomorphism between the tangent and cotangent bundles of M, such that vector fields and one-forms can be identified. To illustrate this notion, we give some examples of Morin collections of vector fields in the end of Section 2.

We remark that in the maximal case, that is, when we have a Morin collection of m vector fields defined on an m-dimensional manifold, our definition of A_k-type singularities is equivalent to that A_k-type singularities presented by Saji *et al.* [17].

Let $L \in \mathbb{R}P^{n-1}$ be a straight line in $\mathbb{R}^n$ and let $\pi_L : \mathbb{R}^n \to L$ be the orthogonal projection to L. In [5], T. Fukuda applied Morse theory and well known properties of the singular sets $A_k(f)$ of a Morin map $f : M \to \mathbb{R}^n$ to study critical points of mappings $\pi_L \circ f : M \to L$ and their restrictions to singular sets $\pi_L \circ f|_{A_k(f)}$ and $\pi_L \circ f|_{\overline{A_k(f)}}$. Similarly, in Sections 3 and 4, we investigate the zeros of a generic one-form

$$\xi(x) = \sum_{i=1}^{n} a_i \omega_i(x)$$

associated to a Morin collection of n smooth one-forms $\omega = \{\omega_i\}_{1\le i\le n}$. We verify that ξ, $\xi_{|_{A_k(\omega)}}$ and $\xi_{|_{\overline{A_k(\omega)}}}$ have properties that are similar to that of generic orthogonal projections $\pi_L \circ f(x)$ associated to Morin maps f.

More precisely, let $a = (a_1, \dots, a_n) \in \mathbb{R}^n \smallsetminus \{\vec{0}\}$ and let $\omega = \{\omega_i\}_{1\le i\le n}$ be a Morin collection of smooth one-forms on M, in Section 3 we prove that the zero set of $\xi(x) = \sum_{i=1}^{n} a_i \omega_i(x)$ is contained in $\Sigma^1(\omega)$ (Lemma 3.1) and, for almost every $a \in \mathbb{R}^n \smallsetminus \{\vec{0}\}$, the zero set of $\xi_{|_{\Sigma^k(\omega)}}$ does not intercept $\Sigma^{k+2}(\omega)$, for $k = 0, \dots, n-2$ (Lemmas 3.6 and 3.7). Moreover, we present necessary and sufficient conditions for a zero of $\xi_{|_{\Sigma^{k+1}(\omega)}}$ to be a zero of $\xi_{|_{\Sigma^k(\omega)}}$, for $k = 0, \dots, n-1$ (Lemmas 3.2 and 3.3). In Section 4, we prove that generically the one-form $\xi(x)$ and its restrictions $\xi_{|_{\Sigma^k(\omega)}}$ and $\xi_{|_{A_k(\omega)}}$ admit only non-degenerate zeros (Lemmas 4.6, 4.7, 4.8 and 4.12). In Lemmas 4.9, 4.10 and 4.11, we give conditions for a non-degenerate zero of $\xi_{|_{\Sigma^{k+1}(\omega)}}$ to be a non-degenerate zero of $\xi_{|_{\Sigma^k(\omega)}}$, for $k = 0, \dots, n-1$.

As a consequence of these results, we end the paper with Theorem 4.13 whose proof uses the classical Poincaré-Hopf Theorem for one-forms.

2. Morin singularities of collections of one-forms

Let $0 < n \le m$ be integer numbers and let M be an m-dimensional smooth manifold with cotangent space at $x \in M$ denoted by T_x^*M. We define the "n-cotangent bundle" of M by

$$T^*M^n = \{(x, \varphi_1, \dots, \varphi_n) \mid x \in M;\ \varphi_i \in T_x^*M, i = 1, \dots, n\},$$

which is an $m(n+1)$-dimensional smooth manifold locally diffeomorphic to $U \times M_{m,n}(\mathbb{R})$, where $U \subset \mathbb{R}^m$ is an open set and $M_{m,n}(\mathbb{R})$ denotes the set of real matrices of size $m \times n$.

Lemma 2.1. *Let $T^*M^{n,n-1} \subset T^*M^n$ be defined by*

$$T^*M^{n,n-1} = \{(x, \varphi_1, \dots, \varphi_n) \in T^*M^n \mid \operatorname{rank}(\varphi_1, \dots, \varphi_n) = n-1\}.$$

Then $T^*M^{n,n-1}$ is smooth a submanifold of T^*M^n of dimension $n(m+1)-1$.

Proof. Let $M_{m,n}^{n-1}(\mathbb{R})$ be the smooth submanifold of $M_{m,n}(\mathbb{R})$ of codimension $m-n+1$ consisting of the matrices with rank equal to $n-1$. The set $T^*M^{n,n-1}$ is locally diffeomorphic to $U \times M_{m,n}^{n-1}(\mathbb{R})$, where $U \subset \mathbb{R}^m$ is an open subset. Thus, $T^*M^{n,n-1}$ is a smooth submanifold of T^*M^n of dimension $n(m+1)-1$. □

Let $\omega = \{\omega_i\}_{1\le i\le n}$ be a collection of n smooth one-forms on M, we will consider the smooth map $\omega : M \to T^*M^n$ defined by

$$\omega(x) = (x, \omega_1(x), \dots, \omega_n(x)).$$

Definition 2.2. *We say that $\omega = \{\omega_i\}_{1\le i\le n}$ is a non-degenerate collection if the map $\omega : M \to T^*M^n$ as above satisfies the following conditions:*

(a) $\omega \pitchfork T^*M^{n,n-1}$ *in* T^*M^n,
(b) $\omega^{-1}(T^*M^{n,\le n-2}) = \varnothing$,
where $T^*M^{n,\le n-2} = \{(x,\varphi_1,\dots,\varphi_n) \in T^*M^n \mid \operatorname{rank}(\varphi_1,\dots,\varphi_n) \le n-2\}$.

Notice that this definition implies that if $\omega = \{\omega_i\}_{1\le i\le n}$ is a non-degenerate collection on M, then for each $x \in M$ the rank of $\omega_1(x),\dots,\omega_n(x)$ is either equal to n or equal to $n-1$.

Definition 2.3. *Let* $\omega = \{\omega_i\}_{1\le i\le n}$ *be a non-degenerate collection on* M*. We define the singular set of the collection* ω *as the set* $\Sigma^1(\omega)$ *of points* $x \in M$ *where the rank of* ω *is not maximal, that is*

$$\Sigma^1(\omega) = \{x \in M \mid \operatorname{rank}(\omega_1(x),\dots,\omega_n(x)) = n-1\}.$$

Lemma 2.4. *Let* $\omega = \{\omega_i\}_{1\le i\le n}$ *be a non-degenerate collection on* M*. Then* $\Sigma^1(\omega)$ *is either the empty set or an* $(n-1)$*-dimensional smooth submanifold of* M*.*

Proof. Notice that $\Sigma^1(\omega) = \omega^{-1}(T^*M^{n,n-1})$ and that $\omega \pitchfork T^*M^{n,n-1}$. Thus, if $\Sigma^1(\omega) \ne \varnothing$ then $\Sigma^1(\omega)$ is a smooth submanifold of M of codimension $m-n+1$ and the result follows. □

Let $\omega = \{\omega_i\}_{1\le i\le n}$ be a non-degenerate collection of smooth one-forms defined on an m-dimensional smooth manifold M. If ω satisfies some transversality conditions, we will define the A_k-type singularities of ω, for $k = 1,\dots,n$, in order to decompose the singular set $\Sigma^1(\omega)$ into disjoint submanifolds according to the type of each singular point. Firstly, we define the A_1-type singular points in $\Sigma^1(\omega)$. We will denote by $\Sigma^2(\omega)$ the subset of $\Sigma^1(\omega)$ given by all singular points of ω that are not A_1-type. For each $k = 2,\dots,n$, we repeat this process defining the A_k-type singular points in $\Sigma^k(\omega)$ and denoting by $\Sigma^{k+1}(\omega)$ the subset of $\Sigma^k(\omega)$ given by all singular points of ω that are not A_k-type. To do that, we present in this section an inductive definition of A_k-type Morin singularities of ω.

Remark 2.5. *Let* $S \subset M$ *be a smooth submanifold of* M*. We will adopt the following notation*

$$N_x^*S = \{\psi \in T_x^*M \,|\, \psi(T_xS) = 0\}.$$

Definition 2.6. *Let* $\omega = \{\omega_i\}_{1\le i\le n}$ *be a non-degenerate collection on* M*. Given*

$$(x,\varphi) = (x,\varphi_1,\dots,\varphi_{n-1}),$$

we define the sets

$$T^*_{\Sigma^1}M^{n-1} = \{(x,\varphi) \mid x \in \Sigma^1(\omega); \varphi_1,\dots,\varphi_{n-1} \in T_x^*M\}$$

and

$$\begin{aligned} N^*_{\Sigma^1}M^{n-1} = \{(x,\varphi) \in T^*_{\Sigma^1}M^{n-1} \mid \operatorname{rank}(\varphi_1,\dots,\varphi_{n-1}) = n-1,\\ \dim(\langle\varphi_1,\dots,\varphi_{n-1}\rangle \cap N_x^*\Sigma^1(\omega)) = 1\}, \end{aligned}$$

where $\langle\varphi_1,\dots,\varphi_{n-1}\rangle$ *denotes the subspace of* T_x^*M *spanned by* $\{\varphi_1,\dots,\varphi_{n-1}\}$.

Lemma 2.7. $T^*_{\Sigma^1}M^{n-1}$ *is a smooth manifold of dimension* $m(n-1)+n-1$.

Proof. For a non-degenerate collection ω, we know that $\Sigma^1(\omega)$ is an $(n-1)$-dimensional smooth submanifold of M. Then, for each $(x,\varphi) \in T^*_{\Sigma^1}M^{n-1}$ there exists an open subset $V \subset \mathbb{R}^{n-1}$ such that $T^*_{\Sigma^1}M^{n-1}$ is locally diffeomorphic to $V \times M_{m,n-1}(\mathbb{R})$ near (x,φ) and the result follows. □

Lemma 2.8. $N^*_{\Sigma^1}M^{n-1}$ *is a smooth hypersurface of* $T^*_{\Sigma^1}M^{n-1}$*, that is, a smooth submanifold of dimension* $m(n-1)+n-2$.

Proof. Since ω is non-degenerate, it follows from Lemma 2.4 that $\Sigma^1(\omega)$ is a smooth submanifold of codimension $m-n+1$ of M. Then, for each $p \in \Sigma^1(\omega)$ there exist an open neighborhood $\mathcal{U}$ of p in M and smooth functions $F_1, \ldots, F_{m-n+1} : \mathcal{U} \to \mathbb{R}$ such that

$$\mathcal{U} \cap \Sigma^1(\omega) = \{x \in \mathcal{U} \mid F_1(x) = \ldots = F_{m-n+1}(x) = 0\}$$

with $\operatorname{rank}(dF_1(x), \ldots, dF_{m-n+1}(x)) = m-n+1$, for each $x \in \mathcal{U} \cap \Sigma^1(\omega)$, and

$$N_p^* \Sigma^1(\omega) = \langle dF_1(p), \ldots, dF_{m-n+1}(p) \rangle.$$

If $(p, \tilde{\varphi}) = (p, \tilde{\varphi}_1, \ldots, \tilde{\varphi}_{n-1}) \in N^*_{\Sigma^1} M^{n-1}$ then

$$\operatorname{rank}(\tilde{\varphi}_1, \ldots, \tilde{\varphi}_{n-1}, dF_1(p), \ldots, dF_{m-n+1}(p)) = m-1,$$

since by the definition of $N^*_{\Sigma^1} M^{n-1}$, $\operatorname{rank}(\tilde{\varphi}_1, \ldots, \tilde{\varphi}_{n-1}) = n-1$ and

$$\dim(\langle \tilde{\varphi}_1, \ldots, \tilde{\varphi}_{n-1} \rangle \cap N_p^* \Sigma^1(\omega)) = 1.$$

In this way,

$$\det(dF_1(p), \ldots, dF_{m-n+1}(p), \tilde{\varphi}_1, \ldots, \tilde{\varphi}_{n-1}) = 0$$

and fixing the notation $\tilde{\varphi}_i = (\tilde{\varphi}_i^1, \ldots, \tilde{\varphi}_i^m)$ for $i = 1, \ldots, n-1$, we can assume that the minor

$$\begin{vmatrix} \dfrac{\partial F_1}{\partial x_1}(p) & \cdots & \dfrac{\partial F_{m-n+1}}{\partial x_1}(p) & \tilde{\varphi}_1^1 & \cdots & \tilde{\varphi}_{n-2}^1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial F_1}{\partial x_{m-1}}(p) & \cdots & \dfrac{\partial F_{m-n+1}}{\partial x_{m-1}}(p) & \tilde{\varphi}_1^{m-1} & \cdots & \tilde{\varphi}_{n-2}^{m-1} \end{vmatrix}$$

does not vanish and consequently, that

$$\begin{vmatrix} \dfrac{\partial F_1}{\partial x_1}(x) & \cdots & \dfrac{\partial F_{m-n+1}}{\partial x_1}(x) & \varphi_1^1 & \cdots & \varphi_{n-2}^1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial F_1}{\partial x_{m-1}}(x) & \cdots & \dfrac{\partial F_{m-n+1}}{\partial x_{m-1}}(x) & \varphi_1^{m-1} & \cdots & \varphi_{n-2}^{m-1} \end{vmatrix} \neq 0 \tag{1}$$

for all $(x, \varphi) \in (\Sigma^1(\omega) \cap \mathcal{U}) \times \mathcal{V}$, where $\mathcal{V} \subset \mathbb{R}^{m(n-1)}$ is an open neighborhood of $\tilde{\varphi}$. Thus, $N^*_{\Sigma^1} M^{n-1}$ can be locally given by

$$N^*_{\Sigma^1} M^{n-1} = \{(x, \varphi) \in \mathcal{U} \times \mathcal{V} \mid F_1 = \ldots = F_{m-n+1} = \Delta = 0\},$$

where $\Delta(x, \varphi) = \det(dF_1(x), \ldots, dF_{m-n+1}(x), \varphi_1, \ldots, \varphi_{n-1})$. Let $B(x, \varphi)$ be the square matrix of order m whose columns are given by the coefficients of the one-forms $dF_1(x)$, …, $dF_{m-n+1}(x)$, φ_1, …, φ_{n-1}:

$$B(x, \varphi) = \begin{pmatrix} dF_1(x) & \cdots & dF_{m-n+1}(x) & \varphi_1 & \cdots & \varphi_{n-1} \end{pmatrix}.$$

By Laplace expansion along the last column of $B(x, \varphi)$, we have

$$\Delta(x, \varphi) = \sum_{i=1}^m \varphi_{n-1}^i \operatorname{cof}(\varphi_{n-1}^i, B),$$

where $\operatorname{cof}(\varphi_{n-1}^i, B)$ denotes the cofactor of φ_{n-1}^i in the matrix $B(x, \varphi)$. Thus

$$\frac{\partial \Delta}{\partial \varphi_{n-1}^m}(x, \varphi) = \sum_{i=1}^m \operatorname{cof}(\varphi_{n-1}^i, B) \frac{\partial \varphi_{n-1}^i}{\partial \varphi_{n-1}^m} + \varphi_{n-1}^i \frac{\partial \operatorname{cof}(\varphi_{n-1}^i, B)}{\partial \varphi_{n-1}^m}$$

and since $\mathrm{cof}(\varphi^i_{n-1}, B)$ does not depend on the variable φ^m_{n-1}, we have

$$\frac{\partial\, \mathrm{cof}(\varphi^i_{n-1}, B)}{\partial \varphi^m_{n-1}} = 0, \text{ for } i = 1, \dots, m.$$

Then,

$$\frac{\partial \Delta}{\partial \varphi^m_{n-1}}(x, \varphi) = \mathrm{cof}(\varphi^m_{n-1}, B) \overset{(1)}{\neq} 0,$$

and the derivative of $\Delta(x,\varphi)$ with respect to φ, denoted by $d_\varphi \Delta(x,\varphi)$, does not vanish. This implies that the matrix

$$\begin{bmatrix} dF_1(x) \\ \vdots \\ dF_{m-n+1}(x) \\ d\Delta(x,\varphi) \end{bmatrix} = \begin{bmatrix} d_x F_1(x) & \vdots & \\ \vdots & \vdots & O_{(m-n+1)\times(n-1)} \\ d_x F_{m-n+1}(x) & \vdots & \\ \cdots\cdots\cdots\cdots\cdots & \vdots & \cdots\cdots\cdots\cdots\cdots\cdots \\ d_x \Delta(x,\varphi) & \vdots & d_\varphi \Delta(x,\varphi) \end{bmatrix}$$

has rank $m - n + 2$, where $O_{(m-n+1)\times(n-1)}$ denotes a null matrix. Hence,

$$\mathrm{rank}(dF_1(x), \dots, dF_{m-n+1}(x), d\Delta(x,\varphi)) = m - n + 2,$$

for each $(x,\varphi) \in N^*_{\Sigma^1} M^{n-1} \cap (\mathcal{U} \times \mathcal{V})$. Therefore, $N^*_{\Sigma^1} M^{n-1}$ is a smooth submanifold of $T^*_{\Sigma^1} M^{n-1}$ of dimension $m + m(n-1) - (m-n+2) = m(n-1) + n - 2$. □

Let $\omega = \{\omega_i\}_{1\le i\le n}$ be a non-degenerate collection on M and $\langle \omega_1(x), \dots, \omega_n(x)\rangle$ the subspace of $T^*_x M$ spanned by $\{\omega_1(x), \dots, \omega_n(x)\}$. Then for each $p \in \Sigma^1(\omega)$, $\dim\langle \omega_1(p), \dots, \omega_n(p)\rangle = n-1$, and there exist an open neighborhood $\mathcal{U}_p$ of p in M and a collection $\{\Omega_1, \dots, \Omega_{n-1}\}$ of $n-1$ smooth one-forms on $\mathcal{U}_p$ such that $\{\Omega_1(x), \dots, \Omega_{n-1}(x)\}$ is a basis of $\langle \omega_1(x), \dots, \omega_n(x)\rangle$ for each $x \in \mathcal{U}_p \cap \Sigma^1(\omega)$. Let $\Omega^1 : \mathcal{U}_p \cap \Sigma^1(\omega) \to T^*_{\Sigma^1} M^{n-1}$ be the map given by

$$\Omega^1(x) = (x, \Omega_1(x), \dots, \Omega_{n-1}(x));$$

we define:

Definition 2.9. *We say that collection $\omega = \{\omega_i\}_{1\le i\le n}$ satisfies the "condition I_1" if for each $p \in \Sigma^1(\omega)$ there exist an open neighborhood $\mathcal{U}_p$ of p in M and a map $\Omega^1 : \mathcal{U}_p \cap \Sigma^1(\omega) \to T^*_{\Sigma^1} M^{n-1}$ as defined above, such that on $\mathcal{U}_p$ the following properties hold:*
(a) $\Omega^1 \pitchfork N^*_{\Sigma^1} M^{n-1}$ *in* $T^*_{\Sigma^1} M^{n-1}$,
(b) $(\Omega^1)^{-1}(N^*_{\Sigma^1} M^{n-1,\ge 2}) = \varnothing$,
where

$$N^*_{\Sigma^1} M^{n-1,\ge 2} = \{(x,\varphi) \in T^*_{\Sigma^1} M^{n-1} \mid \mathrm{rank}(\varphi_1, \dots, \varphi_{n-1}) = n-1, \dim(\langle \varphi_1, \dots, \varphi_{n-1}\rangle \cap N^*_x \Sigma^1(\omega)) \ge 2\}.$$

Notice that if ω satisfies the condition I_1, then for each $x \in \Sigma^1(\omega) \cap \mathcal{U}_p$,

$$\dim(\langle \Omega_1(x), \dots, \Omega_{n-1}(x)\rangle \cap N^*_x \Sigma^1(\omega))$$

is either equal to 0 or equal to 1. We will prove in Proposition 2.23 that this dimension and the condition I_1 do not depend on the choice of the basis $\{\Omega_1, \dots, \Omega_{n-1}\}$.

Definition 2.10. *Let $\omega = \{\omega_i\}_{1\le i\le n}$ be a non-degenerate collection that satisfies the condition I_1. Given $p \in \Sigma^1(\omega)$, consider an open neighborhood $\mathcal{U}_p$ of p in M and a map*

$$\Omega^1(x) = (x, \Omega_1(x), \dots, \Omega_{n-1}(x))$$

as in Definition 2.9. We define the sets $A_1(\omega)$ and $\Sigma^2(\omega)$ as follows:

(*a*) *We say that* $x \in \mathcal{U}_p$ *belongs to* $A_1(\omega)$ *if* $x \in \Sigma^1(\omega)$ *and*

$$\dim(\langle \Omega_1(x), \dots, \Omega_{n-1}(x)\rangle \cap N_x^*\Sigma^1(\omega)) = 0.$$

(*b*) *We say that* $x \in \mathcal{U}_p$ *belongs to* $\Sigma^2(\omega)$ *if* $x \in \Sigma^1(\omega) \setminus A_1(\omega)$, *that is, if* $x \in \Sigma^1(\omega)$ *and*

$$\dim(\langle \Omega_1(x), \dots, \Omega_{n-1}(x)\rangle \cap N_x^*\Sigma^1(\omega)) = 1.$$

Then, for each $p \in \Sigma^1(\omega)$ *we may write*

$$A_1(\omega) \cap \mathcal{U}_p = \{x \in \Sigma^1(\omega) \cap \mathcal{U}_p \mid \dim(\langle \Omega_1(x), \dots, \Omega_{n-1}(x)\rangle \cap N_x^*\Sigma^1(\omega)) = 0\};$$
$$\Sigma^2(\omega) \cap \mathcal{U}_p = \{x \in \Sigma^1(\omega) \cap \mathcal{U}_p \mid \dim(\langle \Omega_1(x), \dots, \Omega_{n-1}(x)\rangle \cap N_x^*\Sigma^1(\omega)) = 1\};$$

and we have

$$A_1(\omega) = \bigcup_{p \in \Sigma^1(\omega)} (A_1(\omega) \cap \mathcal{U}_p) \quad \text{and} \quad \Sigma^2(\omega) = \bigcup_{p \in \Sigma^1(\omega)} \left(\Sigma^2(\omega) \cap \mathcal{U}_p\right).$$

Definition 2.11. *Let* $\omega = \{\omega_i\}_{1 \le i \le n}$ *be a non-degenerate collection on* M *that satisfies the condition* I_1. *We say that* $x \in M$ *is an* A_1*-type Morin singularity of* ω *if* $x \in A_1(\omega)$.

Lemma 2.12. *Let* $\omega = \{\omega_i\}_{1 \le i \le n}$ *be a non-degenerate collection on* M *that satisfies the condition* I_1. *Then* $\Sigma^2(\omega) \subset \Sigma^1(\omega)$ *and* $\Sigma^2(\omega)$ *is either the empty set or an* $(n-2)$*-dimensional smooth submanifold of* M.

Proof. Notice that, locally, $\Sigma^2(\omega) = (\Omega^1)^{-1}(N^*_{\Sigma^1}M^{n-1})$ and $\Omega^1 \pitchfork N^*_{\Sigma^1}M^{n-1}$. Thus, if $\Sigma^2(\omega) \neq \varnothing$ then $\Sigma^2(\omega)$ is a smooth submanifold of $\Sigma^1(\omega)$ of codimension 1 and the result follows. □

Lemma 2.13. *Let* $\omega = \{\omega_i\}_{1 \le i \le n}$ *be a non-degenerate collection on* M *that satisfies the condition* I_1. *For each* $p \in \Sigma^1(\omega)$,

$$p \in \Sigma^2(\omega) \Leftrightarrow \dim(\langle \omega_1(p), \dots, \omega_n(p)\rangle \cap N_p^*\Sigma^1(\omega)) = 1.$$

Proof. Given $p \in \Sigma^1(\omega)$, we can consider a neighborhood $\mathcal{U}_p$ of p in M and a map

$$\Omega^1(x) = (x, \Omega_1(x), \dots, \Omega_{n-1}(x)),$$

as in Definition 2.9, such that $\langle \Omega_1(p), \dots, \Omega_{n-1}(p)\rangle = \langle \omega_1(p), \dots, \omega_n(p)\rangle$. By Definition 2.10 (*b*), $p \in \Sigma^2(\omega)$ if and only if $\dim(\langle \Omega_1(p), \dots, \Omega_{n-1}(p)\rangle \cap N_p^*\Sigma^1(\omega)) = 1$. Thus, $p \in \Sigma^2(\omega)$ if and only if $\dim(\langle \omega_1(p), \dots, \omega_n(p)\rangle \cap N_p^*\Sigma^1(\omega)) = 1$. □

Remark 2.14. *The following results are used in the formulation of an inductive definition of* A_k*-type Morin singularities of* $\omega = \{\omega_i\}_{1 \le i \le n}$, *for* $k = 2, \dots, n$.

Let $3 \le k \le n$ *be an integer number and* $\omega = \{\omega_i\}_{1 \le i \le n}$ *a non-degenerate collection on* M *with singular set* $\Sigma^1(\omega)$. *Let us suppose that, for every* $i = 2, \dots, k-1$, $\Sigma^i(\omega)$ *is a smooth submanifold of* M *such that:*

(*a*) $\Sigma^i(\omega) \subset \Sigma^{i-1}(\omega) \subset \ldots \subset \Sigma^1(\omega)$;
(*b*) $\Sigma^i(\omega)$ *is the empty set or an* $(n-i)$*-dimensional smooth submanifold of* M;
(*c*) *For each* $p \in \Sigma^{i-1}(\omega)$, *we have*

$$p \in \Sigma^i(\omega) \Leftrightarrow \dim(\langle \omega_1(p), \dots, \omega_n(p)\rangle \cap N_p^*\Sigma^{i-1}(\omega)) = i-1.$$

Notice that in Lemmas 2.12 and 2.13 we have already proved that if $\omega = \{\omega_i\}_{1 \le i \le n}$ *satisfies the condition* I_1, *then the above hypothesis holds for* $k = 3$, *that is,* $\Sigma^2(\omega)$ *is a smooth submanifold of* M *satisfying* (*a*), (*b*) *and* (*c*). *Now, we assume that this hypothesis holds for every* $i = 2, \dots, k-1$, *with* $k > 3$, *and we will prove that it also holds for* $i = k$ *if* $\omega = \{\omega_i\}_{1 \le i \le n}$ *satisfies the "condition* I_{k-1}*" that will be given in Definition 2.18.*

Definition 2.15. *Let $r = n - k + 1$ and $(x, \varphi) = (x, \varphi_1, \dots, \varphi_r)$, we define the sets*

$$T^*_{\Sigma^{k-1}} M^r = \{(x, \varphi) \mid x \in \Sigma^{k-1}(\omega); \varphi_1, \dots, \varphi_r \in T^*_x M\}$$

and

$$\begin{aligned} N^*_{\Sigma^{k-1}} M^r = \{(x, \varphi) \in T^*_{\Sigma^{k-1}} M^r \mid \ & \operatorname{rank}(\varphi_1, \dots, \varphi_r) = r, \\ & \dim(\langle \varphi_1, \dots, \varphi_r \rangle \cap N^*_x \Sigma^{k-1}(\omega)) = 1\}, \end{aligned}$$

where $\langle \varphi_1, \dots, \varphi_r \rangle$ denotes the subspace of $T^*_x M$ spanned by $\{\varphi_1, \dots, \varphi_r\}$.

Lemma 2.16. *$T^*_{\Sigma^{k-1}} M^r$ is a smooth manifold of dimension $mr + r$.*

Proof. Analogously to the proof of Lemma 2.7. □

Lemma 2.17. *$N^*_{\Sigma^{k-1}} M^r$ is a smooth hypersurface of $T^*_{\Sigma^{k-1}} M^r$, that is, a smooth submanifold of dimension $mr + r - 1$.*

Proof. Analogously to the proof of Lemma 2.8. □

By hypothesis, for each $p \in \Sigma^{k-1}(\omega)$, we have that

$$\dim(\langle \omega_1(p), \dots, \omega_n(p) \rangle \cap N^*_p \Sigma^{k-2}(\omega)) = k - 2$$

and $\dim\langle \omega_1(p), \dots, \omega_n(p) \rangle = n-1$. Then, there exist an open neighborhood $\mathcal{U}_p$ of p in M and a collection $\{\Omega_1, \dots, \Omega_r\}$ of $r = n-k+1$ smooth one-forms on $\mathcal{U}_p$ such that $\{\Omega_1(x), \dots, \Omega_r(x)\}$ is a basis of a vector subspace complementary to $\langle \omega_1(x), \dots, \omega_n(x) \rangle \cap N^*_x \Sigma^{k-2}(\omega)$ in $\langle \omega_1(x), \dots, \omega_n(x) \rangle$ for each $x \in \mathcal{U}_p \cap \Sigma^{k-1}(\omega)$. That is, for each $x \in \mathcal{U}_p \cap \Sigma^{k-1}(\omega)$ we have that

$$\langle \Omega_1(x), \dots, \Omega_r(x) \rangle \oplus \big(\langle \omega_1(x), \dots, \omega_n(x) \rangle \cap N^*_x \Sigma^{k-2}(\omega) \big)$$

is equal to $\langle \omega_1(x), \dots, \omega_n(x) \rangle$. Let $\Omega^{k-1} : \mathcal{U}_p \cap \Sigma^{k-1}(\omega) \to T^*_{\Sigma^{k-1}} M^r$ be the map given by

$$\Omega^{k-1}(x) = (x, \Omega_1(x), \dots, \Omega_r(x)),$$

we define:

Definition 2.18. *We say that collection $\omega = \{\omega_i\}_{1 \le i \le n}$ satisfies the "condition I_{k-1}", if for each $p \in \Sigma^{k-1}(\omega)$ there exist an open neighborhood $\mathcal{U}_p$ of p in M and a map*

$$\Omega^{k-1} : \mathcal{U}_p \cap \Sigma^{k-1}(\omega) \to T^*_{\Sigma^{k-1}} M^r$$

as defined above, such that on $\mathcal{U}_p$ the following properties hold:
(a) $\Omega^{k-1} \pitchfork N^*_{\Sigma^{k-1}} M^r$ *in* $T^*_{\Sigma^{k-1}} M^r$;
(b) $(\Omega^{k-1})^{-1}(N^*_{\Sigma^{k-1}} M^{r, \ge 2}) = \varnothing$;
where

$$N^*_{\Sigma^{k-1}} M^{r, \ge 2} = \{(x, \varphi) \in T^*_{\Sigma^{k-1}} M^r \mid \operatorname{rank}(\varphi_1, \dots, \varphi_r) = r, \dim(\langle \varphi_1, \dots, \varphi_r \rangle \cap N^*_x \Sigma^{k-1}(\omega)) \ge 2\}.$$

Notice that if ω satisfies the condition I_{k-1}, then for each $x \in \Sigma^{k-1}(\omega) \cap \mathcal{U}_p$,

$$\dim(\langle \Omega_1(x), \dots, \Omega_r(x) \rangle \cap N^*_x \Sigma^{k-1}(\omega))$$

is either equal to 0 or equal to 1. We will prove in Proposition 2.23 that this dimension and the condition I_{k-1} do not depend on the choice of the basis $\{\Omega_1, \dots, \Omega_r\}$.

Definition 2.19. *Let $\omega = \{\omega_i\}_{1 \le i \le n}$ be a non-degenerate collection that satisfies the condition I_{k-1}. Given $p \in \Sigma^{k-1}(\omega)$, consider an open neighborhood $\mathcal{U}_p$ of p in M and a map*

$$\Omega^{k-1}(x) = (x, \Omega_1(x), \dots, \Omega_r(x))$$

as in Definition 2.18. We define the sets $A_{k-1}(\omega)$ and $\Sigma^k(\omega)$ as follows:

(a) *We say that* $x \in \mathcal{U}_p$ *belongs to* $A_{k-1}(\omega)$ *if* $x \in \Sigma^{k-1}(\omega)$ *and*

$$\dim(\langle \Omega_1(x), \dots, \Omega_r(x)\rangle \cap N_x^*\Sigma^{k-1}(\omega)) = 0.$$

(b) *We say that* $x \in \mathcal{U}_p$ *belongs to* $\Sigma^k(\omega)$ *if* $x \in \Sigma^{k-1}(\omega) \smallsetminus A_{k-1}(\omega)$*, that is, if* $x \in \Sigma^{k-1}(\omega)$ *and*

$$\dim(\langle \Omega_1(x), \dots, \Omega_r(x)\rangle \cap N_x^*\Sigma^{k-1}(\omega)) = 1.$$

Then, for each $p \in \Sigma^{k-1}(\omega)$ *we may write*

$$\begin{aligned} A_{k-1}(\omega) \cap \mathcal{U}_p &= \{x \in \Sigma^{k-1}(\omega) \cap \mathcal{U}_p \mid \dim(\langle \Omega_1(x), \dots, \Omega_r(x)\rangle \cap N_x^*\Sigma^{k-1}(\omega)) = 0\};\\ \Sigma^k(\omega) \cap \mathcal{U}_p &= \{x \in \Sigma^{k-1}(\omega) \cap \mathcal{U}_p \mid \dim(\langle \Omega_1(x), \dots, \Omega_r(x)\rangle \cap N_x^*\Sigma^{k-1}(\omega)) = 1\};\end{aligned}$$

and we have

$$A_{k-1}(\omega) = \bigcup_{p \in \Sigma^{k-1}(\omega)} (A_{k-1}(\omega) \cap \mathcal{U}_p) \quad \textit{and} \quad \Sigma^k(\omega) = \bigcup_{p \in \Sigma^{k-1}(\omega)} \left(\Sigma^k(\omega) \cap \mathcal{U}_p\right).$$

Lemma 2.20. *Under the hypothesis of Remark 2.14, let* $\omega = \{\omega_i\}_{1 \le i \le n}$ *be a non-degenerate collection on* M *that satisfies the condition* I_{k-1}*. Then* $\Sigma^k(\omega) \subset \Sigma^{k-1}(\omega)$ *and* $\Sigma^k(\omega)$ *is either the empty set or an* $(n-k)$*-dimensional smooth submanifold of* M*.*

Proof. Analogously to the proof of Lemma 2.12. □

Lemma 2.21. *Under the hypothesis of Remark 2.14, let* $\omega = \{\omega_i\}_{1 \le i \le n}$ *be a non-degenerate collection on* M *that satisfies the condition* I_{k-1}*. For each* $p \in \Sigma^{k-1}(\omega)$*,*

$$p \in \Sigma^k(\omega) \Leftrightarrow \dim(\langle \omega_1(p), \dots, \omega_n(p)\rangle \cap N_p^*\Sigma^{k-1}(\omega)) = k-1.$$

Proof. We have that $\Sigma^{k-1}(\omega) \subset \Sigma^{k-2}(\omega)$ and for each $p \in \Sigma^{k-1}(\omega)$:

(i) $N_p^*\Sigma^{k-2}(\omega) \subset N_p^*\Sigma^{k-1}(\omega)$ (see Remark 2.5);
(ii) $\dim(\langle \omega_1(p), \dots, \omega_n(p)\rangle \cap N_p^*\Sigma^{k-2}(\omega)) = k-2$;
(iii) There exist an open neighborhood $\mathcal{U}_p$ of p in M and a collection $\{\Omega_1(x), \dots, \Omega_r(x)\}$ of $r = n-k+1$ smooth one-forms on $\mathcal{U}_p$ such that, for each $x \in \mathcal{U}_p \cap \Sigma^{k-1}(\omega)$, $\langle \omega_1(x), \dots, \omega_n(x)\rangle$ is equal to

$$\langle \Omega_1(x), \dots, \Omega_r(x)\rangle \oplus \left(\langle \omega_1(x), \dots, \omega_n(x)\rangle \cap N_x^*\Sigma^{k-2}(\omega)\right).$$

For clearer notations, let us denote

$$\langle \bar\omega(x)\rangle = \langle \omega_1(x), \dots, \omega_n(x)\rangle \quad \text{and} \quad \langle \bar\Omega^{k-1}(x)\rangle = \langle \Omega_1(x), \dots, \Omega_r(x)\rangle.$$

Then,

$$\begin{aligned} p \in \Sigma^k(\omega) &\overset{(\text{Def. } 2.19)}{\Leftrightarrow} \dim\left(\langle \bar\Omega^{k-1}(p)\rangle \cap N_p^*\Sigma^{k-1}(\omega)\right) = 1\\ &\overset{(i),(iii)}{\Leftrightarrow} \dim\left(\langle \bar\omega(p)\rangle \cap N_p^*\Sigma^{k-1}(\omega)\right) - \dim\left(\langle \bar\omega(p)\rangle \cap N_p^*\Sigma^{k-2}(\omega)\right) = 1\\ &\overset{(ii)}{\Leftrightarrow} \dim\left(\langle \bar\omega(p)\rangle \cap N_p^*\Sigma^{k-1}(\omega)\right) - (k-2) = 1\\ &\Leftrightarrow \dim\left(\langle \bar\omega(p)\rangle \cap N_p^*\Sigma^{k-1}(\omega)\right) = k-1.\end{aligned}$$

□

According to Lemmas 2.20 and 2.21, if the hypothesis of Remark 2.14 holds for every $i = 2, \dots, k-1$ and $\omega = \{\omega_i\}_{1 \le i \le n}$ satisfies the condition I_{k-1}, then this hypothesis will hold for $i = 2, \dots, k$. In other words, we can state the following result.

Theorem 2.22. *Let* $\omega = \{\omega_i\}_{1 \le i \le n}$ *be a non-degenerate collection on* M*. If* ω *satisfies the conditions* I_j*, for* $j = 1, \dots, n-1$*, then for every* $k = 1, \dots, n$ *we have that*

(a) $\Sigma^k(\omega) \subset \Sigma^{k-1}(\omega) \subset \dots \subset \Sigma^2(\omega) \subset \Sigma^1(\omega)$*;*

(*b*) $\Sigma^k(\omega)$ *is the empty set or an* $(n-k)$*-dimensional smooth submanifold of* M*;*
(*c*) *Let* $k > 1$*. For each* $p \in \Sigma^{k-1}(\omega)$*,*

$$p \in \Sigma^k(\omega) \Leftrightarrow \dim(\langle \omega_1(p), \dots, \omega_n(p)\rangle \cap N_p^*\Sigma^{k-1}(\omega)) = k-1.$$

The following proposition shows that Definitions 2.9, 2.10, 2.18 and 2.19 do not depend on the choice of the bases $\{\Omega_1(x), \dots, \Omega_{n-1}(x)\}$ and $\{\Omega_1(x), \dots, \Omega_r(x)\}$. The first part (a) provides equations that define the submanifolds $\Sigma^k(\omega)$ locally. We use these local equations to demonstrate part (b). The proof can be found in Appendix A.

Proposition 2.23.

(*a*) *Let* $p \in \Sigma^{k-1}(\omega)$*. There are an open neighborhood* $\mathcal{U}$ *of* p *in* M *and smooth functions* $F_i : \mathcal{U} \to \mathbb{R}$*,* $i = 1, \dots, m-r$*, such that*

$$\mathcal{U} \cap \Sigma^{k-1}(\omega) = \{x \in \mathcal{U} \mid F_1(x) = \dots = F_{m-r}(x) = 0\}$$

with $\operatorname{rank}(dF_1(x), \dots, dF_{m-r}(x)) = m-r$ *for* $x \in \mathcal{U} \cap \Sigma^{k-1}(\omega)$*, and there is a collection* $\{\Omega_1(x), \dots, \Omega_r(x)\}$ *of* r *smooth one-forms defined on* $\mathcal{U}$ *which is a basis of a vector subspace complementary to* $\langle \bar{\omega}(x)\rangle \cap N_x^*\Sigma^{k-2}(\omega)$ *in* $\langle \bar{\omega}(x)\rangle$ *for each* $x \in \mathcal{U} \cap \Sigma^{k-1}(\omega)$*. Let*

$$\Delta_k(x) = \det(dF_1, \dots, dF_{m-r}, \Omega_1, \dots, \Omega_r)(x).$$

Then ω *satisfies the condition* I_{k-1} *on* $\mathcal{U}$ *if and only if the following properties hold for each* $x \in \mathcal{U} \cap \Sigma^{k-1}(\omega)$*:*

(*i*) $\dim\langle \Omega_1(x), \dots, \Omega_r(x)\rangle \cap N_x^*\Sigma^{k-1}(\omega) = 0$ *or* 1*;*

(*ii*) *if* $\dim\langle \Omega_1(x), \dots, \Omega_r(x)\rangle \cap N_x^*\Sigma^{k-1}(\omega) = 1$ *(or equivalently* $\Delta_k(x) = 0$*), then*

$$\operatorname{rank}(dF_1(x), \dots, dF_{m-r}(x), d\Delta_k(x)) = m-r+1.$$

In this case, $\Sigma^k(\omega)$ *can be locally defined as*

$$\mathcal{U} \cap \Sigma^k(\omega) = \{x \in \mathcal{U} \mid F_1(x) = \dots = F_{m-r}(x) = \Delta_k(x) = 0\}.$$

(*b*) *The definitions of* $\Sigma^1(\omega)$*,* $\Sigma^k(\omega)$ *and* $A_{k-1}(\omega)$ *do not depend on the choice of the basis* $\{\Omega_1, \dots, \Omega_{n-k+1}\}$*, for every* $k = 2, \dots, n$*.*

Remark 2.24. *It is not difficult to see that, for every* $k = 1, \dots, n$*,* $\Sigma^k(\omega)$ *is a closed submanifold of* M *such that*

$$\Sigma^k(\omega) = A_k(\omega) \cup \Sigma^{k+1}(\omega) = \bigcup_{i=k}^n A_i(\omega).$$

Furthermore, $A_k(\omega) = \Sigma^k(\omega) \smallsetminus \Sigma^{k+1}(\omega)$*. Then, the singular sets* $A_k(\omega)$ *are* $(n-k)$*-dimensional submanifolds of* M *such that* $\overline{A_k(\omega)} = \Sigma^k(\omega)$*.*

Finally, based on the previous considerations, we define:

Definition 2.25. *Let* $\omega = \{\omega_i\}_{1\le i\le n}$ *be a non-degenerate collection on* M *that satisfies the condition* I_j*, for* $j = 1, \dots, n-1$*. For each* $k \in \{1, \dots, n\}$*, we say that* $x \in M$ *is an* A_k*-type Morin singularity of* ω *if* $x \in A_k(\omega)$*.*

Definition 2.26. *Let* $\omega = \{\omega_i\}_{1\le i\le n}$ *be a collection of* n *smooth one-forms on* M*, with* $0 < n \le m$*. We call* ω *a Morin collection if* ω *is non-degenerate and it satisfies the condition* I_j*, for* $j = 1, \dots, n-1$*.*

Remark 2.27. *By Definition 2.26, if* $\omega = \{\omega_i\}_{1\le i\le n}$ *is a Morin collection then* ω *admits only* A_k*-type singular points for* $k = 1, \dots, n$*.*

As we mentioned in Section 1, fixed a Riemannian metric on M, we can consider vector fields instead of one-forms and define the notion of Morin collection of n vector fields analogously to the definition of Morin collection of n one-forms:

Definition 2.28. *Let $V = \{V_i\}_{1\leq i\leq n}$ be a collection of n smooth vector fields on M, with $0 < n \leq m$. We call V a Morin collection if V is non-degenerate and it satisfies the condition I_j, for $j = 1,\dots,n-1$.*

Next, we present some examples of Morin collections of vector fields.

Example 2.29. *Let $f : M^m \to \mathbb{R}^n$ be a smooth Morin map defined on an m-dimensional Riemannian manifold M, with $m \geq n$. The collection of n vector fields $V(x) = \{\nabla f_1(x),\dots,\nabla f_n(x)\}$ given by the gradients of the coordinate functions of f is, clearly, a Morin collection of vector fields whose singular points are the same as the singular points of f. That is, $A_k(V) = A_k(f)$, for $k = 1,\dots,n$.*

Example 2.30. *Let $a \in \mathbb{R}$ be a regular value of a C^2 mapping $f : \mathbb{R}^3 \to \mathbb{R}$. Suppose that $M = f^{-1}(a)$ and consider $V = \{V_1, V_2\}$ be a collection of 2 vector fields on M, given by*

$$\begin{array}{rcl} V_1(x) & = & (-f_{x_2}(x), f_{x_1}(x), 0); \\ V_2(x) & = & (-f_{x_3}(x), 0, f_{x_1}(x)). \end{array}$$

Since a is a regular value of f, we have that $\nabla f(x) = (f_{x_1}(x), f_{x_2}(x), f_{x_3}(x)) \neq \vec{0}$, $\forall x \in M$. Thus, $\operatorname{rank}(V_1(x), V_2(x))$ is either equal to 2 or equal to 1 . The singular points of V are the points $x \in M$ where $\operatorname{rank}(V_1(x), V_2(x)) = 1$, that is,

$$\Sigma^1(V) = \{x \in M \mid f_{x_1}(x) = 0\}$$

and $V = \{V_1, V_2\}$ is non-degenerate if and only if $\operatorname{rank}(\nabla f(x), \nabla f_{x_1}(x)) = 2$ for each $x \in \Sigma^1(V)$. In this case, $\Sigma^1(V)$ is a submanifold of M of dimension 1. Let $x \in \Sigma^1(V)$ be a singular point of V, then the space $\langle V_1(x), V_2(x)\rangle$ is spanned by the vector $e_1 = (1,0,0)$ and $x \in A_2(V)$ if and only if

$$\operatorname{rank}(\nabla f(x), \nabla f_{x_1}(x), e_1) < 3,$$

that is, if and only if $\Delta_2 := f_{x_2} f_{x_1x_3} - f_{x_3} f_{x_1x_2}$ vanishes at x. Moreover, V satisfies the condition I_1 if and only if $\operatorname{rank}(\nabla f(x), \nabla f_{x_1}(x), \nabla\Delta_2(x)) = 3$ for $x \in A_2(V)$. In this case, $A_2(V)$ is a submanifold of M of dimension 0. Therefore, $V = \{V_1, V_2\}$ is a Morin collection of 2 vector fields if and only if $\operatorname{rank}(\nabla f(x), \nabla f_{x_1}(x)) = 2$ on the singular set $\Sigma^1(V) = \{x \in M \mid f_{x_1}(x) = 0\}$ and $\det(\nabla f(x), \nabla f_{x_1}(x), \nabla\Delta_2(x)) \neq 0$ on $A_2(V) = \{x \in M \mid f_{x_1}(x) = 0, \Delta_2(x) = 0\}$.

Example 2.31. *Let us apply Example 2.30 to the collection of 2 vector fields $V = \{V_1, V_2\}$ defined on the torus $\mathrm{T} := f^{-1}(R^2)$, where R^2 is a regular value of*

$$f(x_1, x_2, x_3) = (\sqrt{x_2^2 + x_3^2} - a)^2 + (x_1 + x_2)^2,$$

with $a > R$. Then, one can verify that $\Sigma^1(V) = \{x \in \mathrm{T} \,|\, x_1 + x_2 = 0\}$, that is,

$$\Sigma^1(V) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid \sqrt{x_2^2 + x_3^2} - a)^2 = R^2\}$$

and $\operatorname{rank}(\nabla f(x), \nabla f_{x_1}(x))$ is equal to

$$\operatorname{rank}\begin{bmatrix} 0 & \dfrac{2x_2(\sqrt{x_2^2 + x_3^2} - a)}{\sqrt{x_2^2 + x_3^2}} & \dfrac{2x_3(\sqrt{x_2^2 + x_3^2} - a)}{\sqrt{x_2^2 + x_3^2}} \\ 1 & 1 & 0 \end{bmatrix},$$

which is 2 *for all* $x \in \mathrm{T} \cap \Sigma^1(V)$. *Moreover,*

$$\Delta_2(x) = \frac{-4x_3(\sqrt{x_2^2+x_3^2}-a)}{\sqrt{x_2^2+x_3^2}},$$

such that

$$A_2(V) = \{x \in \mathrm{T} \,|\, x_1 + x_2 = 0; x_3 = 0\},$$

which is the set given by the points $(-a-R, a+R, 0)$, $(a+R, -a-R, 0)$, $(-a+R, a-R, 0)$ *and* $(a-R, -a+R, 0)$. *It is not difficult to see that* $\operatorname{rank}(\nabla f(x), \nabla f_{x_1}(x), \nabla\Delta_2(x)) = 3, \forall x \in \mathrm{T} \cap A_2(V)$. *Therefore, the collection* $V = \{V_1, V_2\}$ *given by*

$$\begin{array}{rcl} V_1(x) & = & \left(\frac{-2x_2(\sqrt{x_2^2+x_3^2}-a)}{\sqrt{x_2^2+x_3^2}} - 2(x_1+x_2), 2(x_1+x_2), 0\right); \\ V_2(x) & = & \left(\frac{-2x_3(\sqrt{x_2^2+x_3^2}-a)}{\sqrt{x_2^2+x_3^2}}, 0, 2(x_1+x_2)\right), \end{array}$$

is a Morin collection of 2 *vector fields defined on the torus* T *which admits singular points of type* A_1 *and* A_2.

Example 2.32. *Let* $a \in \mathbb{R}$ *be a regular value of a* C^2 *mapping* $f : \mathbb{R}^3 \to \mathbb{R}$. *Suppose that* $M = f^{-1}(a)$ *and consider* $\overline{W_1}$ *and* $\overline{W_2}$ *be the orthogonal projections of* $e_2 = (0,1,0)$ *and* $e_3 = (0,0,1)$ *over* T_xM *given by*

$$\begin{array}{rcl} \overline{W_1} & = & e_2 - \left\langle e_2, \frac{\nabla f}{|\nabla f|}\right\rangle \frac{\nabla f}{|\nabla f|}; \\ \overline{W_2} & = & e_3 - \left\langle e_3, \frac{\nabla f}{|\nabla f|}\right\rangle \frac{\nabla f}{|\nabla f|}. \end{array}$$

Let $W = \{W_1, W_2\}$ *be the collection of* 2 *vector fields defined by* $W_1 = \|\nabla f\|^2\,\overline{W_1}$ *and* $W_2 = \|\nabla f\|^2\,\overline{W_2}$, *that is,*

$$\begin{array}{rcl} W_1 & = & (-f_{x_1}f_{x_2}, f_{x_1}^2 + f_{x_3}^2, -f_{x_2}f_{x_3}); \\ W_2 & = & (-f_{x_1}f_{x_3}, -f_{x_2}f_{x_3}, f_{x_1}^2 + f_{x_2}^2). \end{array}$$

In this case, W_1 *and* W_2 *are gradients vector fields, that is,* W *is a collection of* 2 *gradient vector fields. It is not difficult to see that* $\operatorname{rank}(W_1(x), W_2(x))$ *is either equal to* 2 *or equal to* 1, *and the singular set of* W *is* $\Sigma^1(W) = \{x \in M \mid f_{x_1}(x) = 0\}$. *Let* $x \in \Sigma^1(W)$ *be a singular point of* W, *then the space* $\langle W_1(x), W_2(x)\rangle$ *is spanned by the vector* $(0, f_{x_3}, -f_{x_2})$, *such that* $A_2(W) = \{x \in M \mid f_{x_1}(x) = 0, f_{x_1x_1}(x) = 0\}$. *Therefore,* $W = \{W_1, W_2\}$ *is a Morin collection of* 2 *vector fields if and only if* $\operatorname{rank}(\nabla f(x), \nabla f_{x_1}(x)) = 2$ *on the singular set* $\Sigma^1(W)$ *and* $\det(\nabla f(x), \nabla f_{x_1}(x), \nabla f_{x_1x_1}(x)) \neq 0$ *on* $A_2(W)$.

Example 2.33. *Let us apply Example 2.32 to the collection of vector fields* $W = \{W_1, W_2\}$ *defined on the torus* $\mathrm{T} := f^{-1}(R^2)$ *of Example 2.31. In this situation, one can verify that* $\Sigma^1(W)$ *is the same singular set as* $\Sigma^1(V)$ *in the Example 2.31. Moreover,* $\operatorname{rank}(\nabla f(x), \nabla f_{x_1}(x)) = 2$ *for every* $x \in \Sigma^1(W)$. *However, since* $f_{x_1x_1}(x) = 2$ *for every* $x \in \Sigma^1(W)$, W *does not admits singular points of type* A_2. *That is,* W *is Morin collection of* 2 *vector fields on* T *which admits only Morin singularities of type* A_1.

Example 2.34. *Let us consider the collections* $V = \{V_1, V_2\}$ *and* $W = \{W_1, W_2\}$ *from Examples 2.30 and 2.32 defined on the unit sphere* $M := f^{-1}(1)$, *where* $f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2$. *We know that the singular sets of* V *and* W *are the same, that is,* $\Sigma^1(V) = \Sigma^1(W) = \{x \in M \,|\, x_1 = 0\}$ *and* $\operatorname{rank}(\nabla f(x), \nabla f_{x_1}(x)) = 2$ *for all singular point* x. *However,* $\Delta_2(x) = 0, \forall x \in \Sigma^1(V)$, *such that* $\nabla\Delta_2 \equiv \vec{0}$. *On the other hand,* $f_{x_1x_1}(x) \neq 0, \forall x \in \Sigma^1(W)$, *such that* $A_2(W) = \varnothing$. *Therefore,*

V is not a Morin collection and W is a Morin collection that admits only Morin singularities of type A_1.

Example 2.35. *In the Example 2.34, if we consider $f(x_1, x_2, x_3) = x_1^2 - x_1x_2 + x_3^2$ then one can verify that V and W are both Morin collections of 2 vector fields that admits only Morin singularities of type A_1. Let us consider the case where V of Example 2.30 is defined on $M := f^{-1}(-1)$ and $f(x_1, x_2, x_3) = x_1^2 - x_1x_2 + x_3^2$. It is easy to see that -1 is a regular value of f and $\Sigma^1(V) = \{x \in M \mid 2x_1 - x_2 = 0\}$. That is,*

$$\Sigma^1(V) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 - x_1x_2 + x_3^2 + 1 = 0; 2x_1 - x_2 = 0\}$$

and $\operatorname{rank}(\nabla f(x), \nabla f_{x_1}(x))$ *is equal to*

$$\operatorname{rank}\begin{bmatrix} (2x_1 - x_2) & -x_1 & 2x_3 \\ 2 & -1 & 0 \end{bmatrix}$$

which is 2, for all $x \in M \cap \Sigma^1(V)$. Moreover, $\Delta_2(x) = 2x_3$ and

$$A_2(V) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 - x_1x_2 + x_3^2 + 1 = 0; 2x_1 - x_2 = 0; x_3 = 0\}$$

which is the set given by the points $(1, 2, 0)$ and $(-1, -2, 0)$. We also have that

$$\det(\nabla f(x), \nabla f_{x_1}(x), \nabla \Delta_2(x))$$

is equal to

$$\det\begin{bmatrix} (2x_1 - x_2) & -x_1 & 2x_3 \\ 2 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = 4x_1$$

which is equal to ± 4 for each $x \in A_2(V)$. That is, $\operatorname{rank}(\nabla f(x), \nabla f_{x_1}(x), \nabla \Delta_2(x)) = 3$, *for all $x \in M \cap A_2(V)$. Therefore, the collection $V = \{V_1, V_2\}$ given by*

$$\begin{array}{rcl} V_1(x) & = & (x_1, 2x_1 - x_2, 0)\,; \\ V_2(x) & = & (-2x_3, 0, 2x_1 - x_2)\,. \end{array}$$

is a Morin collection of 2 vector fields defined on M which admits singular points of type A_1 and A_2.

3. Zeros of a generic one-form $\xi(x)$ associated to a Morin collection of one-forms

Let $a = (a_1, \ldots, a_n) \in \mathbb{R}^n \setminus \{\vec{0}\}$ and let $\omega = \{\omega_i\}_{1 \le i \le n}$ be a Morin collection of n smooth one-forms defined on an m-dimensional manifold M. In this section, we will consider the one-form $\xi(x) = \sum_{i=1}^{n} a_i\omega_i(x)$ defined on M and we will prove some properties of the zeros of ξ and its restrictions to the singular sets of ω. We will consider the notation $\langle\bar{\omega}(x)\rangle = \langle\omega_1(x), \ldots, \omega_n(x)\rangle$.

Lemma 3.1. *If p is a zero of the one-form ξ then $p \in \Sigma^1(\omega)$ and p is a zero of $\xi_{|_{\Sigma^1(\omega)}}$.*

Proof. Suppose that $\xi(p) = 0$. So $\operatorname{rank}(\omega_1(p), \ldots, \omega_n(p)) \le n - 1$, since $a \ne \vec{0}$. However, the collection ω is non-degenerate, thus $\operatorname{rank}(\omega_1(p), \ldots, \omega_n(p)) = n-1$. That is, $p \in \Sigma^1(\omega)$. Moreover, $\xi(p) = 0$ implies that $T_pM \subset \ker(\xi(p))$ and since $T_p\Sigma_1(\omega) \subset T_pM$, we conclude that p is a zero of $\xi_{|_{\Sigma^1(\omega)}} = 0$. □

Lemma 3.2. *If $p \in A_{k+1}(\omega)$, then for each $k = 0, \ldots, n-2$, p is a zero of $\xi_{|_{\Sigma^{k+1}(\omega)}}$ if and only if p is a zero of $\xi_{|_{\Sigma^k(\omega)}}$.*

Proof. Suppose that $p \in A_{k+1}(\omega)$ and that, locally, we have:

$$\begin{aligned} \mathcal{U} \cap \Sigma^k(\omega) &= \{x \in \mathcal{U} \,|\, F_1(x) = \ldots = F_{m-n+1}(x) = \Delta_2(x) = \ldots = \Delta_k(x) = 0\}; \\ \mathcal{U} \cap \Sigma^{k+1}(\omega) &= \{x \in \mathcal{U} \,|\, F_1(x) = \ldots = F_{m-n+1}(x) = \Delta_2(x) = \ldots = \Delta_{k+1}(x) = 0\}; \end{aligned}$$

for an open neighborhood $\mathcal{U}$ of p in M. If p is a zero of the restriction $\xi_{|_{\Sigma^k(\omega)}}$ then $\xi(p) \in N_p^*\Sigma^k(\omega) = \langle dF_1(p), \ldots, dF_{m-n+1}(p), d\Delta_2(p), \ldots, d\Delta_k(p)\rangle$. In particular, $\xi(p) \in N_p^*\Sigma^{k+1}(\omega)$, therefore p is a zero of $\xi_{|_{\Sigma^{k+1}(\omega)}}$.

On the other hand, if p is a zero of $\xi_{|_{\Sigma^{k+1}(\omega)}}$ then $\xi(p) \in N_p^*\Sigma^{k+1}(\omega) \cap \langle \bar{\omega}(p)\rangle$.

Since $p \in A_{k+1}(\omega)$, we have that $p \in \Sigma_{k+1}(\omega) \smallsetminus \Sigma_{k+2}(\omega)$, thus

$$\begin{cases} \dim(\langle \bar{\omega}(p)\rangle \cap N_p^*\Sigma^k(\omega)) = k; \\ \dim(\langle \bar{\Omega}^{k+1}(p)\rangle \cap N_p^*\Sigma^{k+1}(\omega)) = 0; \end{cases}$$

where $\bar{\Omega}^{k+1}(p)$ represents a smooth basis for a vector subspace complementary to $\langle \bar{\omega}(p)\rangle \cap N_p^*\Sigma^k(\omega)$ in $\langle \bar{\omega}(p)\rangle$. Since $\dim(N_p^*\Sigma^k(\omega)) = m-n+k$, $\dim(N_p^*\Sigma^{k+1}(\omega)) = m-n+k+1$ and $N_p^*\Sigma^k(\omega) \subset N_p^*\Sigma^{k+1}(\omega)$, we have

$$\dim(\langle \bar{\omega}(p)\rangle \cap N_p^*\Sigma^{k+1}(\omega)) = \dim(\langle \bar{\omega}(p)\rangle \cap N_p^*\Sigma^k(\omega)) = k.$$

Thus, $\langle \bar{\omega}(p)\rangle \cap N_p^*\Sigma^k(\omega) = \langle \bar{\omega}(p)\rangle \cap N_p^*\Sigma^{k+1}(\omega)$. Therefore, $\xi(p) \in N_p^*\Sigma^k(\omega)$, that is, p is a zero of $\xi_{|_{\Sigma^k(\omega)}}$. □

Lemma 3.3. *If $p \in A_n(\omega)$ then p is a zero of the restriction $\xi_{|_{\Sigma^{n-1}(\omega)}}$.*

Proof. Analogously to Lemma 3.2, we consider local equations of $\Sigma^n(\omega)$:

$$\mathcal{U} \cap \Sigma^n(\omega) = \{x \in \mathcal{U} \,|\, F_1(x) = \ldots = F_{m-n+1}(x) = \Delta_2(x) = \ldots = \Delta_n(x) = 0\},$$

with $N_x^*\Sigma^n(\omega) = \langle dF_1(x), \ldots, dF_{m-n+1}(x), d\Delta_2(x), \ldots, d\Delta_n(x)\rangle$. Since $A_n(\omega) = \Sigma^n(\omega)$, if $p \in A_n(\omega)$ then

$$\dim(\langle \bar{\omega}(p)\rangle \cap N_p^*\Sigma^{n-1}(\omega)) = n-1.$$

Thus, $\langle \bar{\omega}(p)\rangle \subset N_p^*\Sigma^{n-1}(\omega)$ and consequently, $\xi(p) \in N_p^*\Sigma^{n-1}(\omega)$. Therefore, p is a zero of $\xi_{|_{\Sigma^{n-1}(\omega)}}$. □

Remark 3.4. *If $p \in \Sigma^1(\omega)$ then* $\operatorname{rank}(\omega_1(p), \ldots, \omega_n(p)) = n-1$ *and, writing $\omega_i = (\omega_i^1, \ldots, \omega_i^m)$, we can assume that*

$$\boldsymbol{M}(x) = \begin{vmatrix} \omega_1^1(x) & \omega_2^1(x) & \cdots & \omega_{n-1}^1(x) \\ \vdots & \vdots & \ddots & \vdots \\ \omega_1^{n-1}(x) & \omega_2^{n-1}(x) & \cdots & \omega_{n-1}^{n-1}(x) \end{vmatrix} \neq 0, \tag{2}$$

for all x in an open neighborhood $\mathcal{U}$ of p in M. In particular, if $p \in \mathcal{U}$ is a singular point of ξ then $a_n \neq 0$, otherwise, we would have $a_1 = \ldots = a_{n-1} = a_n = 0$. We will use this fact in next results.

Lemma 3.5. *Let $p \in \Sigma^1(\omega)$ such that $\boldsymbol{M}(p) \neq 0$. Then $\xi(p) = 0$ if and only if $\sum_{i=1}^{n} a_i \omega_i^j(p) = 0$, for every $j = 1, \ldots, n-1$.*

Proof. It follows easily from the definition of $\Sigma^1(\omega)$ and ξ. □

Lemma 3.6. *Let $Z(\xi)$ be the zero set of the one-form ξ. Then for almost every $a \in \mathbb{R}^n \setminus \{\vec{0}\}$, $Z(\xi) \cap \Sigma^2(\omega) = \emptyset$.*

Proof. Let $\mathcal{U}$ be an open subset of M on which $\mathbf{M}(x) \neq 0$ and

$$\mathcal{U} \cap \Sigma^2(\omega) = \{x \in \mathcal{U} \,|\, F_1(x) = \ldots = F_{m-n+1}(x) = \Delta_2(x) = 0\},$$

with $\operatorname{rank}(dF_1(x), \ldots, dF_{m-n+1}(x), d\Delta_2(x)) = m-n+2$, for each $x \in \Sigma^2(\omega) \cap \mathcal{U}$. Let us consider $F : \mathcal{U} \times \mathbb{R}^n \setminus \{\vec{0}\} \to \mathbb{R}^{m+1}$ the mapping defined by

$$F(x,a) = (F_1(x), \ldots, F_{m-n+1}(x), \Delta_2(x), \sum_{i=1}^n a_i \omega_i^1(x), \ldots, \sum_{i=1}^n a_i \omega_i^{n-1}(x)).$$

By Lemma 3.5, if $x \in \Sigma^1(\omega)$ then

$$\sum_{i=1}^n a_i \omega_i(x) = 0 \Leftrightarrow \sum_{i=1}^n a_i \omega_i^j(x) = 0, \forall j = 1, \ldots, n-1.$$

Thus, if $(x,a) \in F^{-1}(\vec{0})$ we have that $x \in Z(\xi) \cap \Sigma^2(\omega)$. Furthermore, the Jacobian matrix of F at a point $(x,a) \in F^{-1}(\vec{0})$:

$$\left[\begin{array}{c|cccc}
dF_1(x) & & & & \\
\vdots & & O_{(m-n+2)\times n} & & \\
dF_{m-n+1}(x) & & & & \\
d\Delta_2(x) & & & & \\
\hline
 & \omega_1^1(x) & \cdots & \omega_{n-1}^1(x) & \omega_n^1(x) \\
 & \omega_1^2(x) & \cdots & \omega_{n-1}^2(x) & \omega_n^2(x) \\
(*) & \vdots & \ddots & \vdots & \vdots \\
 & \omega_1^{n-1}(x) & \cdots & \omega_{n-1}^{n-1}(x) & \omega_n^{n-1}(x)
\end{array}\right]$$

has rank $m+1$. That is, $\vec{0}$ is regular value of F and $F^{-1}(\vec{0})$ is a submanifold of dimension $n-1$. Let $\pi : F^{-1}(\vec{0}) \to \mathbb{R}^n \setminus \{\vec{0}\}$ be the projection over $\mathbb{R}^n \setminus \{\vec{0}\}$ given by $\pi(x,a) = a$, by Sard's Theorem, a is regular value of π for almost every $a \in \mathbb{R}^n \setminus \{\vec{0}\}$. Therefore, $\pi^{-1}(a) \cap F^{-1}(\vec{0}) = \emptyset$ for almost every $a \in \mathbb{R}^n \setminus \{\vec{0}\}$. However, $\pi^{-1}(a) \cap F^{-1}(\vec{0}) = \{(x,a) \in \mathcal{U} \times \{a\} : x \in Z(\xi) \cap \Sigma^2(\omega)\}$. Thus, $Z(\xi) \cap \Sigma^2(\omega) = \emptyset$ for almost every $a \in \mathbb{R}^n \setminus \{\vec{0}\}$. ☐

Lemma 3.7. *Let $Z(\xi|_{\Sigma^k(\omega)})$ be the zero set of the restriction of the one-form ξ to $\Sigma^k(\omega)$, with $k \geq 1$. Then for almost every $a \in \mathbb{R}^n \setminus \{\vec{0}\}$, $Z(\xi|_{\Sigma^k(\omega)}) \cap \Sigma^{k+2}(\omega) = \emptyset$.*

Proof. For each $k = 1, \ldots, n-2$, let $\mathcal{U}$ be an open subset of M on which

$$\mathcal{U} \cap \Sigma^k(\omega) = \{x \in \mathcal{U} \,|\, F_1(x) = \ldots = F_{m-n+k}(x) = 0\},$$

with $\operatorname{rank}(dF_1(x), \ldots, dF_{m-n+k}(x)) = m-n+k$, for all $x \in \mathcal{U} \cap \Sigma^k(\omega)$ and

$$\mathcal{U} \cap \Sigma^{k+2}(\omega) = \{x \in \mathcal{U} \,|\, F_1(x) = \ldots = F_{m-n+k+2}(x) = 0\},$$

with $\operatorname{rank}(dF_1(x), \ldots, dF_{m-n+k+2}(x)) = m-n+k+2$, for all $x \in \mathcal{U} \cap \Sigma^{k+2}(\omega)$.

By Szafraniec's characterization (see [19, p. 196]) adapted to one-forms, x is a zero of the restriction $\xi|_{\Sigma^k(\omega)}$ if and only if there exists $(\lambda_1, \ldots, \lambda_{m-n+k}) \in \mathbb{R}^{m-n+k}$ such that

$$\xi(x) = \sum_{j=1}^{m-n+k} \lambda_j dF_j(x).$$

Let us write $\xi(x) = (\xi_1(x), \dots, \xi_m(x))$, where $\xi_s(x) = \sum_{i=1}^{n} a_i \omega_i^s(x)$, $s = 1, \dots, m$, we define

$$N_s(x,a,\lambda) := \xi_s(x) - \sum_{j=1}^{m-n+k} \lambda_j \frac{\partial F_j}{\partial x_s}(x),$$

such that $\xi_{|_{\Sigma^k(\omega)}}(x) = 0$ if and only if $N_s(x,a,\lambda) = 0$, for all $s = 1, \dots, m$.

Let $F : \mathcal{U} \times \mathbb{R}^n \smallsetminus \{\vec{0}\} \times \mathbb{R}^{m-n+k} \to \mathbb{R}^{2m-n+k+2}$ be the mapping defined by

$$F(x,a,\lambda) = (F_1, \dots, F_{m-n+k+2}, N_1, \dots, N_m),$$

if $(x,a,\lambda) \in F^{-1}(\vec{0})$ then $x \in Z(\xi_{|_{\Sigma^k(\omega)}}) \cap \Sigma^{k+2}(\omega)$ and the Jacobian matrix of F at (x,a,λ):

$$\left[\begin{array}{c|cc} dF_1(x) & & \\ \vdots & & O_{(m-n+k+2)\times(m+k)} \\ dF_{m-n+k+2}(x) & & \\ \hline d_xN_1(x,a,\lambda) & & \\ \vdots & B_{m\times n} & C_{m\times(m-n+k)} \\ d_xN_m(x,a,\lambda) & & \end{array}\right]$$

has rank $2m-n+k+1$, where $O_{(m-n+k+2)\times(m+k)}$ is a null matrix, $B_{m\times n}$ is a matrix whose columns vectors are given by the coefficients of the one-forms $\omega_1(x), \dots, \omega_n(x)$ of the collection ω:

$$B_{m\times n} = \begin{bmatrix} \omega_1^1(x) & \cdots & \omega_n^1(x) \\ \vdots & \ddots & \vdots \\ \omega_1^m(x) & \cdots & \omega_n^m(x) \end{bmatrix}$$

and $C_{m\times(m-n+k)}$ is the matrix whose columns vectors are, up to sign, the coefficients of the derivatives $dF_1, \dots, dF_{m-n+k}$ with respect to x:

$$C_{m\times(m-n+k)} = \begin{bmatrix} -\dfrac{\partial F_1}{\partial x_1}(x) & \cdots & -\dfrac{\partial F_{m-n+k}}{\partial x_1}(x) \\ \vdots & \ddots & \vdots \\ -\dfrac{\partial F_1}{\partial x_m}(x) & \cdots & -\dfrac{\partial F_{m-n+k}}{\partial x_m}(x) \end{bmatrix}.$$

Notice that, if $(x,a,\lambda) \in F^{-1}(\vec{0})$ then $x \in \Sigma^{k+1}(\omega)$ and, by Lemma 2.21,

$$\dim(\langle \bar{\omega}(x) \rangle \cap N_x^*\Sigma^k(\omega)) = k.$$

Thus, $\dim(\langle \bar{\omega}(x) \rangle + N_x^*\Sigma^k(\omega)) = m-1$. Therefore,

$$\operatorname{rank}\left[\begin{array}{c|c} B_{m\times n} & C_{m\times(m-n+k)} \end{array}\right] = m-1$$

and the Jacobian matrix of F at (x,a,λ) has rank $2m-n+k+1$. That is, $F^{-1}(\vec{0})$ has dimension less or equal to $n-1$. Let $\pi : F^{-1}(\vec{0}) \to \mathbb{R}^n \smallsetminus \{\vec{0}\}$ be the projection over $\mathbb{R}^n \smallsetminus \{\vec{0}\}$, that is, $\pi(x,a,\lambda) = a$. By Sard's Theorem, a is regular value of π for almost every $a \in \mathbb{R}^n \smallsetminus \{\vec{0}\}$. Therefore, $\pi^{-1}(a) \cap F^{-1}(\vec{0}) = \varnothing$ for almost every $a \in \mathbb{R}^n \smallsetminus \{\vec{0}\}$. However,

$$\pi^{-1}(a) \cap F^{-1}(\vec{0}) = \{(x,a,\lambda) \in \mathcal{U} \times \{a\} \times \mathbb{R}^{m-n+k} \mid x \in Z(\xi_{|_{\Sigma^k(\omega)}}) \cap \Sigma^{k+2}(\omega)\}.$$

Thus, $Z(\xi_{|_{\Sigma^k(\omega)}}) \cap \Sigma^{k+2}(\omega) = \varnothing$ for almost every $a \in \mathbb{R}^n \smallsetminus \{\vec{0}\}$. □

4. Non-degenerate zeros of a generic one-form $\xi(x)$ associated to a Morin collection of one-forms

In this section we will verify that, generically, the one-form $\xi(x)$ and its restrictions $\xi_{|_{\Sigma^k(\omega)}}$, $\xi_{|_{A_k(\omega)}}$ admit only non-degenerate zeros. Furthermore, we will see how these non-degenerate zeros can be related. Then, we end the paper with our main result (Theorem 4.13).

We start with some technical lemmas.

Lemma 4.1. *Let A be a square matrix of order m given by:*

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ a_{21} & \cdots & a_{2m} \\ \vdots & \cdots & \vdots \\ a_{m1} & \cdots & a_{mm} \end{bmatrix}.$$

If there exist $(\lambda_1, \dots, \lambda_m) \in \mathbb{R}^m \smallsetminus \{\vec{0}\}$ such that $\sum_{j=1}^{m} \lambda_j a_{ij} = 0$, $i = 1, \dots, m$, then

$$\lambda_j \operatorname{cof}(a_{ik}) - \lambda_k \operatorname{cof}(a_{ij}) = 0, \ \forall j, k = 1, \dots, m.$$

Lemma 4.2. *Let us consider the matrix*

$$M_i(x) = \begin{bmatrix} \omega_1^1(x) & \cdots & \omega_{n-1}^1(x) & \omega_n^1(x) \\ \vdots & \ddots & \vdots & \vdots \\ \omega_1^{n-1}(x) & \cdots & \omega_{n-1}^{n-1}(x) & \omega_n^{n-1}(x) \\ \omega_1^i(x) & \cdots & \omega_{n-1}^i(x) & \omega_n^i(x) \end{bmatrix}.$$

If x is a zero of ξ then for $\ell \in \{1, \dots, n-1\}$, $j \in \{1, \dots, n-1, i\}$ and $i \in \{n, \dots, m\}$, we have

$$a_n \operatorname{cof}(\omega_\ell^j, M_i) = a_\ell \operatorname{cof}(\omega_n^j, M_i).$$

Proof. This result is a consequence of Lemma 4.1 applied to the matrix $A = M_i(x)$, where $a_{\ell j} = \omega_j^\ell(x)$, for $j = 1, \dots, n$ and $\ell = 1, \dots, n-1, i$. It is enough to take $(\lambda_1, \dots, \lambda_n) = (a_1, \dots, a_n)$. $\square$

Lemma 4.3. *Let $\mathcal{U} \subset \mathbb{R}^m$ be an open set and let $H : \mathcal{U} \times \mathbb{R}^n \smallsetminus \{\vec{0}\} \to \mathbb{R}^m$ be a smooth mapping given by $H(x, a) = (h_1(x, a), \dots, h_m(x, a))$. If*

$$\operatorname{rank}(dh_1(x, a), \dots, dh_m(x, a)) = m, \forall (x, a) \in H^{-1}(\vec{0})$$

then $\operatorname{rank}(d_x h_1(x, a), \dots, d_x h_m(x, a)) = m$ *for almost every $a \in \mathbb{R}^n \smallsetminus \{\vec{0}\}$.*

In the previous section we proved that every zero of ξ belongs to $\Sigma^1(\omega)$. Next, we will show that, generically, such zeros belong to $A_1(\omega)$ and they are non-degenerate. To do that, we must find explicit equations that define the manifolds $T^* M^{n,n-1}$ and $\Sigma^1(\omega)$ locally.

Lemma 4.4. *Let $(p, \tilde{\varphi}) \in T^* M^{n,n-1}$, it is possible to exhibit, explicitly, functions $\mathrm{m}_i(x, \varphi) : \tilde{\mathcal{U}} \to \mathbb{R}$, $i = n, \dots, m$, defined on an open neighborhood $\tilde{\mathcal{U}}$ of $(p, \tilde{\varphi})$ in $T^* M^n$, such that, locally*

$$T^* M^{n,n-1} = \left\{ (x, \varphi) \in \tilde{\mathcal{U}} \mid \mathrm{m}_n = \ldots = \mathrm{m}_m = 0 \right\}$$

with $\operatorname{rank}(d\mathrm{m}_n, \dots, d\mathrm{m}_m) = m - n + 1$, *for all $(x, \varphi) \in T^* M^{n,n-1} \cap \tilde{\mathcal{U}}$.*

Proof. Let $(p,\tilde{\varphi}) \in T^*M^{n,n-1}$, we may assume that

$$m(\varphi) = \begin{vmatrix} \varphi_1^1 & \varphi_2^1 & \cdots & \varphi_{n-1}^1 \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_1^{n-1} & \varphi_2^{n-1} & \cdots & \varphi_{n-1}^{n-1} \end{vmatrix} \neq 0$$

for (x,φ) in an open neighborhood $\tilde{\mathcal{U}}$ of $(p,\tilde{\varphi})$ in T^*M^n. In this situation, $T^*M^{n,n-1}$ can be locally defined as

$$T^*M^{n,n-1} = \left\{(x,\varphi) \in \tilde{\mathcal{U}} \mid m_n = \ldots = m_m = 0\right\},$$

where $m_i := m_i(\varphi)$ is the determinant

$$m_i(\varphi) = \begin{vmatrix} \varphi_1^1 & \varphi_2^1 & \cdots & \varphi_{n-1}^1 & \varphi_n^1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \varphi_1^{n-1} & \varphi_2^{n-1} & \cdots & \varphi_{n-1}^{n-1} & \varphi_n^{n-1} \\ \varphi_1^i & \varphi_2^i & \cdots & \varphi_{n-1}^i & \varphi_n^i \end{vmatrix}, \; i = n,\ldots,m.$$

Let us verify that $\operatorname{rank}(dm_n,\ldots,dm_m) = m-n+1$ in $(T^*M^{n,n-1}) \cap \tilde{\mathcal{U}}$.

For clearer notations, consider $I = \{1,\ldots,n\}$ and $I_i = \{1,\ldots,n-1,i\}$ for each $i \in \{n,\ldots,m\}$. Then

$$dm_i(\varphi) = \sum_{j\in I,\ell\in I_i} \operatorname{cof}(\varphi_j^\ell, m_i)\, d\varphi_j^\ell, \tag{3}$$

where $\operatorname{cof}(\varphi_j^\ell, m_i)$ is the cofactor of φ_j^ℓ in the matrix

$$\begin{bmatrix} \varphi_1^1 & \varphi_2^1 & \cdots & \varphi_{n-1}^1 & \varphi_n^1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \varphi_1^{n-1} & \varphi_2^{n-1} & \cdots & \varphi_{n-1}^{n-1} & \varphi_n^{n-1} \\ \varphi_1^i & \varphi_2^i & \cdots & \varphi_{n-1}^i & \varphi_n^i \end{bmatrix}$$

and

$$d\varphi_j^\ell = \left(\frac{\partial\varphi_j^\ell}{\partial\varphi_1^1},\ldots,\frac{\partial\varphi_j^\ell}{\partial\varphi_1^m},\frac{\partial\varphi_j^\ell}{\partial\varphi_2^1},\ldots,\frac{\partial\varphi_j^\ell}{\partial\varphi_2^m},\ldots,\frac{\partial\varphi_j^\ell}{\partial\varphi_n^1},\ldots,\frac{\partial\varphi_j^\ell}{\partial\varphi_n^m}\right)$$

is the vector whose coordinate at the position $(j-1)m+\ell$ is equal to 1 and all the others are zero. In particular, since $i \in \{n,\ldots,m\}$,

$$d\varphi_n^i = (0,\ldots,0,\underbrace{0,\ldots,\overset{i}{1},\ldots,0}_{m-n+1}) \in \underbrace{(\mathbb{R}^m)^* \times \ldots \times (\mathbb{R}^m)^*}_{n \text{ times}}$$

and the $m-n+1$ last coordinates of $d\varphi_j^\ell$ are zero for all $j \neq n$ or $\ell \neq i$. Moreover,

$$\operatorname{cof}(\varphi_n^i, m_i) = m(\varphi) \neq 0, \quad \text{for} \quad i = n,\ldots,m.$$

Thus,

$$\frac{\partial(m_n,\ldots,m_m)}{\partial(\varphi_n^n,\ldots,\varphi_n^m)} = \begin{vmatrix} \operatorname{cof}(\varphi_n^n, m_n) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \operatorname{cof}(\varphi_n^m, m_m) \end{vmatrix}.$$

That is, for all $(x,\varphi) \in (T^*M^{n,n-1}) \cap \tilde{\mathcal{U}}$, we have

$$\frac{\partial(m_n,\ldots,m_m)}{\partial(\varphi_n^n,\ldots,\varphi_n^m)} = m(\varphi)^{(m-n+1)} \begin{vmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{vmatrix} \neq 0. \tag{4}$$

Therefore, $\operatorname{rank}(m_n,\ldots,m_m) = m-n+1$ for all $(x,\varphi) \in (T^*M^{n,n-1}) \cap \tilde{\mathcal{U}}$. □

Lemma 4.5. *Let $p \in \Sigma^1(\omega)$ be a singular point of ω, it is possible to exhibit, explicitly, functions $\mathbf{M}_i(x) : \mathcal{U} \to \mathbb{R}$, $i = n, \dots, m$, defined on an open neighborhood $\mathcal{U}$ of p in M, such that, locally*

$$\mathcal{U} \cap \Sigma^1(\omega) = \{x \in \mathcal{U} \mid \mathbf{M}_n(x) = \dots = \mathbf{M}_m(x) = 0\}$$

with $\operatorname{rank}(d\mathbf{M}_n(x), \dots, d\mathbf{M}_m(x)) = m - n + 1$, *for all $x \in \Sigma^1(\omega) \cap \mathcal{U}$.*

Proof. Let $\omega = \{\omega_i\}_{1 \le i \le n}$ be a Morin collection of one-forms and let $p \in \Sigma^1(\omega)$. By Remark 3.4, we can consider $\mathcal{U}$ an open neighborhood of p in M, where $\mathbf{M}(x) \neq 0$. Thus, in this neighborhood the set $\Sigma^1(\omega)$ can be defined as

$$\mathcal{U} \cap \Sigma^1(\omega) = \{x \in \mathcal{U} \mid \mathbf{M}_n = \dots = \mathbf{M}_m = 0\},$$

where $\mathbf{M}_i := \mathbf{M}_i(x)$ is the determinant

$$\mathbf{M}_i(x) = \begin{vmatrix} \omega_1^1(x) & \omega_2^1(x) & \cdots & \omega_{n-1}^1(x) & \omega_n^1(x) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \omega_1^{n-1}(x) & \omega_2^{n-1}(x) & \cdots & \omega_{n-1}^{n-1}(x) & \omega_n^{n-1}(x) \\ \omega_1^i(x) & \omega_2^i(x) & \cdots & \omega_{n-1}^i(x) & \omega_n^i(x) \end{vmatrix} \tag{5}$$

for $i = n, \dots, m$.

Let $G(\omega) = \{(x, \omega_1(x), \dots, \omega_n(x)) \mid x \in M\}$ be the graph of the collection ω. For each $x \in \Sigma^1(\omega) \cap \mathcal{U}$, we have that $G(\omega) \pitchfork T^*M^{n,n-1}$ at $(x, \omega(x))$. Then, the equations that define $G(\omega)$ and $T^*M^{n,n-1}$ locally are independent at $(x, \omega(x))$. By similar arguments to that used in the proof of Lemma 4.4, it follows that the functions $\mathbf{M}_n(x), \dots, \mathbf{M}_m(x)$ are independent at x, that is, for all $x \in \Sigma^1(\omega) \cap \mathcal{U}$, $\operatorname{rank}(d\mathbf{M}_n(x), \dots, d\mathbf{M}_m(x)) = m - n + 1$. $\square$

Lemma 4.6. *For almost every $a \in \mathbb{R}^n \setminus \{\vec{0}\}$, the one-form $\xi(x) = \sum_{i=1}^{n} a_i \omega_i(x)$ admits only non-degenerate zeros. Moreover, such zeros belong to $A_1(\omega)$.*

Proof. Suppose that $p \in M$ is a zero of ξ. Then, by Lemmas 3.1 and 3.6, for almost every $a \in \mathbb{R}^n \setminus \{\vec{0}\}$ we have that $p \in \Sigma^1(\omega) \setminus \Sigma^2(\omega)$, that is, $p \in A_1(\omega)$. Assume that $\mathbf{M}(x) \neq 0$ in an open neighborhood $\mathcal{U}$ of p in M (see Remark 3.4) such that

$$\mathcal{U} \cap \Sigma^1(\omega) = \{x \in \mathcal{U} : \mathbf{M}_n(x) = \dots = \mathbf{M}_m(x) = 0\}.$$

Let us write

$$\xi_s(x) = \sum_{i=1}^{n} a_i \omega_i^s(x), \ s = 1, \dots, m$$

and let us consider the mapping $F : \mathcal{U} \times \mathbb{R}^n \setminus \{\vec{0}\} \to \mathbb{R}^m$ defined by

$$F(x,a) = (\mathbf{M}_n(x), \dots, \mathbf{M}_m(x), \xi_1(x), \dots, \xi_{n-1}(x)).$$

Its Jacobian matrix at a point (x, a) is given by:

$$\operatorname{Jac} F(x,a) = \left[\begin{array}{c|cccc} d_x\mathbf{M}_n(x) & & & & \\ \vdots & & O_{(m-n)\times n} & & \\ d_x\mathbf{M}_m(x) & & & & \\ \hline d_x\xi_1(x) & \omega_1^1(x) & \cdots & \omega_{n-1}^1(x) & \omega_n^1(x) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ d_x\xi_{n-1}(x) & \omega_1^{n-1}(x) & \cdots & \omega_{n-1}^{n-1}(x) & \omega_n^{n-1}(x) \end{array}\right].$$

Notice that, by Lemma 3.5, $F^{-1}(\vec{0})$ corresponds to the zeros of ξ on $\Sigma^1(\omega)\cap\mathcal{U}$. Since $\mathbf{M}(x)\neq 0$ and $\operatorname{rank}(d\mathbf{M}_n(x),\ldots,d\mathbf{M}_m(x)) = m-n+1$ for all $x\in\Sigma^1(\omega)\cap\mathcal{U}$, then $\operatorname{rank}(\operatorname{Jac}F(x,a)) = m$ for all $(x,a)\in F^{-1}(\vec{0})$. Thus, $\dim F^{-1}(\vec{0}) = n$.

Let $\pi : F^{-1}(\vec{0}) \to \mathbb{R}^n\setminus\{\vec{0}\}$ be the projection $\pi(x,a) = a$, by Sard's Theorem, almost every $a\in\mathbb{R}^n\setminus\{\vec{0}\}$ is a regular value of π and $\dim(\pi^{-1}(a)\cap F^{-1}(\vec{0})) = 0$. That is, for almost every a, the zeros of ξ are isolated in $\Sigma^1(\omega)$. Let us proof that, moreover, these zeros are non-degenerate.

Since $\operatorname{rank}(\operatorname{Jac}F(x,a)) = m$, for all $(x,a)\in F^{-1}(\vec{0})$, then by Lemma 4.3 we have that

$$\operatorname{rank}(d_x\mathbf{M}_n(p),\ldots,d_x\mathbf{M}_m(p),d_x\xi_1(p),\ldots,d_x\xi_{n-1}(p)) = m,$$

which happens if and only if $\operatorname{rank}(B) = m$, where B is the matrix

$$B = \begin{bmatrix} d_x\xi_1(p) \\ \vdots \\ d_x\xi_{n-1}(p) \\ a_n d_x\mathbf{M}_n(p) \\ \vdots \\ a_n d_x\mathbf{M}_m(p) \end{bmatrix}$$

whose row vectors we will denote by $R_i, i = 1,\ldots,m$ (by Remark 3.4, $a_n\neq 0$).

Let us denote $I = \{1,\ldots,n\}$ and $I_i = \{1,\ldots,n-1,i\}$ for each $i\in\{n,\ldots,m\}$. By Equation (5), we can write

$$d\mathbf{M}_i(x) = \sum_{\ell\in I, j\in I_i} \operatorname{cof}(\omega_\ell^j(x),M_i)\,d\omega_\ell^j(x)$$

and by Lemma 4.2,

$$d\mathbf{M}_i(p) = \sum_{\ell\in I, j\in I_i} \frac{a_\ell}{a_n}\operatorname{cof}(\omega_n^j(p),M_i)\,d\omega_\ell^j(p).$$

Thus,

$$\begin{aligned} a_n d\mathbf{M}_i(p) &= \sum_{\ell\in I, j\in I_i} a_\ell \operatorname{cof}(\omega_n^j(p),M_i)\,d\omega_\ell^j(p) \\ &= \sum_{j\in I_i}\operatorname{cof}(\omega_n^j(p),M_i)\left[\sum_{\ell\in I} a_\ell\, d\omega_\ell^j(p)\right] \\ &= \sum_{j\in I_i}\operatorname{cof}(\omega_n^j(p),M_i)\,[d_x\xi_j(p)] \\ &= \operatorname{cof}(\omega_n^i(p),M_i)\,[d_x\xi_i(p)] + \sum_{j\in I_i\setminus\{i\}}\operatorname{cof}(\omega_n^j(p),M_i)\,[d_x\xi_j(p)]. \end{aligned}$$

Notice that, $\operatorname{cof}(\omega_n^i(p),M_i) = \mathbf{M}(p)\neq 0$, for all $i = n,\ldots,m$. Then, for each $i = n,\ldots,m$, we replace the i^{th} row R_i of matrix B by

$$\frac{1}{\operatorname{cof}(\omega_n^i(p),M_i)}\left(R_i - \sum_{j=1}^{n-1}\operatorname{cof}(\omega_n^j(p),M_i)R_j\right)$$

such that we obtain the matrix of maximal rank:

$$\begin{bmatrix} d_x\xi_1(p) \\ \vdots \\ d_x\xi_{n-1}(p) \\ d_x\xi_n(p) \\ \vdots \\ d_x\xi_m(p) \end{bmatrix}.$$

Therefore, the zeros of $\xi(x)$ are non-degenerate. □

Lemma 4.7. *For almost every $a \in \mathbb{R}^n \setminus \{\vec{0}\}$, the one-form $\xi_{|_{A_k(\omega)}}$ admits only non-degenerate zeros, $k \geq 2$.*

Proof. Suppose that $\xi_{|_{A_k(\omega)}}(p) = 0$. By Proposition 2.23 (a) and Lemma 4.5, we can consider $\mathcal{U}$ an open neighborhood of p in M where $\mathbf{M}(x) \neq 0$ and on which the respective singular sets $(k = 2, \dots, n)$ can be locally defined as

$$\mathcal{U} \cap \Sigma^k(\omega) = \{x \in \mathcal{U} : \mathbf{M}_n(x) = \dots = \mathbf{M}_m(x) = \Delta_2(x) = \dots = \Delta_k(x) = 0\},$$

with $\operatorname{rank}(d\mathbf{M}_n, \dots, d\mathbf{M}_m, d\Delta_2, \dots, d\Delta_k) = m - n + k$, $\forall x \in \Sigma^k(\omega) \cap \mathcal{U}$.

Analogously to the proof of Lemma 3.7, by Szafraniec's characterization (see [19, p. 196]), x is a zero of the restriction $\xi_{|_{\Sigma^k(\omega)}}$ if and only if there exists $(\lambda_n, \dots, \lambda_m, \beta_2, \dots, \beta_k) \in \mathbb{R}^{m-n+k}$ such that

$$\xi(x) = \sum_{j=n}^{m} \lambda_j d\mathbf{M}_j(x) + \sum_{\ell=2}^{k} \beta_\ell d\Delta_\ell(x).$$

Let us consider the functions

$$N_s(x, a, \lambda, \beta) := \xi_s(x) - \sum_{j=n}^{m} \lambda_j \frac{\partial \mathbf{M}_j}{\partial x_s}(x) - \sum_{\ell=2}^{k} \beta_\ell \frac{\partial \Delta_\ell}{\partial x_s}(x), \;\; s = 1, \dots, m,$$

and let $G : \mathcal{U} \setminus \{\Delta_{k+1} = 0\} \times \mathbb{R}^n \setminus \{\vec{0}\} \times \mathbb{R}^{m-n+k} \to \mathbb{R}^{2m-n+k}$ be the mapping given by

$$G(x, a, \lambda, \beta) = (\mathbf{M}_n, \dots, \mathbf{M}_m, \Delta_2, \dots, \Delta_k, N_1, \dots, N_m).$$

Analogously to the proof of Lemma 4.6, if $(x, a, \lambda, \beta) \in G^{-1}(\vec{0})$ then $x \in A_k(\omega) \cap Z(\xi_{|_{\Sigma^k(\omega)}})$. On the other hand, if $x \in A_k(\omega)$ then

$$\dim(\langle \bar{\omega}(x) \rangle \cap N_x^* \Sigma^{k-1}(\omega)) = k - 1$$

and $\dim(\langle \bar{\omega}(x) \rangle \cap N_x^* \Sigma^k(\omega)) = k - 1$, such that $\dim(\langle \bar{\omega}(x) \rangle + N_x^* \Sigma^k(\omega)) = m$. This implies that the Jacobian matrix of G has maximal rank at every $(x, a, \lambda, \beta) \in G^{-1}(\vec{0})$. Thus $\dim G^{-1}(\vec{0}) = n$.

Let $\pi : G^{-1}(\vec{0}) \to \mathbb{R}^n \setminus \{\vec{0}\}$ be the projection $\pi(x, a, \lambda, \beta) = a$, then for almost every $a \in \mathbb{R}^n \setminus \{\vec{0}\}$, $\dim(\pi^{-1}(a) \cap G^{-1}(\vec{0})) = 0$ and $\pi^{-1}(a) \pitchfork G^{-1}(\vec{0})$. Therefore, the zeros of $\xi_{|_{A_k(\omega)}}$ are non-degenerate. □

Lemma 4.8. *For almost every $a \in \mathbb{R}^n \setminus \{\vec{0}\}$, the one-form $\xi_{|_{A_1(\omega)}}$ admits only non-degenerate zeros.*

Proof. This proof follows analogously the proof of Lemma 4.7. □

By Lemma 3.2, if $p \in A_{k+1}(\omega)$, then p is a zero of $\xi_{|_{\Sigma^{k+1}(\omega)}}$ if and only if p is a zero of $\xi_{|_{\Sigma^k(\omega)}}$. The next results state that this relation also holds for non-degenerate zeros.

Lemma 4.9. *Let $p \in A_1(\omega)$ be a zero of $\xi_{|_{\Sigma^1(\omega)}}$, then p is a non-degenerate zero of $\xi_{|_{\Sigma^1(\omega)}}$ if and only if p is a non-degenerate zero of ξ.*

Proof. Let $p \in A_1(\omega)$ be a zero of the restriction $\xi_{|_{\Sigma^1(\omega)}}$ and let $\mathcal{U}$ be an open neighborhood of p in M at which $\mathbf{M}(x) \neq 0$, $\forall x \in \mathcal{U}$ and $\mathcal{U} \cap \Sigma^1(\omega) = \{x \in \mathcal{U} : \mathbf{M}_n(x) = \dots = \mathbf{M}_m(x) = 0\}$. By Szafraniec's characterization ([19, p. 196]), $\exists!(\lambda_n, \dots, \lambda_m) \in \mathbb{R}^{m-n+1}$, such that

$$\xi(p) + \sum_{i=n}^{m} \lambda_i d\mathbf{M}_i(p) = 0.$$

Furthermore, p is a non-degenerate zero of $\xi|_{\Sigma^1(\omega)}$ if and only if the matrix

$$
\left[\begin{array}{c|ccc}
 & \frac{\partial \mathbf{M}_n}{\partial x_1}(p) & \cdots & \frac{\partial \mathbf{M}_m}{\partial x_1}(p) \\
\mathrm{Jac}\left(\xi + \sum_{i=n}^{m} \lambda_i d\mathbf{M}_i\right)(p) & \vdots & \ddots & \vdots \\
 & \frac{\partial \mathbf{M}_n}{\partial x_m}(p) & \cdots & \frac{\partial \mathbf{M}_m}{\partial x_m}(p) \\
\hline
d_x\mathbf{M}_n(p) & & & \\
\vdots & & O_{(m-n+1)} & \\
d_x\mathbf{M}_m(p) & & &
\end{array}\right] \tag{6}
$$

is non-singular. Since $\xi(p) = 0$, then $p \in \Sigma^1(\omega) \cap \mathcal{U}$ and $\sum_{i=n}^{m} \lambda_i d\mathbf{M}_i(p) = \vec{0}$. Thus,

$$\lambda_n = \ldots = \lambda_m = 0,$$

and writing $\xi = (\xi_1, \ldots, \xi_m)$ we have that the Matrix (6) is non-singular if and only if the matrix

$$
\left[\begin{array}{c|ccc}
d_x\xi_1(p) & \frac{\partial \mathbf{M}_n}{\partial x_1}(p) & \cdots & \frac{\partial \mathbf{M}_m}{\partial x_1}(p) \\
\vdots & \vdots & \ddots & \vdots \\
d_x\xi_m(p) & \frac{\partial \mathbf{M}_n}{\partial x_m}(p) & \cdots & \frac{\partial \mathbf{M}_m}{\partial x_m}(p) \\
\hline
a_n d_x\mathbf{M}_n(p) & & & \\
\vdots & & O_{(m-n+1)} & \\
a_n d_x\mathbf{M}_m(p) & & &
\end{array}\right] \tag{7}
$$

is non-singular (by Remark 3.4, $a_n \neq 0$). Moreover, by Equation (5) and Lemma 4.2, we can write

$$
\begin{aligned}
a_n d_x\mathbf{M}_i(p) &= a_n \sum_{\ell \in I, j \in I_i} \mathrm{cof}(\omega_\ell^j(p), M_i) d\omega_\ell^j(p) \\
&= \sum_{\ell \in I, j \in I_i} a_\ell \, \mathrm{cof}(\omega_n^j(p), M_i) d\omega_\ell^j(p) \\
&= \sum_{j \in I_i} \mathrm{cof}(\omega_n^j(p), M_i) \left[\sum_{\ell \in I} a_\ell d\omega_\ell^j(p)\right] \\
&= \sum_{j \in I_i} \mathrm{cof}(\omega_n^j(p), M_i) \left[d_x\xi_j(p)\right].
\end{aligned}
$$

Let us denote the m first row vectors of Matrix (7) by $L_j, j = 1, \dots, m$, and let us denote the $m - n + 1$ last row vectors of Matrix (7) by $R_i, i = n, \dots, m$:

$$\begin{aligned} L_j &= \left(d_x\xi_j(p), \frac{\partial \mathbf{M}_n}{\partial x_j}(p), \dots, \frac{\partial \mathbf{M}_m}{\partial x_j}(p) \right); \\ R_i &= \left(a_n \frac{\partial \mathbf{M}_i}{\partial x_1}(p), \dots, a_n \frac{\partial \mathbf{M}_i}{\partial x_m}(p), \vec{0} \right). \end{aligned}$$

Then, replacing each row vector R_i, $i = n, \dots, m$, by $R_i - \sum_{j\in I_i} \operatorname{cof}(\omega_n^j, M_i) L_j$, we obtain

$$R_i = \left(\underbrace{0, \dots\ 0}_{m \text{ times}}, - \sum_{j\in I_i} \operatorname{cof}(\omega_n^j, M_i) \frac{\partial \mathbf{M}_n}{\partial x_j}, \dots, - \sum_{j\in I_i} \operatorname{cof}(\omega_n^j, M_i) \frac{\partial \mathbf{M}_m}{\partial x_j} \right)$$

and the Matrix (7) becomes:

$$\left[\begin{array}{c|ccc} d_x\xi_1(p) & \dfrac{\partial \mathbf{M}_n}{\partial x_1}(p) & \cdots & \dfrac{\partial \mathbf{M}_m}{\partial x_1}(p) \\ \vdots & \vdots & \ddots & \vdots \\ d_x\xi_m(p) & \dfrac{\partial \mathbf{M}_n}{\partial x_m}(p) & \cdots & \dfrac{\partial \mathbf{M}_m}{\partial x_m}(p) \\ \cdots\ \cdots\ \cdots & \cdots\ \cdots & \cdots\ \cdots & \cdots\ \cdots \\ O_{(m-n+1)\times m} & & \mathbf{M}'_{(m-n+1)} & \end{array} \right] \tag{8}$$

where $\mathbf{M}'_{(m-n+1)} = -\left(\ m_{ij}\ \right)_{n\le i,j\le m}$ is the matrix given by

$$m_{ij} = \sum_{k\in I_i} \operatorname{cof}(\omega_n^k, M_i) \frac{\partial \mathbf{M}_j}{\partial x_k},\ i, j = n, \dots, m. \tag{9}$$

Next, we will verify that the matrix $\mathbf{M}'$ is non-singular. Since $p \in A_1(\omega)$, then

$$\dim(\langle \bar{\omega}(p) \rangle \cap N_p^*\Sigma^1(\omega)) = 0$$

and $\dim(\langle \bar{\omega}(p) \rangle \oplus N_p^*\Sigma^1(\omega)) = m$. Since $\mathbf{M}(p) \neq 0$, $\{\omega_1(p), \dots, \omega_{n-1}(p)\}$ is a basis of the space $\langle \bar{\omega}(p) \rangle$ and, consequently, the matrix

$$\begin{bmatrix} \omega_1^1(p) & \cdots & \omega_1^{n-1}(p) & \omega_1^n(p) & \cdots & \omega_1^m(p) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \omega_{n-1}^1(p) & \cdots & \omega_{n-1}^{n-1}(p) & \omega_{n-1}^n(p) & \cdots & \omega_{n-1}^m(p) \\ \dfrac{\partial \mathbf{M}_n}{\partial x_1}(p) & \cdots & \dfrac{\partial \mathbf{M}_n}{\partial x_{n-1}}(p) & \dfrac{\partial \mathbf{M}_n}{\partial x_n}(p) & \cdots & \dfrac{\partial \mathbf{M}_n}{\partial x_m}(p) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial \mathbf{M}_m}{\partial x_1}(p) & \cdots & \dfrac{\partial \mathbf{M}_m}{\partial x_{n-1}}(p) & \dfrac{\partial \mathbf{M}_m}{\partial x_n}(p) & \cdots & \dfrac{\partial \mathbf{M}_m}{\partial x_m}(p) \end{bmatrix} \tag{10}$$

has maximal rank. Let us denote the row vectors of Matrix (10) by L'_j, $j=1,\dots,m$. Then, for $j=1,\dots,n-1$, we replace L'_j by

$$\sum_{k=1}^{n-1}\operatorname{cof}(\omega_k^j,M)L'_k=\left(\sum_{k=1}^{n-1}\operatorname{cof}(\omega_k^j,M)\omega_k^1,\dots,\sum_{k=1}^{n-1}\operatorname{cof}(\omega_k^j,M)\omega_k^m\right). \tag{11}$$

It is not difficult to verify that

$$\sum_{k=1}^{n-1}\operatorname{cof}(\omega_k^j,M)\omega_k^\ell=\begin{cases}\mathbf{M}, & \ell=j;\\ 0 & \ell=1,\dots,n-1 \text{ and } \ell\neq j;\\ -\operatorname{cof}(\omega_n^j,\mathbf{M}_\ell), & \ell=n,\dots,m.\end{cases}$$

Thus, Matrix (10) becomes

$$\begin{bmatrix}
\mathbf{M} & \cdots & 0 & \vdots & -\operatorname{cof}(\omega_n^1,\mathbf{M}_n) & \cdots & -\operatorname{cof}(\omega_n^1,\mathbf{M}_m)\\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\
0 & \cdots & \mathbf{M} & \vdots & -\operatorname{cof}(\omega_n^{n-1},\mathbf{M}_n) & \cdots & -\operatorname{cof}(\omega_n^{n-1},\mathbf{M}_m)\\
\cdots & \cdots & \cdots & \vdots & \cdots\ \cdots\ \cdots & \cdots & \cdots\ \cdots\ \cdots\\
\dfrac{\partial\mathbf{M}_n}{\partial x_1} & \cdots & \dfrac{\partial\mathbf{M}_n}{\partial x_{n-1}} & \vdots & \dfrac{\partial\mathbf{M}_n}{\partial x_p} & \cdots & \dfrac{\partial\mathbf{M}_n}{\partial x_m}\\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\
\dfrac{\partial\mathbf{M}_m}{\partial x_1} & \cdots & \dfrac{\partial\mathbf{M}_m}{\partial x_{n-1}} & \vdots & \dfrac{\partial\mathbf{M}_m}{\partial x_p} & \cdots & \dfrac{\partial\mathbf{M}_m}{\partial x_m}
\end{bmatrix} \tag{12}$$

that still has maximal rank. Now, let us denote the first $n-1$ row vectors of Matrix (12) by L''_j, for $j=1,\dots,n-1$, and let us consider the following expression for $j=n,\dots,m$,

$$\begin{aligned}
&\mathbf{M}L'_j-\sum_{k=1}^{n-1}\frac{\partial\mathbf{M}_j}{\partial x_k}L''_k\\
&=\mathbf{M}\left(\frac{\partial\mathbf{M}_j}{\partial x_1},\dots,\frac{\partial\mathbf{M}_j}{\partial x_{n-1}},\frac{\partial\mathbf{M}_j}{\partial x_n},\dots,\frac{\partial\mathbf{M}_j}{\partial x_m}\right)\\
&+\left(-\mathbf{M}\frac{\partial\mathbf{M}_j}{\partial x_1},\dots,-\mathbf{M}\frac{\partial\mathbf{M}_j}{\partial x_{n-1}},\sum_{k=1}^{n-1}\frac{\partial\mathbf{M}_j}{\partial x_k}\operatorname{cof}(\omega_n^k,M_n),\dots,\sum_{k=1}^{n-1}\frac{\partial\mathbf{M}_j}{\partial x_k}\operatorname{cof}(\omega_n^k,M_m)\right)\\
&=\left(0,\dots,0,\sum_{k=1}^{n-1}\frac{\partial\mathbf{M}_j}{\partial x_k}\operatorname{cof}(\omega_n^k,M_n)+\mathbf{M}\frac{\partial\mathbf{M}_j}{\partial x_n},\dots,\sum_{k=1}^{n-1}\frac{\partial\mathbf{M}_j}{\partial x_k}\operatorname{cof}(\omega_n^k,M_m)+\mathbf{M}\frac{\partial\mathbf{M}_j}{\partial x_m}\right).
\end{aligned}$$

Notice that $\mathbf{M}=\operatorname{cof}(\omega_n^i,\mathbf{M}_i)$, for $i=n,\dots,m$. Then the expression

$$\mathbf{M}L'_j-\sum_{k=1}^{n-1}\frac{\partial\mathbf{M}_j}{\partial x_k}L''_k \tag{13}$$

is equal to

$$\left(0,\dots,0,\sum_{k\in I_n}\frac{\partial\mathbf{M}_j}{\partial x_k}\operatorname{cof}(\omega_n^k,M_n),\dots,\sum_{k\in I_m}\frac{\partial\mathbf{M}_j}{\partial x_k}\operatorname{cof}(\omega_n^k,M_m)\right).$$

Thus, by Equation (9), we obtain

$$\mathbf{M}L'_j-\sum_{k=1}^{n-1}\frac{\partial\mathbf{M}_j}{\partial x_k}L''_k=(0,\dots,0,m_{nj},\dots,m_{mj}).$$

In this way, we replace the row L'_j in Matrix (12) by (13) for $j = n, \dots, m$, and the matrix obtained

$$
\begin{bmatrix}
\mathbf{M} & \cdots & 0 & \vdots & -\operatorname{cof}(\omega_n^1, \mathbf{M}_n) & \cdots & -\operatorname{cof}(\omega_n^1, \mathbf{M}_m) \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \mathbf{M} & \vdots & -\operatorname{cof}(\omega_n^{n-1}, \mathbf{M}_n) & \cdots & -\operatorname{cof}(\omega_n^{n-1}, \mathbf{M}_m) \\
\cdots & \cdots & \cdots & \vdots & \cdots\ \cdots\ \cdots & \cdots & \cdots\ \cdots\ \cdots \\
 & & & \vdots & & & \\
 & O_{(n-1)} & & \vdots & & (-\mathbf{M}')^t & \\
 & & & \vdots & & &
\end{bmatrix}
\tag{14}
$$

also is non-singular. Then, since $\mathbf{M} \neq 0$, we have that $\det \mathbf{M}' \neq 0$. Thus, we can conclude that Matrix (7) is non-singular if and only if Matrix (8) is non-singular, which occurs if and only if

$$\det \begin{bmatrix} d_x\xi_1(p) \\ \vdots \\ d_x\xi_m(p) \end{bmatrix} \neq 0.$$

In other words, p will be a non-degenerate zero of $\xi_{|_{\Sigma^1(\omega)}}$ if and only if p is a non-degenerate zero of ξ. □

Lemma 4.10. *Let $p \in A_{k+1}(\omega)$ be a zero of $\xi_{|_{\Sigma^{k+1}(\omega)}}$. Then, for almost every $a \in \mathbb{R}^n \setminus \{\vec{0}\}$, p is a non-degenerate zero of $\xi_{|_{\Sigma^{k+1}(\omega)}}$ if and only if p is a non-degenerate zero of $\xi_{|_{\Sigma^k(\omega)}}$.*

Proof. Let $p \in A_{k+1}(\omega)$ be a zero of $\xi_{|_{\Sigma^{k+1}(\omega)}}$ and let $\mathcal{U}$ be an open neighborhood of p in M at which $\mathbf{M}(x) \neq 0$, $\forall x \in \mathcal{U}$ and the singular sets $\Sigma^k(\omega)$ $(k = 2, \dots, n)$ are defined by $\mathcal{U} \cap \Sigma^k(\omega) = \{x \in \mathcal{U} : \mathbf{M}_n(x) = \dots = \mathbf{M}_m(x) = \Delta_2(x) = \dots = \Delta_k(x) = 0\}$. By Szafraniec's characterization ([19, p. 196]), p is a zero of the restriction $\xi_{|_{\Sigma^{k+1}(\omega)}}$ if and only if there exists a unique $(\lambda_n, \dots, \lambda_m, \beta_2, \dots, \beta_{k+1}) \in \mathbb{R}^{m-n+k+1}$ such that

$$\xi(p) + \sum_{i=n}^{m} \lambda_i d\mathbf{M}_i(p) + \sum_{j=2}^{k+1} \beta_j d\Delta_j(p) = 0. \tag{15}$$

Since p is a zero of $\xi_{|_{\Sigma^k(\omega)}}$, we have $\beta_{k+1} = 0$. Moreover, also by Szafraniec's characterization, for $\ell = k, k+1$, p is a non-degenerate zero of $\xi_{|_{\Sigma^\ell(\omega)}}$ if and only if the determinant of the following matrix does not vanish at p:

$$
J_\ell =
\begin{bmatrix}
 & \vdots & \frac{\partial \mathbf{M}_n}{\partial x_1} & \cdots & \frac{\partial \mathbf{M}_m}{\partial x_1} & \frac{\partial \Delta_2}{\partial x_1} & \cdots & \frac{\partial \Delta_\ell}{\partial x_1} \\
\operatorname{Jac}_x\left(\xi + \sum_{i=n}^{m} \lambda_i d\mathbf{M}_i + \sum_{j=2}^{k} \beta_j d\Delta_j\right) & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
 & \vdots & \frac{\partial \mathbf{M}_n}{\partial x_m} & \cdots & \frac{\partial \mathbf{M}_m}{\partial x_m} & \frac{\partial \Delta_2}{\partial x_m} & \cdots & \frac{\partial \Delta_\ell}{\partial x_m} \\
\cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots & \vdots & \cdots & \cdots & \cdots\ \cdots & \cdots & \cdots & \cdots \\
d_x\mathbf{M}_n & \vdots & & & & & & \\
\vdots & \vdots & & & & & & \\
d_x\mathbf{M}_m & \vdots & & & & & & \\
d_x\Delta_2 & \vdots & & & & O_{(m-n+\ell)} & & \\
\vdots & \vdots & & & & & & \\
d_x\Delta_\ell & \vdots & & & & & &
\end{bmatrix}.
\tag{16}
$$

Thus, to prove the lemma it is enough to show that the Matrix J_{k+1} is non-singular at p if and only if the Matrix J_k is non-singular at p.

Notice that the Jacobian matrix with respect to x

$$\mathrm{Jac}_x\left(\xi + \sum_{i=n}^{m} \lambda_i d\mathbf{M}_i + \sum_{j=2}^{k} \beta_j d\Delta_j\right) \tag{17}$$

is a submatrix of both Matrices J_{k+1} and J_k, and recall that, for x in an open neighborhood of p, $\Delta_{k+1} = \det(d\mathbf{M}_n, \ldots, d\mathbf{M}_m, d\Delta_2, \ldots, d\Delta_k, \Omega_1, \ldots, \Omega_{n-k})$, where $\{\Omega_1(x), \ldots, \Omega_{n-k}(x)\}$ is a basis of a vector subspace complementary to $\langle\bar{\omega}(x)\rangle \cap N_x^*\Sigma^{k-1}(\omega)$ in $\langle\bar{\omega}(x)\rangle$. That is,

$$\langle\bar{\omega}(x)\rangle = \langle\Omega_1(x), \ldots, \Omega_{n-k}(x)\rangle \oplus (\langle\bar{\omega}(x)\rangle \cap N_x^*\Sigma^{k-1}(\omega)).$$

Since, for almost every a, $\xi_{|_{\Sigma^{k-1}(\omega)}}(p) \neq 0$ then $\xi(p) \in \langle\bar{\omega}(p)\rangle \setminus N_p^*\Sigma^{k-1}(\omega)$ and there exists $(\mu_1, \ldots, \mu_{n-k}) \in \mathbb{R}^{n-k} \setminus \{\vec{0}\}$ such that $\xi(p) = \sum_{i=1}^{n-k} \mu_i \Omega_i(p) + \varphi(p)$, for some $\varphi(p) \in N_p^*\Sigma^{k-1}(\omega)$, where $\varphi(p) = \sum_{i=n}^{m} \tilde{\lambda}_i d\mathbf{M}_i(p) + \sum_{j=2}^{k-1} \tilde{\beta}_j d\Delta_j(p)$. Then, equation (15) can be written as:

$$\sum_{i=1}^{n-k} \mu_i \Omega_i(p) + \sum_{i=n}^{m} (\lambda_i + \tilde{\lambda}_i) d\mathbf{M}_i(p) + \sum_{j=2}^{k-1} (\beta_j + \tilde{\beta}_j) d\Delta_j(p) + \beta_k d\Delta_k(p) = 0. \tag{18}$$

Let us consider the mapping

$$H(x) = \sum_{i=1}^{n-k} \mu_i \Omega_i(x) + \sum_{i=n}^{m} (\lambda_i + \tilde{\lambda}_i) d\mathbf{M}_i(x) + \sum_{j=2}^{k-1} (\beta_j + \tilde{\beta}_j) d\Delta_j(x) + \beta_k d\Delta_k(x),$$

defined on $\mathcal{U}$. The Jacobian matrix of $H(x)$ is given by:

$$\begin{bmatrix} \sum_{i=1}^{n-k} \mu_i d_x \Omega_i^1 + \sum_{i=n}^{m} (\lambda_i + \tilde{\lambda}_i) d_x \dfrac{\partial \mathbf{M}_i}{\partial x_1} + \sum_{j=2}^{k-1} (\beta_j + \tilde{\beta}_j) d_x \dfrac{\partial \Delta_j}{\partial x_1} + \beta_k d_x \dfrac{\partial \Delta_k}{\partial x_1} \\ \vdots \\ \sum_{i=1}^{n-k} \mu_i d_x \Omega_i^m + \sum_{i=n}^{m} (\lambda_i + \tilde{\lambda}_i) d_x \dfrac{\partial \mathbf{M}_i}{\partial x_m} + \sum_{j=2}^{k-1} (\beta_j + \tilde{\beta}_j) d_x \dfrac{\partial \Delta_j}{\partial x_m} + \beta_k d_x \dfrac{\partial \Delta_k}{\partial x_m} \end{bmatrix}. \tag{19}$$

To apply Lemma 4.1, fix the notation: $A_i(x) = (a_{1i}(x), \ldots, a_{mi}(x))$, where

$$A_i(x) := \begin{cases} \Omega_i(x), & i = 1, \ldots, n-k; \\ d\mathbf{M}_i(x), & i = n, \ldots, m; \end{cases}$$

$$A_{n-k+j-1}(x) := d\Delta_j(x), \quad j = 2, \ldots, k;$$

$$\alpha_i := \begin{cases} \mu_i, & i = 1, \ldots, n-k; \ (\text{we can suppose } \alpha_1 \neq 0, \ \text{since } \xi(p) \neq \varphi(p)) \\ (\lambda_i + \tilde{\lambda}_i), & i = n, \ldots, m; \end{cases}$$

$$\alpha_{n-k+j-1} := (\beta_j + \tilde{\beta}_j), \quad j = 2, \ldots, k; \quad (\tilde{\beta}_k = 0).$$

In this way, equation (18) can be written as $\sum_{i=1}^{m} \alpha_i A_i(p) = 0$ which implies that

$$\sum_{i=1}^{m} \alpha_i a_{ji}(p) = 0, \ \forall j = 1, \ldots, m.$$

We also have that

$$\begin{aligned} \Delta_{k+1} &= \det(A_n, \ldots, A_m, A_{n-k+1}, \ldots, A_{n-1}, A_1, \ldots, A_{n-k}) \\ &= (-1)^{\varepsilon} \det(A_1, \ldots, A_m) \end{aligned}$$

where ε is either equal to zero or equal to 1, depending on the number of required permutations between the columns of the matrix A to obtain Δ_{k+1}. Thus, by Lemma 4.1,

$$
\begin{aligned}
\alpha_1(-1)^{\varepsilon} d\Delta_{k+1} &\overset{\alpha_1\neq 0}{=} \alpha_1 \sum_{i,j=1}^{m} \operatorname{cof}(a_{ij}) da_{ij} \\
&= \sum_{i=1}^{m}\left(\alpha_1 \operatorname{cof}(a_{i1}) da_{i1} + \sum_{j=2}^{m} \alpha_j \operatorname{cof}(a_{i1}) da_{ij}\right) \\
&= \sum_{i=1}^{m} \operatorname{cof}(a_{i1}) \left[\sum_{j=1}^{m} \alpha_j da_{ij}\right] \\
&= \sum_{i=1}^{m} \operatorname{cof}(a_{i1}) \mathcal{L}_i
\end{aligned}
\tag{20}
$$

where $\mathcal{L}_i, i = 1, \ldots, m$, denote the rows of the Jacobian matrix (19) at p. If we denote by $\tilde{L}_i, i = 1, \ldots, m$, the row vectors of Jacobian matrix (17) at p, then we can verify that

$$
\sum_{i=1}^{m} \operatorname{cof}(a_{i1}) \mathcal{L}_i = \sum_{i=1}^{m} \operatorname{cof}(a_{i1}) \tilde{L}_i. \tag{21}
$$

Let us denote the first m row vectors of Matrix J_{k+1} in (16) by $L_i, i = 1, \ldots, m$, and its last row vector by $L_{\Delta_{k+1}}$. By equations (20) at p and (21), if we replace $L_{\Delta_{k+1}}$ by

$$
(-1)^{\varepsilon} \alpha_1 L_{\Delta_{k+1}} - \sum_{i=1}^{m} \operatorname{cof}(a_{i1}) L_i, \tag{22}
$$

we obtain

$$
\left[
\begin{array}{c|ccccccc|c}
 & \frac{\partial \mathbf{M}_n}{\partial x_1} & \cdots & \frac{\partial \mathbf{M}_m}{\partial x_1} & \frac{\partial \Delta_2}{\partial x_1} & \cdots & \frac{\partial \Delta_k}{\partial x_1} & & \frac{\partial \Delta_{k+1}}{\partial x_1} \\
\operatorname{Jac}\left(\xi + \sum_{i=n}^{m} \lambda_i d\mathbf{M}_i + \sum_{j=2}^{k} \beta_j d\Delta_j\right) & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & & \vdots \\
 & \frac{\partial \mathbf{M}_n}{\partial x_m} & \cdots & \frac{\partial \mathbf{M}_m}{\partial x_m} & \frac{\partial \Delta_2}{\partial x_m} & \cdots & \frac{\partial \Delta_k}{\partial x_m} & & \frac{\partial \Delta_{k+1}}{\partial x_m} \\
\hline
d_x\mathbf{M}_n & & & & & & & & 0 \\
\vdots & & & & & & & & \vdots \\
d_x\mathbf{M}_m & & & & & & & & 0 \\
d_x\Delta_2 & & & & O_{(m-n+k)} & & & & 0 \\
\vdots & & & & & & & & \vdots \\
d_x\Delta_k & & & & & & & & 0 \\
\hline
\vec{0} & & & & \vec{0} & & & & \gamma_{\widetilde{k+1}}
\end{array}
\right]. \tag{23}
$$

Let us show that $\gamma_{\widetilde{k+1}}(p) \neq 0$. We have

$$
\begin{aligned}
\gamma_{\widetilde{k+1}} &\overset{(22)}{=} -\sum_{i=1}^{m} \operatorname{cof}(a_{i1}) \frac{\partial \Delta_{k+1}}{\partial x_i} \\
&= -\det(d\Delta_{k+1}, A_2, \ldots, A_m) \\
&= -\det(d\Delta_{k+1}, \Omega_2, \ldots, \Omega_{n-k}, d\Delta_2, \ldots, d\Delta_k, d\mathbf{M}_n, \ldots, d\mathbf{M}_m).
\end{aligned}
$$

Suppose that $\gamma_{\widetilde{k+1}} = 0$. Since each one of the sets $\{\Omega_2(p), \ldots, \Omega_{n-k}(p)\}$ and

$$
\{d\Delta_{k+1}(p), d\Delta_2(p), \ldots, d\Delta_k(p), d\mathbf{M}_n(p), \ldots, d\mathbf{M}_m(p)\}
$$

consist of linearly independent vectors, there exists $j \in \{2, \ldots, n-k\}$ such that $\Omega_j(p) \in N_p^*\Sigma^{k+1}(\omega)$. Suppose that $j = n-k$, that is,

$$\Omega_{n-k}(p) \in N_p^*\Sigma^{k+1}(\omega) = \langle d\mathbf{M}_n, \ldots, d\mathbf{M}_m, d\Delta_2, \ldots, d\Delta_k, d\Delta_{k+1}\rangle.$$

Since $\xi_{|_{\Sigma^{k+1}}}(p) = 0$, we have $\xi(p) \in N_p^*\Sigma^{k+1}(\omega)$. Then,

$$\sum_{i=1}^{n-k} \mu_i \Omega_i + \underbrace{\sum_{i=n}^{m} \tilde{\lambda}_i d\mathbf{M}_i + \sum_{j=2}^{k-1} \tilde{\beta}_j d\Delta_j}_{\in N_p^*\Sigma^{k+1}(\omega)} \in N_p^*\Sigma^{k+1}(\omega)$$

$$\Rightarrow \sum_{i=1}^{n-k-1} \mu_i \Omega_i = \sum_{i=1}^{n-k} \mu_i \Omega_i - \mu_{n-k}\Omega_{n-k} \in N_p^*\Sigma^{k+1}(\omega).$$

Thus, $\sum_{i=1}^{n-k-1} \mu_i \Omega_i$ and $\mu_{n-k}\Omega_{n-k}$ are linearly independent vectors in the vector subspace

$$\langle \Omega_1, \ldots, \Omega_{n-k}\rangle \cap N_p^*\Sigma^{k+1}(\omega),$$

which implies that

$$\dim\left(\langle \Omega_1(p), \ldots, \Omega_{n-k}(p)\rangle \cap N_p^*\Sigma^{k+1}(\omega)\right) \geq 2.$$

Consequently, since $\langle \bar{\omega}\rangle = \langle \Omega_1, \ldots, \Omega_{n-k}\rangle \oplus \left(\langle \bar{\omega}\rangle \cap N_p^*\Sigma^{k-1}(\omega)\right)$ we have that

$$\dim\left(\langle \bar{\omega}(p)\rangle \cap N_p^*\Sigma^{k+1}(\omega)\right) \geq 2 + (k-1) = k+1,$$

which means that $p \in \Sigma^{k+2}(\omega)$. But this contradicts the hypothesis that $p \in A_{k+1}(\omega)$, since as we know $\Sigma^{k+2}(\omega) = \Sigma^{k+1}(\omega) \setminus A_{k+1}(\omega)$. Therefore $\gamma_{\tilde{k+1}}(p) \neq 0$, and we conclude that the Matrix J_{k+1} is non-singular at p if and only if the Matrix (23) is non-singular at p, which occurs if and only if the Matrix J_k is non-singular at the point p.

□

Lemma 4.11. *For almost every $a \in \mathbb{R}^n \setminus \{\vec{0}\}$, if $p \in A_n(\omega)$ then p is a non-degenerate zero of $\xi_{|_{\Sigma^{n-1}(\omega)}}$.*

Proof. We know that if $p \in A_n(\omega)$ then $\xi_{|_{\Sigma^{n-1}(\omega)}}(p) = 0$. By Szafraniec's characterization [20, p.149-151], p is a non-degenerate zero of $\xi_{|_{\Sigma^{n-1}(\omega)}}$ if and only if the following conditions hold:

(*i*) $\Delta(p) = \det(d\mathbf{M}_n, \ldots, d\mathbf{M}_m, d\Delta_2, \ldots, d\Delta_{n-1}, \xi)(p) = 0$;

(*ii*) $\det(d\mathbf{M}_n, \ldots, d\mathbf{M}_m, d\Delta_2, \ldots, d\Delta_{n-1}, d\Delta)(p) \neq 0$.

Condition (*i*) is clearly satisfied, since $\xi_{|_{\Sigma^{n-1}(\omega)}}(p) = 0$. Let us verify that condition (*ii*) also holds.

For each $x \in \Sigma^{n-1}(\omega)$ in an open neighborhood $\mathcal{U}$ of p in M, let $\{\Omega'(x)\}$ be a smooth basis for a vector subspace complementary to $\langle \bar{\omega}(x)\rangle \cap N_x^*\Sigma^{n-2}(\omega)$ in the vector space $\langle \bar{\omega}(x)\rangle$. Since $\xi(x) \in \langle \bar{\omega}(x)\rangle$, we have

$$\xi(x) = \lambda(x)\Omega'(x) + \varphi(x),$$

where $\lambda(x) \in \mathbb{R}$ and $\varphi(x) \in \langle \bar{\omega}(x)\rangle \cap N_x^*\Sigma^{n-2}(\omega)$, $\forall x \in \mathcal{U} \cap \Sigma^{n-1}(\omega)$.

In particular, if $x \in A_n(\omega)$, we know that, for almost every $a \in \mathbb{R}^n \setminus \{\vec{0}\}$, $\xi_{|_{\Sigma^{n-2}(\omega)}}(x) \neq 0$ and, consequently, $\xi(x) \notin N_x^*\Sigma^{n-2}(\omega)$. Thus $\lambda(p) \neq 0$. For all $x \in \mathcal{U} \cap \Sigma^{n-1}(\omega)$, we obtain

$$\begin{aligned} \Delta(x) &= \det(d\mathbf{M}_n, \ldots, d\mathbf{M}_m, d\Delta_2, \ldots, d\Delta_{n-1}, \lambda\Omega' + \varphi)(x) \\ &= \lambda(x)\det(d\mathbf{M}_n, \ldots, d\mathbf{M}_m, d\Delta_2, \ldots, d\Delta_{n-1}, \Omega')(x) \\ &= \lambda(x)\Delta_n(x), \end{aligned}$$

with $\Delta_n(p) = 0$ and $\lambda(p) \neq 0$. Then, we have

$$\begin{aligned}&\langle d\mathbf{M}_n(p), \dots, d\mathbf{M}_m(p), d\Delta_2(p), \dots, d\Delta_{n-1}(p), d\Delta(p)\rangle \\ &= \langle d\mathbf{M}_n(p), \dots, d\mathbf{M}_m(p), d\Delta_2(p), \dots, d\Delta_{n-1}(p), d(\lambda\Delta_n)(p)\rangle\end{aligned}$$

(see Lemma A.1). However, $d(\lambda\Delta_n)(x) = d\lambda(x)\Delta_n(x) + \lambda(x)d\Delta_n(x)$, $\Delta_n(p) = 0$ and $\lambda(p) \neq 0$. Thus,

$$\begin{aligned}&\langle d\mathbf{M}_n(p), \dots, d\mathbf{M}_m(p), d\Delta_2(p), \dots, d\Delta_{n-1}(p), d\Delta(p)\rangle \\ &= \langle d\mathbf{M}_n(p), \dots, d\mathbf{M}_m(p), d\Delta_2(p), \dots, d\Delta_{n-1}(p), d\Delta_n(p)\rangle.\end{aligned}$$

Therefore, $\det(d\mathbf{M}_n(p), \dots, d\mathbf{M}_m(p), d\Delta_2(p), \dots, d\Delta_{n-1}(p), d\Delta(p)) \neq 0$. □

Lemma 4.12. *For almost every $a \in \mathbb{R}^n \setminus \{\vec{0}\}$, the one-form $\xi_{|_{\Sigma^k(\omega)}}$ admits only non-degenerate zeros, $k \geq 1$.*

Proof. Suppose that $\xi_{|_{\Sigma^k(\omega)}}(p) = 0$. Then, for almost every $a \in \mathbb{R}^n \setminus \{\vec{0}\}$, $p \in A_k(\omega) \cup A_{k+1}(\omega)$ since $Z(\xi_{|_{\Sigma^k(\omega)}}) \cap \Sigma^{k+2}(\omega) = \varnothing$ by Lemma 3.7 and $\Sigma^k(\omega) = A_k(\omega) \cup A_{k+1}(\omega) \cup \Sigma^{k+2}(\omega)$.

If $p \in A_k(\omega)$ then $\xi_{|_{A_k(\omega)}}(p) = 0$. Since $\xi_{|_{A_k(\omega)}}$ admits only non-degenerate zeros and $A_k(\omega) \subset \Sigma^k(\omega)$ is an open subset, we conclude that p is a non-degenerate zero of $\xi_{|_{\Sigma^k(\omega)}}$.

If $p \in A_{k+1}(\omega)$ and $k < n-1$ then $\xi_{|_{\Sigma^{k+1}(\omega)}}(p) = 0$. In particular, since $A_{k+1}(\omega) \subset \Sigma^{k+1}(\omega)$ is an open subset then $\xi_{|_{A_{k+1}(\omega)}}(p) = 0$. By Lemmas 4.8 and 4.7, $\xi_{|_{A_{k+1}(\omega)}}$ admits only non-degenerate zeros, and since $A_{k+1}(\omega)$ is an open set of $\Sigma^{k+1}(\omega)$, we conclude that p is a non-degenerate zero of $\xi_{|_{\Sigma^{k+1}(\omega)}}$. Therefore, by Lemma 4.10, p is non-degenerate zero of $\xi_{|_{\Sigma^k(\omega)}}$. Finally, if $p \in A_n(\omega)$, by Lemma 4.11, p is a non-degenerate zero of $\xi_{|_{\Sigma^{n-1}(\omega)}}$.

□

Theorem 4.13. *Let $\omega = \{\omega_i\}_{1 \leq i \leq n}$ be a Morin collection of smooth one-forms defined on an m-dimensional compact manifold M. Then,*

$$\chi(M) \equiv \sum_{k=1}^{n} \chi(\overline{A_k(\omega)}) \mod 2.$$

Proof. Let us denote by $Z(\varphi)$ the set of zeros of a one-form φ and let us denote by $\#Z(\varphi)$ the number of elements of this set, whenever $Z(\varphi)$ is finite. Let

$$\xi(x) = \sum_{i=1}^{n} a_i\omega_i(x)$$

be a one-form with $a = (a_1, \dots, a_n) \in \mathbb{R}^n \setminus \{\vec{0}\}$ satisfying the generic conditions of the previous lemmas of Sections 3 and 4.

Since M is compact and the submanifolds $\Sigma^k(\omega)$ are closed in M, by the Poincaré-Hopf Theorem for one-forms we obtain

- $\chi(M) \equiv \#Z(\xi) \mod 2$;
- $\chi(\overline{A_k(\omega)}) = \chi(\Sigma^k(\omega)) \equiv \#Z(\xi_{|_{\Sigma^k(\omega)}}) \mod 2$, for $k = 1, \dots, n-1$;
- $\chi(\overline{A_n(\omega)}) = \chi(\Sigma^n(\omega)) \equiv \#Z(\xi_{|_{\Sigma^n(\omega)}}) \mod 2$.

By Lemma 3.1, if $p \in Z(\xi)$ then $p \in \Sigma^1(\omega)$ and $\xi_{|_{\Sigma^1(\omega)}}(p) = 0$. Moreover, by Lemma 3.6, $Z(\xi) \cap \Sigma^2(\omega) = \varnothing$. Thus $p \in A_1(\omega)$. On the other hand, Lemma 3.2 shows that if

$$p \in Z(\xi_{|_{\Sigma^1(\omega)}}) \cap A_1(\omega),$$

then p is also a zero of the one-form ξ. Thus,

$$\#Z(\xi) \equiv \#Z(\xi_{|_{\Sigma^1(\omega)}} \cap A_1(\omega)) \mod 2.$$

By Lemma 3.7, if $p \in Z(\xi_{|_{\Sigma^k(\omega)}})$ then $p \notin \Sigma^{k+2}(\omega)$. Thus, $p \in A_k(\omega) \cup A_{k+1}(\omega)$ and, for $k = 1, \dots, n-1$, we have

$$\#Z(\xi_{|_{\Sigma^k(\omega)}}) \equiv \#Z(\xi_{|_{\Sigma^k(\omega)}} \cap A_k(\omega)) + \#Z(\xi_{|_{\Sigma^k(\omega)}} \cap A_{k+1}(\omega)) \mod 2.$$

By Lemma 3.2, we also have

$$\#Z(\xi_{|_{\Sigma^k(\omega)}} \cap A_{k+1}(\omega)) = \#Z(\xi_{|_{\Sigma^{k+1}(\omega)}} \cap A_{k+1}(\omega))$$

and by Lemma 3.3,

$$\#A_n(\omega) = \#Z(\xi_{|_{\Sigma^{n-1}(\omega)}} \cap A_n(\omega)).$$

Then,

- $\chi(M) \equiv \#Z(\xi_{|_{\Sigma^1(\omega)}} \cap A_1(\omega)) \mod 2$;
- For $k = 1, \dots, n-1$,

$$\chi(\overline{A_k(\omega)}) \equiv \#Z(\xi_{|_{\Sigma^k(\omega)}} \cap A_k(\omega)) + \#Z(\xi_{|_{\Sigma^{k+1}(\omega)}} \cap A_{k+1}(\omega)) \mod 2;$$

- $\chi(\overline{A_n(\omega)}) = \#Z(\xi_{|_{\Sigma^{n-1}(\omega)}} \cap A_n(\omega))$.

Therefore,

$$\begin{aligned} \chi(M) + \sum_{k=1}^{n} \chi(\overline{A_k(\omega)}) &\equiv 2\#Z(\xi_{|_{\Sigma^1(\omega)}} \cap A_1(\omega)) \\ &+ 2\#Z(\xi_{|_{\Sigma^2(\omega)}} \cap A_2(\omega)) + \dots \\ &+ 2\#Z(\xi_{|_{\Sigma^{n-1}(\omega)}} \cap A_{n-1}(\omega)) \\ &+ 2\#Z(\xi_{|_{\Sigma^{n-1}(\omega)}} \cap A_n(\omega)) \mod 2 \\ &\equiv 0 \mod 2. \end{aligned}$$

□

As for the definition of Morin collection of n one-forms, the results presented in Sections 3 and 4 of this paper also can be naturally adapted to the context of collections of n vector fields. In particular, the main theorems that have been used, as the Poincaré-Hopf Theorem and the Szafraniec's characterizations, have their respective versions for vector fields.

Finally, we end the paper with a very simple example. Let us verify that Theorem 4.13 indeed holds for the Morin collection of 2 vector fields $V = \{V_1, V_2\}$ presented in the Example 2.31. To do that, it is enough to see that the torus T is a compact manifold with $\chi(T) = 0$. Moreover, $\overline{A_1(V)} = \Sigma^1(V)$ is given by two circles in $\mathbb{R}^3$ and $\overline{A_2(V)}$ consists of four points, such that $\chi(\overline{A_1(V)}) = 0$ and $\chi(\overline{A_2(V)}) = 4$. Therefore,

$$\chi(T) \equiv \chi(\overline{A_1(V)}) + \chi(\overline{A_2(V)}) \mod 2.$$

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Appendix

A. Proof of Preposition 2.23

Proof of Proposition 2.23, part (a). Firstly, let us show that if $\bar{x} \in \mathcal{U} \cap \Sigma^{k-1}(\omega)$ such that $\Omega^{k-1}(\bar{x}) \in N^*_{\Sigma^{k-1}} M^r$, then the following conditions are equivalent:

(I) $\operatorname{rank}(dF_1(\bar{x}), \dots, dF_{m-r}(\bar{x}), d\Delta_k(\bar{x})) = m - r + 1$;
(II) $\Omega^{k-1} \pitchfork N^*_{\Sigma^{k-1}} M^r$ in $T^*_{\Sigma^{k-1}} M^r$ at $\bar{x}$.

Let $\Omega^{k-1}(\bar{x}) \in \mathcal{U} \times \mathcal{V}$. By the proof of Lemma 2.17, $N^*_{\Sigma^{k-1}} M^r$ can be locally given by independent equations as follows

$$N^*_{\Sigma^{k-1}} M^r = \{(x, \varphi) \in \mathcal{U} \times \mathcal{V} \mid F_1 = \dots = F_{m-r} = \Delta = 0\},$$

where $\Delta(x, \varphi) = \det(dF_1(x), \dots, dF_{m-r}(x), \varphi_1, \dots, \varphi_r)$and $\mathcal{V} \subset \mathbb{R}^{mr}$ is an open set. Let

$$G(\Omega^{k-1}) = \{(x, \Omega_1(x), \dots, \Omega_r(x)) \mid x \in \mathcal{U} \cap \Sigma^{k-1}(\omega)\}$$

be the restriction of the graph of $(\Omega_1(x), \dots, \Omega_r(x))$ to $\mathcal{U} \cap \Sigma^{k-1}(\omega)$, $G(\Omega^{k-1})$ can be locally given by

$$\begin{aligned} G(\Omega^{k-1}) = \{(x, \varphi) \in T^* M^r \mid F_1(x) = \dots = F_{m-r}(x) = 0; \\ \Omega_i^j(x) - \varphi_i^j = 0, i = 1, \dots, r \text{ and } j = 1, \dots, m\}, \end{aligned}$$

where $T^* M^r$ denotes the r-cotangent bundle of M, $\Omega_i(x) = (\Omega_i^1(x), \dots, \Omega_i^m(x))$ and $\varphi_i = (\varphi_i^1, \dots, \varphi_i^m)$ for $i = 1, \dots, r$. In particular, the local equations of $G(\Omega^{k-1})$ are clearly independent and $\dim G(\Omega^{k-1}) = r$. Let (x, φ) be local coordinates in $T^* M^r$, with $x = (x_1, \dots, x_m)$ and

$$\varphi = (\varphi_1^1, \dots, \varphi_1^m, \varphi_2^1, \dots, \varphi_2^m, \dots, \varphi_r^1, \dots, \varphi_r^m),$$

let us consider the derivatives of the local equations of $N^*_{\Sigma^{k-1}} M^r$ and $G(\Omega^{k-1})$ with respect to (x, φ). We will denote the derivative with respect to x by d_x and the derivative with respect to φ by d_φ, then we have

$$d\left(\Omega_i^j(x) - \varphi_i^j\right) = \left(d_x \Omega_i^j(x)\ , -d_\varphi \varphi_i^j\right), \tag{24}$$

for $i = 1, \dots, r$ and $j = 1, \dots, m$, where $d_\varphi \varphi_i^j = (0, \dots, 0, 1, 0, \dots, 0)$ is the vector whose $m(i-1)+j^{th}$ entry is equal to 1 and the others are zero. By Lagrange's rules the determinant

$$\Delta(x, \varphi) = \det(dF_1(x), \dots, dF_{m-r}(x), \varphi_1, \dots, \varphi_r)$$

can be written as

$$\Delta(x, \varphi) = \sum_I F_I(x) N_I(\varphi)$$

for $I = \{i_1, \dots, i_r\} \subset \{1, \dots, m\}$, where

$$N_I(\varphi) = \begin{vmatrix} \varphi_1^{i_1} & \dots & \varphi_r^{i_1} \\ \vdots & \ddots & \vdots \\ \varphi_1^{i_r} & \dots & \varphi_r^{i_r} \end{vmatrix} \tag{25}$$

is the minor obtained from the matrix

$$\begin{bmatrix} \varphi_1^1 & \dots & \varphi_r^1 \\ \vdots & \ddots & \vdots \\ \varphi_1^m & \dots & \varphi_r^m \end{bmatrix}$$

taking the lines $i_1, \ldots, i_r$, and

$$F_I(x) = \pm \begin{vmatrix} \dfrac{\partial F_1}{\partial x_{k_1}}(x) & \ldots & \dfrac{\partial F_{m-r}}{\partial x_{k_1}}(x) \\ \vdots & \ddots & \vdots \\ \dfrac{\partial F_1}{\partial x_{k_{m-r}}}(x) & \ldots & \dfrac{\partial F_{m-r}}{\partial x_{k_{m-r}}}(x) \end{vmatrix} \tag{26}$$

is, up to sign, the minor obtained from the matrix $(dF_1(x)\ldots dF_{m-r}(x))$ removing the lines $i_1, \ldots, i_r$, that is, $\{k_1, \ldots, k_{m-r}\} = \{1, \ldots, m\} \setminus I$. Therefore,

$$d\Delta(x,\varphi) = \Big(\sum_I N_I(\varphi) d_x F_I(x) \ , \ \sum_I F_I(x) d_\varphi N_I(\varphi) \ \Big).$$

Notice that $\Omega^{k-1} \pitchfork N^*_{\Sigma^{k-1}} M^r$ in $T^*_{\Sigma^{k-1}} M^r$ at the point $x \in \mathcal{U} \cap \Sigma^{k-1}(\omega)$ if and only if

$$G(\Omega^{k-1}) \pitchfork N^*_{\Sigma^{k-1}} M^r \quad \text{in} \quad T^*_{\Sigma^{k-1}} M^r \quad \text{at} \quad (x, \Omega_1(x), \ldots, \Omega_r(x)).$$

Let π_1 be the projection of the cotangent space of T^*M^r over the cotangent space of $T^*_{\Sigma^{k-1}} M^r$:

$$\begin{array}{rccc} \pi_1 : & T^*_{(x,\varphi)}(T^*M^r) & \longrightarrow & T^*_{(x,\varphi)}(T^*_{\Sigma^{k-1}} M^r) \\ & (\psi(x), \varphi_1, \ldots, \varphi_r) & \longmapsto & (\pi(\psi(x)), \varphi_1, \ldots, \varphi_r) \end{array}$$

where π denotes the restriction to $T_x \Sigma^{k-1}(\omega)$, that is, $\pi(\psi(x)) = \psi(x)|_{T_x\Sigma^{k-1}(\omega)}$. By Equation (24),

$$\pi_1 \left(d(\Omega_i^j(x) - \varphi_i^j) \right) = \left(\pi(d_x \Omega_i^j(x)) \ , -d_\varphi \varphi_i^j \right),$$

for $i = 1, \ldots, r$ and $j = 1, \ldots, m$. We also have that

$$\pi_1 \left(d\Delta(x,\varphi) \right) = \left(\ \pi \left(\sum_I N_I(\varphi) d_x F_I(x) \right) \ , \ \sum_I F_I(x) d_\varphi N_I(\varphi) \ \right).$$

Then, $G(\Omega^{k-1}) \pitchfork N^*_{\Sigma^{k-1}} M^r$ in $T^*_{\Sigma^{k-1}} M^r$ at $(x, \Omega_1(x), \ldots, \Omega_r(x))$ such that

$$(x, \Omega_1(x), \ldots, \Omega_r(x)) \in N^*_{\Sigma^{k-1}} M^r$$

if and only if the matrix

$$\left[\begin{array}{c:c} \pi(d_x \Omega_1^1(x)) & \\ \vdots & \\ \pi(d_x \Omega_1^m(x)) & -Id_{mr} \\ \vdots & \\ \pi(d_x \Omega_r^m(x)) & \\ \cdots\cdots\cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots\cdots \\ \pi\left(\sum_I N_I(\varphi) d_x F_I(x) \right) & \sum_I F_I(x) d_\varphi N_I(\varphi) \end{array} \right] \tag{27}$$

has maximal rank at x. By the expression of $N_I(\varphi)$ in (25), we have

$$d_\varphi N_I(\varphi) = \sum_{i,j} \operatorname{cof}(\varphi_i^j) d_\varphi \varphi_i^j, \tag{28}$$

for $i = 1, \ldots, r$, $j \in I$ and $\operatorname{cof}(\varphi_i^j)$ denoting the cofactor of φ_i^j in the matrix

$$\begin{bmatrix} \varphi_1^{i_1} & \ldots & \varphi_r^{i_1} \\ \vdots & \ddots & \vdots \\ \varphi_1^{i_r} & \ldots & \varphi_r^{i_r} \end{bmatrix}.$$

Let $d = C_{m,r} = \dfrac{m!}{r!(m-r)!}$, we will denote by $I_1, \ldots, I_d$ the subsets of $\{1, \ldots, m\}$ containing exactly r elements. By equation (28),

$$\sum_I F_I(x) d_\varphi N_I(\varphi) = \sum_{\ell=1}^{d} F_{I_\ell}(x) \left(\sum_{i=1}^{r} \sum_{j \in I_\ell} \operatorname{cof}(\varphi_i^j) d_\varphi \varphi_i^j \right)$$

and,

$$\begin{aligned}
&\sum_{\ell=1}^{d} F_{I_\ell}(x) \left(\sum_{i=1}^{r} \sum_{j \in I_\ell} \operatorname{cof}(\varphi_i^j) d_\varphi \varphi_i^j \right) \\
&= \sum_{i=1}^{r} \left[F_{I_1}(x) \left(\sum_{j \in I_1} \operatorname{cof}(\varphi_i^j) d_\varphi \varphi_i^j \right) + \ldots + F_{I_d}(x) \left(\sum_{j \in I_d} \operatorname{cof}(\varphi_i^j) d_\varphi \varphi_i^j \right) \right] \\
&= \sum_{i=1}^{r} \left[\left(\sum_{I:\, 1 \in I} F_I(x) \right) \operatorname{cof}(\varphi_i^1) d_\varphi \varphi_i^1 + \ldots + \left(\sum_{I:\, m \in I} F_I(x) \right) \operatorname{cof}(\varphi_i^m) d_\varphi \varphi_i^m \right] \\
&= \sum_{i=1}^{r} \left[\sum_{j=1}^{m} \left(\sum_{I:\, j \in I} F_I(x) \right) \operatorname{cof}(\varphi_i^j) d_\varphi \varphi_i^j \right].
\end{aligned}$$

Thus, for $i = 1, \ldots, r$ and $j = 1, \ldots, m$, we can write

$$\sum_I F_I(x) d_\varphi N_I(\varphi) = \sum_{i,j} \beta_i^j(x, \varphi) d_\varphi \varphi_i^j, \tag{29}$$

where

$$\beta_i^j(x, \varphi) = \left(\sum_{I:\, j \in I} F_I(x) \right) \operatorname{cof}(\varphi_i^j).$$

We will denote the rows of the Matrix (27) by $R_i^j = \left(\pi(d_x \Omega_i^j(x)) \;, -d_\varphi \varphi_i^j \right)$, for $i = 1, \ldots, r$ and $j = 1, \ldots, m$, and we denote the last row of the Matrix (27) by R_Δ. Replacing the row R_Δ by

$$R_\Delta + \sum_{i,j} \beta_i^j(x, \varphi) R_i^j$$

for $i = 1, \ldots, r$ and $j = 1, \ldots, m$, we obtain a new matrix

$$\left[\begin{array}{c|c}
\pi(d_x \Omega_1^1(x)) & \\
\vdots & -Id_{mr} \\
\pi(d_x \Omega_r^m(x)) & \\
\hline
R'_\Delta & R''_\Delta
\end{array} \right] \tag{30}$$

which has rank equal to the rank of the Matrix (27), where

$$R''_\Delta = \sum_I F_I(x) d_\varphi N_I(\varphi) + \sum_{i,j} \beta_i^j(x, \varphi)(-d_\varphi \varphi_i^j) \overset{(29)}{=} \vec{0}$$

and

$$\begin{aligned}
R'_\Delta \quad &= \pi \left(\sum_I N_I(\varphi) d_x F_I(x) \right) + \sum_{i,j} \beta_i^j(x, \varphi) \pi \left(d_x \Omega_i^j(x) \right) \\
&= \pi \left(\sum_I N_I(\varphi) d_x F_I(x) + \sum_{i,j} \beta_i^j(x, \varphi) d_x \Omega_i^j(x) \right).
\end{aligned}$$

Notice that for each $\bar{x} \in \mathcal{U} \cap \Sigma^{k-1}(\omega)$, we have $\Omega_i^j(\bar{x}) = \varphi_i^j$. In this case, Equation (29) implies that

$$\sum_{i,j} \beta_i^j(\bar{x},\varphi) d_x\Omega_i^j(\bar{x}) = \sum_{i,j} \beta_i^j(\bar{x},\Omega^{k-1}(\bar{x})) d_x\Omega_i^j(\bar{x}) = \sum_I F_I(\bar{x}) d_x N_I(\Omega^{k-1}(\bar{x})).$$

Thus, at $\bar{x}$

$$R'_\Delta = \pi\left(\sum_I N_I(\Omega^{k-1}(\bar{x})) d_x F_I(\bar{x}) + \sum_I F_I(\bar{x}) d_x N_I(\Omega^{k-1}(\bar{x}))\right) = \pi(d\Delta_k(\bar{x}))$$

and the Matrix (30) is equal to

$$\left[\begin{array}{c:c} \pi(d_x\Omega_1^1(\bar{x})) & \\ \vdots & -Id_{mr} \\ \pi(d_x\Omega_r^m(\bar{x})) & \\ \hdashline \pi(d\Delta_k(\bar{x})) & \vec{0} \end{array}\right].$$

Thus, for each $\bar{x} \in \mathcal{U} \cap \Sigma^{k-1}(\omega)$ such that $\Omega^{k-1}(\bar{x}) \in N^*_{\Sigma^{k-1}} M^r$, $\Omega^{k-1} \pitchfork N^*_{\Sigma^{k-1}} M^r$ in $T^*_{\Sigma^{k-1}} M^r$ at $\bar{x}$ if and only if $\pi(d\Delta_k(\bar{x})) \neq 0$, that is, the restriction of $d\Delta_k(\bar{x})$ to $T_{\bar{x}}\Sigma^{k-1}(\omega)$ is not zero, which means that $d\Delta_k(\bar{x}) \notin \langle dF_1(\bar{x}), \dots, dF_{m-r}(\bar{x})\rangle$, or equivalently

$$\operatorname{rank}(dF_1(\bar{x}), \dots, dF_{m-r}(\bar{x}), d\Delta_k(\bar{x})) = m - r + 1.$$

Now suppose that ω satisfies the condition I_{k-1} on $\mathcal{U}$. By property (b) of Definition 2.18, we have that $\dim\langle\Omega_1(x), \dots, \Omega_r(x)\rangle \cap N_x^*\Sigma^{k-1}(\omega)$ is either equal to 0 or equal to 1 for each $x \in \mathcal{U} \cap \Sigma^{k-1}(\omega)$. If $\dim\langle\Omega_1(x), \dots, \Omega_r(x)\rangle \cap N_x^*\Sigma^{k-1}(\omega) = 1$, then $x \in \mathcal{U} \cap \Sigma^k(\omega)$ and $\Delta_k(x) = 0$. In this case, the transversality given by property (a) of Definition 2.18 implies that

$$\operatorname{rank}(dF_1(x), \dots, dF_{m-r}(x), d\Delta_k(x)) = m - r + 1.$$

On the other hand, we assume that properties (i) and (ii) hold for each $x \in \mathcal{U} \cap \Sigma^{k-1}(\omega)$. By property (i), the property (b) of Definition 2.18 holds on $\mathcal{U}$. If

$$\dim\langle\Omega_1(x), \dots, \Omega_r(x)\rangle \cap N_x^*\Sigma^{k-1}(\omega) = 0,$$

then $\Omega^{k-1}(x)$ does not intersect $N^*_{\Sigma^{k-1}} M^r$, thus $\Omega^{k-1} \pitchfork N^*_{\Sigma^{k-1}} M^r$ in $T^*_{\Sigma^{k-1}} M^r$ at x. If

$$\dim\langle\Omega_1(x), \dots, \Omega_r(x)\rangle \cap N_x^*\Sigma^{k-1}(\omega) = 1,$$

then $x \in \mathcal{U} \cap \Sigma^k(\omega)$ by Definition 2.19 and $\operatorname{rank}(dF_1(x), \dots, dF_{m-r}(x), d\Delta_k(x)) = m - r + 1$ by property (ii). Thus $\Omega^{k-1} \pitchfork N^*_{\Sigma^{k-1}} M^r$ in $T^*_{\Sigma^{k-1}} M^r$ at x and ω satisfies the condition I_{k-1} on $\mathcal{U}$.

By the previous arguments and Definition 2.19, if ω satisfies the condition I_{k-1} on $\mathcal{U}$ then $\mathcal{U} \cap \Sigma^k(\omega) = \{x \in \mathcal{U} \mid F_1(x) = \ldots = F_{m-r}(x) = \Delta_k(x) = 0\}$. ☐

The following technical lemma will be used in the proof of Proposition 2.23, part (b).

Lemma A.1. *Let $f_i : \mathcal{V} \subset \mathbb{R}^\ell \to \mathbb{R}, i = 1, \dots, s$ be smooth functions defined on an open subset of $\mathbb{R}^\ell$. Let $M \subset \mathbb{R}^\ell$ be a manifold locally given by $M = \{x \in \mathcal{V} \mid f_1(x) = \ldots = f_s(x) = 0\}$, with* $\operatorname{rank}(df_1(x), \dots, df_s(x)) = s$*, for all $x \in M \cap \mathcal{V}$. If $g, h : \mathcal{V} \subset \mathbb{R}^\ell \to \mathbb{R}$ are smooth functions such that $g(x) = \lambda(x)h(x)$, for all $x \in M \cap \mathcal{V}$ and some smooth function $\lambda : \mathcal{V} \to \mathbb{R}$, then:*

(i) *If $\lambda(x) \neq 0$ and $x \in M$ then $g(x) = 0 \Leftrightarrow h(x) = 0$.*
(ii) *If $\lambda(x) \neq 0$, $x \in M$ and $h(x) = 0$ then*

$$\langle df_1(x), \dots, df_s(x), dg(x)\rangle = \langle df_1(x), \dots, df_s(x), dh(x)\rangle.$$

Proof of Proposition 2.23, part (b). Firstly, notice that the definition of $\Sigma^1(\omega)$ does not depend on the choice of any basis. Then, assume that the definition of $\Sigma^i(\omega)$ does not depend on the choice of the basis $\{\Omega_1(x), \dots, \Omega_{n-i+1}(x)\}$ for every $i = 2, \dots, k-1$. As considered in part (a), for each $p \in \Sigma^{k-1}(\omega)$, there is an open neighborhood $\mathcal{U}$ of p in M such that

$$\begin{aligned} \mathcal{U} \cap \Sigma^1(\omega) &= \{x \in \mathcal{U} : F_1(x) = \dots = F_{m-n+1}(x) = 0\}, \\ \mathcal{U} \cap \Sigma^{k-1}(\omega) &= \{x \in \mathcal{U} : F_1(x) = \dots = F_{m-n+1}(x) = \Delta_2(x) = \dots = \Delta_{k-1}(x) = 0\}, \\ \mathcal{U} \cap \Sigma^k(\omega) &= \{x \in \mathcal{U} : F_1(x) = \dots = F_{m-n+1}(x) = \Delta_2(x) = \dots = \Delta_k(x) = 0\}, \end{aligned}$$

with $\operatorname{rank}(dF_1(x), \dots, dF_{m-n+1}(x), d\Delta_2(x), \dots, d\Delta_{k-1}(x)) = m-n+k-1$, for $x \in \mathcal{U} \cap \Sigma^{k-1}(\omega)$ and $\operatorname{rank}(dF_1(x), \dots, dF_{m-n+1}(x), d\Delta_2(x), \dots, d\Delta_k(x)) = m-n+k$, for $x \in \mathcal{U} \cap \Sigma^k(\omega)$. Let us recall that

$$\Delta_k(x) = \det(dF_1, \dots, dF_{m-n+1}, d\Delta_2, \dots, d\Delta_{k-1}, \Omega_1, \dots, \Omega_{n-k+1})(x),$$

where $\{\Omega_1(x), \dots, \Omega_{n-k+1}(x)\}$ is a collection of $n-k+1$ smooth one-forms defined on $\mathcal{U}$ which is a basis of a vector subspace complementary to $\langle\bar{\omega}(x)\rangle \cap N_x^*\Sigma^{k-2}(\omega)$ in $\langle\bar{\omega}(x)\rangle$ for each $x \in \mathcal{U} \cap \Sigma^{k-1}(\omega)$.

Let us consider $\{\tilde{\Omega}_1(x), \dots, \tilde{\Omega}_{n-k+1}(x)\}$ a collection of $n-k+1$ smooth one-forms defined on $\mathcal{U}$ such that for each $x \in \mathcal{U} \cap \Sigma^{k-1}(\omega)$, $\{\tilde{\Omega}_1(x), \dots, \tilde{\Omega}_{n-k+1}(x)\}$ is another basis of a vector subspace complementary to $\langle\bar{\omega}(x)\rangle \cap N_x^*\Sigma^{k-2}(\omega)$ in $\langle\bar{\omega}(x)\rangle$. Then,

$$\langle\bar{\omega}(x)\rangle = \big(\langle\bar{\omega}(x)\rangle \cap N_x^*\Sigma^{k-2}(\omega)\big) \oplus \langle\tilde{\Omega}_1(x), \dots, \tilde{\Omega}_{n-k+1}(x)\rangle$$

and

$$\dim(\langle\tilde{\Omega}_1(x), \dots, \tilde{\Omega}_{n-k+1}(x)\rangle \cap N_x^*\Sigma^{k-1}(\omega))$$

is either equal to 0 or equal to 1, for $x \in \mathcal{U} \cap \Sigma^{k-1}(\omega)$. Moreover,

$$\begin{cases} \tilde{\Omega}_1(x) = \displaystyle\sum_{\ell=1}^{n-k+1} a_{\ell 1}(x)\Omega_\ell(x) + \varphi_1(x) \\ \tilde{\Omega}_2(x) = \displaystyle\sum_{\ell=1}^{n-k+1} a_{\ell 2}(x)\Omega_\ell(x) + \varphi_2(x) \\ \vdots \\ \tilde{\Omega}_{n-k+1}(x) = \displaystyle\sum_{\ell=1}^{n-k+1} a_{\ell(n-k+1)}(x)\Omega_\ell(x) + \varphi_{n-k+1}(x) \end{cases}$$

where $a_{ij}(x) \in \mathbb{R}$ and $\varphi_j(x) \in \langle\bar{\omega}(x)\rangle \cap N_x^*\Sigma^{k-2}(\omega)$, for $j = 1, \dots, n-k+1$. We will show that for each $x \in \mathcal{U} \cap \Sigma^{k-1}(\omega)$,

$$\det(A(x)) = \begin{vmatrix} a_{11}(x) & a_{12}(x) & \cdots & a_{1(n-k+1)}(x) \\ \vdots & \vdots & \ddots & \vdots \\ a_{(n-k+1)1}(x) & a_{(n-k+1)2}(x) & \cdots & a_{(n-k+1)(n-k+1)}(x) \end{vmatrix} \neq 0.$$

Suppose that the statement is false, that is, $\det(A(x)) = 0$. This means that the columns of matrix $A(x)$ are linearly dependent. So we can suppose without loss of generality that the first column of $A(x)$ can be written as a linear combination of the others columns:

$$(a_{11}(x), \dots, a_{(n-k+1)1}(x)) = \sum_{s=2}^{n-k+1} \lambda_s (a_{1s}(x), \dots, a_{(n-k+1)s}(x)),$$

where $\lambda_s \in \mathbb{R}$, for $s = 2, \dots, n-k+1$. Thus, removing x in the notation, we have

$$\tilde{\Omega}_1 = \sum_{\ell=1}^{n-k+1} a_{\ell 1}\Omega_\ell + \varphi_1 \quad \Rightarrow \tilde{\Omega}_1 = \sum_{\ell=1}^{n-k+1}\left(\sum_{s=2}^{n-k+1}\lambda_s a_{\ell s}\right)\Omega_\ell + \varphi_1$$

$$\Rightarrow \tilde{\Omega}_1 = \sum_{s=2}^{n-k+1}\lambda_s\left(\sum_{\ell=1}^{n-k+1} a_{\ell s}\Omega_\ell\right) + \varphi_1$$

then

$$\begin{aligned}\tilde{\Omega}_1 - \sum_{s=2}^{n-k+1}\lambda_s\tilde{\Omega}_s &= \left[\sum_{s=2}^{n-k+1}\lambda_s\left(\sum_{\ell=1}^{n-k+1} a_{\ell s}\Omega_\ell\right) + \varphi_1\right] - \sum_{s=2}^{n-k+1}\lambda_s\left(\sum_{\ell=1}^{n-k+1} a_{\ell s}\Omega_\ell + \varphi_s\right)\\ &= \varphi_1 - \sum_{s=2}^{n-k+1}\lambda_s\varphi_s.\end{aligned}$$

This means that

$$\tilde{\Omega}_1 - \sum_{s=2}^{n-k+1}\lambda_s\tilde{\Omega}_s \in \left(\langle\bar{\omega}\rangle \cap N_x^*\Sigma^{k-2}(\omega)\right) \cap \langle\tilde{\Omega}_1, \dots, \tilde{\Omega}_{n-k+1}\rangle = \{0\},$$

that is, $\tilde{\Omega}_1(x), \dots, \tilde{\Omega}_{n-k+1}(x)$ are linearly dependent. However, this contradicts the initial assumption that $\{\tilde{\Omega}_1(x), \dots, \tilde{\Omega}_{n-k+1}(x)\}$ is a basis of a vector subspace for each x in $\mathcal{U} \cap \Sigma^{k-1}(\omega)$. Therefore, $\det(A(x)) \neq 0$.

Let ${}^tA(x)$ be the transpose of matrix $A(x)$. For each $x \in \mathcal{U} \cap \Sigma^{k-1}(\omega)$, we have $\det({}^tA(x)) = \det(A(x)) \neq 0$ and, removing x in the notation,

$$\begin{aligned}&\det(dF_1, \dots, dF_{m-n+1}, d\Delta_2, \dots, d\Delta_{k-1}, \tilde{\Omega}_1, \dots, \tilde{\Omega}_{n-k+1})\\ &= \det(dF_1, \dots, dF_{m-n+1}, d\Delta_2, \dots, d\Delta_{k-1}, \sum_{\ell=1}^{n-k+1} a_{\ell 1}\Omega_\ell, \dots, \sum_{\ell=1}^{n-k+1} a_{\ell(n-k+1)}\Omega_\ell)\\ &= \det({}^tA)\det(dF_1, \dots, dF_{m-n+1}, d\Delta_2, \dots, d\Delta_{k-1}, \Omega_1, \dots, \Omega_{n-k+1}).\end{aligned} \tag{31}$$

Thus, for $x \in \mathcal{U} \cap \Sigma^{k-1}(\omega)$ we have that $\dim(\langle\tilde{\Omega}_1(x), \dots, \tilde{\Omega}_{n-k+1}(x)\rangle \cap N_x^*\Sigma^{k-1}(\omega))$ is equal to $\dim(\langle\Omega_1(x), \dots, \Omega_{n-k+1}(x)\rangle \cap N_x^*\Sigma^{k-1}(\omega))$. In particular, if $x \in \mathcal{U} \cap \Sigma^k(\omega)$ then $\Delta_k(x) = 0$ and

$$\tilde{\Delta}_k(x) = \det(dF_1, \dots, dF_{m-n+1}, d\Delta_2, \dots, d\Delta_{k-1}, \tilde{\Omega}_1, \dots, \tilde{\Omega}_{n-k+1}) = 0$$

such that, by statement (ii) of Lemma A.1,

$$\begin{aligned}&\langle dF_1(x), \dots, dF_{m-n+1}(x), d\Delta_2(x), \dots, d\Delta_{k-1}(x), d\Delta_k(x)\rangle\\ &= \langle dF_1(x), \dots, dF_{m-n+1}(x), d\Delta_2(x), \dots, d\Delta_{k-1}(x), d\tilde{\Delta}_k(x)\rangle,\end{aligned}$$

which implies that

$$\operatorname{rank}(dF_1(x), \dots, dF_{m-n+1}(x), d\Delta_2(x), \dots, d\Delta_{k-1}(x), d\tilde{\Delta}_k(x))$$

is equal to $m - n + k$. Therefore, the condition I_{k-1} and the definition of $\Sigma^k(\omega)$ do not depend on the choice of the basis $\{\Omega_1(x), \dots, \Omega_{n-k+1}(x)\}$.

Since $A_k(\omega) = \Sigma^k(\omega) \smallsetminus \Sigma^{k+1}(\omega)$ for $k = 1, \dots, n$, we conclude that $A_k(\omega)$ also does not depend on the choice of the basis. $\square$

References

[1] Ando, Y. On local structures of the singularities A_k D_k and E_k of smooth maps. *Trans. Amer. Math. Soc. 331*, 2 (1992), 639–651.

[2] Bobowik, J., and Szafraniec, Z. Counting signed swallowtails of polynomial selfmaps of $\mathbb{R}^3$. *Manuscripta Mathematica 151*, 1 (2016), 209–221. DOI: 10.1007/s00229-016-0832-4

[3] Dutertre, N., and Fukui, T. On the topology of stable maps. *J. Math. Soc. Japan 66*, 1 (2014), 161–203.

[4] Èliašberg, J. M. On singularities of folding type. *Mathematics of the USSR-Izvestiya 4*, 5 (1970), 1119. DOI: 10.1070/im1970v004n05abeh000946

[5] Fukuda, T. Topology of folds, cusps and Morin singularities. In *A fête of topology*. Academic Press, Boston, MA, 1988, pp. 331–353. DOI: 10.1016/b978-0-12-480440-1.50019-8

[6] Inaba, K., Ishikawa, M., Kawashima, M., and Nguyen, T. T. On linear deformations of Brieskorn singularities of two variables into generic maps. *Tohoku Math. J. (2) 69*, 1 (2017), 85–111. DOI: 10.2748/tmj/1493172130

[7] Kalmár, B., and Terpai, T. Characteristic classes and existence of singular maps. *Trans. Amer. Math. Soc. 364*, 7 (2012), 3751–3779. DOI: 10.1090/s0002-9947-2012-05544-9

[8] Krzyżanowska, I., and Szafraniec, Z. On polynomial mappings from the plane to the plane. *J. Math. Soc. Japan 66*, 3 (07 2014), 805–818. DOI: 10.2969/jmsj/06630805

[9] Morin, B. Formes canoniques des singularités d'une application différentiable. *Comptes Rendus Hebdomadaires des séances de l'Académie des Sciences 260*, 25 (1965), 6503–6506. DOI: 10.5962/bhl.part.22031

[10] Quine, J. R. A global theorem for singularities of maps between oriented 2-manifolds. *Trans. Amer. Math. Soc. 236* (1978), 307–314. DOI: 10.1090/s0002-9947-1978-0474378-x

[11] Ruiz, C. M. An alternative proof of a formula on Euler characteristics of singularity sets of a Morin map $M \to \mathbb{R}^n$ based on Morse theory. *Topology and its Applications 234*, (2018), 92–103. DOI: 10.1016/j.topol.2017.11.012

[12] Saeki, O. Studying the topology of Morin singularities from a global viewpoint. *Math. Proc. Cambridge Philos. Soc. 117*, 2 (1995), 223–235. DOI: 10.1017/s0305004100073072

[13] Saeki, O., and Sakuma, K. Maps with only Morin singularities and the Hopf invariant one problem. *Math. Proc. Cambridge Philos. Soc. 124*, 3 (1998), 501–511. DOI: 10.1017/s0305004197002478

[14] Saji, K. Criteria for Morin singularities for maps into lower dimensions, and applications. *Real and complex singularities, Contemp. Math.,Amer. Math. Soc., Providence, RI 675* , (2016), 315–336. DOI: 10.1090/conm/675/13599

[15] Saji, K. Criteria for Morin singularities into higher dimensions. *RIMS Kôkyûroku Bessatsu B55.* (2016), 203–222.

[16] Saji, K. Isotopy of Morin singularities. *Houston J. Math. 42* (2016), 499–519.

[17] Saji, K., Umehara, M., Yamada, K. An index formula for a bundle homomorphism of the tangent bundle into a vector bundle of the same rank, and its applications. *J. Math. Soc. Japan 69* (2017), 417–457. DOI: 10.2969/jmsj/06910417

[18] Szabó, E., Szűcs, A., and Terpai, T. On bordism and cobordism groups of Morin maps. *J. Singul. 1* (2010), 134–145.

[19] Szafraniec, Z. The Euler characteristic of algebraic complete intersections. *J. Reine Angew. Math. 397* (1989), 194–201. DOI: 10.1515/crll.1989.397.194

[20] Szafraniec, Z. A formula for the Euler characteristic of a real algebraic manifold. *Manuscripta Math. 85*, 3-4 (1994), 345–360. DOI: 10.1007/bf02568203

[21] Szücs, A. On the cobordism groups of cooriented, codimension one Morin maps. *J. Singul. 4* (2012), 196–205. DOI: 10.5427/jsing.2012.4m

Camila M. Ruiz, Departamento de Matemática Aplicada, Instituto de Ciências Tecnológias e Exatas, Universidade Federal do Triângulo Mineiro, 38064-200 Uberaba - MG, Brasil

Email address: camila.ruiz@uftm.edu.br

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DEFORMATION RETRACTS TO INTERSECTIONS OF WHITNEY STRATIFICATIONS

SAURABH TRIVEDI AND DAVID TROTMAN

Abstract. We give a counterexample to a conjecture of Eyral on the existence of deformation retracts to intersections of Whitney stratifications embedded in a smooth manifold. We then prove that the conjecture holds if the stratifications are definable in some o-minimal structure without assuming any regularity conditions. Moreover, we also show that the conjecture holds for Whitney stratifications if they intersect transversally.

1. Introduction

In [2] Eyral proved the existence of deformation retracts to intersections of Whitney stratifications sitting inside a compact real analytic manifold, and used the result to prove connectivity properties of such intersections. He later used these results to find examples of global rectified homotopical depths and proved a conjecture of Grothendieck on homotopical depth; see [3].

In proving his results Eyral exploits the triangulability properties of compact real analytic manifolds. He then conjectures the existence of deformation retracts to intersections of Whitney stratifications embedded in any non-compact smooth manifold. More precisely, he conjectures the following statement:

Conjecture 5.2 in [2]. Let M be a smooth manifold, A and B be two closed subsets of M and C a closed subset of B. Suppose that there exist a Whitney stratification of B adapted to C (i.e. C is a union of strata of B) and a Whitney stratification of A whose strata intersect the strata of C transversally. Then, there exists a neighbourhood W of $A \cap B$ in B such that the couple $(A\cap B, (A\cap B)\setminus(A\cap B\cap C))$ is a strong deformation retract of the couple $(W, W\setminus(W\cap C))$.

He further conjectures that certain pairs of intersection of Whitney stratifications embedded in any smooth manifold (not necessarily compact) are highly connected and claims that this can be proved using the above conjecture; see Conjecture 5.1 in [2].

We show by a simple counterexample that Conjecture 5.2 is false in general; see Figure 1.

Let $M = \mathbb{R}^2$, $B = x$-axis. Let A be the graph of

$$y = \begin{cases} x^3 \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

in $\mathbb{R}^2$. Choose C such that $A \cap C$ is empty. Then $A \cap B$ is an infinite (double) sequence of points. Any neighbourhood W of $A \cap B$ will have a component containing the origin and infinitely many points of $A \cap B$. There exists no retraction of W onto $A \cap B$ since the image of a connected set under a continuous map is also connected. Thus $A \cap B$ is not a (strong deformation) retract of any such W.

We remark that though the function y above is not smooth, a smooth counterexample can easily be given. For example we can take A to be the graph of

$$y = \begin{cases} e^{-1/x^2} \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}.$$

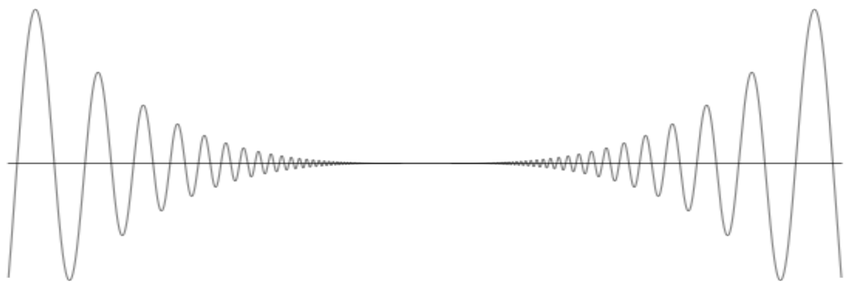

FIGURE 1.

In this article we show that the conjecture holds without the hypothesis of Whitney regularity of stratifications if the strata are assumed to be definable in some o-minimal structure. Furthermore, the conjecture also holds with the extra assumption of transverse intersection of the Whitney stratifications of A and B.

2. Triangulations of definable sets and Whitney stratifications

In this section we recall the definition of triangulations and present some of the results about triangulability of stratifications and definable sets in o-minimal structures. For definitions of definable sets and o-minimal structures we refer the reader to van den Dries [16]. For definitions of stratifications and the regularity conditions (a) and (b) of Whitney we refer to the Ph.D. thesis of the second author [15]. In addition, we assume that the stratifications are locally compact to avoid pathologies. For definitions of simplices, open simplices, simplicial complexes and polytopes we refer to Munkres [10].

Recall that a topological set X is said to be triangulable if there exists a simplicial complex K and a homeomorphism $\phi : |K| \to X$, where $|K|$ is the polytope of K. The simplicial complex K is then said to be a triangulation of X. We remark that we allow K to be a simplicial complex with an infinite number of simplices and recall that if K is finite then the polytope $|K|$ is compact and conversely if $A \subset |K|$ is compact, then $A \subset |K_0|$ for some finite subcomplex K_0 of K; see Lemma 2.5 in Munkres [10]. This implies that if a triangulable set X is compact then its triangulation is finite. It is well known that any smooth manifold is triangulable; see Part 2 of Munkres [9].

We know that the definable sets in any o-minimal structure can be triangulated. Let us recall the precise statement on triangulations of definable sets; a proof of this result can be found in Coste [1] or van den Dries [16].

In what follows by a definable set we mean a set definable in an o-minimal structure $\mathcal{D}$ over $\mathbb{R}$. Let us mention that definable sets admit definable Whitney stratifications, i.e. every definable set can be stratified into finitely many connected definable submanifolds called strata, such that every pair of adjacent strata satisfies Whitney (b)-regularity; see [11] for a proof.

Theorem 2.1. *Let $A \subset \mathbb{R}^n$ be a compact definable set and $\{B_1, \dots, B_k\}$ be definable subsets of A. Then, there exists a definable homeomorphism $\Phi : |K| \to A$ from a finite simplicial complex K onto A such that each B_i is a union of images of open simplices of K under Φ.*

From this it immediately follows that:

Theorem 2.2. *Let Σ be a definable stratification of a definable subset V of a compact definable set A in $\mathbb{R}^n$, then there exists a definable triangulation $\Phi : |K| \to A$ of A such that every stratum of Σ is a union of images of open simplices of K under Φ.*

In short, every definable stratification can be triangulated.

Although the results are stated for compact definable subsets of $\mathbb{R}^n$, non-compact definable subsets can also be triangulated. This can be seen as follows:

Let A be a definable subset (non-necessarily compact) of $\mathbb{R}^n$ and $B_1, \dots, B_k$ be definable subsets of B. Take the compactification $\mathbb{P}^n$ of $\mathbb{R}^n$ and consider A as an embedded subset of $\mathbb{P}^n$. We know that A considered as an embedded subset of $\mathbb{P}^n$ is still a definable subset of some $\mathbb{R}^N$ where $\mathbb{P}^n$ embeds. We remark here that such an embedding of a non-definable set might not be triangulable, a typical example is the embedding of the graph of $\sin(x)$ in $\mathbb{P}^2$. We can then apply the above results to $\mathbb{P}^n$ and obtain a triangulation of A.

Thus, we have:

Theorem 2.3. *Let Σ be a definable stratification of a definable subset V of a definable set (not necessarily compact) A in $\mathbb{R}^n$, then there exists a triangulation $\Phi : |K| \to A$ of A such that every stratum of Σ is a union of images of open simplices of K under Φ.*

Furthermore, any abstract stratified set can be embedded as a Whitney regular stratified subanalytic set (semialgebraic if the set is compact) in a Euclidean space, see Noirel [12]. Also, Shiota [14] showed that every locally compact Whitney stratified set is homeomorphic to a subanalytic set. Then, one can use the theorem of Hironaka [6] or alternatively Hardt [5] to triangulate the subanalytic set and pull it back to obtain a triangulation of a given abstract stratified set. Also, Mather [8] proved that every Whitney stratified set admits the structure of an abstract stratified set. It follows from this that Whitney stratified sets in $\mathbb{R}^n$ are triangulable. Let us mention this result in the following theorem:

Theorem 2.4. *Let M be a smooth manifold and A be a closed subset of M admitting a Whitney (b) regular stratification. Then, there exists a simplicial complex K whose polytope is homeomorphic to M and such that every stratum of A is a union of images of open simplices of K under the homeomorphism.*

At this point we would like to mention that Goresky [4] also proved that any abstract stratified set in the sense of Mather [8] is triangulable. But, it is not clear whether Goresky's idea works for non-compact stratified sets, for Goresky uses Hudson's [7] notion of "Euclidean polyhedra" to define a triangulation and polyhedra of Hudson only have finitely many simplices and so are compact. Hironaka's or Hardt's triangulation works for non-compact sets too.

We give an example of a Whitney (a)-regular stratification which is not triangulable; see Figure 2. Consider the set X given by the closure of the graph of $\sin(1/x)$ for $x > 0$ in $\mathbb{R}^2$. Stratify it with three strata, the limiting points of the interval in the y-axis, the open interval in the y-axis and the graph of $\sin(1/x)$ for $x > 0$. This is a Whitney (a)-regular stratification, but is not triangulable since it is not path-connected. The set is not a definable set. Furthermore, the intersection U of the x-axis, which is a transverse intersection with the set X, does not have any open set in the x-axis that retracts to the set U.

3. Existence of neighbourhoods of subcomplexes

In this section we will show how to use barycentric subdivisions of a simplicial complex to obtain neighbourhoods of subcomplexes. These neighbourhoods will be used to construct deformation retracts in the next section. The construction of the neighbourhoods is standard but we describe it here for the sake of clarity.

We first of all recall the definition of the 'join' of two simplicial complexes. Let K_1 and K_2 be two simplicial complexes. The join of K_1 and K_2, denoted by $K_1 * K_2$ is the simplicial complex spanned by the vertices of K_1 and K_2 together.

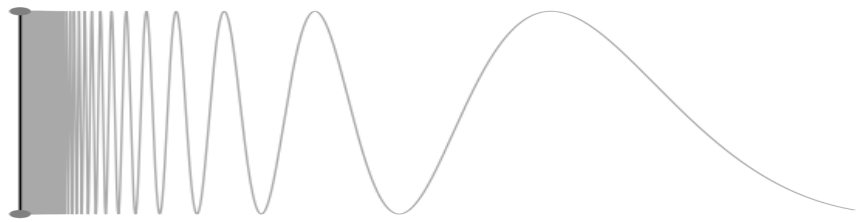

FIGURE 2.

Let K be a simplicial complex and L be a subcomplex of K. Denote by $|K|$ the polytope of K and that of L by $|L|$. Let K' be the first barycentric subdivision of K. Since L is a subcomplex this subdivision induces a subdivision of L. Recall that the polytope does not change after the subdivision. Denote by $K \div L$ the subcomplex of K' generated by the vertices of K' that are not in L'. Then, there is a natural embedding of $|K|$ onto $|L * K \div L|$. This embedding allows us to write elements of $|K|$ as tuples (x, t, y) where $t \in [0, 1]$, $x \in |K \div L|$ and $y \in |L|$. Define

$$\mathcal{N}(L) = \{p = (x, t, y) \in |K| | t \in [0, 1), x \in |K \div L| \text{ and } y \in |L|\}.$$

It is then easy to see that $\mathcal{N}(L)$ is an open neighbourhood in $|K|$ of $|L|$. For example, see the picture below.

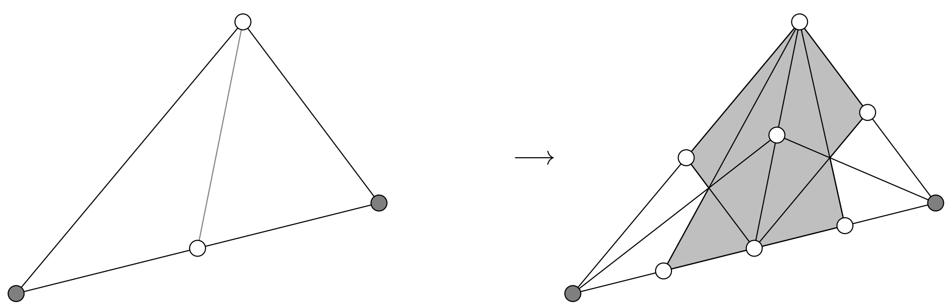

FIGURE 3.

In the picture above the complex $|K|$ is the full triangle while $|L|$ consists of the simplex spanned by the white vertices on the left side of the picture above. The constructed neighbourhood of $|L|$ is the shaded region. It is clear that $|K \div L|$ is what remains after deleting the shaded region.

Moreover, for $t_0 \in (0, 1]$ the set

$$\mathcal{N}(L, t_0) = \{p = (x, t, y) \in |K| | t \in [0, t_0), x \in |K \div L| \text{ and } y \in |L|\}.$$

also defines a neighbourhood of L in K. By varying t_0 we get a system of neighbourhoods of L in K. Notice that we can also construct neighbourhood of an open subcomplex of K. Here, by an open subcomplex we mean a union of open simplices of K. More precisely, given an open subcomplex, we can first take the union of its open simplices with their boundaries to get a subcomplex of K. We can then follow the steps to find a neighbourhood of this subcomplex which also works as a neighbourhood of the open subcomplex we started with.

Finally, if $\phi : |K| \to X$ is a triangulation of a topological space X such that a subspace $Y \subset X$ is the image of a subcomplex L of K under ϕ, then the image of $\mathcal{N}(L, t_0)$ under ϕ is a neighbourhood of Y in X. The closure of $N(L, t_0)$ in $|K|$ is said to be a closed neighbourhood of L in K. In the rest of the article, by a neighbourhood $\mathcal{N}(Y)$ of a subspace of a triangulable space X we mean the image of $\mathcal{N}(L, t_0)$ (for some $t_0 \in [0, 1)$)under the homeomorphism of the triangulation of X. It is then easy to see that Y is a deformation retract of $\mathcal{N}(Y)$. Moreover every neighbourhood of Y in X contains a neighbourhood of type $\mathcal{N}(Y)$; see Proposition 1.4 and 1.5 of Eyral [2].

4. Construction of deformation retracts

In this section we prove the main results. We first need the following lemma whose proof closely follows the proof of Proposition 1.6 in Eyral [2] and is left to the reader:

Lemma 4.1. *Let X be a triangulable space, Y and Z be two subspaces of X that are images of some open simplices of the triangulation of X and $\mathcal{N}(Y)$ the neighbourhood of Y in X, then there exists a system of neighbourhoods $\{V_\alpha\}$ of Y in X such that, for every α, $(V_\alpha, V_\alpha \setminus Z)$ is a deformation retract of $(\mathcal{N}(Y), \mathcal{N}(Y) \setminus Z)$.*

We prove that:

Theorem 4.2. *Let M be a definable submanifold of $\mathbb{R}^n$ and A, B and $C \subset B$ be closed definable subsets in M. Then, there exists a neighbourhood W of $A \cap B$ in B such that the couple*

$$(A \cap B, (A \cap B) \setminus (A \cap B \cap C))$$

is a strong deformation retract of the couple $(W, W \setminus (W \cap C))$.

Proof. By Theorem 2.3, we can choose a triangulation $\Phi : |K| \to M$ of M such that A and B are union of images of some open simplices of K. Furthermore, K can be chosen in such a way that C is also a union of the image of open simplices of a sub-complex of the complex K' that triangulates B, i.e. $\Phi(K') = B$.

Since finite intersections of definable sets are definable, we can choose a triangulation, a subdivision of K if necessary, of M adapted to A, B, C, $A \cap B$ and $A \cap B \cap C$. Moreover, $(A \cap B) \setminus (A \cap B \cap C)$ is also a definable set. Thus, subdividing K if necessary, we can assume that all these definable sets are unions of images of some open simplices of K under Φ. Now consider the neighbourhood of $W = \mathcal{N}(A \cap B)$ of $A \cap B$ and constructed in the previous section. By Lemma 4.1, it is then clear that $(A \cap B, (A \cap B) \setminus (A \cap B \cap C))$ is a deformation retract of $(W, W \setminus (W \cap C))$. This concludes the proof of the theorem. ☐

Moreover,

Theorem 4.3. *Let M be a smooth manifold and A, B and $C \subset B$ be closed subsets of M. Suppose there exist a Whitney stratification of B adapted to C (i.e. C is a union of strata of B) and a Whitney stratification of A whose strata intersect the strata of B transversally. Then, there exists a neighbourhood W of $A \cap B$ in B such that the couple $(A \cap B, (A \cap B) \setminus (A \cap B \cap C))$ is a strong deformation retract to the couple $(W, W \setminus (W \cap C))$.*

Proof. Since A and B are Whitney stratifications and they intersect transversally, the intersection $A \cap B$ is also a Whitney stratification; see Orro and Trotman [13]. Thus, the union $A \cup B$ admits a Whitney stratification by the strata of $A \cap B$ and their complements in the corresponding strata of A and B. By Theorem 2.4, we can choose a triangulation $\Phi : |K| \to M$ of M such that the strata of $A \cup B$ are union of images of open simplices of K under ϕ. Furthermore, since C is a union of strata of B, K can be chosen in such a way that C is also the image of open simplices of a sub-complex of the complex K' that triangulates B, i.e. $\Phi(K') = B$.

Therefore, we can suitably choose a triangulation, refinement of K if necessary, of M adapted to the strata of A, B, C, $A \cap B$ and $A \cap B \cap C$. That is, every stratum of the five stratifications is a union of images of open simplices of K under Φ. Now consider the neighbourhood

$$W = \mathcal{N}(A \cap B)$$

of $A \cap B$ constructed in the previous section. By Lemma 4.1, it is then clear that

$$(A \cap B, (A \cap B) \setminus (A \cap B \cap C))$$

is a deformation retract of $(W, W \setminus (W \cap C))$. This concludes the proof of the theorem. $\square$

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References

[1] M. Coste, *An Introduction to o-minimal Geometry*, Dip. Mat. Univ. Pisa, Dottorato di Ricerca in Matematica, Istituti Editoriali e Poligrafici Internazionali, Pisa, 2000.
[2] C. Eyral, *Sur l'homotopie des espaces stratifiés*, Internat. Math. Res. Notices (1999), no. 13, 717–734.
[3] ———, *Profondeur homotopique et conjecture de Grothendieck*, Ann. Sci. École Norm. Sup. (4) **33** (2000), no. 6, 823–836. DOI: 10.1016/s0012-9593(00)01054-5
[4] R. M. Goresky, *Triangulation of stratified objects*, Proc. Amer. Math. Soc. **72** (1978), no. 1, 193–200. DOI: 10.1090/s0002-9939-1978-0500991-2
[5] R. M. Hardt, *Triangulation of subanalytic sets and proper light subanalytic maps*, Invent. Math. **38** (1976/77), no. 3, 207–217. DOI: 10.1007/bf01403128
[6] H. Hironaka, *Triangulations of algebraic sets*, Algebraic geometry (Proc. Sympos. Pure Math., Vol. 29, Humboldt State Univ., Arcata, Calif., 1974) (1975), 165–185. DOI: 10.1090/pspum/029/0374131
[7] J. F. P. Hudson, *Piecewise linear topology*, University of Chicago Lecture Notes Inc., New York-Amsterdam, 1969.
[8] J. N. Mather, *Notes on topological stability*, Bull. Amer. Math. Soc. **49** (2012), 475–506. DOI: 10.1090/s0273-0979-2012-01383-6
[9] J. R. Munkres, *Elementary differential topology*, Lectures given at Massachusetts Institute of Technology, Fall, vol. 1961, Princeton University Press, Princeton, N.J., 1963.
[10] ———, *Elements of algebraic topology*, Addison-Wesley Publishing Company, Menlo Park, CA, 1984.
[11] N. Nguyen, S. Trivedi, and D. J. A. Trotman, *A geometric proof of the existence of definable Whitney stratifications*, Illinois J. Math. **58** (2014), no. 2, 381–389. DOI: 10.1215/ijm/1436275489
[12] L. Noirel, *Plongements sous-analytiques d'espaces stratifiés de Thom-Mather*, Ph.D. Thesis, University of Provence (1996).
[13] P. Orro and D. J. A. Trotman, *Regularity of the transverse intersection of two regular stratifications*, London Mathematical Society Lecture Note Series, p. 298–304, Cambridge University Press, 2010. DOI: 10.1017/cbo9780511731983.022
[14] M. Shiota, *Triangulation of subanalytic sets*, Singularities (Warsaw, 1985), Banach Center Publ., vol. 20, PWN, Warsaw, 1988, pp. 385–395. DOI: 10.4064/-20-1-385-395
[15] D. J. A. Trotman, *Whitney stratifications: Fault and detectors*, Ph.D. Thesis, University of Warwick (1977).
[16] L. van den Dries, *Tame topology and o-minimal structures.*, Cambridge: Cambridge University Press, 1998.

Saurabh Trivedi, Indian Statistical Institute North-East Centre, Punioni Solmara, Tezpur 784501, Assam, India & ICMC - Universidade de São Paulo, 13566-590, São Carlos, SP, Brasil

David Trotman, I2M (UMR 7353), Centre de Mathématiques et Informatique, Aix-Marseille Université, 39 rue Joliot-Curie, 13453 Marseille Cedex 13, France.

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COMPARISON OF STRATIFIED-ALGEBRAIC AND TOPOLOGICAL K-THEORY

WOJCIECH KUCHARZ AND KRZYSZTOF KURDYKA

ABSTRACT. Stratified-algebraic vector bundles on real algebraic varieties have many desirable features of algebraic vector bundles but are more flexible. We give a characterization of the compact real algebraic varieties X having the following property: There exists a positive integer r such that for any constant rank topological vector bundle ξ on X, the direct sum of r copies of ξ is isomorphic to a stratified-algebraic vector bundle. In particular, each compact real algebraic variety of dimension at most 8 has this property. Our results are expressed in terms of K-theory.

1. Introduction and main results

In the recent paper [30], we introduced and investigated stratified-algebraic vector bundles on real algebraic varieties. They occupy an intermediate position between algebraic and topological vector bundles. Here we continue the line of research undertaken in [30, 28] and look for new relationships between stratified-algebraic and topological vector bundles. In a broader context, the present paper is also closely related to [5, 16, 23, 24, 26, 27, 29]. All results announced in this section are proved in Section 2.

Throughout this paper the term *real algebraic variety* designates a locally ringed space isomorphic to an algebraic subset of $\mathbb{R}^N$, for some N, endowed with the Zariski topology and the sheaf of real-valued regular functions (such an object is called an affine real algebraic variety in [7]). The class of real algebraic varieties is identical with the class of quasi-projective real varieties, cf. [7, Proposition 3.2.10, Theorem 3.4.4]. Morphisms of real algebraic varieties are called *regular maps*. Each real algebraic variety carries also the Euclidean topology, which is induced by the usual metric on $\mathbb{R}$. Unless explicitly stated otherwise, all topological notions relating to real algebraic varieties refer to the Euclidean topology.

Let $\mathbb{F}$ stand for $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ (the quaternions). All $\mathbb{F}$-vector spaces will be left $\mathbb{F}$-vector spaces. When convenient, $\mathbb{F}$ will be identified with $\mathbb{R}^{d(\mathbb{F})}$, where $d(\mathbb{F}) = \dim_{\mathbb{R}} \mathbb{F}$.

Let X be a real algebraic variety. For any nonnegative integer n, let $\varepsilon^n_X(\mathbb{F})$ denote the standard trivial $\mathbb{F}$-vector bundle on X with total space $X \times \mathbb{F}^n$, where $X \times \mathbb{F}^n$ is regarded as a real algebraic variety. An algebraic $\mathbb{F}$-vector bundle on X is an algebraic $\mathbb{F}$-vector subbundle of $\varepsilon^n_X(\mathbb{F})$ for some n (cf. [7, Chapters 12 and 13] for various characterizations of algebraic $\mathbb{F}$-vector bundles).

We now recall the fundamental notion introduced in [30]. By a *stratification* of X we mean a finite collection $\mathcal{S}$ of pairwise disjoint Zariski locally closed subvarieties whose union is X. Each subvariety in $\mathcal{S}$ is called a stratum of $\mathcal{S}$. A *stratified-algebraic* $\mathbb{F}$*-vector bundle on* X is a

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topological $\mathbb{F}$-vector subbundle ξ of $\varepsilon^n_X(\mathbb{F})$, for some n, such that for some stratification $\mathcal{S}$ of X, the restriction $\xi|_S$ of ξ to each stratum S of $\mathcal{S}$ is an algebraic $\mathbb{F}$-vector subbundle of $\varepsilon^n_S(\mathbb{F})$.

A topological $\mathbb{F}$-vector bundle ξ on X is said to *admit an algebraic structure* if it is isomorphic to an algebraic $\mathbb{F}$-vector bundle on X. Similarly, ξ is said to *admit a stratified-algebraic structure* if it is isomorphic to a stratified-algebraic $\mathbb{F}$-vector bundle on X. These two types of $\mathbb{F}$-vector bundles have been extensively investigated in [3, 4, 6, 7, 9, 10, 11, 13] and [30, 28], respectively. In general, their behaviors are quite different, cf. [30, Example 1.11]. Here we further develop the direction of research initiated in [30, 28]. It is convenient to bring into play Grothendieck groups.

Denote by $K_{\mathbb{F}}(X)$ the Grothendieck group of topological $\mathbb{F}$-vector bundles on X. For any topological $\mathbb{F}$-vector bundle ξ on X, let $[\![\xi]\!]$ denote its class in $K_{\mathbb{F}}(X)$. Since X has the homotopy type of a compact polyhedron [7, pp. 217, 225], it follows that the abelian group $K_{\mathbb{F}}(X)$ is finitely generated (cf. [21, Exercise III.7.5] or the spectral sequence in [2, 15]). Let $K_{\mathbb{F}\text{-str}}(X)$ be the subgroup of $K_{\mathbb{F}}(X)$ generated by the classes of all $\mathbb{F}$-vector bundles admitting a stratified-algebraic structure.

If the variety X is compact, then the group $K_{\mathbb{F}\text{-str}}(X)$ contains complete information on $\mathbb{F}$-vector bundles on X admitting a stratified-algebraic structure. More precisely, we have the following.

Theorem 1.1 ([30, Corollary 3.14]). *Let X be a compact real algebraic variety. A topological $\mathbb{F}$-vector bundle ξ on X admits a stratified-algebraic structure if and only if the class $[\![\xi]\!]$ is in $K_{\mathbb{F}\text{-str}}(X)$.*

In other words, with notation as in Theorem 1.1, ξ admits a stratified-algebraic structure if and only if there exists a stratified-algebraic $\mathbb{F}$-vector bundle η on X such that the direct sum $\xi \oplus \eta$ admits a stratified-algebraic structure.

For our purposes it is convenient to distinguish some vector bundles by imposing a suitable condition on their rank. For any topological $\mathbb{F}$-vector bundle ξ on X, we regard $\operatorname{rank}\xi$ (the rank of ξ) as a function $\operatorname{rank}\xi\colon X \to \mathbb{Z}$, which assigns to every point x in X the dimension of the fiber of ξ over x. Clearly, $\operatorname{rank}\xi$ is a constant function on each connected component of X.

We say that ξ has *property* (rk) if for every integer d, the set $\{x \in X \mid (\operatorname{rank}\xi)(x) = d\}$ is algebraically constructible. Recall that a subset of X is said to be algebraically constructible if it belongs to the Boolean algebra generated by the Zariski closed subsets of X. It readily follows that each stratified-algebraic $\mathbb{F}$-vector bundle on X has property (rk). Thus property (rk) is a necessary condition for ξ to admit a stratified-algebraic structure.

We now give a simple example to illustrate the role of property (rk). The real algebraic curve

$$C = \{(x, y) \in \mathbb{R}^2 : x^2(x^2 - 1)(x^2 - 4) + y^2 = 0\}$$

is irreducible with singular locus $\{(0,0)\}$. It has three connected components, the singleton $\{(0,0)\}$ and two ovals. Clearly, every algebraic $\mathbb{F}$-vector bundle on C has constant rank, while the rank function of a topological $\mathbb{F}$-vector bundle on C may take three distinct values. On the other hand, the rank function of a stratified-algebraic vector bundle on C need not be constant, but must be constant on $C \setminus \{(0,0)\}$.

Returning to the general case, denote by $K^{(\mathrm{rk})}_{\mathbb{F}}(X)$ the subgroup of $K_{\mathbb{F}}(X)$ generated by the classes of all topological $\mathbb{F}$-vector bundles having property (rk). By construction,

$$K_{\mathbb{F}\text{-str}}(X) \subseteq K^{(\mathrm{rk})}_{\mathbb{F}}(X).$$

Since the group $K_{\mathbb{F}}(X)$ is finitely generated, so is the quotient group

$$\Gamma_{\mathbb{F}}(X) \coloneqq K^{(\mathrm{rk})}_{\mathbb{F}}(X)/K_{\mathbb{F}\text{-str}}(X).$$

Thus the group $\Gamma_{\mathbb{F}}(X)$ is finite if and only if $r\Gamma_{\mathbb{F}}(X) = 0$ for some positive integer r. In the present paper the group $\Gamma_{\mathbb{F}}(X)$ is the main object of investigation.

For any $\mathbb{F}$-vector bundle ξ on X and any positive integer r, we denote by

$$\xi(r) = \xi \oplus \cdots \oplus \xi$$

the r-fold direct sum. The following preliminary result shows that our approach here is consistent with that of [28].

Proposition 1.2. *Let X be a compact real algebraic variety. For a positive integer r, the following conditions are equivalent:*

(a) *The group $\Gamma_{\mathbb{F}}(X)$ is finite and $r\Gamma_{\mathbb{F}}(X) = 0$.*
(b) *For each topological $\mathbb{F}$-vector bundle ξ on X having property* (rk)*, the $\mathbb{F}$-vector bundle $\xi(r)$ admits a stratified-algebraic structure.*
(c) *For each topological $\mathbb{F}$-vector bundle η on X having constant rank, the $\mathbb{F}$-vector bundle $\eta(r)$ admits a stratified-algebraic structure.*

In [28, Conjecture C], it is suggested that the group $\Gamma_{\mathbb{F}}(X)$ is always finite (for X compact). We show here that the finiteness of the group $\Gamma_{\mathbb{F}}(X)$ is equivalent to a certain condition involving cohomology classes of a special kind. For any nonnegative integer k, we defined in [30] a subgroup $H^{2k}_{\mathbb{C}\text{-str}}(X;\mathbb{Z})$ of the cohomology group $H^{2k}(X;\mathbb{Z})$. For the convenience of the reader, the definition and basic properties of $H^{2k}_{\mathbb{C}\text{-str}}(X;\mathbb{Z})$ are recalled in Section 2.

Theorem 1.3. *For any compact real algebraic variety X, the following conditions are equivalent:*

(a) *The group $\Gamma_{\mathbb{F}}(X)$ is finite.*
(b) *The quotient group $H^{4k}(X;\mathbb{Z})/H^{4k}_{\mathbb{C}\text{-str}}(X;\mathbb{Z})$ is finite for every positive integer k satisfying $8k - 2 < \dim X$.*

Since the groups $H^{2k}_{\mathbb{C}\text{-str}}(-;\mathbb{Z})$ are hard to compute, it is worthwhile to give a simple topological criterion for the finiteness of the group $\Gamma_{\mathbb{F}}(X)$. To this end some preparation is required.

For any positive integer d, let $\mathbb{S}^d$ denote the unit d-sphere

$$\mathbb{S}^d = \{(u_0, \dots, u_d) \in \mathbb{R}^{d+1} \mid u_0^2 + \cdots + u_d^2 = 1\}.$$

Let s_d be a generator of the cohomology group $H^d(\mathbb{S}^d;\mathbb{Z}) \cong \mathbb{Z}$. A cohomology class u in $H^d(\Omega;\mathbb{Z})$, where Ω is an arbitrary topological space, is said to be *spherical* if $u = h^*(s_d)$ for some continuous map $h\colon \Omega \to \mathbb{S}^d$. Denote by $H^d_{\mathrm{sph}}(\Omega;\mathbb{Z})$ the subgroup of $H^d(\Omega;\mathbb{Z})$ generated by all spherical cohomology classes. In general a cohomology class in $H^d_{\mathrm{sph}}(\Omega;\mathbb{Z})$ need not be spherical.

Theorem 1.4. *Let X be a compact real algebraic variety. If the quotient group*

$$H^{4k}(X;\mathbb{Z})/H^{4k}_{\mathrm{sph}}(X;\mathbb{Z})$$

is finite for every positive integer k satisfying $8k - 2 < \dim X$, then the group $\Gamma_{\mathbb{F}}(X)$ is finite.

As a consequence we obtain the following.

Corollary 1.5. *Let X be a compact real algebraic variety. If each connected component of X is homotopically equivalent to $\mathbb{S}^{d_1} \times \cdots \times \mathbb{S}^{d_n}$ for some positive integers $d_1, \dots, d_n$, then the group $\Gamma_{\mathbb{F}}(X)$ is finite.*

Proof. Since $H^l_{\mathrm{sph}}(X;\mathbb{Z}) = H^l(X;\mathbb{Z})$ for every positive integer l, it suffices to make use of Theorem 1.4. □

It is interesting to compare Corollary 1.5 with related, previously known, results. If

$$X = X_1 \times \cdots \times X_n,$$

where each X_i is a compact real algebraic variety homotopically equivalent to $\mathbb{S}^{d_i}$ for $1 \le i \le n$, then $\Gamma_{\mathbb{F}}(X) = 0$ for $\mathbb{F} = \mathbb{C}$ and $\mathbb{F} = \mathbb{H}$, and $2\Gamma_{\mathbb{R}}(X) = 0$, cf. [30, Theorem 1.10]. On the other hand, there exists a nonsingular real algebraic variety X diffeomorphic to the n-fold product $\mathbb{S}^1 \times \cdots \times \mathbb{S}^1$, $n > d(\mathbb{F})$, such that $\Gamma_{\mathbb{F}}(X) \neq 0$, cf. [30, Example 7.10].

For any compact real algebraic variety X, the equality $H^l(X;\mathbb{Z}) = 0$ holds if $l > \dim X$, cf. [7, p. 217]. Hence, in view of either Theorem 1.3 or Theorem 1.4, the group $\Gamma_{\mathbb{F}}(X)$ is finite for $\dim X \le 6$. This is extended below to $\dim X \le 8$. Actually, we obtain a result containing additional information.

Denote by $e(\mathbb{F})$ the integer satisfying $d(\mathbb{F}) = 2^{e(\mathbb{F})}$, that is,

$$e(\mathbb{F}) = \begin{cases} 0 & \text{if } \mathbb{F} = \mathbb{R} \\ 1 & \text{if } \mathbb{F} = \mathbb{C} \\ 2 & \text{if } \mathbb{F} = \mathbb{H}. \end{cases}$$

Given a nonnegative integer n, set

$$a(n) = \min\{l \in \mathbb{Z} \mid l \ge 0,\ 2^l \ge n\},$$
$$a(n,\mathbb{F}) = \max\{0, a(n) - e(\mathbb{F})\}.$$

It is conjectured in [28] that $2^{a(\dim X,\mathbb{F})}\Gamma_{\mathbb{F}}(X) = 0$ for every compact real algebraic variety X. This conjecture is confirmed in [28] for varieties of dimension not exceeding 5. Using different methods, we get the following.

Theorem 1.6. *For any compact real algebraic variety X of dimension at most 8, the group $\Gamma_{\mathbb{F}}(X)$ is finite and*

$$2^{a(\dim X,\mathbb{F})+a(X)}\Gamma_{\mathbb{F}}(X) = 0,$$

where $a(X) = 0$ if $\dim X \le 7$ and $a(X) = 2$ if $\dim X = 8$.

We are not able to decide whether Theorem 1.6 holds with $a(X) = 0$ for $\dim X = 8$.

In Section 2 we establish relationships between the groups $H^{2k}_{\mathrm{sph}}(-;\mathbb{Z})$ and $H^{2k}_{\mathbb{C}\text{-str}}(-;\mathbb{Z})$ for $k \ge 1$. This leads to the proofs of Theorems 1.3 and 1.4. Along the way we obtain closely related results, Theorems 2.14, 2.15 and 2.16, which are of independent interest. Noteworthy is also Theorem 2.13, which plays a key role in the proof of Theorem 1.6. In Section 3 we investigate topological $\mathbb{C}$-line bundles admitting a stratified-algebraic structure.

Notation. Given two $\mathbb{F}$-vector bundles ξ and η on the same topological space, we will write $\xi \cong \eta$ to indicate that they are isomorphic.

2. Stratified-algebraic versus topological vector bundles

To begin with we establish a connection between vector bundles having property (rk) and those of constant rank.

Lemma 2.1. *Let X be a real algebraic variety and let ξ be a topological $\mathbb{F}$-vector bundle on X. If ξ has property* (rk)*, then there exists a stratified-algebraic $\mathbb{F}$-vector bundle η on X such that the direct sum $\xi \oplus \eta$ is of constant rank.*

Proof. Since X has the homotopy type of a compact polyhedron [7, pp. 217, 225], we may assume that ξ is a topological $\mathbb{F}$-vector subbundle of $\varepsilon_X^n(\mathbb{F})$ for some positive integer n. Assume that ξ has property (rk). By definition, for each integer d satisfying $0 \leq d \leq n$, the set

$$R(d) = \{x \in X \mid (\operatorname{rank} \xi)(x) = d\}$$

is algebraically constructible. Thus $R(d)$ is the union of a finite collection of pairwise disjoint Zariski locally closed subvarieties of X. In particular, there exists a stratification $\mathcal{S}$ of X such that each set $R(d)$ is the union of some strata of $\mathcal{S}$. Furthermore, each nonempty set $R(d)$ is the union of some connected components of X. It follows that we can find a topological $\mathbb{F}$-vector subbundle η of $\varepsilon_X^n(\mathbb{F})$ whose restriction $\eta|_{R(d)}$ is the trivial $\mathbb{F}$-vector subbundle of $\varepsilon_{R(d)}^n(\mathbb{F})$ with total space $R(d) \times (\mathbb{F}^{n-d} \times \{0\})$, where $\mathbb{F}^{n-d} \times \{0\} \subseteq \mathbb{F}^n$. By construction, η is a stratified-algebraic $\mathbb{F}$-vector bundle and the direct sum $\xi \oplus \eta$ is of rank n. □

In particular, if $K_{\mathbb{F}}^{(\mathrm{crk})}(X)$ is the subgroup of $K_{\mathbb{F}}(X)$ generated by the classes of all topological $\mathbb{F}$-vector bundles of constant rank, then

$$K_{\mathbb{F}\text{-str}}(X) + K_{\mathbb{F}}^{(\mathrm{crk})}(X) = K_{\mathbb{F}}^{(\mathrm{rk})}(X).$$

Hence the group $\Gamma_{\mathbb{F}}(X)$ is isomorphic to the quotient group

$$K_{\mathbb{F}}^{(\mathrm{crk})}(X)/K_{\mathbb{F}\text{-str}}(X) \cap K_{\mathbb{F}}^{(\mathrm{crk})}(X).$$

Proof of Proposition 1.2. Obviously, (b) implies (a). According to Theorem 1.1, (a) implies (b). Hence, in view of Lemma 2.1, (a) and (c) are equivalent. □

Let X be a real algebraic variety. Let $\mathbb{K}$ be a subfield of $\mathbb{F}$, where $\mathbb{K}$ (as $\mathbb{F}$) stands for $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. Any $\mathbb{F}$-vector bundle ξ on X can be regarded as a $\mathbb{K}$-vector bundle, which is indicated by $\xi_{\mathbb{K}}$. In particular, $\xi_{\mathbb{K}} = \xi$ if $\mathbb{K} = \mathbb{F}$. Furthermore, $\xi_{\mathbb{R}} = (\xi_{\mathbb{K}})_{\mathbb{R}}$. If the $\mathbb{F}$-vector bundle ξ admits a stratified-algebraic structure, then so does the $\mathbb{K}$-vector bundle $\xi_{\mathbb{K}}$.

The following result will be frequently referred to.

Theorem 2.2. *Let X be a compact real algebraic variety. A topological $\mathbb{F}$-vector bundle ξ on X admits a stratified-algebraic structure if and only if the $\mathbb{K}$-vector bundle $\xi_{\mathbb{K}}$ admits a stratified-algebraic structure.*

Proof. The proof for $\mathbb{K} = \mathbb{R}$, rather involved, is given in [30, Theorem 1.7]. The general case follows since $\xi_{\mathbb{R}} = (\xi_{\mathbb{K}})_{\mathbb{R}}$. □

We will also make use of the extension of scalars construction. Let X be a real algebraic variety. Any $\mathbb{K}$-vector bundle ξ on X gives rise to the $\mathbb{F}$-vector bundle $\mathbb{F} \otimes \xi$ on X. Here $\mathbb{F} \otimes \xi = \xi$ if $\mathbb{K} = \mathbb{F}$, $\mathbb{C} \otimes \xi$ is the complexification of ξ if $\mathbb{K} = \mathbb{R}$, and $\mathbb{H} \otimes \xi$ is the quaternionization of ξ if $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. If the $\mathbb{K}$-vector bundle ξ admits a stratified-algebraic structure, then so does the $\mathbb{F}$-vector bundle $\mathbb{F} \otimes \xi$.

For any $\mathbb{C}$-vector bundle ξ, let $\bar{\xi}$ denote the conjugate bundle, cf. [31]. Note that $\bar{\xi}_{\mathbb{R}} \cong \xi_{\mathbb{R}}$. Furthermore, for the $\mathbb{H}$-vector bundle $\mathbb{H} \otimes \xi$, we have

$$(\mathbb{H} \otimes \xi)_{\mathbb{C}} \cong \xi \oplus \bar{\xi}.$$

Lemma 2.3. *Let X be a compact real algebraic variety and let ξ be a topological $\mathbb{C}$-vector bundle on X. For any positive integer q, the $\mathbb{H}$-vector bundle $(\mathbb{H} \otimes \xi)(q)$ admits a stratified-algebraic structure if and only if so does the $\mathbb{C}$-vector bundle $\xi(2q)$.*

Proof. Since
$$((\mathbb{H}\otimes\xi)(q))_{\mathbb{C}} \cong (\mathbb{H}\otimes\xi)_{\mathbb{C}}(q) \cong (\xi\oplus\bar{\xi})(q)$$
and
$$((\xi\oplus\bar{\xi})(q))_{\mathbb{R}} \cong (\xi_{\mathbb{R}}\oplus\bar{\xi}_{\mathbb{R}})(q) \cong (\xi_{\mathbb{R}}\oplus\xi_{\mathbb{R}})(q) \cong (\xi(2q))_{\mathbb{R}},$$
we get
$$((\mathbb{H}\otimes\xi)(q))_{\mathbb{R}} \cong \xi(2q))_{\mathbb{R}}.$$
The proof is complete in view of Theorem 2.2. □

For any $\mathbb{R}$-vector bundle ξ, we have $(\mathbb{C}\otimes\xi)_{\mathbb{R}} \cong \xi\oplus\xi$.

Lemma 2.4. *Let X be a compact real algebraic variety and let ξ be a topological $\mathbb{R}$-vector bundle on X. For any positive integer q, the $\mathbb{C}$-vector bundle $(\mathbb{C}\otimes\xi)(q)$ admits a stratified-algebraic structure if and only if so does the $\mathbb{R}$-vector bundle $\xi(2q)$.*

Proof. Since
$$((\mathbb{C}\otimes\xi)(q))_{\mathbb{R}} \cong (\mathbb{C}\otimes\xi)_{\mathbb{R}}(q) \cong (\xi\oplus\xi)(q) \cong \xi(2q),$$
the proof is complete in view of Theorem 2.2. □

For the convenience of the reader we recall the definition and basic properties of stratified-$\mathbb{C}$-algebraic cohomology classes, introduced and investigated in [30].

Let V be a compact nonsingular real algebraic variety. A *nonsingular projective complexification* of V is a pair $(\mathbb{V},\iota)$, where $\mathbb{V}$ is a nonsingular projective scheme over $\mathbb{R}$ and $\iota\colon V\to\mathbb{V}(\mathbb{C})$ is an injective map such that $\mathbb{V}(\mathbb{R})$ is Zariski dense in $\mathbb{V}$, $\iota(V)=\mathbb{V}(\mathbb{R})$ and ι induces a biregular isomorphism between V and $\mathbb{V}(\mathbb{R})$. Here the set $\mathbb{V}(\mathbb{R})$ of real points of $\mathbb{V}$ is regarded as a subset of the set $\mathbb{V}(\mathbb{C})$ of complex points of $\mathbb{V}$. The existence of $(\mathbb{V},\iota)$ follows form Hironaka's theorem on resolution of singularities [19] (cf. also [22] for a very readable exposition). We identify $\mathbb{V}(\mathbb{C})$ with the set of complex points of the scheme $\mathbb{V}_{\mathbb{C}} := \mathbb{V}\times_{\operatorname{Spec}\mathbb{R}}\operatorname{Spec}\mathbb{C}$ over $\mathbb{C}$. For any nonnegative integer k, denote by $H^{2k}_{\mathrm{alg}}(\mathbb{V}(\mathbb{C});\mathbb{Z})$ the subgroup of $H^{2k}(\mathbb{V}(\mathbb{C});\mathbb{Z})$ that consists of the cohomology classes corresponding to algebraic cycles (defined over $\mathbb{C}$) on $\mathbb{V}_{\mathbb{C}}$ of codimension k, cf. [14] or [17, Chapter 19]. The subgroup
$$H^{2k}_{\mathbb{C}\text{-alg}}(V;\mathbb{Z}) := \iota^*(H^{2k}_{\mathrm{alg}}(\mathbb{V}(\mathbb{C});\mathbb{Z}))$$
of $H^{2k}(V;\mathbb{Z})$ does not depend on the choice of $(\mathbb{V};\iota)$, cf. [6]. Cohomology classes in $H^{2k}_{\mathbb{C}\text{-alg}}(V;\mathbb{Z})$ are called $\mathbb{C}$-*algebraic*. The groups $H^{2k}_{\mathbb{C}\text{-alg}}(-;\mathbb{Z})$ are subtle invariants with numerous applications, cf. [6, 8, 11, 13, 25].

Let X and Y be real algebraic varieties. A map $f\colon X\to Y$ is said to be *stratified-regular* if it is continuous and for some stratification $\mathcal{S}$ of X, the restriction $f|_S\colon S\to Y$ of f to each stratum S of $\mathcal{S}$ is a regular map. A cohomology class u in $H^{2k}(X;\mathbb{Z})$ is said to be *stratified-$\mathbb{C}$-algebraic* if there exists a stratified-regular map $\varphi\colon X\to V$, into a compact nonsingular real algebraic variety V, such that $u=\varphi^*(v)$ for some cohomology class v in $H^{2k}_{\mathbb{C}\text{-alg}}(V;\mathbb{Z})$. The set $H^{2k}_{\mathbb{C}\text{-str}}(X;\mathbb{Z})$ of all stratified-$\mathbb{C}$-algebraic cohomology classes in $H^{2k}(X;\mathbb{Z})$ forms a subgroup. The direct sum
$$H^{\mathrm{even}}_{\mathbb{C}\text{-str}}(X;\mathbb{Z}) := \bigoplus_{k\geq 0} H^{2k}_{\mathbb{C}\text{-str}}(X;\mathbb{Z})$$
is a subring of the ring
$$H^{\mathrm{even}}(X;\mathbb{Z}) := \bigoplus_{k\geq 0} H^{2k}(X;\mathbb{Z}).$$
If ξ is a stratified-algebraic $\mathbb{C}$-vector bundle on X, then the kth Chern class $c_k(\xi)$ of ξ is in $H^{2k}_{\mathbb{C}\text{-str}}(X;\mathbb{Z})$ for every nonnegative integer k. The reader can find proofs of these facts in [30].

For any topological $\mathbb{F}$-vector bundle ξ on X, one can interpret $\operatorname{rank}\xi$ as an element of $H^0(X;\mathbb{Z})$. Then the following holds.

Lemma 2.5. *Let X be a real algebraic variety and let ξ be a topological $\mathbb{F}$-vector bundle on X. If ξ has property* (rk)*, then* $\operatorname{rank}\xi$ *is in* $H^0_{\mathbb{C}\text{-str}}(X;\mathbb{Z})$.

Proof. Assume that the $\mathbb{F}$-vector bundle ξ has property (rk). We make use of the notation introduced in the proof of Lemma 2.1. Furthermore, we regard $V = \{0, \ldots, n\}$ as a real algebraic variety and $\operatorname{rank}\xi$ as a map $\operatorname{rank}\xi\colon X \to V$. Then $\operatorname{rank}\xi$ is a stratified-regular map. Note that $\operatorname{rank}\xi$ interpreted as a cohomology class in $H^0(X;\mathbb{Z})$ coincides with $(\operatorname{rank}\xi)^*(v)$, where v is the cohomology class in $H^0(V;\mathbb{Z})$ whose restriction to the singleton $\{i\}$ is equal to 1 in $H^0(\{i\};\mathbb{Z})$ for every i in V. Since $H^0_{\mathbb{C}\text{-alg}}(V;\mathbb{Z}) = H^0(V;\mathbb{Z})$, the cohomology class $(\operatorname{rank}\xi)^*(v)$ is in $H^0_{\mathbb{C}\text{-str}}(X;\mathbb{Z})$, as required. □

The following observation will prove to be useful.

Proposition 2.6. *Let X be a compact real algebraic variety. For a topological $\mathbb{C}$-vector bundle ξ on X, the following conditions are equivalent:*

(a) *There exists a positive integer r such that the $\mathbb{C}$-vector bundle $\xi(r)$ admits a stratified-algebraic structure.*
(b) *The $\mathbb{C}$-vector bundle ξ has property* (rk) *and for every positive integer j, there exists a positive integer b_j such that the cohomology class $b_j c_j(\xi)$ is in $H^{2j}_{\mathbb{C}\text{-str}}(X;\mathbb{Z})$.*

Proof. Assume that condition (a) is satisfied. Then $\xi(r)$ has property (rk) and hence ξ has it as well. Furthermore, the total Chern class $c(\xi(r))$ is in $H^{\text{even}}_{\mathbb{C}\text{-str}}(X;\mathbb{Z})$. We have

$$c(\xi(r)) = c(\xi) \smile \cdots \smile c(\xi),$$

where the right-hand-side is the r-fold cup product. In particular, $c_1(\xi(r)) = rc_1(\xi)$ is in $H^2_{\mathbb{C}\text{-str}}(X;\mathbb{Z})$. By induction, for every positive integer j, we can find a positive integer b_j such that the cohomology class $b_j c_j(\xi)$ is in $H^{2j}_{\mathbb{C}\text{-str}}(X;\mathbb{Z})$. Thus (a) implies (b).

Now assume that condition (b) is satisfied. Since ξ has property (rk), by Lemma 2.5, $\operatorname{rank}\xi$ is in $H^0_{\mathbb{C}\text{-str}}(X;\mathbb{Z})$. Hence (b) implies that the Chern character $\operatorname{ch}(\xi)$ is in $H^{\text{even}}_{\mathbb{C}\text{-str}}(X;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$. Consequently, for some positive integer r, the class $r[\![\xi]\!] = [\![\xi(r)]\!]$ is in $K_{\mathbb{C}\text{-str}}(X)$, cf. [30, Proposition 8.9]. According to Theorem 1.1, the $\mathbb{C}$-vector bundle $\xi(r)$ admits a stratified-algebraic structure. Thus (b) implies (a), which completes the proof. □

We now collect some results on spherical cohomology classes. Every compact real algebraic variety is triangulable [7, p. 217] and hence a result due to Serre can be stated as follows.

Proposition 2.7 ([32, p. 289, Propoposition 2′])**.** *Let X be a compact real algebraic variety. Then there exists a positive integer a such that for every positive integer d satisfying*

$$\dim X \leq 2d - 2$$

and every cohomology class u in $H^d(X;\mathbb{Z})$, the cohomology class au is spherical. In particular, the inclusion

$$aH^d(X;\mathbb{Z}) \subseteq H^d_{\text{sph}}(X;\mathbb{Z})$$

holds for such a and d.

Let X and Y be real algebraic varieties. A map $f\colon X \to Y$ is said to be *continuous rational* if it is continuous and its restriction to some Zariski open and dense subvariety of X is a regular map. Assuming that the variety X is nonsingular, the map f is continuous rational if and only if it is stratified-regular, cf. [23, Proposition 8] and [30, Remark 2.3].

Lemma 2.8. *Let X be a compact nonsingular real algebraic variety and let d be a positive integer. For any continuous map $h\colon X \to \mathbb{S}^d$ and any continuous map $\varphi\colon \mathbb{S}^d \to \mathbb{S}^d$ of (topological) degree 2, the composite map $\varphi \circ h\colon X \to \mathbb{S}^d$ is homotopic to a stratified-regular map.*

Proof. We may assume without loss of generality that h is a $\mathcal{C}^\infty$ map. By Sard's theorem, h is transverse to each point in some open subset U of $\mathbb{S}^d$ diffeomorphic to $\mathbb{R}^d$. Let y and z be distinct points in U, and let A be a $\mathcal{C}^\infty$ arc in U joining y and z. Then

$$M \coloneqq h^{-1}(y) \cup h^{-1}(z)$$

is a compact $\mathcal{C}^\infty$ submanifold of X. Furthermore, $B \coloneqq h^{-1}(A)$ is a compact $\mathcal{C}^\infty$ manifold with boundary $\partial B = M$, embedded in X with trivial normal bundle.

Hence, according to [12, Theorem 1.12], there exists a $\mathcal{C}^\infty$ map $F\colon X \to \mathbb{R}^d$ transverse to 0 in $\mathbb{R}^d$ and such that $M = F^{-1}(0)$. By the Weierstrass approximation theorem, the $\mathcal{C}^\infty$ map F can be approximated, in the $\mathcal{C}^\infty$ topology, by a regular map $G\colon X \to \mathbb{R}^d$. If G is sufficiently close to F, then G is transverse to 0 and $V \coloneqq G^{-1}(0)$ is a nonsingular Zariski closed subvariety of X. Furthermore, V is isotopic to M in X, cf. [1, Theorem 20.2].

We can choose a $\mathcal{C}^\infty$ map $\psi\colon \mathbb{S}^d \to \mathbb{S}^d$ of degree 2 that is transverse to y and satisfies $\psi^{-1}(y) = \{y, z\}$. By Hopf's theorem, ψ is homotopic to φ. Consequently, the maps $\varphi \circ h$ and $\psi \circ h$ are homotopic. It suffices to prove that $\psi \circ h$ is homotopic to a stratified-regular map. By construction, the map $\psi \circ h$ is transverse to y and

$$(\psi \circ h)^{-1}(y) = h^{-1}(\psi^{-1}(y)) = h^{-1}(y) \cup h^{-1}(z) = M.$$

Since M is isotopic to V, according to [24, Theorem 2.4], the map $\psi \circ h$ is homotopic to a continuous rational map $f\colon X \to \mathbb{S}^d$. The map f is stratified-regular, the variety X being nonsingular. $\square$

As a consequence, we obtain the following observation.

Remark 2.9. For any compact nonsingular real algebraic variety X, the inclusion

$$2H^{2k}_{\mathrm{sph}}(X;\mathbb{Z}) \subseteq H^{2k}_{\mathbb{C}\text{-str}}(X;\mathbb{Z})$$

holds for every positive integer k. Indeed, it suffices to prove that for any spherical cohomology class u in $H^{2k}(X;\mathbb{Z})$, the cohomology class $2u$ is in $H^{2k}_{\mathbb{C}\text{-str}}(X;\mathbb{Z})$. To this end, let $h\colon X \to \mathbb{S}^{2k}$ be a continuous map with $h^*(s_{2k}) = u$ and let $\varphi\colon \mathbb{S}^{2k} \to \mathbb{S}^{2k}$ be a continuous map of degree 2. Then

$$(\varphi \circ h)^*(s_{2k}) = h^*(\varphi^*(s_{2k})) = h^*(2s_{2k}) = 2u.$$

Recall that $H^{2k}_{\mathbb{C}\text{-alg}}(\mathbb{S}^{2k};\mathbb{Z}) = H^{2k}(\mathbb{S}^{2k};\mathbb{Z})$, cf. [6, Proposition 4.8]. Since, according to Lemma 2.8, the map $\varphi \circ h$ is homotopic to a stratified-regular map, it follows that the cohomology class $2u$ is in $H^{2k}_{\mathbb{C}\text{-str}}(X;\mathbb{Z})$.

It would be interesting to decide whether the nonsingularity of X in Remark 2.9 is essential. Dropping the nonsingularity assumption, we obtain below a weaker but useful result, Lemma 2.12. First some preparation is necessary.

By a *multiblowup* of a real algebraic variety X we mean a regular map $\pi\colon X' \to X$ which is the composition of a finite collection of blowups with nonsingular centers. If C is a Zariski closed subvariety of X and the restriction $\pi_C\colon X' \setminus \pi^{-1}(C) \to X \setminus C$ of π is a biregular isomorphism, then we say that the multiblowup π is *over* C.

A *filtration* of X is a finite sequence $\mathcal{F} = (X_{-1}, X_0, \dots, X_m)$ of Zariski closed subvarieties satisfying

$$\varnothing = X_{-1} \subseteq X_0 \subseteq \cdots \subseteq X_m = X.$$

We will make use of the following result.

Theorem 2.10 ([30, Theorem 5.4])**.** *Let X be a compact real algebraic variety. For a topological $\mathbb{F}$-vector bundle ξ on X, the following conditions are equivalent:*

(a) *The $\mathbb{F}$-vector bundle ξ admits a stratified-algebraic structure.*
(b) *There exists a filtration $\mathcal{F} = (X_{-1}, X_0, \dots, X_m)$ of X, and for each $i = 0, \dots, m$, there exists a multiblowup $\pi_i \colon X'_i \to X_i$ over X_{i-1} such that the pullback $\mathbb{F}$-vector bundle $\pi_i^*(\xi|_{X_i})$ on X'_i admits a stratified-algebraic structure.*

We now derive the following.

Lemma 2.11. *Let X be a compact real algebraic variety. Let d be a positive integer and let θ be a topological $\mathbb{F}$-vector bundle on $\mathbb{S}^d$. For any continuous map $h \colon X \to \mathbb{S}^d$ and any continuous map $\varphi \colon \mathbb{S}^d \to \mathbb{S}^d$ of degree 2, the pullback $\mathbb{F}$-vector bundle $(\varphi \circ h)^*\theta$ on X admits a stratified-algebraic structure.*

Proof. Let $\mathcal{F} = (X_{-1}, X_0, \dots, X_m)$ be a filtration of X such that the variety $X_i \setminus X_{i-1}$ is nonsingular for $0 \le i \le m$. According to Hironaka's theorem on resolution of singularities [19, 22], for each $i = 0, \dots, m$, there exists a multiblowup $\pi_i \colon X'_i \to X_i$ over X_{i-1} with X'_i nonsingular. In view of Theorem 2.10, the $\mathbb{F}$-vector bundle $\xi := (\varphi \circ h)^*\theta$ on X admits a stratified-algebraic structure if and only if the $\mathbb{F}$-vector bundle $\xi_i := \pi_i^*(\xi|_{X_i})$ on X'_i admits a stratified-algebraic structure for $0 \le i \le m$. If $e_i \colon X_i \hookrightarrow X$ is the inclusion map, then

$$\xi_i = \pi_i^*(e_i^*\xi) = \pi_i^*(e_i^*((\varphi \circ h)^*\theta)) = (\varphi \circ h \circ e_i \circ \pi_i)^*\theta.$$

Since the variety X'_i is nonsingular and the map $h \circ e_i \circ \pi_i \colon X'_i \to \mathbb{S}^d$ is continuous, according to Lemma 2.8, the map $\varphi \circ h \circ e_i \circ \pi_i$ is homotopic to a stratified-regular map $f_i \colon X'_i \to \mathbb{S}^d$. In particular, $\xi_i \cong f_i^*\theta$. We may assume that the $\mathbb{F}$-vector bundle θ is algebraic since each topological $\mathbb{F}$-vector bundle on $\mathbb{S}^d$ admits an algebraic structure, cf. [34, Theorem 11.1] and [7, Proposition 12.1.12; pp. 325, 326, 352]. Thus $f_i^*\theta$ is a stratified-algebraic $\mathbb{F}$-vector bundle on X'_i. Consequently, the $\mathbb{F}$-vector bundle ξ_i admits a stratified-algebraic structure, as required. $\square$

Here is the result we have already alluded to in the comment following Remark 2.9.

Lemma 2.12. *For any compact real algebraic variety X, the inclusion*

$$2(k-1)! H^{2k}_{\mathrm{sph}}(X; \mathbb{Z}) \subseteq H^{2k}_{\mathbb{C}\text{-str}}(X; \mathbb{Z}).$$

holds for every positive integer k.

Proof. Let k be a positive integer. It suffices to prove that for every spherical cohomology class u in $H^{2k}(X;\mathbb{Z})$, the cohomology class $2(k-1)!u$ is in $H^{2k}_{\mathbb{C}\text{-str}}(X;\mathbb{Z})$. To this end, let $h \colon X \to \mathbb{S}^{2k}$ be a continuous map with $h^*(s_{2k}) = u$ and let $\varphi \colon \mathbb{S}^{2k} \to \mathbb{S}^{2k}$ be a continuous map of degree 2. Then

$$(\varphi \circ h)^*(s_{2k}) = h^*(\varphi^*(s_{2k})) = h^*(2s_{2k}) = 2u.$$

Now we choose a topological $\mathbb{C}$-vector bundle θ on $\mathbb{S}^{2k}$ with $c_k(\theta) = (k-1)!s_{2k}$, cf. [2, p. 19] or [18, p. 155]. Then

$$c_k((\varphi \circ h)^*\theta) = (\varphi \circ \theta)^*(c_k(\theta)) = (\varphi \circ h)^*((k-1)!s_{2k}) = 2(k-1)!u.$$

According to Lemma 2.11, the $\mathbb{C}$-vector bundle $(\varphi \circ h)^*\theta$ on X admits a stratified-algebraic structure, and hence the cohomology class $2(k-1)!u$ is in $H^{2k}_{\mathbb{C}\text{-str}}(X;\mathbb{Z})$. The proof is complete. $\square$

The following result will be used in the proof of Theorem 1.6 and is also of independent interest.

Theorem 2.13. *Let X be a compact real algebraic variety. Let k be a positive integer and let θ be a topological $\mathbb{F}$-vector bundle on $\mathbb{S}^{2k}$, where $\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \mathbb{H}$. For any continuous map $h\colon X \to \mathbb{S}^{2k}$, the $\mathbb{F}$-vector bundle $h^*\theta \oplus h^*\theta$ on X admits a stratified-algebraic structure.*

Proof. Let $\varphi\colon \mathbb{S}^{2k} \to \mathbb{S}^{2k}$ be a continuous map of degree 2. Then

$$c_k(\varphi^*\theta_{\mathbb{C}}) = \varphi^*(c_k(\theta_{\mathbb{C}})) = 2c_k(\theta_{\mathbb{C}}) = c_k(\theta_{\mathbb{C}} \oplus \theta_{\mathbb{C}}),$$

and hence the $\mathbb{C}$-vector bundles $\varphi^*\theta_{\mathbb{C}}$ and $\theta_{\mathbb{C}} \oplus \theta_{\mathbb{C}}$ on $\mathbb{S}^{2k}$ are stably equivalent, cf. [2, p. 19] or [18, p. 155]. Consequently, the $\mathbb{C}$-vector bundles

$$h^*(\varphi^*\theta_{\mathbb{C}}) = (\varphi \circ h)^*\theta_{\mathbb{C}} \text{ and } h^*(\theta_{\mathbb{C}} \oplus \theta_{\mathbb{C}}) = (h^*\theta \oplus h^*\theta)_{\mathbb{C}}$$

on X are stably equivalent as well. By Lemma 2.11, the $\mathbb{C}$-vector bundle $(\varphi \circ h)^*\theta_{\mathbb{C}}$ admits a stratified-algebraic structure. Hence, according to Theorem 1.1, the $\mathbb{C}$-vector bundle $(h^*\theta \oplus h^*\theta)_{\mathbb{C}}$ admits a stratified-algebraic structure. Now the proof is complete in view of Theorem 2.2. □

The next three theorems are crucial for the proof of Theorem 1.3. We first consider $\mathbb{H}$-vector bundles. Note that for any $\mathbb{H}$-vector bundle ξ, we have $c_l(\xi_{\mathbb{C}}) = 0$ for every odd positive integer l.

Theorem 2.14. *Let X be a compact real algebraic variety. For a topological $\mathbb{H}$-vector bundle ξ on X, the following conditions are equivalent:*

(a) *There exists a positive integer r such that the $\mathbb{H}$-vector bundle $\xi(r)$ admits a stratified-algebraic structure.*
(b) *The $\mathbb{H}$-vector bundle ξ has property* (rk) *and there exists a positive integer a such that the cohomology class $ac_{2k}(\xi_{\mathbb{C}})$ is in $H^{4k}_{\mathbb{C}\text{-str}}(X;\mathbb{Z})$ for every positive integer k satisfying $8k - 2 < \dim X$.*

Proof. If condition (a) is satisfied, then the $\mathbb{C}$-vector bundle $\xi_{\mathbb{C}}(r)$ admits a stratified-algebraic structure, being isomorphic to $(\xi(r))_{\mathbb{C}}$. Thus condition (b) holds in view of Proposition 2.6.

Now assume that condition (b) is satisfied. By Proposition 2.7 and Lemma 2.12, there exists a positive integer b such that the cohomology class $bc_{2k}(\xi_{\mathbb{C}})$ is in $H^{4k}_{\mathbb{C}\text{-str}}(X;\mathbb{Z})$ for every positive integer k. Furthermore, $c_l(\xi_{\mathbb{C}}) = 0$ for every odd positive integer l. Hence, according to Proposition 2.6, there exists a positive integer r such that the $\mathbb{C}$-vector bundle $\xi_{\mathbb{C}}(r)$ admits a stratified-algebraic structure. Since the $\mathbb{C}$-vector bundles $\xi_{\mathbb{C}}(r)$ and $(\xi(r))_{\mathbb{C}}$ are isomorphic, by Theorem 2.2, the $\mathbb{H}$-vector bundle $\xi(r)$ admits a stratified-algebraic structure. Thus (b) implies (a). The proof is complete. □

Recall that for any topological $\mathbb{C}$-vector bundle ξ, the equality $c_k(\bar{\xi}) = (-1)^k c_k(\xi)$ holds for every nonnegative integer k, cf. [31, p. 168].

Theorem 2.15. *Let X be a compact real algebraic variety. For a topological $\mathbb{C}$-vector bundle ξ on X, the following conditions are equivalent:*

(a) *There exists a positive integer r such that the $\mathbb{C}$-vector bundle $\xi(r)$ admits a stratified-algebraic structure.*
(b) *The $\mathbb{C}$-vector bundle ξ has property* (rk) *and there exists a positive integer a such that the cohomology class $ac_{2k}(\xi \oplus \bar{\xi})$ is in $H^{4k}_{\mathbb{C}\text{-str}}(X;\mathbb{Z})$ for every positive integer k satisfying $8k - 2 < \dim X$.*

Proof. Since $(\mathbb{H} \otimes \xi)_{\mathbb{C}} \cong \xi \oplus \bar{\xi}$, the equality $c_l((\mathbb{H} \otimes \xi)_{\mathbb{C}}) = c_l(\xi \oplus \bar{\xi})$ holds for every nonnegative integer l. Furthermore, the $\mathbb{C}$-vector bundle ξ has property (rk) if and only if the $\mathbb{H}$-vector bundle $\mathbb{H} \otimes \xi$ has it. Hence the proof is complete in view of Lemma 2.3 and Theorem 2.14. □

Let ξ be an $\mathbb{R}$-vector bundle. Recall that for any nonnegative integer k, the kth Pontryagin class of ξ is defined by $p_k(\xi) = (-1)^k c_{2k}(\mathbb{C} \otimes \xi)$.

Theorem 2.16. *Let X be a compact real algebraic variety. For a topological $\mathbb{R}$-vector bundle ξ on X, the following conditions are equivalent:*

(a) *There exists a positive integer r such that the $\mathbb{R}$-vector bundle $\xi(r)$ admits a stratified-algebraic structure.*

(b) *The $\mathbb{R}$-vector bundle ξ has property* (rk) *and there exists a positive integer a such that the cohomology class $ap_k(\xi)$ is in $H^{4k}_{\mathbb{C}\text{-str}}(X;\mathbb{Z})$ for every positive integer k satisfying $8k - 2 < \dim X$.*

Proof. Assume that condition (a) is satisfied. Then the $\mathbb{R}$-vector bundle $\xi(r)$ has property (rk) and hence ξ has it as well. Furthermore, the $\mathbb{C}$-vector bundle $(\mathbb{C} \otimes \xi)(r)$ admits a stratified-algebraic structure, being isomorphic to $\mathbb{C} \otimes \xi(r)$. According to Proposition 2.6, for every positive integer j, there exists a positive integer b_j such that the cohomology class $b_j c_j(\mathbb{C} \otimes \xi)$ is in $H^{2j}_{\mathbb{C}\text{-str}}(X;\mathbb{Z})$. In particular, (a) implies (b) in view of the definition of $p_k(\xi)$.

Now assume that condition (b) is satisfied. By Proposition 2.7 and Lemma 2.12, there exists a positive integer b such that the cohomology class $bc_{2k}(\mathbb{C} \otimes \xi)$ is in $H^{4k}_{\mathbb{C}\text{-str}}(X;\mathbb{Z})$ for every positive integer k. Recall that $2c_l(\mathbb{C} \otimes \xi) = 0$ for every odd positive integer l, cf. [31, p. 174]. Hence, according to Proposition 2.6, the $\mathbb{C}$-vector bundle $(\mathbb{C} \otimes \xi)(q)$ admits a stratified-algebraic structure for some positive integer q. In view of Lemma 2.4, the $\mathbb{R}$-vector bundle $\xi(2q)$ admits a stratified-algebraic structure. Thus (b) implies (a). The proof is complete. □

We need one more technical result.

Lemma 2.17. *Let X be a compact real algebraic variety. If the group $\Gamma_{\mathbb{C}}(X)$ is finite, then the quotient group $H^{2j}(X;\mathbb{Z})/H^{2j}_{\mathbb{C}\text{-str}}(X;\mathbb{Z})$ is finite for every positive integer j. If the group $\Gamma_{\mathbb{F}}(X)$ is finite, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{H}$, then the quotient group $H^{4k}(X;\mathbb{Z})/H^{4k}_{\mathbb{C}\text{-str}}(X;\mathbb{Z})$ is finite for every positive integer k.*

Proof. Recall that the cohomology group $H^*(X;\mathbb{Z})$ is finitely generated, the variety X being triangulable.

There exists a positive integer b such that for every positive integer j and every cohomology class u in $H^{2j}(X;\mathbb{Z})$, one can find a topological $\mathbb{C}$-vector bundle ξ on X with

$$c_i(\xi) = 0 \text{ for } 1 \leq i \leq j-1 \text{ and } c_j(\xi) = bu,$$

cf. [2, p. 19] or [18, p. 155, Theorem A]. We can choose such a $\mathbb{C}$-vector bundle ξ of constant rank.

Assume that the group $\Gamma_{\mathbb{C}}(X)$ is finite and $r\Gamma_{\mathbb{C}}(X) = 0$ for some positive integer r. Then the $\mathbb{C}$-vector bundle $\xi(r)$ admits a stratified-algebraic structure, and hence the cohomology class

$$c_j(\xi(r)) = rc_j(\xi) = rbu$$

is in $H^{2j}_{\mathbb{C}\text{-str}}(X;\mathbb{Z})$. Thus the quotient group $H^{2j}(X;\mathbb{Z})/H^{2j}_{\mathbb{C}\text{-str}}(X;\mathbb{Z})$ is finite, as asserted.

Note that the complexification $\mathbb{C} \otimes \xi_{\mathbb{R}}$ of the $\mathbb{R}$-vector bundle $\xi_{\mathbb{R}}$ satisfies

$$\mathbb{C} \otimes \xi_{\mathbb{R}} \cong \xi \oplus \bar{\xi}.$$

Similarly, for the quaternionization $\mathbb{H} \otimes \xi$ of the $\mathbb{C}$-vector bundle ξ, we have

$$(\mathbb{H} \otimes \xi)_{\mathbb{C}} \cong \xi \oplus \bar{\xi}.$$

If the group $\Gamma_{\mathbb{R}}(X)$ is finite and $q\Gamma_{\mathbb{R}}(X) = 0$ for some positive integer q, then the $\mathbb{R}$-vector bundle $\xi_{\mathbb{R}}(q)$ admits a stratified-algebraic structure, and hence so do the $\mathbb{C}$-vector bundles

$$\mathbb{C} \otimes \xi_{\mathbb{R}}(q) \cong (\mathbb{C} \otimes \xi_{\mathbb{R}})(q) \cong (\xi \oplus \bar{\xi})(q).$$

If the group $\Gamma_{\mathbb{H}}(X)$ is finite and $q\Gamma_{\mathbb{H}}(X) = 0$, then the $\mathbb{H}$-vector bundle $(\mathbb{H} \otimes \xi)(q)$ admits a stratified-algebraic structure, and hence so do the $\mathbb{C}$-vector bundles

$$((\mathbb{H} \otimes \xi)(q))_{\mathbb{C}} \cong (\mathbb{H} \otimes \xi)_{\mathbb{C}}(q) \cong (\xi \oplus \bar{\xi})(q).$$

Consequently, if $q\Gamma_{\mathbb{F}}(X) = 0$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{H}$, then the Chern class $c_j((\xi \oplus \bar{\xi})(q))$ is in $H^{2j}_{\mathbb{C}\text{-str}}(X;\mathbb{Z})$. Now suppose that $j = 2k$, where k is a positive integer. Then

$$c_i(\xi \oplus \bar{\xi}) = 0 \text{ for } 1 \leq i \leq 2k-1 \text{ and } c_{2k}(\xi \oplus \bar{\xi}) = 2c_{2k}(\xi) = 2bu,$$

which implies the equality

$$c_{2k}((\xi \oplus \bar{\xi})(q)) = qc_{2k}(\xi \oplus \bar{\xi}) = 2qbu.$$

Thus the cohomology class $2qbu$ is in $H^{4k}_{\mathbb{C}\text{-str}}(X;\mathbb{Z})$. In conclusion, the quotient group

$$H^{4k}(X;\mathbb{Z})/H^{4k}_{\mathbb{C}\text{-str}}(X;\mathbb{Z})$$

is finite. The proof is complete. □

We are now ready to prove the theorems announced in Section 1.

Proof of Theorem 1.3. In view of Lemma 2.17, condition (a) implies (b). By combining Theorems 2.14, 2.15 and 2.16, we conclude that (b) implies (a). □

Proof of Theorem 1.4. It suffices to make use of Theorem 1.3 and Lemma 2.12. □

Proof of Theorem 1.6. Let $n = \dim X$. According to Proposition 1.2, it suffices to prove that for any topological $\mathbb{F}$-vector bundle ξ of constant positive rank on X, the $\mathbb{F}$-vector bundle $\xi(r)$ admits a stratified-algebraic structure, where

$$r = \begin{cases} 2^{a(n,\mathbb{F})} & \text{if } n \leq 7 \\ 2^{a(n,\mathbb{F})+2} & \text{if } n = 8. \end{cases}$$

If $n \leq d(\mathbb{F})$, then $a(n,\mathbb{F}) = 1$ and the $\mathbb{F}$-vector bundle $\xi(1) = \xi$ admits a stratified-algebraic structure, cf. [30, Corollary 3.6].

Henceforth we assume that $n \geq d(\mathbb{F}) + 1$.

The rest of the proof is divided into three steps.

Case 1. Suppose that $\mathbb{F} = \mathbb{H}$.

The 4-sphere $\mathbb{S}^4$ can be identified (as a topological space) with the quaternionic projective line $\mathbb{P}^1(\mathbb{H})$. Let θ be the $\mathbb{H}$-line bundle on $\mathbb{S}^4$ corresponding to the tautological $\mathbb{H}$-line bundle on $\mathbb{P}^1(\mathbb{H})$. Since $5 \leq n \leq 8$, we have $a(n,\mathbb{H}) = 1$.

First suppose that $5 \leq n \leq 7$. Then ξ can be expressed as $\xi = \lambda \oplus \varepsilon$, where λ and ε are topological $\mathbb{H}$-vector bundles, $\operatorname{rank} \lambda = 1$ and ε is trivial, cf. [20, p. 99]. For the same reason, the $\mathbb{H}$-vector bundle $\lambda \oplus \lambda$ has a nowhere vanishing continuous section. Thus the $\mathbb{H}$-line bundle λ is generated by two continuous sections. It follows that we can find a continuous map $h\colon X \to \mathbb{S}^4$ with $\lambda \cong h^*\theta$. According to Theorem 2.13, the $\mathbb{H}$-vector bundle $\lambda \oplus \lambda = \lambda(2)$ admits a stratified-algebraic structure. Since $\xi(2) \cong \lambda(2) \oplus \varepsilon(2)$, the $\mathbb{H}$-vector bundle $\xi(2)$ admits a stratified-algebraic structure, as required.

Now suppose that $n = 8$. It remains to prove that the $\mathbb{H}$-vector bundle $\xi(8)$ admits a stratified-algebraic structure. This can be done as follows. Let $\mathcal{F} = (X_{-1}, X_0, \ldots, X_m)$ be a filtration of X such that the variety $X_i \setminus X_{i-1}$ is nonsingular of pure dimension for $0 \leq i \leq m$. According to Hironaka's theorem on resolution of singularities [19, 22], for each $i = 0, \ldots, m$, there exists a multiblowup $\pi_i\colon X'_i \to X_i$ over X_{i-1} with X'_i nonsingular of pure dimension. Consider the pullback $\mathbb{H}$-vector bundle $\xi_i := \pi_i^*(\xi|_{X_i})$ on X_i'. According to Theorem 2.10, it suffices to prove that the $\mathbb{H}$-vector bundle $\xi_i(8)$ admits a stratified-algebraic structure. If $\dim X_i' \leq 7$, we already established a stronger result, namely, $\xi_i(2)$ admits a stratified-algebraic structure. If $\dim X_i' = 8$, we choose a finite subset A_i of X_i' whose intersection with each connected component of X_i' consists of one point. Let $\sigma_i \colon X_i'' \to X_i'$ be the blowup of X_i' with center A_i. We can replace $\pi_i \colon X_i' \to X_i$ by the composite map $\sigma_i \circ \pi_i \colon X_i'' \to X_i$ and replace the $\mathbb{H}$-vector bundle ξ_i on X_i' by the $\mathbb{H}$-vector bundle $(\sigma_i \circ \pi_i)^*(\xi|_{X_i})$ on X_i''. Note that X_i'' is a compact nonsingular real algebraic variety of pure dimension 8, and each connected component of X_i'' is nonorientable as a $\mathcal{C}^\infty$ manifold. Thus in order to simplify notation we may assume that the variety X is nonsingular of pure dimension 8, and each connected component of X is nonorientable as a $\mathcal{C}^\infty$ manifold. The last condition implies the equality $2H^8(X;\mathbb{Z}) = 0$. Since $c_l(\xi_\mathbb{C}) = 0$ for every odd positive integer l, we get

$$c_4((\xi(4))_\mathbb{C}) = c_4(\xi_\mathbb{C}(4)) = 4c_4(\xi_\mathbb{C}) + 6c_2(\xi_\mathbb{C}) \smile c_2(\xi_\mathbb{C}) = 0$$

in $H^8(X;\mathbb{Z})$. The $\mathbb{H}$-vector bundle $\xi(4)$ can be expressed as the direct sum of a topological $\mathbb{H}$-vector bundle η of rank 2 and a trivial $\mathbb{H}$-vector bundle, cf. [20, p. 99]. Then

$$c_4(\eta_\mathbb{C}) = c_4((\xi(4))_\mathbb{C}) = 0.$$

Recall that $c_4(\eta_\mathbb{C})$ is the Euler class $e(\eta_\mathbb{R})$ of the oriented $\mathbb{R}$-vector bundle $\eta_\mathbb{R} = (\eta_\mathbb{C})_\mathbb{R}$, cf. [31, p. 159]. Interpreting $e(\eta_\mathbb{R})$ as an obstruction, we conclude that the $\mathbb{H}$-vector bundle η has a nowhere vanishing continuous section, cf. [31, pp. 139, 140, 147] and [33]. Consequently, the $\mathbb{H}$-vector bundle $\xi(4)$ can be expressed as $\xi(4) = \mu \oplus \delta$, where μ and δ are topological $\mathbb{H}$-vector bundles, $\operatorname{rank} \mu = 1$ and δ is trivial.

Since $\xi(8) \cong \mu(2) \oplus \delta(2)$, it suffices to prove that the $\mathbb{H}$-vector bundle $\mu(2)$ admits a stratified-algebraic structure. Note that

$$c_4((\mu(2))_\mathbb{C}) = c_4((\xi(8))_\mathbb{C}) = 8c_4(\xi_\mathbb{C}) + 28c_2(\xi_\mathbb{C}) \smile c_2(\xi_\mathbb{C}) = 0$$

in $H^8(X;\mathbb{Z})$. Now, interpreting $c_4(\mu(2)) = e((\mu(2))_\mathbb{R})$ as an obstruction, we get a nowhere vanishing continuous section of $\mu(2)$. In other words, the $\mathbb{H}$-line bundle μ is generated by two continuous sections. It follows that we can find a continuous map $g \colon X \to \mathbb{S}^4$ with $\mu \cong g^*\theta$. According to Theorem 2.13, the $\mathbb{H}$-vector bundle $\mu \oplus \mu = \mu(2)$ admits a stratified-algebraic structure. The proof of Case 1 is complete.

Case 2. Suppose that $\mathbb{F} = \mathbb{C}$.

Since $n \geq 3$, we have $a(n, \mathbb{C}) = a(n, \mathbb{H}) + 1$. Hence it suffices to apply Case 1 and Lemma 2.3 to the $\mathbb{H}$-vector bundle $\mathbb{H} \otimes \xi$.

Case 3. Suppose that $\mathbb{F} = \mathbb{R}$.

Since $n \geq 2$, we have $a(n, \mathbb{R}) = a(n, \mathbb{C}) + 1$. Hence it suffices to apply Case 2 and Lemma 2.4 to the $\mathbb{C}$-vector bundle $\mathbb{C} \otimes \xi$.

The proof is complete. □

3. Line bundles

In this short section we concentrate our attention on $\mathbb{C}$-line bundles. For any real algebraic variety X, let $\mathrm{VB}^1_\mathbb{C}(X)$ denote the group of isomorphism classes of topological $\mathbb{C}$-line bundles on X (with operation induced by tensor product). Let $\mathrm{VB}^1_{\mathbb{C}\text{-str}}(X)$ be the subgroup of $\mathrm{VB}^1_\mathbb{C}(X)$ consisting of the isomorphism classes of all $\mathbb{C}$-line bundles admitting a stratified-algebraic structure. Since X has the homotopy type of a compact polyhedron [7, pp. 217, 225], the group $\mathrm{VB}^1_\mathbb{C}(X)$

is finitely generated, being isomorphic to the cohomology group $H^2(X;\mathbb{Z})$. In particular, the quotient group

$$\Gamma^1_{\mathbb{C}}(X) \coloneqq \mathrm{VB}^1_{\mathbb{C}}(X)/\,\mathrm{VB}^1_{\mathbb{C}\text{-str}}(X)$$

is finitely generated. Thus the group $\Gamma^1_{\mathbb{C}}(X)$ is finite if and only if $r\Gamma^1_{\mathbb{C}}(X) = 0$ for some positive integer r. Furthermore, the latter condition holds if and only if for every topological $\mathbb{C}$-line bundle λ on X its rth tensor power $\lambda^{\otimes r}$ admits a stratified-algebraic structure.

Proposition 3.1. *Let X be a real algebraic variety. For any topological $\mathbb{C}$-line bundle λ on X and positive integer r, if $\lambda(r)$ admits a stratified-algebraic structure, then so does $\lambda^{\otimes r}$.*

Proof. If the $\mathbb{C}$-vector bundle $\lambda(r)$ admits a stratified-algebraic structure, then so does the $\mathbb{C}$-line bundle $\det \lambda(r)$, cf. [30, Proposition 3.15]. Here $\det \lambda(r)$ stands for the rth exterior power of $\lambda(r)$. The proof is complete since the $\mathbb{C}$-line bundles $\det \lambda(r)$ and $\lambda^{\otimes r}$ are isomorphic. □

As a consequence, we obtain the following.

Corollary 3.2. *Let X be a compact real algebraic variety. If r is a positive integer and $r\Gamma_{\mathbb{C}}(X) = 0$, then $r\Gamma^1_{\mathbb{C}}(X) = 0$.*

Proof. It suffices to make use of Propositions 1.2 and 3.1. □

Corollary 3.3. *For any compact real algebraic variety X of dimension at most 8, the group $\Gamma^1_{\mathbb{C}}(X)$ is finite and*

$$2^{a(\dim X,\mathbb{C})+a(X)}\Gamma^1_{\mathbb{C}}(X) = 0,$$

where $a(X) = 0$ if $\dim X \leq 7$ and $a(X) = 2$ if $\dim X = 8$.

Proof. This follows from Theorem 1.6 and Corollary 3.2. □

A different proof of Corollary 3.3 for varieties of dimension at most 5 is given in [28]. It is plausible that $2\Gamma^1_{\mathbb{C}}(X) = 0$ for every compact real algebraic variety X, cf. [28, Conjecture B, Proposition 1.5]. This is confirmed by Corollary 3.3 for $\dim X \leq 4$. Without restrictions on the dimension of X we have the following.

Theorem 3.4. *Let X be a compact real algebraic variety with $H^2_{\mathrm{sph}}(X;\mathbb{Z}) = H^2(X;\mathbb{Z})$. Then the group $\Gamma^1_{\mathbb{C}}(X)$ is finite and $2\Gamma^1_{\mathbb{C}}(X) = 0$.*

Proof. According to Lemma 2.12, $2H^2(X;\mathbb{Z}) \subseteq H^2_{\mathbb{C}\text{-str}}(X;\mathbb{Z})$. Hence for any topological $\mathbb{C}$-line bundle λ on X, the Chern class $c_1(\lambda^{\otimes 2}) = 2c_1(\lambda)$ is in $H^2_{\mathbb{C}\text{-str}}(X;\mathbb{Z})$.

In view of [30, Proposition 8.6], the $\mathbb{C}$-line bundle $\lambda^{\otimes 2}$ admits a stratified-algebraic structure. Thus $2\Gamma^1_{\mathbb{C}}(X) = 0$, as asserted. □

The following special case is of interest.

Corollary 3.5. *Let X be a compact real algebraic variety. If each connected component of X is homotopically equivalent to $\mathbb{S}^{d_1} \times \cdots \times \mathbb{S}^{d_n}$ for some positive integers $d_1, \ldots, d_n$, then the group $\Gamma^1_{\mathbb{C}}(X)$ is finite and $2\Gamma^1_{\mathbb{C}}(X) = 0$.*

Proof. Since $H^2_{\mathrm{sph}}(X;\mathbb{Z}) = H^2(X;\mathbb{Z})$, it suffices to apply Theorem 3.4. □

According to [30, Example 7.10], there exists a nonsingular real algebraic variety X diffeomorphic to the n-fold product $\mathbb{S}^1 \times \cdots \times \mathbb{S}^1$, $n \geq 3$, with $\Gamma^1_{\mathbb{C}}(X) \neq 0$.

References

[1] Ralph Abraham and Joel Robbin. *Transversal mappings and flows*. Benjamin, 1967.

[2] Michael Francis Atiyah and Friedrich Hirzebruch. Vector bundles and homogenous spaces. *Proc. Symp. in Pure Math., Amer. Math. Soc.*, 3:7–38, 1961.

[3] Riccardo Benedetti and Alberto Tognoli. On real algebraic vector bundles. *Bull. Sci. Math.*, 104:89–112, 1980.

[4] Riccardo Benedetti and Alberto Tognoli. *Remarks and counterexamples in the theory of real algebraic vector bundles and cycles*, pages 198–211. Number 959 in Lecture Notes in Math. Springer, 1982. DOI: 10.1007/bfb0062255.

[5] Marcin Bilski, Wojciech Kucharz, Anna Valette, and Guillaume Valette. Vector bundles and regulous maps. *Math Z.*, 275:403–418, 2013. DOI: 10.1007/s00209-012-1141-6.

[6] Jacek Bochnak, Michael Buchner, and Wojciech Kucharz. Vector bundles over real algebraic varieties. *K-Theory*, 3:271–298., 1989. DOI: 10.1007/bf00533373.

[7] Jacek Bochnak, Michel Coste, and Marie-Françoise Roy. *Real Algebraic Geometry*, volume 36 of *Ergeb. der Math. und ihrer Grenzgeb. Folge 3*. Springer, 1998.

[8] Jacek Bochnak and Wojciech Kucharz. On real algebraic morphisms into even-dimensional spheres. *Ann. of Math.*, (3) 128:415–433, 1988. DOI: 10.2307/1971431.

[9] Jacek Bochnak and Wojciech Kucharz. K-theory of real algebraic surfaces and threefolds. *Math. Proc. Cambridge Philos. Soc.*, 106:471–480, 1989. DOI: 10.1017/s0305004100068213.

[10] Jacek Bochnak and Wojciech Kucharz. Vector bundles on a product of real cubic curves. *K-Theory*, 6:487–497, 1992. DOI: 10.1007/bf00997378.

[11] Jacek Bochnak and Wojciech Kucharz. Elliptic curves and real algebraic morphisms. *J. Algebraic Geometry*, 2:635–666, 1993.

[12] Jacek Bochnak and Wojciech Kucharz. Complete intersections in differential topology and analytic geometry. *Boll. Un. Mat. Ital. B*, 10:1019–1041, 1996.

[13] Jacek Bochnak, Wojciech Kucharz, and Robert Silhol. Morphisms, line bundles and moduli spaces in real algebraic geometry. *Inst. Hautes Études Sci. Publ. Math.*, 86:5–65, 1997. DOI: 10.1007/bf02698900.

[14] Armand Borel and André Haefliger. La classe d'homologie fondamentale d'un espace analytique. *Bull. Soc. Math. France*, 89:461–513, 1961. DOI: 10.24033/bsmf.1571.

[15] Eldon Dyer. *Cohomology Theories*. Benjamin, 1969.

[16] Goulwen Fichou, Johannes Huisman, Frédéric Mangolte, and Jean-Philippe Monnier. Fonctions régulues. *J. Reine Angew. Math.*, 718:103–151, 2016. DOI: 10.1515/crelle-2014-0034.

[17] William Fulton. *Intersection Theory*, volume 2 of *Ergeb. der Math. und ihrer Grenzgeb. Folge 3*. Springer, 1984.

[18] Phillip Griffiths and John Adams. *Topics in Algebraic and Analytic Geometry*. Number 13 in Mathematical Notes. Princeton University Press, 1974.

[19] Heisuke Hironaka. Resolution of singularities of an algebraic variety over a field of characteristic zero. *Ann. of Math.*, 79:109–326, 1964. DOI: 10.2307/1970486.

[20] Dale Husemoller. *Fibre Bundles*. Springer, 1975.

[21] Max Karoubi. *K-Thoery*. Number 226 in Grundlehren der Mathematischen Wissenschaften. Springer, 1978.

[22] János Kollár. *Lectures on Resolution of Singularities*, volume 166 of *Ann. of Math. Studies*. Princeton University Press, 2007.

[23] János Kollár and Krzysztof Nowak. Continuous rational functions on real and p-adic varieties. *Math. Z.*, 279:85–97, 2015.

[24] Wojciech Kucharz. Rational maps in real algebraic geometry. *Adv. Geom.*, 9:517–539, 2009.

[25] Wojciech Kucharz. Complex cycles on algebraic models of smooth manifolds. *Math. Ann.*, 346:829–856, 2010. DOI: 10.1007/s00208-009-0413-x.

[26] Wojciech Kucharz. Approximation by continuous rational maps into spheres. *J. Eur. Math. Soc. (JEMS)*, 16:1555–1569, 2014. DOI: 10.4171/jems/469.

[27] Wojciech Kucharz. Continuous rational maps into the unit 2-sphere. *Arch. Math.*, 102:257–261, 2014. DOI: 10.1007/s00013-014-0627-3.

[28] Wojciech Kucharz. Some conjectures on stratified-algebraic vector bundles. *J. Singul.*, 12:92–104, 2015. DOI: 105427/jsing.2015.12.

[29] Wojciech Kucharz. Continuous rational maps into spheres. *Math. Z.*, 283:1201–1215, 2016. DOI: 10.1007/s00209-016-1639-4.

[30] Wojciech Kucharz and Krzysztof Kurdyka. Stratified-algebraic vector bundles. *J. Reine Angew. Math.*, 745:105–154, 2018. DOI: 10.1515/crelle-2015-0105.

[31] John Willard Milnor and James D. Stasheff. *Characteristic Classes*. Number 76 in Ann. of Math. Studies. Princeton University Press, 1974.

[32] Jean-Pierre Serre. Groupes d'homologie et classes des groupes abeliens. *Ann. of Math.*, 58:258–294, 1953. DOI: 10.2307/1969789.

[33] Norman Steenrod. *The Topology of Fibre Bundles*. Princeton University Press, 1951.

[34] Richard G. Swan. Topological examples of projective modules. *Trans. Amer. Math. Soc.*, 230:201–234, 1977. DOI: 10.1090/s0002-9947-1977-0448350-9.

Wojciech Kucharz, Institute of Mathematics, Faculty of Mathematics and Computer Science, Jagiellonian University, ul. Łojasiewicza 6, 30-348 Kraków, Poland

Email address: `Wojciech.Kucharz@im.uj.edu.pl`

Krzysztof Kurdyka, Laboratoire de Mathématiques, UMR 5127 du CNRS, Université Savoie Mont Blanc, Campus Scientifique, 73 376 Le Bourget-du-Lac Cedex
France

Email address: `kurdyka@univ-savoie.fr`

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THE EMBEDDED NASH PROBLEM OF BIRATIONAL MODELS OF RATIONAL TRIPLE SINGULARITIES

B. KARADENIZ, H. MOURTADA, C. PLÉNAT, AND M. TOSUN

1. Introduction

Given a variety X defined over an algebraically closed field of characteristic 0, we are often not able to exhibit an explicit resolution of its singularities; on the other hand there are infinitely many resolutions of singularities of X giving extra information which is not intrinsic to the singularity. The need for understanding the information which is common to all the resolutions of singularities of a given space X led Nash (in [22]) to study the arc space of X. See also [6, 24] for more details. This paper follows this line of thoughts. The difference here is that we are interested in the embedded resolutions of singularities of $X \subset \mathbb{A}^n$.

For this purpose, we replace the arc space X_∞ of X with the jet schemes of X: the arc space X_∞ of X is the space of germs of formal curves drawn on X. The jet schemes are a family of finite dimensional schemes indexed by integers which approximate the infinite dimensional arc space; for $m \in \mathbf{N}$, the m-th jet scheme X_m of X, can be thought of as the space of arcs in the ambient space $\mathbb{A}^n$ whose "contact" with X is greater or equal to $m+1$; this gives the intuition why these schemes should detect information about embedded resolutions of singularities. The main question considered in this paper is: can we construct an embedded resolution of singularities from the jet schemes of $X \subset \mathbb{A}^n$? More precisely, we ask the following much less optimistic question:

($\star$) *Can one construct an embedded resolution of singularities of $X \subset \mathbb{A}^n$ from the irreducible components of the space X_m^{Sing} of jets centered at the singular locus of $X \subset \mathbb{A}^n$?*

This question is studied in [18, 17, 15, 20]. In [20], the authors proved that the irreducible components of the jet schemes centered at the singular locus of a rational double point surface singularity (known also as "simple singularities" in the literature) give a minimal embedded resolution by a birational toric modification of the ambient space. Equivalently, a certain natural family of the irreducible components of the jet schemes of X centered at the singular point 0 X_m^0 is in bijection with the divisorial valuations whose center is a toric divisor on every toric embedded resolution; this bijection is actually a conceptual correspondence since one can associate with any irreducible component of X_m^0 a divisorial valuation centered at the origin of $\mathbb{A}^n$ (see [5]).

In general, such a statement is hopeless: indeed, even for an irreducible plane curve singularity (say, for the cusp $\{y^2-x^3=0\} \subset \mathbb{A}^2$), the irreducible components of the jet schemes centered at the origin give divisorial valuations which do not appear in the minimal embedded resolution of the curve singularity (in that case, the minimal embedded resolution makes sense and is unique).

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The answer to ($\star$) is no in general. Indeed, consider the three-dimensional variety defined by

$$X = \{x^2 + y^2 + z^2 + w^5 = 0\} \subset \mathbb{A}^4.$$

It has an isolated singularity at the origin 0. On the one hand, by a direct computation, we see that the jet schemes X_m^0 centered at 0 are irreducible for every $m \geq 1$. On the other hand, we have two exceptional (irreducible) divisors that appear on every embedded resolution of the singularity (at least those which correspond to the two essential divisors appearing in the abstract resolution of the origin 0) of X; these are the divisors associated with the monomial valuations on $k[x, y, z, w]$ defined by the vectors $(1, 1, 1, 1)$ and $(2, 2, 2, 1)$. The valuation associated with the vector $(2, 2, 2, 1)$ does not correspond to any of the schemes X_m^0 with $m \geq 1$. Note that this example is one of the counterexamples to the Nash problem given in [12]; note also that the Nash correspondence is bijective in dimension 2 [8, 9] but there are many counter-examples in higher dimension ([11, 7]). This suggests that a reasonable frame to study the question ($\star$) is the surface singularities.

In this paper we study the question ($\star$) for a family of hypersurface singularities whose normalizations are rational triple point singularities (RTP-singularities, for short). These hypersurfaces are classified in [1] and are called the non-isolated forms of RTP-singularities. We prove that, for this class of singularities, the answer to ($\star$) is positive. When X is of that type, we determine again a natural family of irreducible components of X_m^{Sing}, $m \geq 1$ whose associated divisorial valuations are monomial, hence defined by some vectors in $\mathbb{N}^3$. For all of the non-isolated forms of RTP-singularities except when X is of type $B_{k-1,2l-1}$, we show that these vectors give a regular subdivision Σ of the dual Newton fan of X and hence a nonsingular toric variety Z_Σ; since our singularities are Newton non-degenerate [27, 2, 1], this gives a birational toric morphism $Z_\Sigma \longrightarrow \mathbb{A}^3$ which is an embedded resolution of $X \subset \mathbb{A}^3$; the irreducible components of the exceptional divisor correspond to the natural set of irreducible components of X_m^{Sing}.

When X is of type $B_{k-1,2l-1}$, we again build a toric embedded resolution from the irreducible components of the jet schemes which does not factor through the toric map associated with the dual Newton fan (such resolutions of non-degenerate singularities also appear when one considers an embedded resolution in family [14]). This again shows mysteriously that the jet schemes tell us something about the "minimality" of the embedded resolution, as in the case of rational double point singularities.

The paper is organized as follows: Section 2 present a reminder on RTP-singularities. Section 3 is devoted to jet schemes and how one can associate a divisorial valuation with a component of the jet schemes; it also contains a summary of the approach to the embedded resolutions which will be constructed in the sequel. Each of the remaining sections is devoted to a class of RTP-singularities (given in the table of contents above): we compute each of the jet schemes and present the results in the jet graph (see Section 3). We then give the toric embedded resolution which comes from the jet schemes. We give the explicit computations with all details for the classes $E_{6,0}$ and $A_{k-1,l-1,m-1}$. For the other classes, except a subclass of the type B, we proceed similarly, so we present here only the results of the computations. The case $B_{k-1,2l-1}$ with $k \geq l$ is treated in detail as its behavior is completely different from the other cases. This is related to the fact that the abstract toric resolution of $B_{k-1,2l-1}$ which is obtained from a subdivision of the two dimensional cones of the dual Newton fan is not minimal [1].

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2. RTP-singularities

Let X denote a germ of a surface $(X,0) \subset (\mathbb{C}^N,0)$ having a singularity at 0. We say that the singularity of X is rational if $H^1(\tilde{X},\mathcal{O}_{\tilde{X}}) = 0$ where $\pi : \tilde{X} \longrightarrow X$ is a resolution of X. This definition does not depend on the resolution π. It is well known that the rational singularities of complex surfaces have nice combinatorial properties which can be computed via their resolutions. In [3], the rational singularities of multiplicity 3 are classified by their dual graphs associated with the irreducible components of the minimal resolutions. For short, we call RTP-singularities this class of rational singularities. They are among the surface singularities defined in $\mathbb{C}^4$ and, each of which is defined by three equations given in [26]. The classification problem of rational singularities of multiplicity $m \geq 3$ is well studied in [13] and [25].

In [1], the authors obtain the equations of a class of hypersurfaces in $\mathbb{C}^3$ having nonisolated singularities obtained by projecting the equations of RTP-singularities to a generic hyperplane in $\mathbb{C}^4$ and, they call them the non-isolated forms of RTP-singularities since the normalizations of these hypersurfaces in $\mathbb{C}^3$ are exactly the RTP-singularities. They also show that:

Theorem 2.1. *The RTP-singularities are non-degenerate with respect to their Newton polyhedron. In particular, they can be resolved by a toric birational map $Z \longrightarrow \mathbb{C}^4$.*

In [1], the dual graph of the minimal resolution for all RTP-singularities, except those of type $B_{k-1,2l-1}$ for $k \geq l$ (see Section 6) are constructed by refining the dual Newton fan of the corresponding non-isolated forms of RTP-singularity (see also [23, 27]). In the case of the nonisolated form of a rational singularity of type $B_{k-1,2l-1}$ with $k \geq l$, the resolution obtained by the subdivision of the corresponding dual Newton fan is not minimal: consider the vectors $R := (2l-2,2,2k+1)$, $Q := (2k-l+2,1,2k-l+2)$, $P := (l-1,1,l-1)$, $V := (2k-l,1,2k-l+1)$ and $U := (l-1,1,l)$ coming out in the subdivision of the dual Newton fan of that singularity:

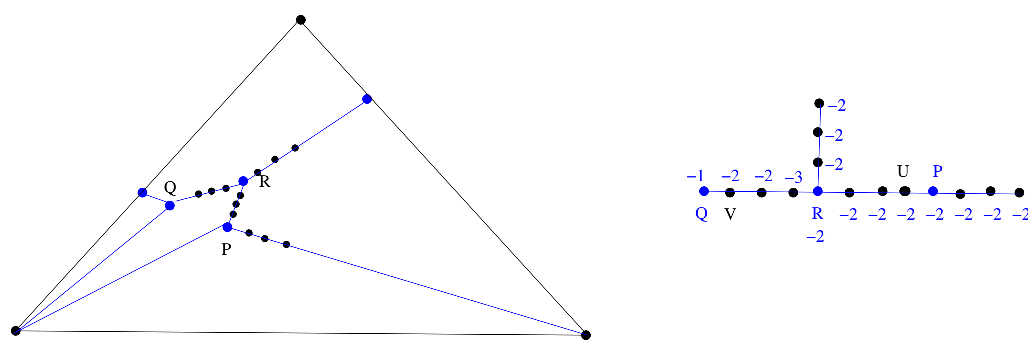


Figure 1. Dual Newton fan of a $B_{k-1,2l-1}$ singularity (with $k \geq l$), and its dual (abstract) resolution graph

Using [23], one can compute the self-intersections of the irreducible components of the exceptional divisors corresponding to these vectors; they are given by the number decorating the dual graph given on the right-hand side. We omit the genus decorations which are all 0 in this case. The exceptional component corresponding to the vector Q has self-intersection (-1); by Castelnuovo's criterion, (cf. for example [10], chapter V), that component can be contracted to a nonsingular point without creating singularities. If we continue to contract each (-1)-curve and neighboring components accordingly we obtain a (-3)-curve on the segment $[QR]$ and the dual graph of the minimal resolution of the RTP-singularity of type $B_{k-1,2l-1}$, $k \geq l$.

3. Jet schemes

Let k be an algebraically closed field of arbitrary characteristic and X be a k-algebraic variety. For $m \in \mathbb{N}$, the jet scheme X_m is the scheme representing the functor

$$F_m\colon \begin{array}{c} k\text{-}Schemes \to Sets \\ Spec(A) \mapsto Hom_k\left(Spec\left(A[t]/(t^{m+1})\right), X\right) \end{array}$$

where A is a k-algebra. The closed points of X_m are in bijection with the $k[t]/(t^{m+1})$ points of X. By definition, we have $X_0 = X$. Moreover, for $m, p \in \mathbb{N}$ with $m > p$, we have a canonical projection $\pi_{m,p} : X_m \longrightarrow X_p$ which is induced by the surjection $A[t]/(t^{m+1}) \longrightarrow A[t]/(t^{p+1})$. These morphisms are affine and verify $\pi_{m,p} \circ \pi_{q,m} = \pi_{q,p}$ for $p < m < q$; they define a projective system whose limit is a scheme that we denote X_∞ and which is called the arc space of X. Let us denote the canonical projection $\pi_{m,0} : X_m \longrightarrow X_0$ by π_m and, the canonical morphisms $X_\infty \longrightarrow X_m$ by Ψ_m.

We show here for a surface $X = \{f = 0\} \subset k^3$ (since the varieties that we are considering are defined this way) that the functor of the jet schemes is representable; this explains also how one determines jet schemes. We have

$$X = \mathrm{Spec}\frac{k[x, y, z]}{(f)}.$$

For a $k-$algebra A, an element γ in $F_m(\mathrm{Spec}(A))$ corresponds to a $k-$algebra homomorphism

$$\gamma^* : \frac{k[x, y, z]}{(f)} \longrightarrow \frac{A[t]}{(t^{m+1})}.$$

The data of such a γ is equivalent to the data of

$$\begin{aligned} \gamma^*(x) &= x(t) = x_0 + x_1 t + \cdots + x_m t^m \in A[t]/(t^{m+1}), \\ \gamma^*(y) &= y(t) = y_0 + y_1 t + \cdots + y_m t^m \in A[t]/(t^{m+1}), \\ \gamma^*(z) &= z(t) = z_0 + z_1 t + \cdots + z_m t^m \in A[t]/(t^{m+1}); \end{aligned}$$

such that

$$f(x(t), y(t), z(t)) = F_0 + F_1 t + \cdots + F_m t^m + \cdots = 0 \text{ mod } (t^{m+1}).$$

Here, for $i \geq 0, F_i$ is simply the coefficient of t^i in the expanding of $f(x(t), y(t), z(t))$.

Hence, the data of such a γ is equivalent to the data of $x_j, y_j, z_j \in A$ with $j = 0, \ldots, m$ such that $F_i(x_0, y_0, z_0, \ldots, x_i, y_i, z_i) = 0$ with $i = 0, \ldots, m$. This is equivalent to determining an A-point of the scheme

$$X_m := \mathrm{Spec}\frac{K[x_i, y_i, z_i; i = 0, \ldots, m]}{(F_0, \ldots, F_m)},$$

which then represents the functor F_m and, is by definition the m-th jet scheme of X.

From now on, we assume that X is a surface in $\mathbb{C}^3$ defined by $\{f(x, y, z) = 0\}$ and Y is a subvariety of X. Let $m \in \mathbb{N}$ We denote by $X_m^Y := \pi_m^{-1}(Y)$. We consider a special type of the irreducible components of $X_m^Y, m \in \mathbb{N}$ where Y is the singular locus of X or $Y \subset X$ is a curve contained in a coordinate hyperplane of $\mathbb{C}^3$. To such Y, we associate a divisorial valuation over $\mathbb{C}^3$ with an irreducible component $\mathcal{C}_m \subset X_m^Y$ in the following way.

Let $\psi_m^a : \mathbb{C}^3_\infty \longrightarrow \mathbb{C}^3_m$ be the truncation morphism associated with the ambient space $\mathbb{C}^3$, here the exponent "a" stands for *ambient map* . The morphism ψ_m^a is a trivial fibration, hence ${\psi_m^a}^{-1}(\mathcal{C}_m)$ is an irreducible cylinder in $\mathbb{C}^3_\infty$. Let η be the generic point of ${\psi_m^a}^{-1}(\mathcal{C}_m)$. By

Corollary 2.6 in [5], the map $\nu_{\mathcal{C}_m}:\mathbb{C}[x,y,z]\longrightarrow\mathbb{N}$ defined by $\nu_{\mathcal{C}_m}(h)=\mathrm{ord}_t h\circ\eta$ is a divisorial valuation on $\mathbb{C}^3$.

To each irreducible component $\mathcal{C}_m$ of X_m^Y, let us associate a vector, called the weight vector, in the following way:

$$v(\mathcal{C}_m):=(\nu_{\mathcal{C}_m}(x),\nu_{\mathcal{C}_m}(y),\nu_{\mathcal{C}_m}(z))\in\mathbb{N}^3.$$

Now, we want to characterize the irreducible components of X_m^Y that will allow us to construct an embedded resolution of X: For $p\in\mathbb{N}$, we consider the following cylinder in the arc space:

$$Cont^p(f)=\{\gamma\in\mathbb{C}^3_\infty;\mathrm{ord}_t f\circ\gamma=p\}.$$

Definition 3.1. Let $X:\{f=0\}$ be a surface in $\mathbb{C}^3$ and let Y be a subvariety of X.
(i) The elements of the set:

$$EC(X):=\{\text{Irreducible components } \mathcal{C}_m \text{ of } X_m^Y \text{ such that } {\psi_m^a}^{-1}(\mathcal{C}_m)\cap Cont^{m+1}f\neq\emptyset$$
$$\text{and } v(\mathcal{C}_m)\neq v(\mathcal{C}_{m-1}) \text{ for any component } \mathcal{C}_{m-1} \text{ verifying } \pi_{m,m-1}(\mathcal{C}_m)\subset\mathcal{C}_{m-1}, m\geqslant 1\}$$

are called the *essential components* for X.
(ii) the elements of the set of associated valuations:

$$EV(X):=\{\nu_{\mathcal{C}_m},\ \mathcal{C}_m\in EC(X)\},$$

are called *embedded-valuations* for X.

This means that the elements of $EV(X)$ appear in the embedded toric resolution of X. We will be interested in a subset of $EV(X)$, which gives us an embedded resolution. In the following sections, in order to determine such a subset when X is a non-isolated form of an RTP-singularity, we will study the m-th jet schemes of X, for $m\leq l$ with l large enough. We will encode the structure of these jet schemes by a levelled graph whose vertices correspond to the irreducible components of X_m^Y for an integer m; two vertices at the level m and $m-1$ are joined by an edge if the transition morphism $\pi_{m,m-1}$ sends the corresponding components one into the other [16]. An element of $EV(X)$ corresponding to a component $\mathcal{C}_m\in EC(X)$ is actually a monomial (or toric) valuation (see proposition 2.3 in [20]) and is defined by the vector $v(\mathcal{C}_m)=(a,b,c)$: this means that, for $h=\sum_{\{(i,j,k)\}}a_{(i,j,k)}x^iy^jz^k\in\mathbb{C}[x,y,z]$ we have:

$$\nu_{\mathcal{C}_m}(h)=\min_{\{(i,j,k)|a_{(i,j,k)}\neq 0\}}\{ai+bj+ck\}.$$

By subdividing the first quadrant of $\mathbb{R}^3$ using the vectors $v(\mathcal{C}_m)$ for some $\mathcal{C}_m\in EC(X)$, we obtain a fan Σ whose support is the first quadrant of $\mathbb{R}^3$ and whose one dimensional cones are generated by these $v(\mathcal{C}_m)$'s. Note that one can obtain different fans from a set of vectors in $\mathbb{R}^3$, depending on the way one relies the vertices and, some of them may not be regular, but here we are interested in finding a regular fan. Hence we have a proper birational map $\mu_\Sigma:Z_\Sigma\longrightarrow\mathbb{C}^3$ where Z_Σ is smooth and the irreducible components of the exceptional divisor of μ_Σ correspond to the vectors $v(\mathcal{C}_m)$ that we consider. More precisely, the divisorial valuations corresponding to the irreducible components of the exceptional divisor of μ_Σ are exactly the $\nu_{\mathcal{C}_m}$ associated with the components $\mathcal{C}_m$ that we consider.

We will find such a regular fan Σ for a non-isolated form X of an RTP-singularity which is not of the type $B_{k-1,2l-1}$ (*i.e* we will construct Σ using the vectors of type $v(\mathcal{C}_m)$) that refines the dual Newton fan of $X\subset\mathbb{C}^3$. Thanks to Varchenko's theorem [27], this gives that μ_Σ is an embedded resolution of $X\subset\mathbb{C}^3$. On the other hand, for a $B_{k-1,2l-1}$-singularity, we cannot apply Varchenko's theorem because there is no Σ refining the dual Newton fan as described above; nevertheless we build a regular fan Σ satisfying the properties above and, we prove by studying the total transform of our singularity by μ_Σ that $\mu_\Sigma:Z_\Sigma\longrightarrow\mathbb{C}^3$ is an embedded resolution of the $B_{k-1,2l-1}$ singularity.

4. RTP-singularities of type $E_{6,0}$

The singularity of $X \subset \mathbb{C}^3$ defined by the equation:

$$f(x,y,z) = z^3 + y^3 z + x^2 y^2 = 0$$

is called $E_{6,0}$-type singularity. Its dual Newton fan is given in Figure 2.

In this section, we compute explicitly the m-th jet schemes (for $m \leq 18$) and we determine a subset of $EV(X)$ which gives a regular subdivision of the dual Newton fan as explained in the previous section. We represent the irreducible components as a graph in Figure 3, where we also weight the vertex associated with a component $\mathcal{C}_m$ by the vector $v(\mathcal{C}_m)$ also defined in the previous section. For a component $\mathcal{C}_m$ which projects by the maps $\pi_{m,m-1}$ given in Section 3 on a monomial component (*i.e.* a component whose associated valuation is monomial) $\mathcal{C}_{m-1}$, which is not itself monomial; we also weight the associated vertex by the unique non-monomial equation which, together with the hyperplane coordinates $\mathcal{C}_{m-1}$, defines $\mathcal{C}_m$. That helps for the computations of the irreducible components in the process. Here we do not pay much attention to the edges since they are not relevant for the problem at hand. The arrows in Figure 3 correspond to a component $\mathcal{C}_m$ such that the inverse image of a dense open set in it gives an irreducible component for every $n \geq m$. First let us fix some notations:

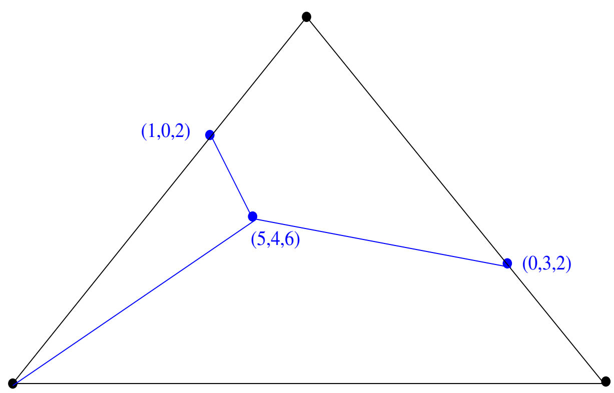


Figure 2. Dual Newton fan of $E_{6,0}$ singularity

Notation: Let

$$(*) \quad f\Big(\sum_{i=0}^{m} x_i t^i, \sum_{i=0}^{m} y_i t^i, \sum_{i=0}^{m} z_i t^i\Big) = \sum_{i=0}^{i=m} F_i t^i \qquad \mathrm{mod}(t^{m+1}).$$

We know that (e.g. [20]) the m-th jet scheme X_m is defined by the ideal

$$I_m = (F_0, F_1, \ldots, F_m) \subset \mathbb{C}[x_i, y_i, z_i; i = 0, \ldots, m].$$

4.1. **Jet Schemes of $E_{6,0}$.** For $m \geq 1$, we will determine the irreducible components of the space of $m-$jets that projects on the singular locus of X, i.e. the irreducible components of $X_m^{Sing} := \pi_{m,0}^{-1}(V(y_0, z_0)) \subset X_m \subset \mathrm{Spec}(\mathbb{C}[x_i, y_i, z_i; i = 0, \ldots, m]) = \mathbb{C}_m^3$; here $V(I)$ denotes the variety defined by an ideal I and $\mathbb{C}_m^3$ is the m-th jet scheme of the affine three dimensional space $\mathbb{C}^3$; we insist here that when considering X_m^{Sing} for a given m, the symbol $V(I)$ designates the variety defined by an ideal I in $\mathbb{C}_m^3$. Recall that $\pi_{m_0} : X_m \longrightarrow X_0 = X$. We also insist on the fact that we consider only the reduced structure of these schemes.

For $m = 1$, we have $X_1^{Sing} = V(y_0, z_0) \subset \mathrm{Spec}(\mathbb{C}[x_i, y_i, z_i; i = 0, 1])$ because, if we put $y_0 = z_0 = 0$ in the equation $(*)$ we get $F_0 = F_1 = 0$ modulo the ideal (y_0, z_0). Hence, X_1^{Sing} consists of a unique irreducible component, denoted by $\mathcal{C}_{1,1}$. The weight vector of $\mathcal{C}_{1,1}$ is $(0, 1, 1)$.

For $m = 2$, we have $X_2^{Sing} = \pi_{2,1}^{-1}(\mathcal{C}_{1,1})$; this uses the fact $\pi_{2,0} = \pi_{1,0} \circ \pi_{2,1}$. A direct computation using the equation $(*)$ gives:

$$F_2 = x_0^2 y_1^2 \bmod (y_0, z_0).$$

Hence $X_2^{Sing} = V(y_0, z_0, x_0^2 y_1^2) \subset \mathrm{Spec}(\mathbb{C}[x_i, y_i, z_i; i = 0, 1, 2]) = \mathbb{C}_2^3$. We deduce that X_2^{Sing} has two irreducible components $\mathcal{C}_{2,1} := V(y_0, z_0, x_0)$ and $\mathcal{C}_{2,2} := V(y_0, z_0, y_1)$ both are sent via $\pi_{2,1}$ into $\mathcal{C}_{1,1}$; there weight vectors are respectively $(1, 1, 1)$ and $(0, 2, 1)$. These vectors are represented in Figure 3 at the levels $m = 1$ and $m = 2$.

For $m = 3$, using the fact $\pi_{3,0} = \pi_{2,0} \circ \pi_{3,2}$, it is sufficient to study $\pi_{3,2}^{-1}(\mathcal{C}_{2,j})$ with $j = 1, 2$ to understand X_3^{Sing}.

- To find $\pi_{3,2}^{-1}(\mathcal{C}_{2,1})$, we compute F_3 modulo the ideal (x_0, y_0, z_0) and we obtain:

  $$F_3 = z_1^3 \bmod (x_0, y_0, z_0);$$

  Hence we obtain that $\mathcal{C}_{3,1} := \pi_{3,2}^{-1}(\mathcal{C}_{2,1}) = V(x_0, y_0, z_0, z_1)$ is irreducible.
- Similarly, we obtain that $\mathcal{C}_{3,2} := \pi_{3,2}^{-1}(\mathcal{C}_{2,2}) = V(y_0, y_1, z_0, z_1)$ is irreducible.

So we have $X_3^{Sing} = \mathcal{C}_{3,1} \cup \mathcal{C}_{3,2}$ where $\mathcal{C}_{3,1}$ and $\mathcal{C}_{3,2}$ are both irreducible and clearly there is no inclusions between them: indeed, $\mathcal{C}_{3,1}$ is included in $V(x_0)$ but $\mathcal{C}_{3,2}$ is not and, $\mathcal{C}_{3,2}$ is included in $V(y_1)$ but $\mathcal{C}_{3,1}$ is not. We conclude that $\mathcal{C}_{3,1}$ and $\mathcal{C}_{3,2}$ are the irreducible components of X_3^{Sing}. Their associated weight vectors are respectively $(1, 1, 2)$ and $(0, 2, 2)$.

For $m = 4$, as in the previous case, it is sufficient to consider $\pi_{4,3}^{-1}(\mathcal{C}_{3,j})$, with $j = 1, 2$. As the computations go almost in the same way, we just announce what we obtain:

- To determine $\pi_{4,3}^{-1}(\mathcal{C}_{3,1})$ we compute F_4 modulo the ideal (x_0, y_0, z_0, z_1). We have

  $$F_4 = x_1^2 y_1^2 \bmod (x_0, y_0, z_0, z_1).$$

  Hence, $\pi_{4,3}^{-1}(\mathcal{C}_{3,1})$ has 2 irreducible components

  $$\mathcal{C}_{4,1} = V(x_0, y_0, y_1, z_0, z_1) \quad \text{and} \quad \mathcal{C}_{4,2} = V(x_0, y_0, x_1, z_0, z_1).$$
- Similarly we have $\pi_{4,3}^{-1}(\mathcal{C}_{3,2}) = \mathcal{C}_{4,1} \cup \mathcal{C}_{4,3}$ where $\mathcal{C}_{4,3} = V(y_0, y_1, y_2, z_1, z_0)$.

Then we get

$$X_4^{Sing} = \mathcal{C}_{4,1} \cup \mathcal{C}_{4,2} \cup \mathcal{C}_{4,3}$$

which is a decomposition into irreducible varieties. Using a similar argument as in the case of $m = 3$, we conclude that there are no mutual inclusions between these components; hence this is the decomposition into irreducible components. The corresponding weight vectors of $\mathcal{C}_{4,1}, \mathcal{C}_{4,2}$ and $\mathcal{C}_{4,3}$ are respectively $(1, 2, 2), (2, 1, 2)$ and $(0, 3, 2)$. Figure 3 encodes also this information.

For $m = 5$, we have

$$X_5^{Sing} = \pi_{5,4}^{-1}(\mathcal{C}_{4,1}) \cup \pi_{5,4}^{-1}(\mathcal{C}_{4,2}) \cup \pi_{5,4}^{-1}(\mathcal{C}_{4,3}) \subset \mathbb{C}_5^3.$$

- To determine $\pi_{5,4}^{-1}(\mathcal{C}_{4,1})$, we compute F_5 modulo $(x_0, y_0, y_1, z_0, z_1)$ that we find to be 0. We deduce, that $C_{5,1} := \pi_{5,4}^{-1}(\mathcal{C}_{4,1}) = V(x_0, y_0, y_1, z_0, z_1)$ is irreducible. A small attention here is needed: The varieties $\mathcal{C}_{4,1}$ and $\mathcal{C}_{5,1}$ are not the same; they are defined by the same equations but in different rings; they actually define the same valuation on $\mathbb{C}^3$ (see Proposition 2.3 in [20]).
- Computing F_5 modulo the ideal $(x_0, x_1, y_0, z_0, z_1)$, we find $F_5 = y_1^3 z_2 = 0$. So $\pi_{5,4}^{-1}(\mathcal{C}_{4,2})$ is the union of $V(x_0, x_1, y_0, y_1, z_0, z_1)$ and $\mathcal{C}_{5,2} := V(x_0, x_1, y_0, z_1, z_0, z_2)$.
- As for $\pi_{5,4}^{-1}(\mathcal{C}_{4,1})$, computing F_5 modulo $(y_0, y_1, y_2, z_0, z_1)$ we find zero. This gives that $\mathcal{C}_{5,3} := \pi_{5,4}^{-1}(\mathcal{C}_{4,3}) = V(y_0, y_1, z_0, z_1, z_2)$is irreducible.

Hence we obtain
$$X_5^{Sing} = \mathcal{C}_{5,1} \cup \mathcal{C}_{5,2} \cup V(x_0, x_1, y_0, y_1, z_0, z_1) \cup \mathcal{C}_{5,3}.$$
Since $V(x_0, x_1, y_0, y_1, z_0, z_1)$ is included in $\mathcal{C}_{5,1}$, the decomposition
$$X_5^{Sing} = \mathcal{C}_{5,1} \cup \mathcal{C}_{5,2} \cup \mathcal{C}_{5,3}$$
is the decomposition into the irreducible components. Moreover, the weight vectors of $\mathcal{C}_{5,j}$ for $j = 1, 2, 3$ are $(1, 2, 2), (2, 1, 3)$ and $(0, 3, 2)$ respectively.

For $m = 6$, we have
$$X_6^{Sing} = \pi_{6,5}^{-1}(\mathcal{C}_{5,1}) \cup \pi_{6,5}^{-1}(\mathcal{C}_{5,2}) \cup \pi_{6,5}^{-1}(\mathcal{C}_{5,3}) \subset \mathbb{C}_6^3.$$

- To determine $\pi_{6,5}^{-1}(\mathcal{C}_{5,1})$, we compute F_6 modulo the ideal $(x_0, y_0, y_1, z_0, z_1)$ and we find
$$\mathcal{C}_{6,1} := \pi_{6,5}^{-1}(\mathcal{C}_{5,1}) = V(x_0, y_0, y_1, z_0, z_1, z_2^3 + x_1^2 y_2^2) \subset \mathbb{C}_6^3.$$
Notice that $\mathcal{C}_{6,1}$ is isomorphic to the product of an affine space and the hypersurface defined by $\{z_2^3 + x_1^2 y_2^2 = 0\}$; this hypersurface is a Hirzebruch-Jung singularity which is well known to be an irreducible quasi-ordinary singularity [4]; in particular $\mathcal{C}_{6,1}$ is irreducible. Actually, we will see that $\mathcal{C}_{6,1}$ will give rise to an irreducible component of X_6^{Sing} whose weight vector is same as the weight vector associated with $\mathcal{C}_{5,1}$, so it is not an essential component (see definition above): the divisorial valuation associated with it is not monomial while a divisorial valuation associated with an essential component is monomial. Before we continue to study on X_6^{Sing}, let us consider $\pi_{m,6}^{-1}(\mathcal{C}_{6,1})$ for $m \geq 7$:
For this, we will stratify $\mathcal{C}_{6,1}$ into its regular locus and its singular locus which are defined respectively by $x_1 = z_2 = 0$ and $y_2 = z_2 = 0$. The inverse images
$$\pi_{7,6}^{-1}(\mathcal{C}_{6,1} \cap \{x_1 = z_2 = 0\}) \quad \text{and} \quad \pi_{7,6}^{-1}(\mathcal{C}_{6,1} \cap \{y_2 = z_2 = 0\})$$
will give the irreducible components of X_7^{Sing} looking like the irreducible components that we have studied before which are the essential components, so give the new weight vectors. The inverse image of the regular part of $\mathcal{C}_{6,1}$ with respect to $\pi_{m,6}$, with $m \geq 7$ is equal to $\pi_{m,6}^{-1}(\mathcal{C}_{6,1} \cap \{z_2 \neq 0\})$; this latter is defined in $\mathbb{C}_m^3 \cap \{z_2 \neq 0\}$ by the ideal generated by $x_0, y_0, y_1, z_0, z_1, z_2^3 + x_1^2 y_2^2$ and
$$F_j = c_j z_3 z_{j-3} + H_j(x_1, \ldots, x_{j-5}, y_2, \ldots, y_{j-4}, z_3, \ldots, z_{j-3}), c_j \in \mathbb{C}^*$$
for $7 \leq j \leq m$. The functions F_j are linear as we can invert $c_j z_3 \neq 0$. Then the Zariski closure $\overline{\pi_{m,6}^{-1}(\mathcal{C}_{6,1} \cap \{z_2 \neq 0\})}$ is irreducible and, is actually an irreducible component of X_m^{Sing} for every $m \leq 7$. Note that the weight vector of $\overline{\pi_{m,6}^{-1}(\mathcal{C}_{6,1} \cap \{z_2 \neq 0\})}$ is $(1, 2, 2)$ which is same as the one for $\mathcal{C}_{6,1}$ and for $\mathcal{C}_{5,1}$; hence they don't give an essential component.

They are encoded in Figure 3 by the dashed arrow which starts at the vertex weighted by the vector $(1,2,2)$ and the equation $z_2^3+x_1^2y_2^2=0$.

- To determine $\pi_{6,5}^{-1}(\mathcal{C}_{5,2})$, we compute F_6 modulo the ideal $(x_0,x_1,y_0,z_0,z_1,z_2)$ and we find that $F_6=y_1^2(z_2y_1+x_2^2)$. So $\pi_{6,5}^{-1}(\mathcal{C}_{5,2})$ is the union of $\mathcal{C}_{6,2}:=V(x_0,x_1,y_0,y_1,z_0,z_1,z_2)$ and $\mathcal{C}_{6,3}:=V(x_0,x_1,y_0,z_1,z_0,z_2,z_3y_1+x_2^2)$ which are both irreducible. We note that $\pi_{m,6}^{-1}(\mathcal{C}_{6,3})$ is irreducible for every $m\geq 7$ and gives rise to an irreducible component of X_m^{Sing} for every $m\geq 7$. The irreducibility of the inverse image results from the fact that $\mathcal{C}_{6,3}$ is the product of an affine space and an A_1-singularity and the jet schemes of such singularity are irreducible [21, 19] (what applies here for A_1 is also true for any rational singularity). The components of $\pi_{m,6}^{-1}(\mathcal{C}_{6,3})$ are not the essential components, they are associated with non-monomial valuations and they have the same weight vector, namely $(2,1,3)$. They are encoded in Figure 3 (to the most right of the graph) by the dashed arrow which starts at the vertex weighted by the vector $(2,1,3)$ and the equation $x_2^2+z_3y_1=0$.
- To determine $\pi_{6,5}^{-1}(\mathcal{C}_{5,3})$, we compute F_6 modulo the ideal (y_0,y_1,y_2,z_0,z_1) and we find that $F_6=z_2^3+x_0^2y_3^3$. Hence
$$\mathcal{C}_{6,4}:=\pi_{6,5}^{-1}(\mathcal{C}_{5,3})=V(y_0,y_1,y_2,z_0,z_1,z_2^3+x_0^2y_3^2)$$
is irreducible. By the same argument as in the case of $\pi_{m,6}^{-1}(\mathcal{C}_{6,1})$, the inverse images $\pi_{7,6}^{-1}(\mathcal{C}_{6,4}\cap\{x_0=z_2=0\})$ and $\pi_{7,6}^{-1}(\mathcal{C}_{6,4}\cap\{y_3=z_2=0\})$ will give rise to the irreducible components of X_7^{Sing}; the Zariski closure $\overline{\pi_{m,6}^{-1}(\mathcal{C}_{6,4}\cap\{z_2\neq 0\})}$ is irreducible and is actually an irreducible component of X_m^{Sing} for every $m\geq 7$. This is encoded in Figure 3 by the dashed arrow starting at the vertex weighted by the vector $(0,3,2)$ and the equation $z_2^3+x_0^2y_3^2$.

To summarize, we obtain $X_6^{Sing}=\mathcal{C}_{6,1}\cup\mathcal{C}_{6,2}\cup\mathcal{C}_{6,3}\cup\mathcal{C}_{6,4}$ where each $\mathcal{C}_{6,j}$ for $j=1\ldots,4$ is irreducible. Obviously, $\mathcal{C}_{6,2}\subset\mathcal{C}_{6,1}$ and, using the same argument as in the case of $m=3$, we verify that there is no inclusion among the remaining $\mathcal{C}_{6,j}$'s. Hence we get the irreducible decomposition
$$X_6^{Sing}=\mathcal{C}_{6,1}\cup\mathcal{C}_{6,3}\cup\mathcal{C}_{6,4}$$
with the respective weight vectors $(1,2,2),(2,1,3)$ and $(0,3,2)$.

For $m=7$, by the above discussions, we have a stratification
$$X_7^{Sing}=\pi_{7,6}^{-1}(\mathcal{C}_{6,1}\cap\{x_1=z_2=0\})\cup\pi_{7,6}^{-1}(\mathcal{C}_{6,1}\cap\{y_2=z_2=0\})\cup\overline{\pi_{7,6}^{-1}(\mathcal{C}_{6,1}\cap\{z_2\neq 0\})}\cup$$
$$\pi_{7,6}^{-1}(\mathcal{C}_{6,3})\cup\pi_{7,6}^{-1}(\mathcal{C}_{6,4}\cap\{y_3=z_2=0\})\cup\overline{\pi_{m,6}^{-1}(\mathcal{C}_{6,4}\cap\{z_2\neq 0\})}$$
which is the decomposition into irreducible components; indeed, on the one hand using the same argument as for $m=3$, there is no inclusions between $\pi_{7,6}^{-1}(\mathcal{C}_{6,3})$ and the other components; on the other hand, the other components are clearly not equal, this means that there are no inclusions between them because they are irreducible and they have the same dimension (actually codimension 7 in $\mathbb{C}_7^3$).

Note that the codimension is easy to compute since the equations are either hyperplane coordinates in $\mathbb{C}_7^3$ or we consider the closure of a constructible set which is defined by hyperplane coordinates and by linear equations. The weight vectors are respectively $(2,2,3)$, $(1,3,3)$, $(1,2,2)$, $(2,1,3)$, $(0,4,3)$, and $(0,3,3)$. Moreover we have
$$\pi_{7,6}^{-1}(\mathcal{C}_{6,4}\cap\{x_0=z_2=0\})=\pi_{7,6}^{-1}(\mathcal{C}_{6,1}\cap\{y_2=z_2=0\}).$$

We should also note that although $\mathcal{C}_{6,2}$ is not an irreducible component, its inverse image $\pi_{7,6}^{-1}(\mathcal{C}_{6,2})$ which is equal to $\pi_{7,6}^{-1}(\mathcal{C}_{6,1} \cap \{y_2 = z_2 = 0\})$ gives an irreducible component.

We have gone through the arguments which allow to determine all the irreducible components of X_m^{Sing} for $m \leq 18$. This is encoded in Figure 3. Note that 18 is the quasi-degree of the weighted homogeneous polynomial defining our singularity.
One last important thing is that the axis $Y = \{x = z = 0\}$ is drawn on our singularity. We determine the essential components of $X_m^Y, m \geq 0$, we find $V(x_0, z_0) \subset X_0 \subset \mathbb{C}_0^3$ and $V(x_0, z_0, z_1) \subset X_1 \subset \mathbb{C}_1^3$ whose weight vectors are respectively $(1,0,1), (1,0,2)$.

To conclude, the essential components are the irreducible components of X_m^Z (where Z is the singular locus of X or $Z = Y$ is the y-axis) whose defining equations are hyperplane coordinates and, their associated valuations are monomial and determined with their weight vectors. Hence we get the graph in Figure 3 for the jet schemes.

Proposition 4.1. *For an $E_{6,0}$-singularity, the monomial valuations associated with the vectors $(0,1,1), (0,2,1), (1,1,1), (0,3,2), (1,1,2), (1,2,2), (2,1,2), (2,1,3), (2,2,3), (3,2,3), (3,2,4)$, $(3,3,4), (4,3,5), (5,4,6)$ belong to $EV(X)$.*

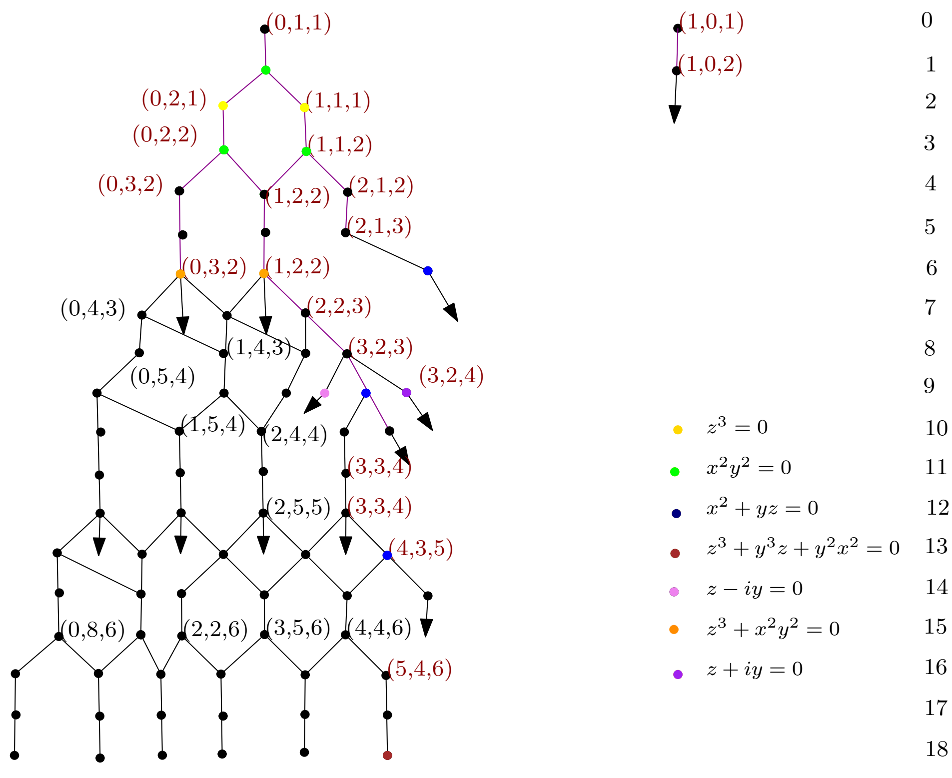


FIGURE 3. Jets schemes of $E_{6,0}$

4.2. **Toric Embedded Resolution of $E_{6,0}$.** Now we are ready to announce the main result for the surface X of type $E_{6,0}$-singularity.

Theorem 4.2. *There exists a toric birational map $\mu_\Sigma : Z_\Sigma \longrightarrow \mathbb{C}^3$ which is an embedded resolution of $X \subset \mathbb{C}^3$ such that the components of the exceptional divisor of μ_Σ correspond to the irreducible components of the m-th jet schemes of X (centered at the singular locus and the intersection of X with the coordinate hyperplane). Moreover this yields a construction (not canonical) of μ_Σ.*

Proof. By [27, 23, 1] (see also [20] for a summary), an embedded resolution of $X \subset \mathbb{C}^3$ can be obtained by constructing a regular subdivision of the dual Newton fan of $X \subset \mathbb{C}^3$. The dual Newton fan Σ for $E_{6,0}$ is presented in Figure 2 .

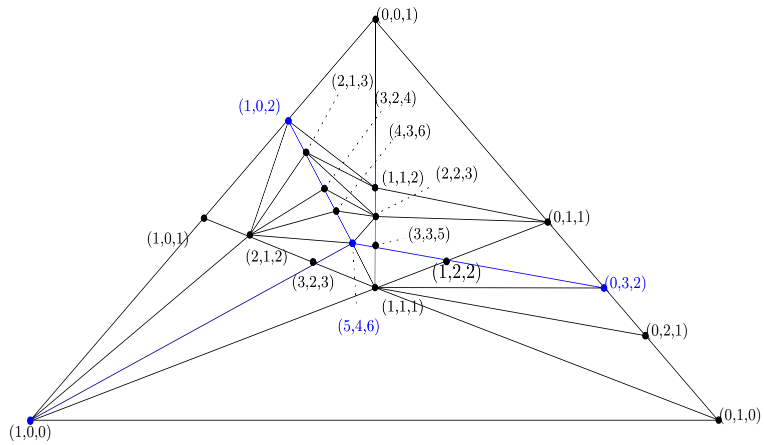


FIGURE 4. An embedded resolution of $E_{6,0}$

In Figure 4, we give a regular subdivision Σ where the rays (cones of dimension 1) are the lines supported by the vectors given in proposition 4.1. To see that this is a regular subdivision, it is sufficient to show that each cone is regular (means that the determinant of the matrix whose columns are any three vectors generating a cone of Σ equals 1). Moreover the 1-dimensional cones (rays) are in bijective correspondence with the components of the exceptional divisors. $\square$

5. RTP-SINGULARITIES OF TYPE $A_{k-1,l-1,m-1}$

The singularity of $X \subset \mathbb{C}^3$ defined by the equations:

- $k \geq \ell \geq m$,
$$z^3 + xz^2 - (x + y^k + y^\ell + y^m)y^k z + y^{2k+\ell} = 0,$$
- $k = \ell < m$,
$$z^3 + (x - y^k)z^2 - (x + y^k + y^m)y^k z + y^{2k+m} = 0.$$

is called $A_{k-1,l-1,m-1}$-type singularity where $k, \ell, m \geq 1$.

5.1. **Jet Schemes and toric Embedded resolution of $A_{k-1,l-1,m-1}$ when $k = l \leq m$.** The singular locus is $\{y = z = 0\}$. So we compute the jets schemes over $\{y = z = 0\}$. The graph representing the irreducible components of the jet schemes of $A_{k-1,l-1,m-1}$ is in Figure 5

Theorem 5.1. *With the preceding notation, the monomial valuations associated with the vectors*

- $(0, 1, 1), (0, 1, 2), \dots (0, 1, k + m)$

- $(s, 1, s), \ldots, (s, 1, m+k-s)$ $1 \leq s \leq k-1$
- $(k, 1, k), \ldots, (k, 1, m)$

belong to $EV(X)$. Moreover, these vectors give a toric birational map $\mu_\Sigma : Z_\Sigma \longrightarrow \mathbb{C}^3$ which is an embedded resolution of $X \subset \mathbb{C}^3$ (in the neighborhood of the origin) such that the components of the exceptional divisor of μ_Σ correspond to the monomial valuations defined by them; hence they correspond to the irreducible components of the m-th jet schemes of X (centered at the singular locus and the intersection of X with the coordinate hyperplanes).

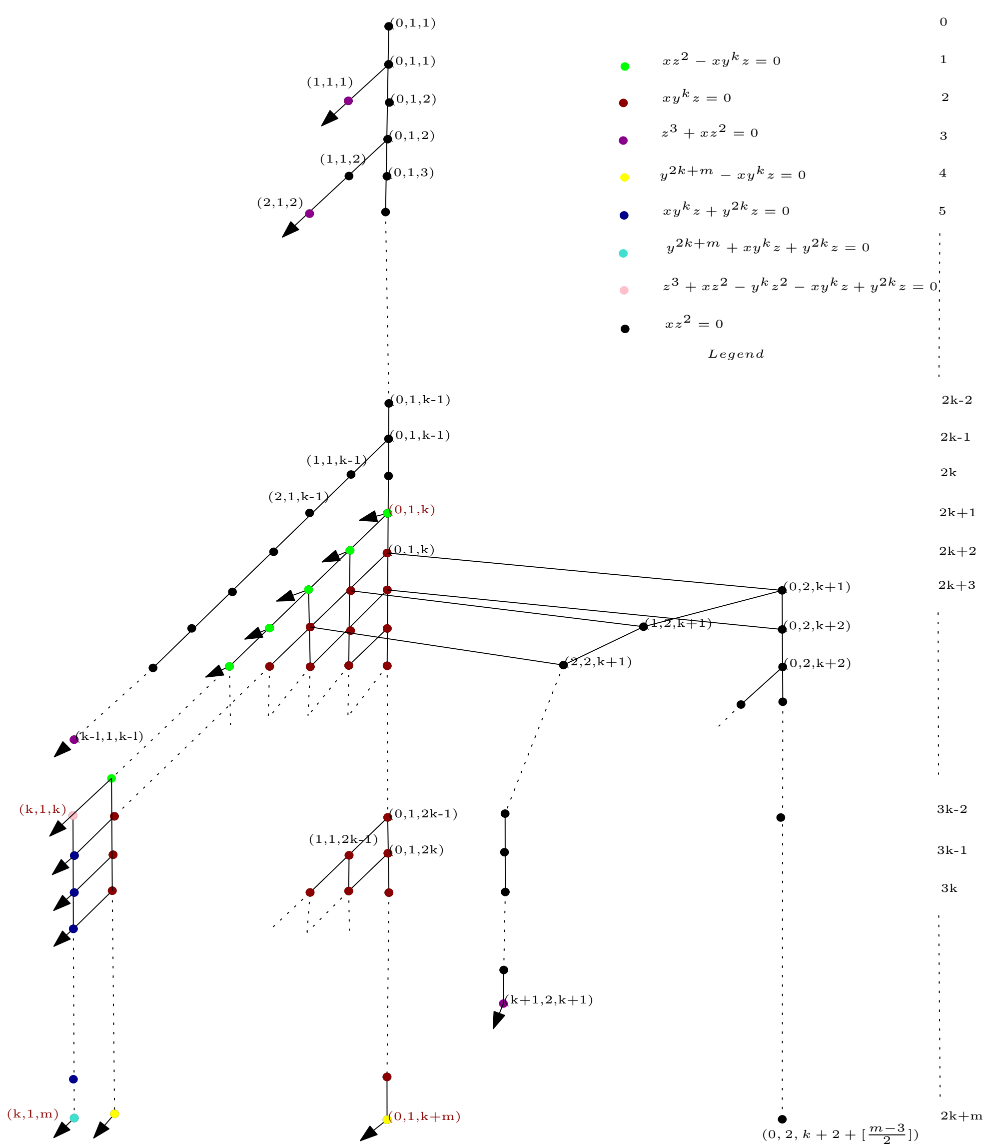


FIGURE 5. Jets schemes of $A_{k-1,k-1,m-1}$

Proof. The first part of the theorem results from the jet graph. Before showing that the given vectors give a simplicial regular decomposition of the dual Newton fan of $A_{k-1,k-1,m-1}$, let us study their positions in the fan:

- for all $0 \le s \le k$, we have $(s,1,k) \in [(0,1,k),(k,1,k)]$
- for all $0 \le s \le k$, we have $(s,1,k+m-s) \in [(0,1,k+m),(k,1,m)]$:

$$\begin{vmatrix} k & 0 & s \\ 1 & 1 & 1 \\ m & k+m & k+m-s \end{vmatrix} = \begin{vmatrix} k & 0 & s \\ 0 & 1 & 1 \\ -k & k+m & -s \end{vmatrix} = 0$$

- the vectors $(\alpha,1,l+\alpha+1)$ for all $0 \le \alpha \le k$ are aligned, for each $0 \le l \le k$.

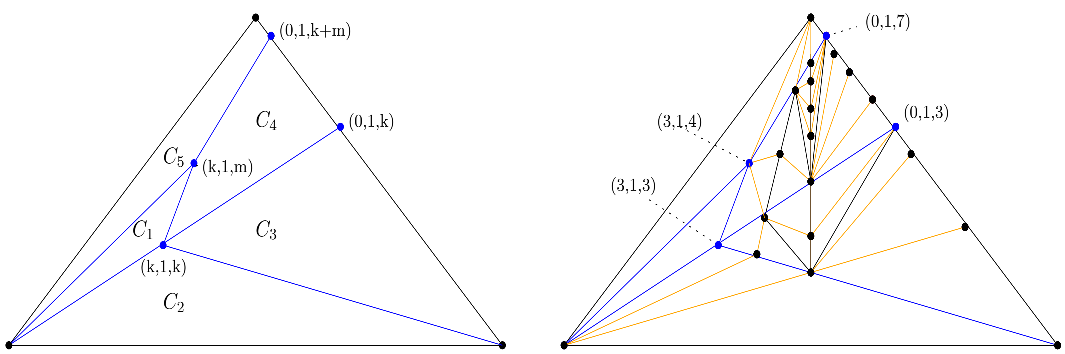


FIGURE 6. Dual Newton fan of $A_{k-1,k-1,m-1}$ and an embedded resolution for $A_{2,2,3}$

Now let us decompose each subcone C_i into regular cones:

Decomposition of C_1: The cone C_1 contains the vectors $(k,1,\beta)$ for $k \le \beta \le m-1$. They are on the skeleton of the fan. For $k \le \beta \le m-1$, we have: $\begin{vmatrix} k & k & 1 \\ 1 & 1 & 0 \\ \beta & \beta+1 & 0 \end{vmatrix} = 1.$

Decomposition of C_2: The cone C_2 contains the vectors $(1,1,1),\ldots,(k,1,k)$ which are on the skeleton. For $0 \le \alpha \le k-1$ we have: $\begin{vmatrix} 1 & \alpha & \alpha+1 \\ 0 & 1 & 1 \\ 0 & \alpha & \alpha+1 \end{vmatrix} = 1.$

Decomposition of C_3: To decompose the cone C_3, we first add successively an edge between the vectors $(k-1,1,k),(k-2,1,k-2),(k-3,1,k),\ldots$ with the last vector being $(0,1,k)$ if k is odd and with $(0,1,0)$ if k is even. Then we obtain that the vectors $(\alpha,1,\alpha),\ldots,(\alpha,1,k)$ are in the same triangles (see Figure 7). Now let us add those vectors and the vectors on the associated edges successively.

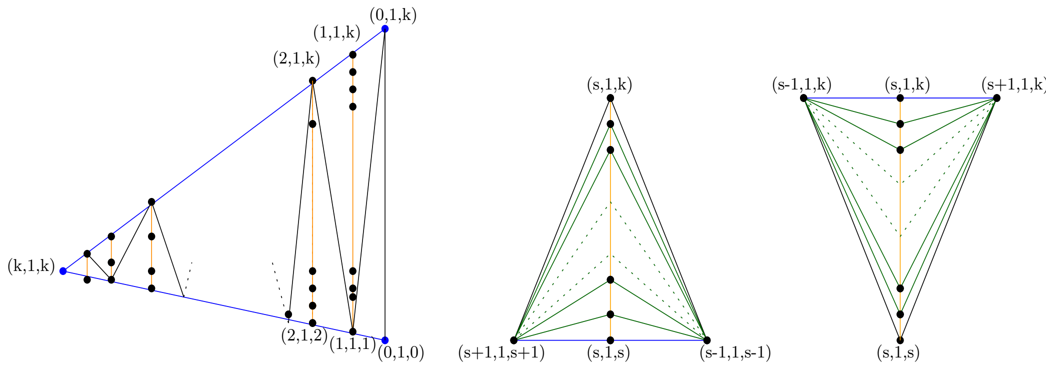


FIGURE 7. Decomposition of the cone C_3 and of its two types of subcones

Each new subcone will be regular as we only have one of the following two cases:

- **Case 1**: for $\alpha \leq \beta \leq k-1$ we have

$$\begin{vmatrix} \alpha-1 & \alpha & \alpha \\ 1 & 1 & 1 \\ k & \beta & \beta+1 \end{vmatrix} = 1 \text{ and } \begin{vmatrix} \alpha+1 & \alpha & \alpha \\ 1 & 1 & 1 \\ k & \beta & \beta+1 \end{vmatrix} = 1$$

- **Case 2**:

$$\begin{vmatrix} \alpha+1 & \alpha & \alpha \\ 1 & 1 & 1 \\ \alpha+1 & \beta & \beta+1 \end{vmatrix} = 1 \text{ and } \begin{vmatrix} \alpha-1 & \alpha & \alpha \\ 1 & 1 & 1 \\ \alpha-1 & \beta & \beta+1 \end{vmatrix} = -1$$

Decomposition of C_4: The cone C_4 is decomposed by adding successively the edges between the vectors $(k,1,m),(k-1,1,k),(k-2,1,m+2),\dots$ with the last vector being $(1,1,k)$ if k is odd and with $(1,1,k+m-1)$ if k is even. Then let us add successively the vectors and the associated edges $(s,1,\alpha)$ for $k \leq \alpha \leq k+m-s$.

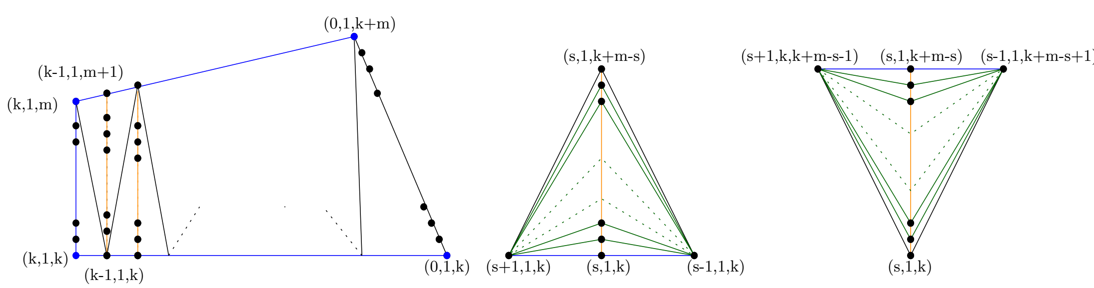


FIGURE 8. Decomposition of the cone C_4 and of its two types of subcones

Each new subcone will be regular as we have: for $0 \leq s \leq k-1$ and for $k \leq \beta \leq k+m-s$,

$$\begin{vmatrix} s-1 & s & s \\ 1 & 1 & 1 \\ k & \beta & \beta+1 \end{vmatrix} = 1 \ \ and \ \ \begin{vmatrix} s+1 & s & s \\ 1 & 1 & 1 \\ k & \beta & \beta+1 \end{vmatrix} = 1$$

or

$$\begin{vmatrix} s+1 & s & s \\ 1 & 1 & 1 \\ k+m-s-1 & \beta & \beta+1 \end{vmatrix} = 1 \ \ and \ \ \begin{vmatrix} s-1 & s & s \\ 1 & 1 & 1 \\ k+m-s+1 & \beta & \beta+1 \end{vmatrix} = 1$$

Decomposition of C_5: The cone C_5 contains the vectors $(s, 1, k+m-s)$ for $0 \leq s \leq k$ which are on the skeleton. For $0 \leq \alpha \leq k$, we have:

$$\begin{vmatrix} 0 & \alpha & \alpha+1 \\ 0 & 1 & 1 \\ 1 & k+m-\alpha & k+m-1-\alpha \end{vmatrix} = 1, \begin{vmatrix} k & 1 & 0 \\ 1 & 0 & 0 \\ m & 0 & 1 \end{vmatrix} = 1.$$

$\square$

5.2. **Jet Schemes and toric Embedded resolution of $A_{k-1,l-1,m-1}$ when $k \geq l \geq m$.** The graph representing the irreducible components of the jet schemes of $A_{k-1,l-1,m-1}$ projecting on the singular locus $\{y = z = 0\}$ is given by Figure 9 below.

Theorem 5.2. *Let $X \subset \mathbb{C}^3$ be a surface of type $A_{k-1,l-1,m-1}$ with $k \geq l \geq m$. The monomial valuations associated with the vectors:*

- $(0, 1, 1), (0, 1, 2), \ldots (0, 1, k+l)$
- $(s, 1, s), \ldots, (s, 1, l+k-s)$ $1 \leq s \leq m-1$
- $(m, 1, m), \ldots, (m, 1, k+l-m)$
- $(m+r, 1, m+r), \ldots, (m+r, 1, k-r)$ *with* $1 \leq r \leq E(\frac{k-m}{2})$

belong to $EV(X)$. Moreover, these vectors give a toric birational map $\mu_\Sigma : Z_\Sigma \longrightarrow \mathbb{C}^3$ which is an embedded resolution of $X \subset \mathbb{C}^3$ (in the neighborhood of the origin) such that the components of the exceptional divisor of μ_Σ correspond to the monomial valuations defined by the vectors; hence they correspond to irreducible components of the m-th jet schemes of X (centered at the singular locus and the intersection of X with the coordinate hyperplanes).

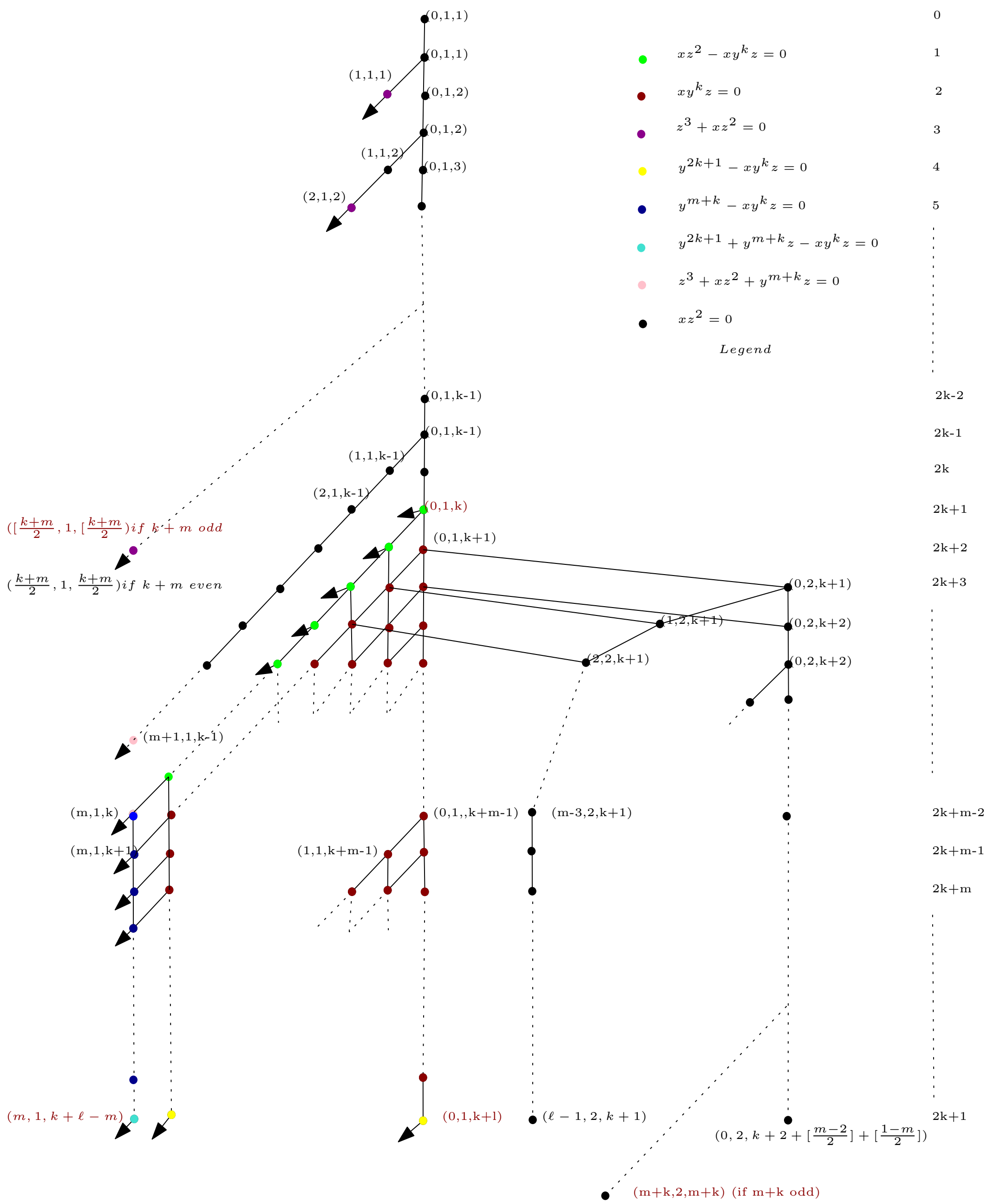


FIGURE 9. Jets schemes of $A_{k-1,l-1,m-1}$

Proof. As above, we first study the positions of the vectors given in theorem 5.2:

- $(m+r,1,k-r) \in [(m+k,2,m+k),(m,1,k)]$:

$$\begin{vmatrix} m+k & m+k & m \\ 1 & 2 & 1 \\ k+r & m+k & k \end{vmatrix} = \begin{vmatrix} r & k & m \\ 0 & 1 & 1 \\ -r & m & k \end{vmatrix} = \begin{vmatrix} r & k & m \\ 0 & 1 & 1 \\ 0 & m+k & k+m \end{vmatrix} = 0$$

- $(\alpha,1,k+l-\alpha) \in [(m,1,k+l-m),(0,1,k+l)]$ for $0 \leq \alpha \leq m$:

$$\begin{vmatrix} m & 0 & \alpha \\ 1 & 1 & 1 \\ k+l-m & k+l & k+l-\alpha \end{vmatrix} = \begin{vmatrix} m & 0 & \alpha \\ 0 & 1 & 0 \\ -m & k+l & -\alpha \end{vmatrix} = 0$$

If $\frac{m+k}{2} \in \mathbb{Z}$, then the dual fan can be decomposed in the same way as for the case $A_{k-1,k-1,m-1}$. Otherwise, we have to show the subcones containing the vector $(m+k,2,m+k)$ are regular. In this case $E(\frac{k-m}{2}) = \frac{k-m-1}{2}$ and $(m+E(\frac{k-m}{2}),1,m+E(\frac{k-m}{2})) = (\frac{k+m-1}{2},1,\frac{k+m-1}{2})$. We have :

$$\begin{vmatrix} \frac{k+m-1}{2} & \frac{k+m-1}{2} & m+k \\ 1 & 1 & 2 \\ \frac{k+m+1}{2} & \frac{k+m-1}{2} & m+k \end{vmatrix} = \begin{vmatrix} 0 & \frac{k+m-1}{2} & m+k \\ 0 & 1 & 2 \\ 1 & \frac{k+m-1}{2} & m+k \end{vmatrix} = 1 \ \textit{and} \ \begin{vmatrix} 1 & \frac{k+m-1}{2} & m+k \\ 0 & 1 & 2 \\ 0 & \frac{k+m-1}{2} & m+k \end{vmatrix} = 1.$$

□

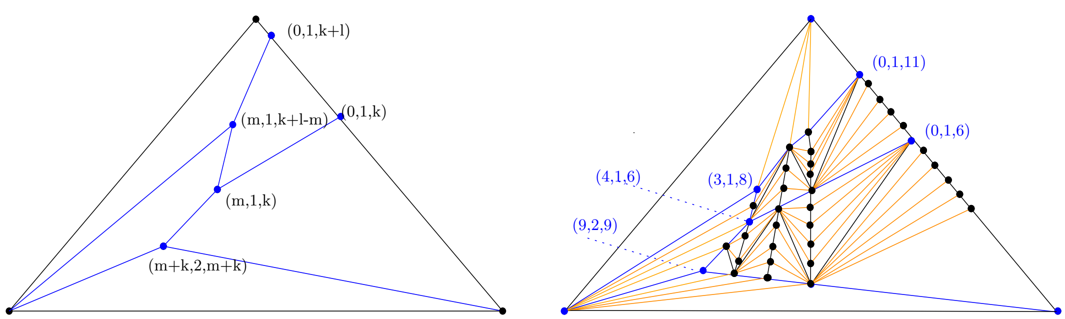


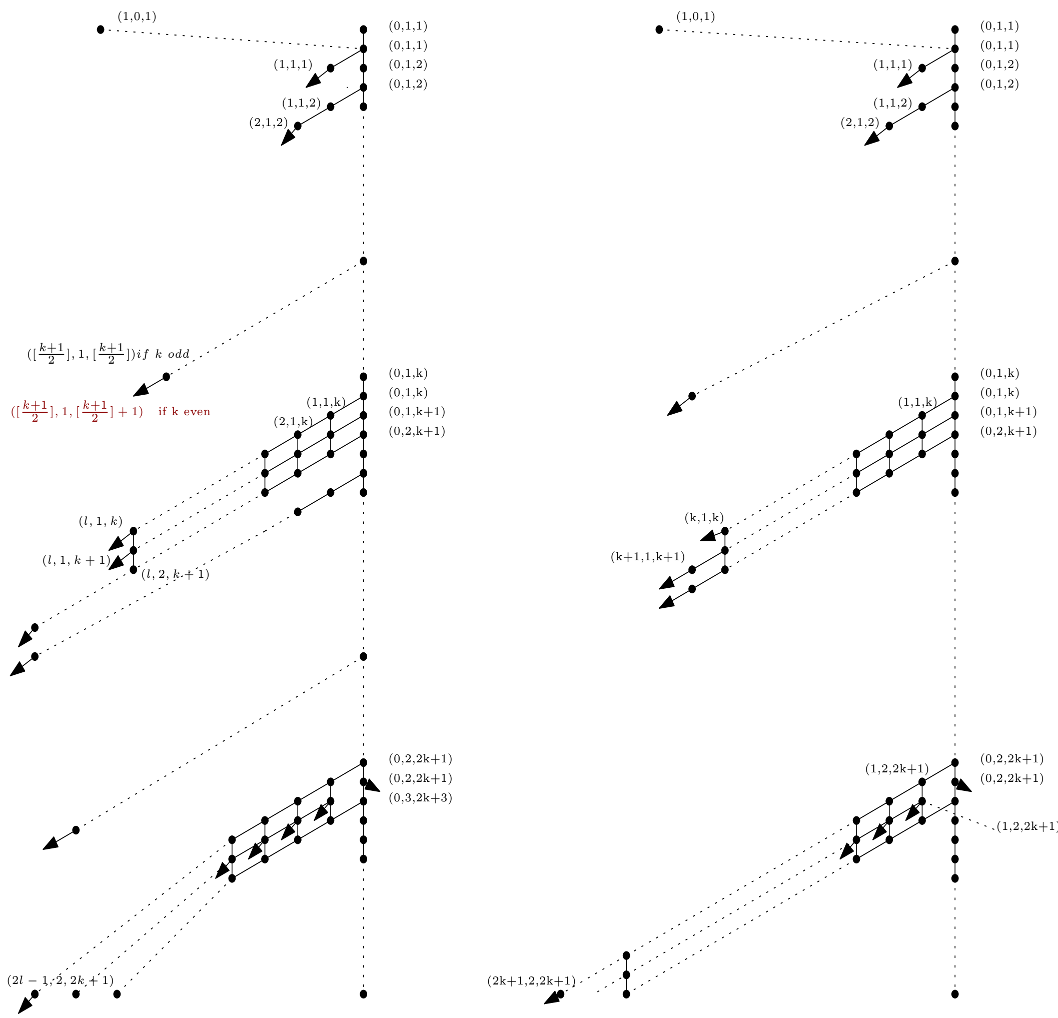
FIGURE 10. Dual Newton fan of $A_{k-1,l-1,m-1}$ with $k \geq l \geq m$, and a resolution of $A_{5,4,2}$

6. Jet Schemes and Toric Embedded Resolution of $B_{k-1,m}$

The singularity of $X \subset \mathbb{C}^3$ defined by the equations:

- $m = 2\ell$,
$$z^3 + xz^2 - (y^{k+1} + y^{\ell})y^k z - xy^{2k+1} = 0,$$
- $m = 2\ell - 1$,
$$z^3 + (x - y^{\ell-1})z^2 - y^{2k+1}z - xy^{2k+1} = 0.$$

is called $B_{k-1,m}$-type singularity with $k \geq 2$ and $m \geq 3$.
In the case where $m = 2l$, the jet schemes and the toric embedded resolution behaves as in the case of $A_{m,k,l}$; so, let's just present the jet graph presenting the irreducible components of the jets schemes projecting over the singular locus $\{y = z = 0\}$ and the axis $\{x = z = 0\}$ included in X:



$l < k+1$ $\qquad$ $l \geq k+1$

FIGURE 11. Jets schemes of $B_{k-1,m}$ when $m = 2l$

Theorem 6.1. *Let $X \subset \mathbb{C}^3$ be a surface of type $B_{k-1,2l}$. The monomial valuations associated with the vectors:*

- $(0,1,1),(0,1,2),\ldots,(0,1,k+1)$
- $(1,1,1),\ldots,(1,1,k+1)$
- $\ldots$
- $(l,1,l),\ldots,(l,1,k+1)$
- $(l+1,1,l+1),\ldots,(l+1,1,k-1)$
- $(l+2,1,l+2),\ldots,(l+1,1,k-2)$
- $\ldots$

- $(E((l+k)/2), 1, E((l+k)/2))$, *and* $(E((l+k)/2), 1, E((l+k)/2)+1)$ *if* $k+l$ *is odd.*
- $(0, 2, 2k+1) \ldots (2l-l, 2, 2k+1)$

belong to $EV(X)$*. Moreover, these vectors give a toric birational map* $\mu_\Sigma : Z_\Sigma \longrightarrow \mathbb{C}^3$ *which is an embedded resolution of* $X \subset \mathbb{C}^3$ *(in the neighborhood of the origin) such that the components of the exceptional divisor of* μ_Σ *correspond to the monomial valuations defined by them; hence they correspond to the irreducible components of the* m*-th jet schemes of* X *(centered at the singular locus and the intersection of* X *with the coordinate hyperplanes).*

The vectors given in the theorem allows us to decompose the corresponding dual Newton fan into regular subcones and find an embedded resolution of the singularity.

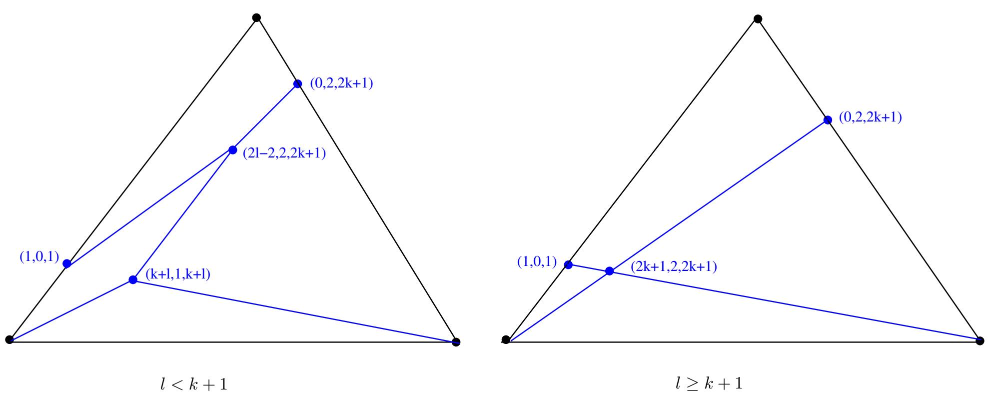


FIGURE 12. Dual Newton fans of $B_{k-1,m}$ when $m = 2l$

Two embedded resolutions for two special cases look as the following:

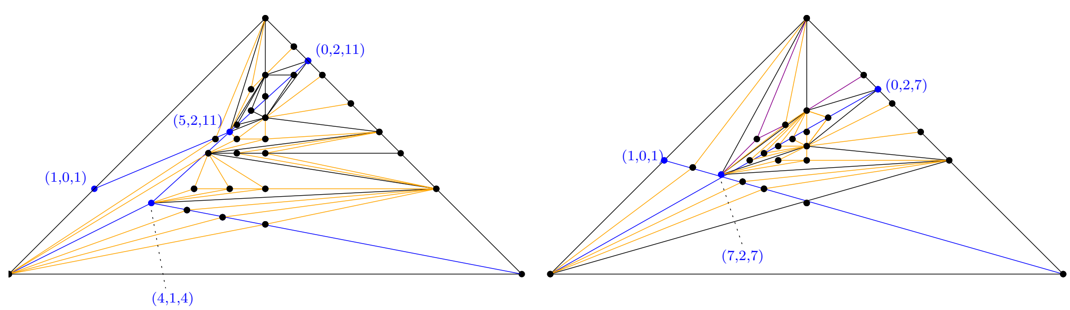


FIGURE 13. Embedded resolution of $B_{4,6}$ and of $B_{2,10}$

In the case of $B_{k-1,2l-1}$, there is an amazing subclass (see Section 2 below) for which the jet schemes give a resolution which is not a subdivision of the dual Newton fan of the singularity. So this case needs to be treated in details. There are two sub-cases to be considered which are the cases $k+1 \leq l$ and $k \geq l$.

Let us first treat the case $k+1 \leq l$: we start by computing the irreducible components of the jet schemes projecting on the singular locus $\{y = z = 0\}$ and the axis $\{x = z = 0\}$ included in X. And, by computing the associated vectors we obtain Figure 14:

Theorem 6.2. *Let X be of type $B_{k-1,2l-1}$ with $k+1 \leq l$. The monomial valuations associated with the vectors:*

- $(0,1,1),(0,1,2),\ldots,(0,1,k+1)$
- $(1,1,1),\ldots,(1,1,k+1)$
- $\ldots$
- $(k,1,k),(k,1,k+1)$
- $(k+1,1,k+1)$
- $(0,2,2k+1)\ldots(2k+1,2,2k+1)$

belong to $EV(X)$. Moreover there exists a toric birational map $\mu_\Sigma : Z_\Sigma \longrightarrow \mathbb{C}^3$ which is an embedded resolution of $X \subset \mathbb{C}^3$ such that the irreducible components of the exceptional divisor of μ_Σ correspond to the irreducible components of the m-th jet schemes of X (centered at the singular locus and the intersection of X with the coordinate hyperplane). This yields a construction of μ_Σ (not canonical).

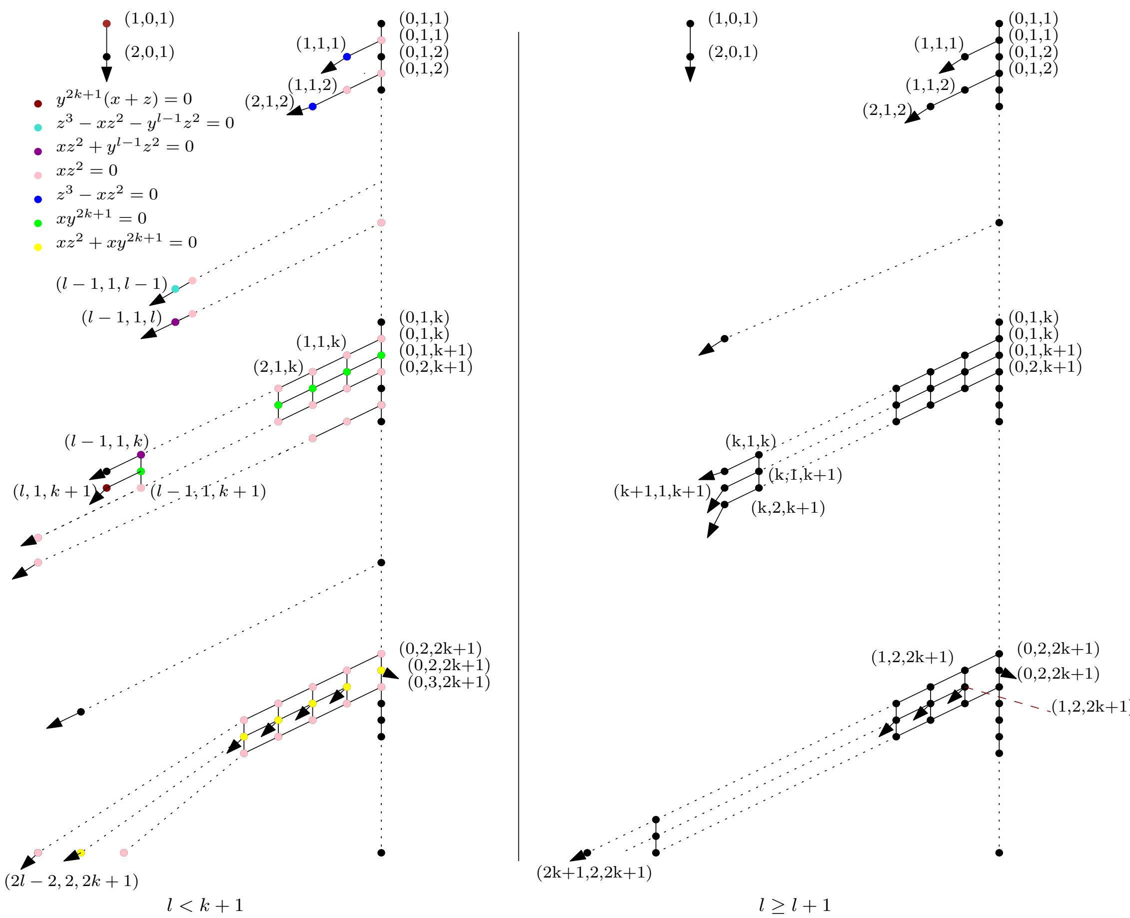


FIGURE 14. Jets schemes of $B_{k-1,2l-1}$

The computations are similar to the case $B_{k-1,2l}$; the associated vectors with the jet schemes give a subdivision of the dual Newton fan, thus an embedded resolution, of the singularity.

Theorem 6.3. *Let X be of type $B_{k-1,2l-1}$ for $l \leq k$. The monomial valuations associated with the vectors*

- $(0,1,1),(0,1,2),\dots,(0,1,k+1)$
- $(1,1,1),\dots,(1,1,k+1)$
- $\dots$
- $(l-1,1,l-1),\dots,(l-1,1,k+1)$
- $(l,1,k+1)$
- $(0,2,2k+1)\dots(2l-2,2,2k+1)$

belong to $EV(X)$. Moreover there exists a toric birational map $\mu_\Sigma : Z_\Sigma \longrightarrow \mathbb{C}^3$ which is an embedded resolution of $X \subset \mathbb{C}^3$ such that the irreducible components of the exceptional divisor of μ_Σ correspond to the irreducible components of the m-th jet schemes of X (centered at the singular locus and the intersection of X with the coordinate hyperplane). This yields a construction of μ_Σ (not canonical).

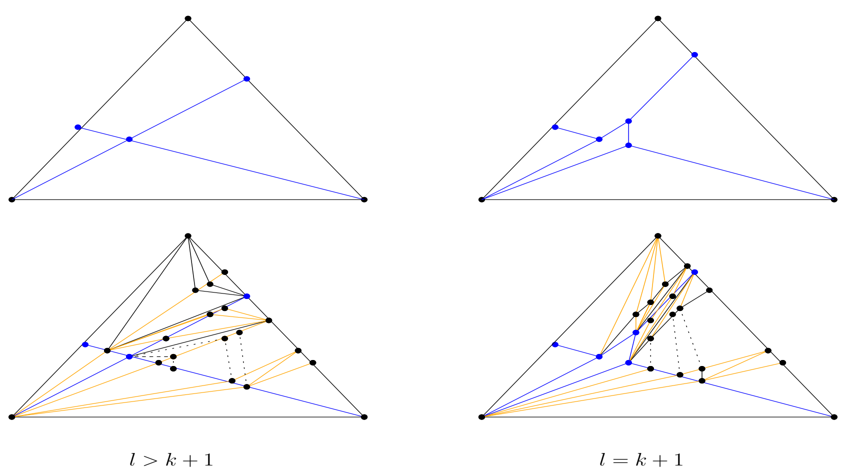


FIGURE 15. Dual Newton fan of $B_{k-1,2l-1}$ for $l > k+1$ (*resp.* $l = k+1$) and an embedded resolution

In the case where X is of type $B_{k-1,2l-1}$ with $l \leq k$, the corresponding dual Newton fan of the singularity is given with the right-hand figure of Figure 16.

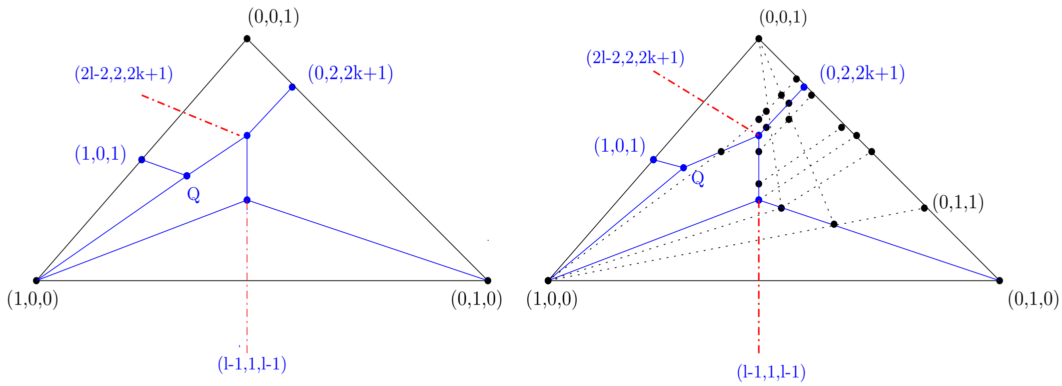


FIGURE 16. Dual Newton fan of $B_{k-1,2l-1}$ with $l \leq k$ and it is with the vectors of Theorem 6.3

Remark 6.4. The set of vectors above does not contain the vector $Q = (2k-l+2, 1, 2k-l+2)$, thus the decomposition obtained by these vectors will not be a regular decomposition of the dual Newton fan of the singularity.

Proof. Consider the polygons

$$J = [(1,0,0),(0,1,0),(0,0,1),(1,0,1),(l,1,k+1),(2l-2,2,2k+1),(l-1,1,l-1)]$$

and

$$K = [(1,0,0),(1,0,1),(l,1,k+1),(2l-2,2,2k+1),(l-1,1,l-1)]$$

in the dual Newton fan of the singularity. In J, the vectors obtained from the jet schemes give a regular subdivision of this polygon (following the computations of $B_{k-1,2l}$). As J is a sub-polygon of the fan, the strict transform of X is regular on these charts. In K, we find a subdivision by adding an edge from $(1,0,0)$ to $(l,1,k+1)$, another edge from $(1,0,0)$ to $(l-1,1,s)$ for $l-1 \leq s \leq k$ and another edge from $(l,1,k+1)$ to $(l-1,1,k)$. In this way, we obtain a regular subdivision of K.

Since K is not compatible with the dual Newton fan, we cannot use Varchenko's theorem to deduce the smoothness of the strict transform of X in the charts corresponding to the subdivision of K by the toric map. So, we should prove this fact:

- For this, let us first consider the cone $[(1,0,0),(l-1,1,s),(l-1,1,s+1)]$ for $l-1 \leq s < k$; the monoidal transformation corresponding to it is:

$$\begin{cases} x = x_1 y_1^{l-1} z_1^{l-1} \\ y = y_1 z_1 \\ z = y_1^{s} z_1^{s+1} \end{cases}$$

Then the total transform of $B_{k-1,2l-1}$ is defined by:

$$\{y_1^{2s+l-1} z_1^{2s+l+1}(y_1^{s-l+1} z_1^{s-l+2} - x_1 - y_1^{2k-s-l+2} z_1^{2k-s-l+1} - x_1 y_1^{2k-2s+1} z_1^{2k+2s-1}) = 0\}$$

The strict transform is smooth and transversal to the exceptional divisors defined by $y_1 = 0$ and $z_1 = 0$.

- Now let us consider the cone $[(1,0,0),(l-1,1,k),(l,1,k+1)]$; the monoidal transformation corresponding to it is:

$$\begin{cases} x = x_1 y_1^{l-1} z_1^{l} \\ y = y_1 z_1 \\ z = y_1^{k} z_1^{k+1} \end{cases}$$

Then the total transform of $B_{k-1,2l-1}$ is :

$$\{y_1^{2k+l-1} z_1^{2k+l+1}(y_1^{k-l+1} z_1^{k-l} - x_1 z_1 - 1 - y_1^{k-l+2} z_1^{k-l+1} - x_1 y_1) = 0\}.$$

The strict transform is smooth and transversal to the exceptional divisors defined by $y_1 = 0$ and $z_1 = 0$.

- Finally let us consider the cone $[(1,0,0),(1,0,1),(l,1,k+1)]$; the monoidal transformation corresponding to it is:

$$\begin{cases} x = x_1 y_1 z_1^{l} \\ y = z_1 \\ z = y_1 z_1^{k+1} \end{cases}$$

Then the total transform of $B_{k-1,2l-1}$ is :

$$\{y_1 z_1^{2k+l+1}(y_1^2 z_1^{k+2-l} - x_1 y_1^2 z_1 - y_1 - z_1^{k-l+1} - x_1) = 0\}.$$

The strict transform is smooth and transversal to the exceptional divisors defined by $y_1 = 0$ and $z_1 = 0$.

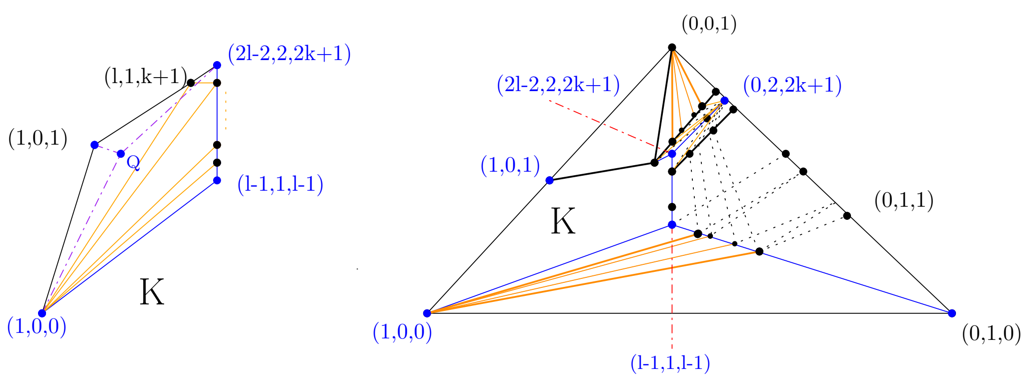


FIGURE 17. The polygon K and an embedded resolution of $B_{k-1,2l-1}$ with $l \leq k$

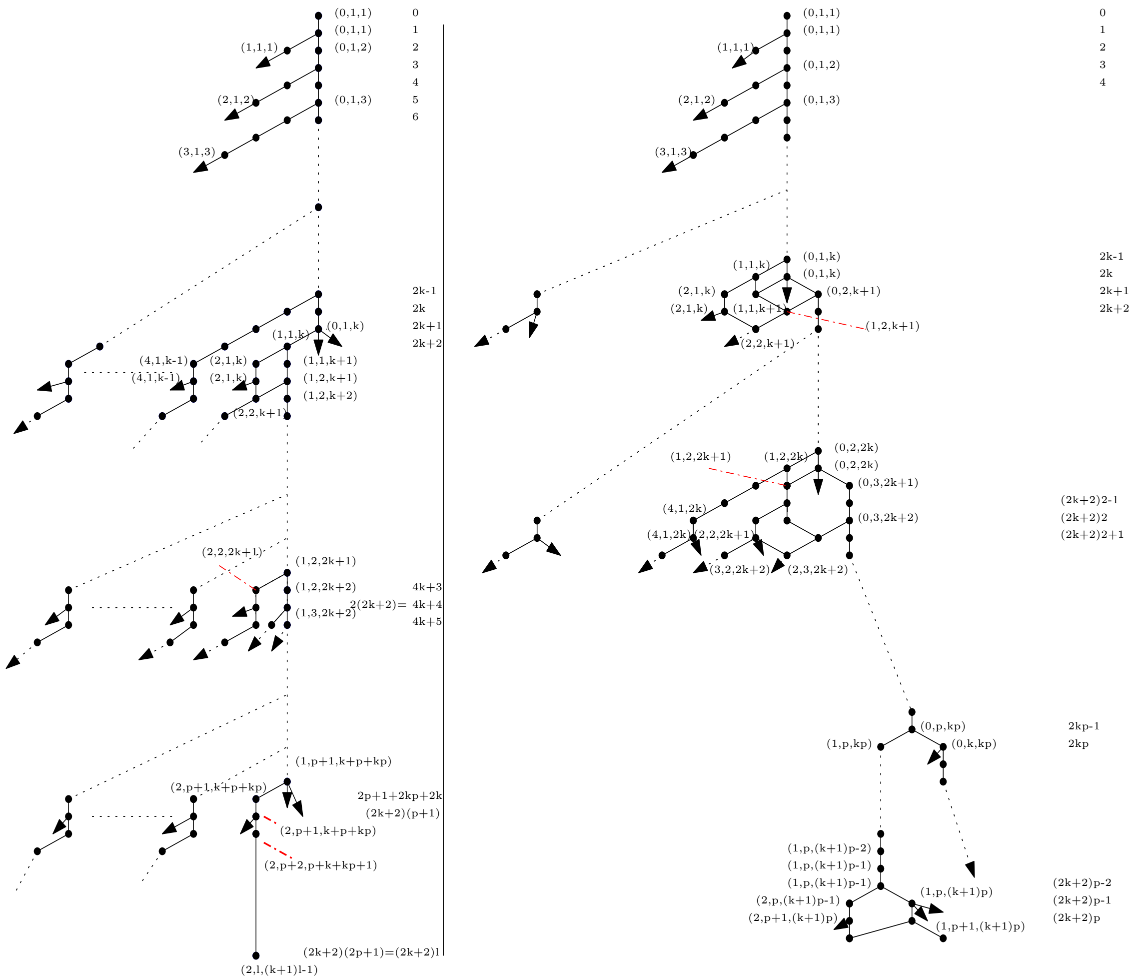
□

7. Jet Schemes and Toric Embedded Resolution of $C_{k-1,l+1}$

The singularity of $X \subset \mathbb{C}^3$ defined by the equation:

$$z^3 + xz^2 - \ell x^{\ell-1} y^{2k} z - (x^{\ell} + y^2) y^{2k} = 0$$

is called $C_{k-1,l+1}$-type singularity where $k \geq 1$ and $\ell \geq 2$. For $k = 3q-1$, we obtain the jet graph given in Figure 18 which represents the irreducible components of the jet schemes of $C_{k-1,l+1}$ projecting on the singular locus $\{y = z = 0\}$.



$C_{k-1,2p+1}$ $\qquad$ $C_{k-1,2p+2}$

FIGURE 18. Jet schemes of $C_{k-1,2p+2}$ with $k = 3q - 1$

Theorem 7.1. *Let X be a surface singularity of type $C_{k-1,l+1}$. The monomial valuations associated with the vectors:*

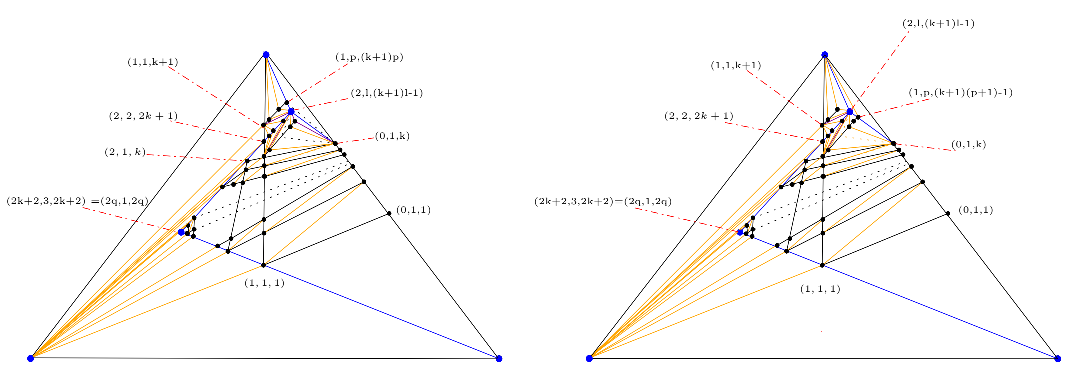
- *for* $k = 3q - 1$ *and* $l = 2p$
  - $(0,1,1),(0,1,2)...(0,1,k)$
  - $(1,1,1),...,(1,1,k),(1,1,k+1)$
  - $(2,1,2),...,(2,1,k)$
  - $(3,1,3),...,(3,1,k-1)$
  - $(4,1,4),...,(4,1,k-1)$
  - $\dots$

- $(2q-1,1,2q-1),(2q-1,1,2q)$
- $(2q,1,2q)$
- $(1,1,k),(1,2,2(k+1)-1),\ldots,(1,p,(k+1)p-1)$
- $(2,1,k),(2,2,2(k+1)-1),(2,3,3(k+1)-1),\ldots,(2,l,(k+1)l-1)$
- $(1,1,k+1),(1,2,2(k+1)),\ldots,(1,p,(k+1)p)$

• *for* $k=3q-1$ *and* $l=2p+1$
- $(0,1,1),(0,1,2)...(0,1,k)$
- $(1,1,1),...,(1,1,k),(1,1,k+1)$
- $(2,1,2),...,(2,1,k)$
- $(3,1,3),...,(3,1,k-1)$
- $(4,1,4),...,(4,1,k-1)$
- $\ldots$
- $(2q-1,1,2q-1),(2q-1,1,2q)$
- $(2q,1,2q)$
- $(1,1,k),(1,2,2(k+1)-1),\ldots,(1,p,(k+1)p-1),(1,p,(k+1)(p+1)-1)$
- $(2,1,k),(2,2,2(k+1)-1),(2,3,3(k+1)-1),\ldots,(2,l,(k+1)l-1)$
- $(1,1,k+1),(1,2,2(k+1)),\ldots,(1,p,(k+1)p)$

belong to $EV(X)$*. Moreover there exists a toric birational map* $\mu_\Sigma : Z_\Sigma \longrightarrow \mathbb{C}^3$ *which is an embedded resolution of* $X\subset\mathbb{C}^3$ *such that the irreducible components of the exceptional divisor of* μ_Σ *correspond to the irreducible components of the* m*-th jet schemes of* X *(centered at the singular locus and the intersection of* X *with the coordinate hyperplane). This yields a construction (not canonical) of* μ_Σ.

The embedded resolutions are represented on the figure below.



$k=3q-1, l=2p$ $\qquad\qquad$ $k=3q-1, l=2p+1$

FIGURE 19. An embedded resolution of $C_{k-1,l+1}$ when $k=3q-1$ and $l=2p$ or $l=2p+1$

8. Jet Schemes and Toric Embedded Resolution of D_{k-1}

The singularity of $X \subset \mathbb{C}^3$ defined by the equation:

$$z^3 + (x + y^{2k})z^2 + (2xy^k - y^2)y^k z + x^2 y^{2k} = 0$$

is called D_{k-1}-type singularity with $k \geq 1$. The jet graph is given in Figure 20 where the irreducible components of the jet schemes of D_{k-1} projecting on the singular locus $\{y = z = 0\}$ and the axis $\{x = z = 0\}$ included in X:

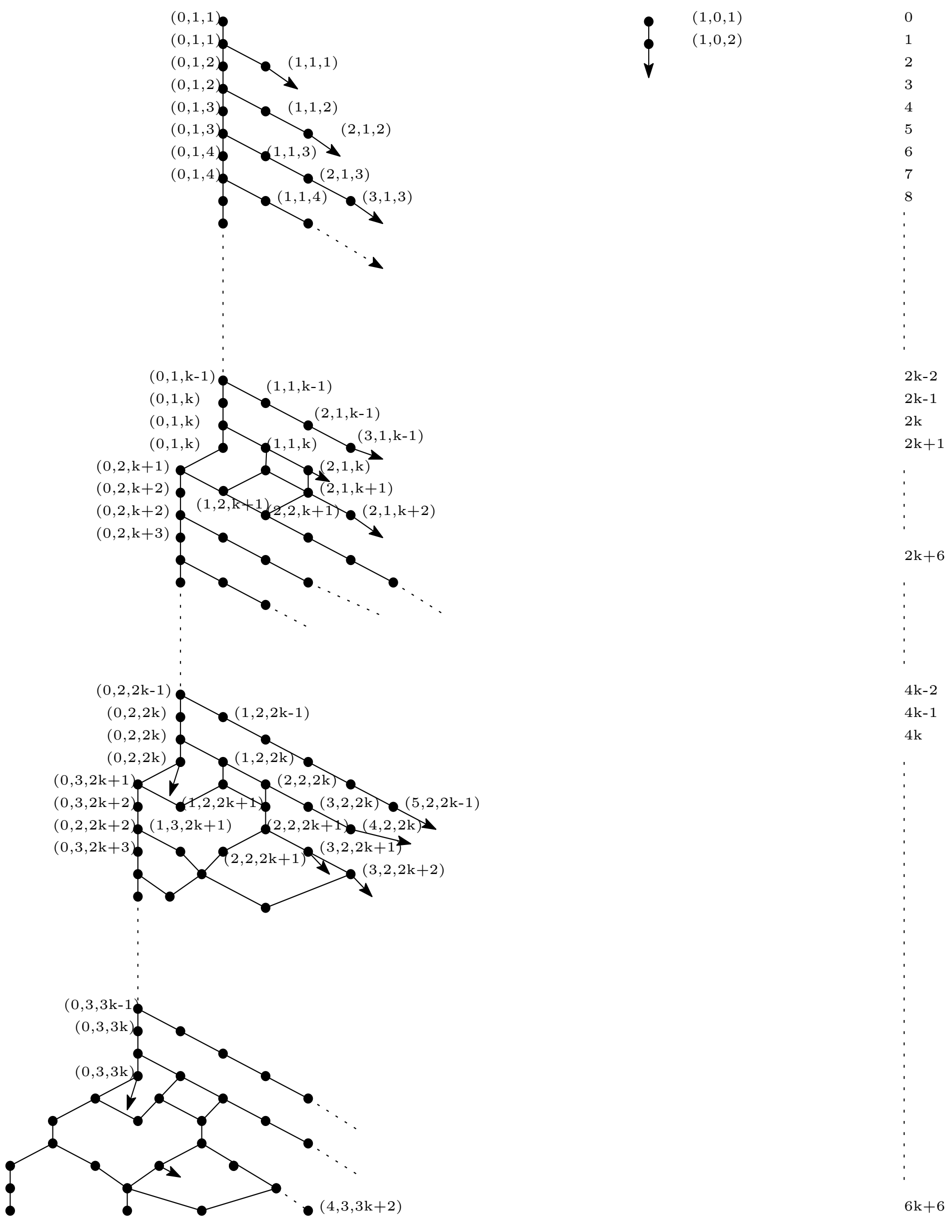


Figure 20. Jet schemes of D_{k-1}

Theorem 8.1. *Let X be a surface singularity of type D_{k-1}. The monomial valuations associated with the following vectors belong to $EV(X)$. Moreover there exists a toric birational map $\mu_\Sigma : Z_\Sigma \longrightarrow \mathbb{C}^3$ which is an embedded resolution of $X \subset \mathbb{C}^3$ such that the irreducible components of the exceptional divisor of μ_Σ correspond to the irreducible components of the m-th jet schemes of X (centered at the singular locus and the intersection of X with the coordinate hyperplane). This yields a construction (not canonical) of μ_Σ.*

- $(1,0,1),(1,0,2)$
- $(0,1,1),(0,1,2)...(0,1,k)$
- $(1,1,1),...,(1,1,k),(2,2,2k+1)),(1,1,k+1)$
- $(2,1,2),...,(2,1,k+2)$
- $(3,1,3),...,(3,1,k-1)$
- $\ldots$
- $(m,1,m),(m,1,m+1),(m,1,m+2)$
- $(m+1,1,m+1)$
- $(3,2,2k+1),(3,2,2k+2)$

When k is odd, we should add two more vectors: $(m+1,1,m+2),(k+2,2,k+2)$, where $m = E(\frac{k}{2})$.

These vectors placed in the dual Newton fan give the regular subdivision:

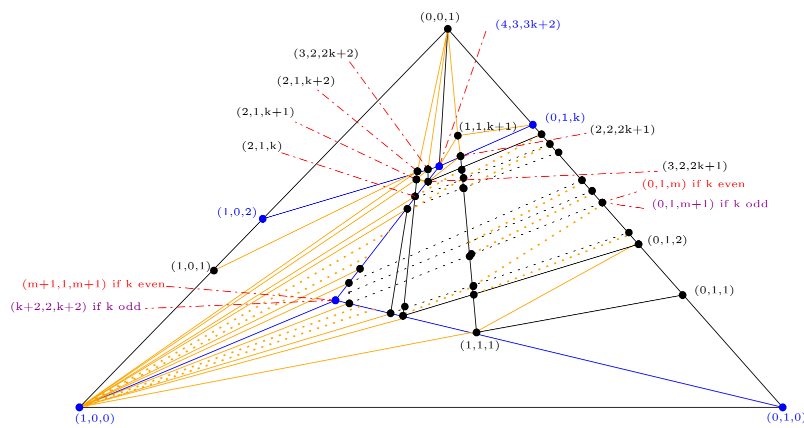


FIGURE 21. Embedded resolutions of D_{k-1} for $k = 2m$ and $k = 2m+1$

9. JET SCHEMES AND TORIC EMBEDDED RESOLUTION OF $E_{7,0}$

The singularity of $X \subset \mathbb{C}^3$ defined by the equation $z^3 + x^2yz + y^4 = 0$ is called an $E_{7,0}$-type singularity. The singular locus is $\{y = z = 0\}$.

Theorem 9.1. *Let X be a surface singularity of type $E_{7,0}$. The monomial valuations associated with the vectors: $\{(0,1,1),(0,2,1),(0,1,2),(0,1,3),$ $(1,1,1),$ $(1,1,2),$ $(1,2,2),$ $(1,2,3),$ $(1,2,4),$ $(2,2,3),$ $(2,3,4),$ $(2,3,5),$ $(3,3,4),$ $(3,4,5),$ $(3,4,6),$ $(4,5,7),$ $(5,6,8)\}$ belong to $EV(X)$. There exists a toric birational map $\mu_\Sigma : Z_\Sigma \longrightarrow \mathbb{C}^3$ which is an embedded resolution of $X \subset \mathbb{C}^3$ such that the irreducible components of the exceptional divisor of μ_Σ correspond to the irreducible components of the m-th jet schemes of X (centered at the singular locus and the intersection of X with the coordinate hyperplane). Moreover this yields a construction (not canonical) of μ_Σ.*

Following almost the same process as in the case of $E_{6,0}$, we continue until $m = 22$ to obtain the following jet graph:

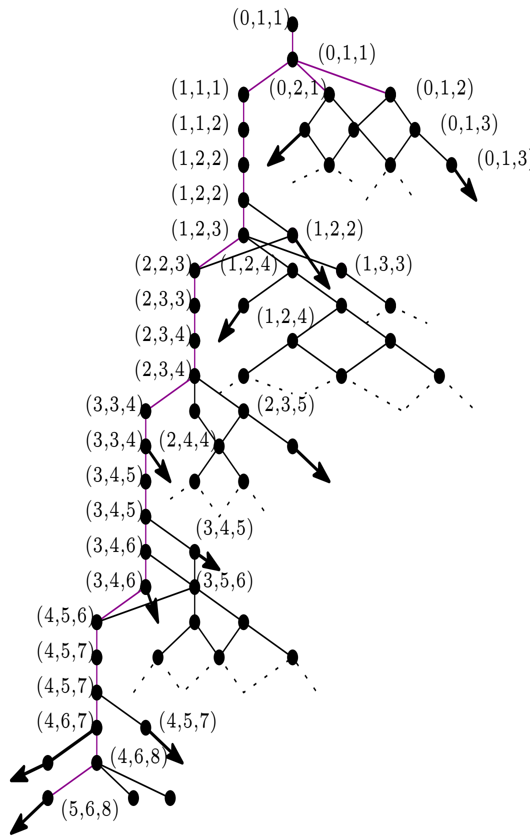


FIGURE 22. Jet schemes of $E_{7,0}$

The vectors corresponding to the irreducible jet schemes give the following subdivision, which is an embedded resolution of X:

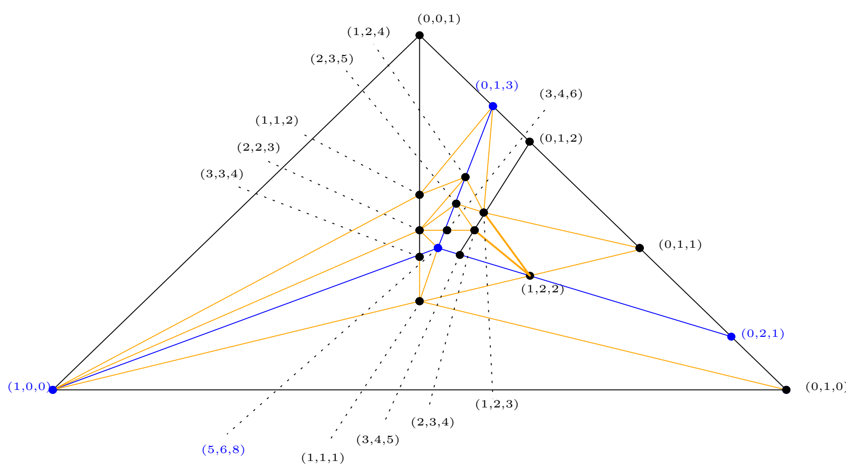


FIGURE 23. An embedded resolution of $E_{7,0}$

10. Jet Schemes and Toric Embedded Resolution of $E_{0,7}$

The singularity of $X \subset \mathbb{C}^3$ defined by the equation:

$$z^3 + y^5 + x^2y^2 = 0$$

is called $E_{0,7}$-type singularity. The singular locus is $\{y = z = 0\}$. The jet graph representing the irreducible jet schemes is obtained as:

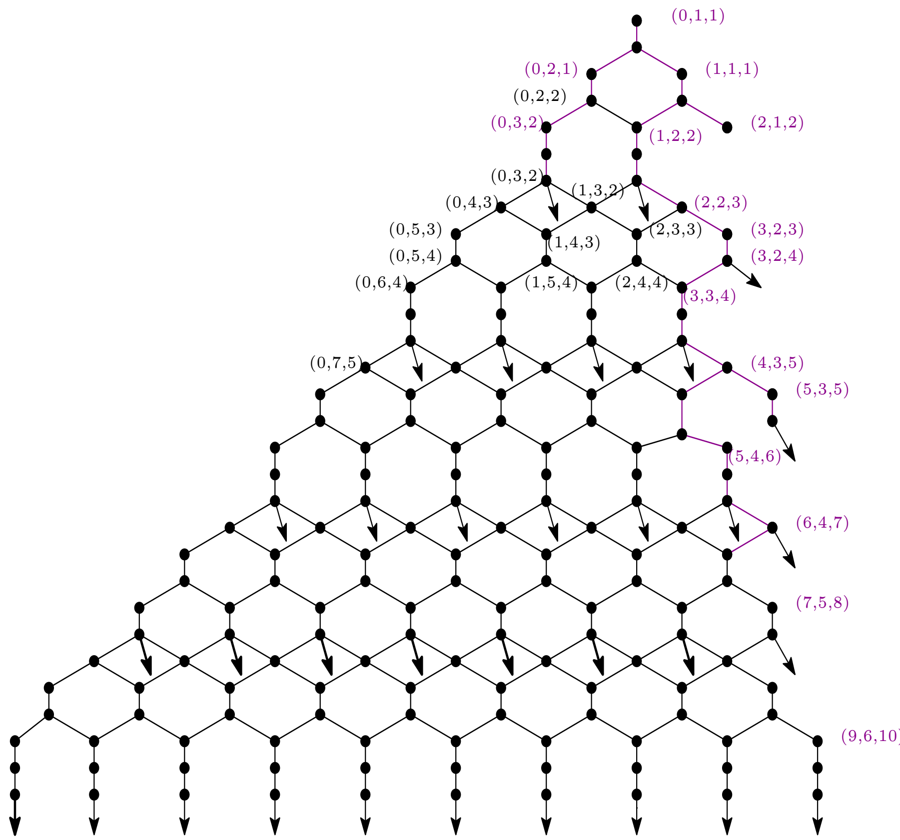


Figure 24. Jet schemes of $E_{0,7}$

Theorem 10.1. *Let X be a surface of type $E_{0,7}$. The monomial valuations associated with the vectors* $\{(0,1,1),(0,2,1),(1,1,1),\ (0,3,2),\ (1,1,2),\ (1,2,2),\ (2,1,2),\ (2,2,3),\ (3,2,3),\ (3,2,4),\ (3,3,4),\ (4,3,5),\ (5,3,5))\ (5,4,6),(6,4,7),(7,5,8),(9,6,10)\}$ *belong to $EV(X)$. There exists a toric birational map $\mu_\Sigma : Z_\Sigma \longrightarrow \mathbb{C}^3$ which is an embedded resolution of $X \subset \mathbb{C}^3$ such that the irreducible components of the exceptional divisor of μ_Σ correspond to the irreducible components of the m-th jet schemes of X (centered at the singular locus and the intersection of X with the coordinate hyperplane). Moreover this yields a construction (not canonical) of μ_Σ.*

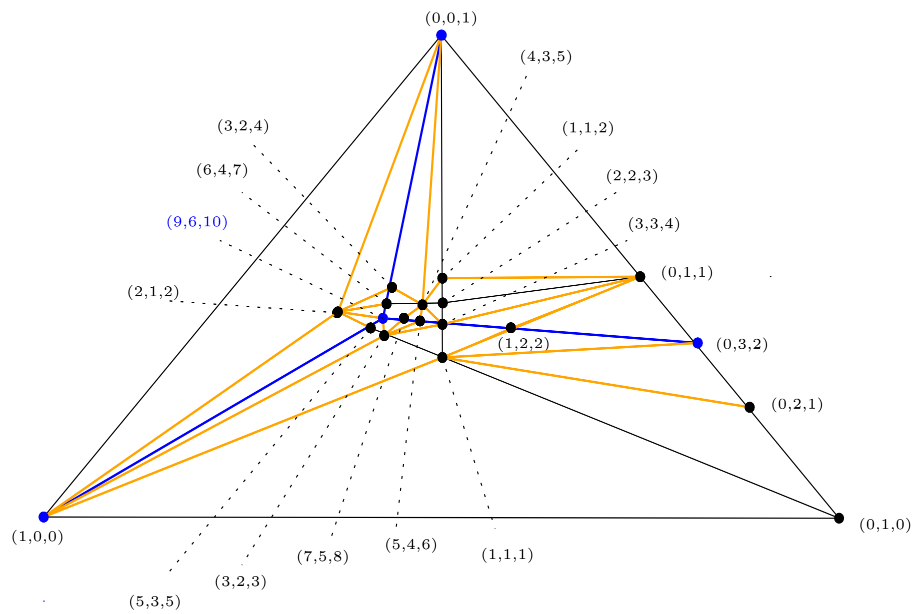


FIGURE 25. An embedded resolution of $E_{0,7}$

11. Jet Schemes and Toric Embedded Resolution of F_{k-1}

The singularity of $X \subset \mathbb{C}^3$ defined by the equation:

$$z^3 + (x + y^{2k})z^2 + 2xy^{2k}z + (x^2 + y^3)y^{2k} = 0$$

is called F_{k-1}-type singularity. The singular locus is $\{y = z = 0\}$.

Theorem 11.1. *Let X be a surface singularity of type F_{k-1}. The monomial valuations associated with the vectors:*

- $(0, 1, 1), \dots, (0, 1, k)$
- $(1, 1, 1), \dots, (1, 1, k + 1)$
- $(2, 1, 2), \dots, (2, 1, k + 1)$
- $(3, 1, 3), \dots, (3, 1, k)$
- $\dots$
- $(a, 1, b)$
- $(2, 2, 2k + 1), (3, 2, 2k + 2), (4, 2, 2k + 1), (6, 2, 2k) \dots (c, 2, d)$
- $(4, 3, 3k + 2), (5, 3, 3k + 2), (7, 3, 3k + 1), (9, 3, 3k) \dots (2k + 3, 3, 2k + 3)$
- $(3k + 2, 3, 3k + 2)$ *if* $k = 3m + 1$

with

- $(a, 1, b) = (\frac{2k+3}{3}, 1, \frac{2k+3}{3})$ *and* $(c, 2, d) = (\frac{4k+6}{3}, 2, \frac{4k+6}{3})$ *if* $k = 3m$ *for* $m \in \mathbb{N}$*;*
- $(a, 1, b) = (\frac{2k+1}{3}, 1, \frac{2k+4}{3})$ *and* $(c, 2, d) = (\frac{4k+2}{3}, 2, \frac{4k+8}{3})$ *if* $k = 3m + 1$ *for* $m \in \mathbb{N}$
- $(a, 1, b) = (\frac{2k-1}{3}, 1, \frac{2k+5}{3})$ *and* $(c, 2, d) = (\frac{4k+4}{3}, 2, \frac{4k+7}{3})$ *if* $k = 3m + 2$ *for* $m \in \mathbb{N}*$

belong to $EV(X)$. Moreover there exists a birational map $\mu_\Sigma : Z_\Sigma \longrightarrow \mathbb{C}^3$ which is an embedded resolution of $X \subset \mathbb{C}^3$ such that the irreducible components of the exceptional divisor of μ_Σ correspond to the irreducible components of the m-th jet schemes of X (centered at the singular locus and the intersection of X with the coordinate hyperplane). Moreover this yields a construction (not canonical) of μ_Σ.

The jet graph representing the irreducible components of the jet schemes projecting on the singular locus is given by:

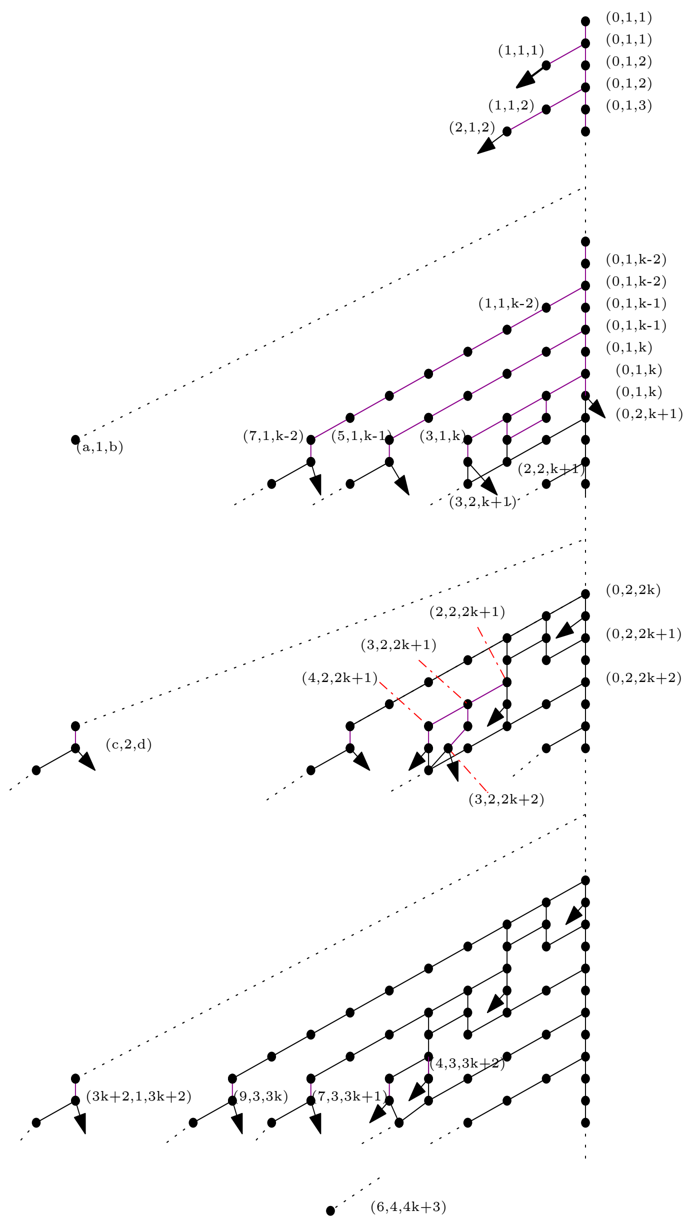


FIGURE 26. Jet schemes of F_{k-1}

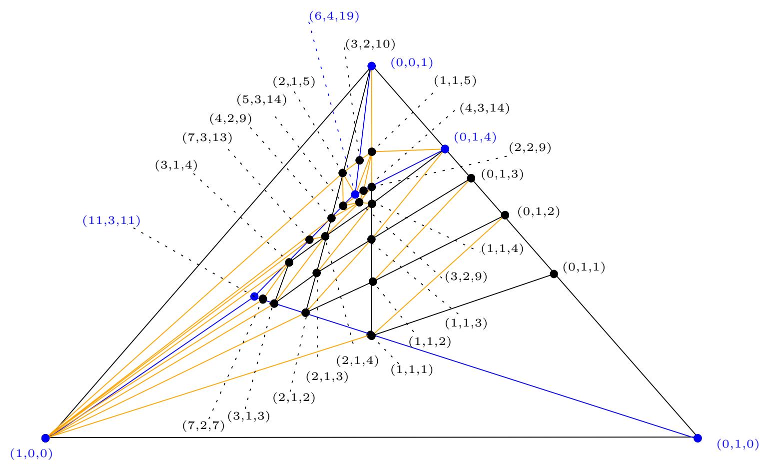


FIGURE 27. An embedded resolution of F_3

12. JET SCHEMES AND TORIC EMBEDDED RESOLUTION OF H_n

The singularity of $X \subset \mathbb{C}^3$ defined by the equation:

- $z^3 + x^2y(x + y^{k-1}) = 0$ where $n = 3k - 1$
- $z^3 + xy^kz + x^3y = 0$ where $n = 3k$
- $z^3 + xy^{k+1}z + x^3y^2 = 0$ where $n = 3k + 1$

is called H_n-type singularity.

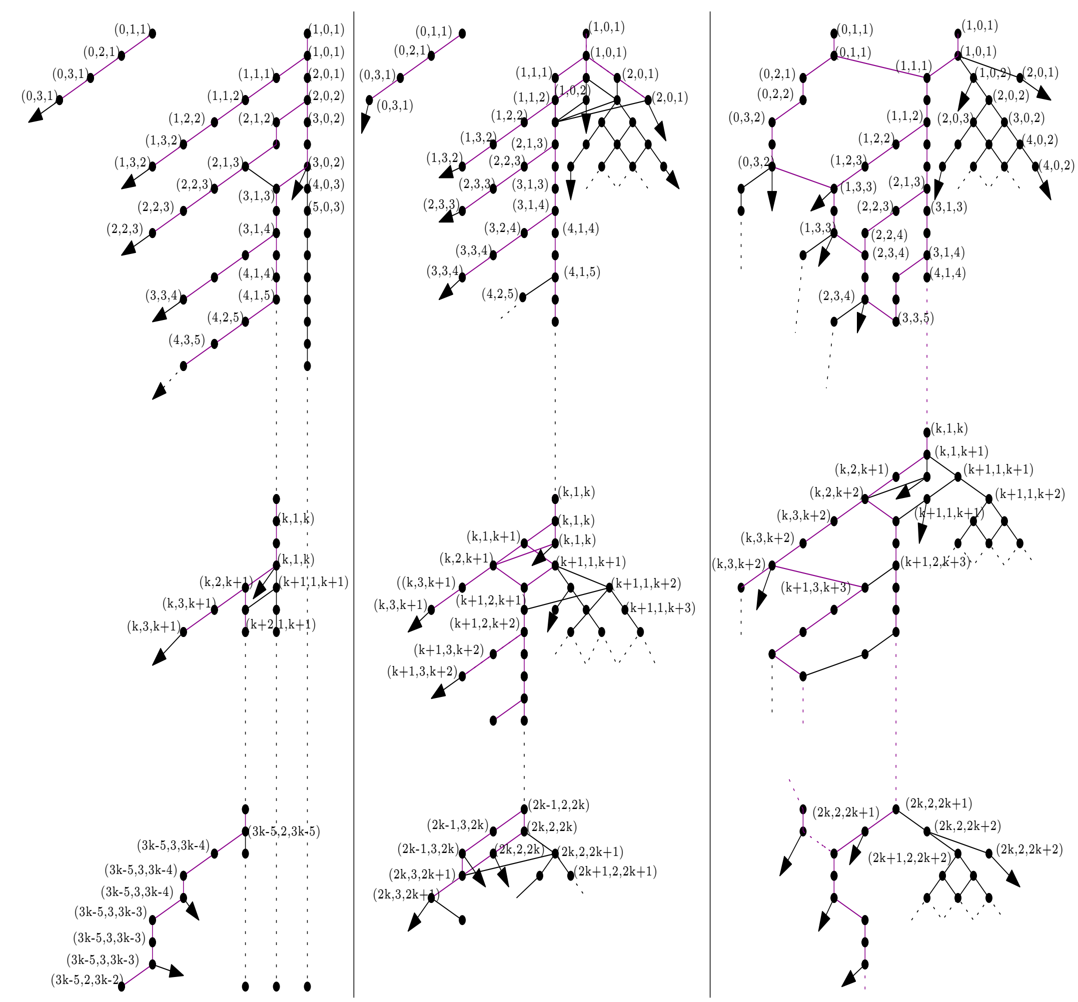
Theorem 12.1. *Let X be a surface of type H_n. The monomial valuations associated with the vectors:*

1. $n = 3k - 1$
   - $(2, 0, 1), (3, 0, 2)$
   - $(0, 1, 1), (1, 1, 2), \dots (k - 1, 1, k)$
   - $(0, 2, 1), (1, 2, 2), \dots (2k - 2, 2, 2k - 1)$
   - $(0, 3, 1), (1, 3, 2), \dots (3k - 3, 3, 3k - 2)$
   - $(1, 0, 1), (1, 1, 1), (2, 1, 2), \dots (k, 1, k)$
2. $n = 3k$
   - $(2, 0, 1))$
   - $(0, 1, 1), (1, 1, 2), \dots (k, 1, k + 1)$
   - $(0, 2, 1), (1, 2, 2), \dots (2k - 1, 2, 2k)$
   - $(0, 3, 1), (1, 3, 2), \dots (3k - 2, 3, 3k - 1)$
   - $(1, 0, 1), (1, 1, 1), (2, 1, 2), \dots (k, 1, k)$
3. $n = 3k - 1$
   - $(0, 1, 1), (1, 1, 2), \dots (k, 1, k + 1)$
   - $(0, 2, 1), (1, 2, 2), \dots (2k, 2, 2k + 1)$

- $(0,3,2),(1,3,3),\dots(3k-1,3,3k+1)$
- $(1,0,1),(1,0,2),(2,0,1)$
- $(1,1,1),(2,1,2),\dots(k,1,k)$

belong to $EV(X)$. Moreover there exists a toric birational map $\mu_\Sigma : Z_\Sigma \longrightarrow \mathbb{C}^3$ which is an embedded resolution of $X \subset \mathbb{C}^3$ such that the irreducible components of the exceptional divisor of μ_Σ correspond to the irreducible components of the m-th jet schemes of X (centered at the singular locus and the intersection of X with the coordinate hyperplane). This yields a construction (not canonical) of μ_Σ.

The tree representing the irreducible components of the jets schemes projecting on the singular locus $\{x=z=0\}$ and the axis $\{y=z=0\}$ included in X is the following:



$n=3k-1$ $n=3k$ $n=3k+1$

FIGURE 28. Jets schemes of H_n

An embedded resolution for each case is represented on the figure below:

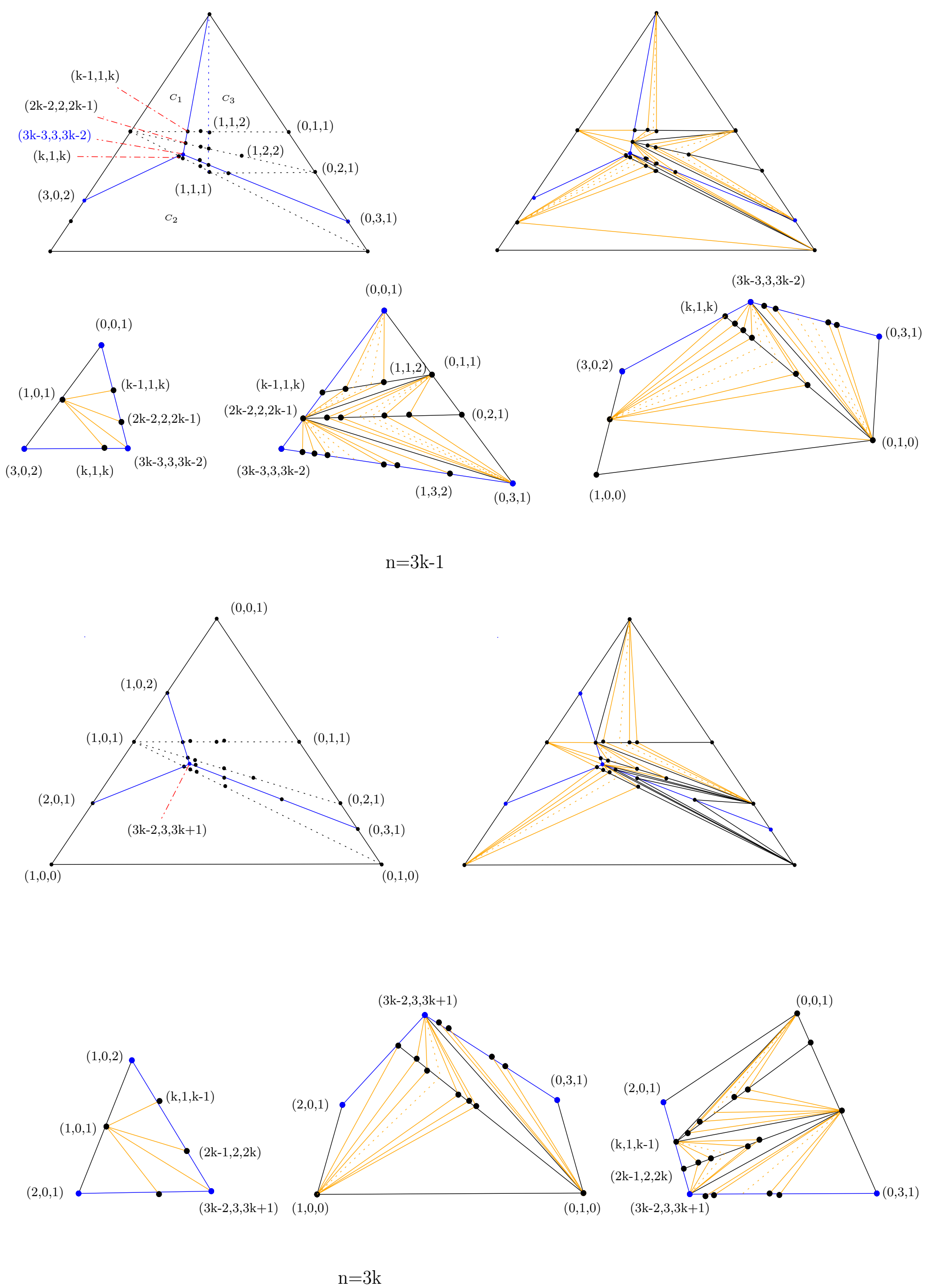
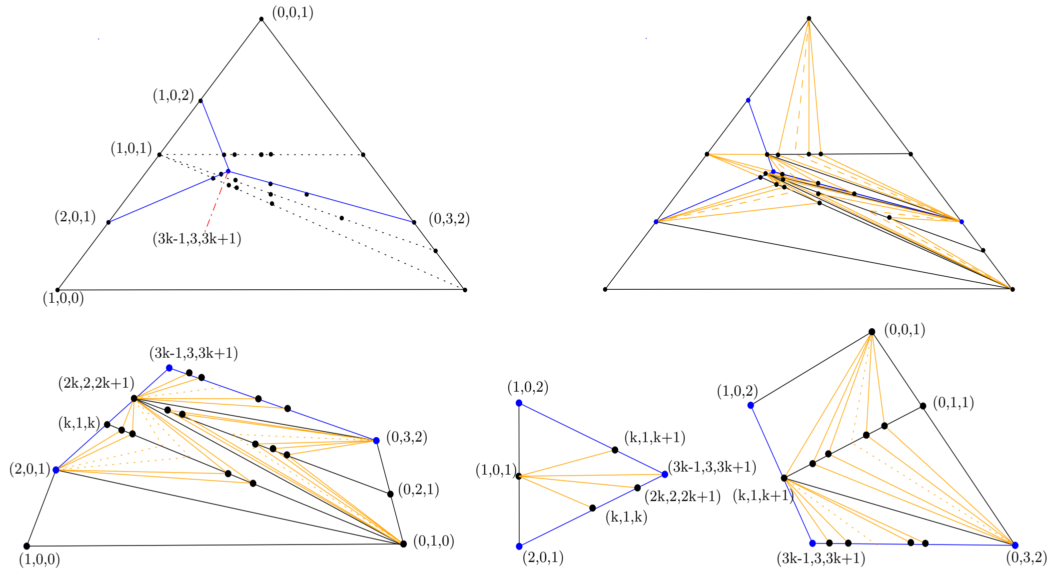


FIGURE 29. Embedded resolutions of H_n



$n = 3k + 1$

FIGURE 30. Embedded resolutions of H_{3k+1}

REFERENCES

[1] A. Altintas Sharland, G. Cevik and M. Tosun, *Nonisolated forms of rational triple singularities*, Rocky Mountain J. Math. 46-2, (2016), pp. 357-388. DOI: 10.1216/rmj-2016-46-2-357

[2] F. Aroca, M. Gomez Morales and K. Shabbir, *Torical modification of Newton non-degenerate ideals*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math.107-1, (2013), pp.221-239. DOI: 10.1007/s13398-012-0100-8

[3] M. Artin, *On isolated rational singularities of surfaces*, Amer. J. Math. 88, (1966), pp.129-136.

[4] H.Cobo and H.Mourtada, *Jet schemes of quasi-ordinary surface singularities*, Nagoya Math. Journal, (2019), pp.1-88. DOI: 10.1017/nmj.2019.26

[5] L.Ein, R.Lazarsfeld, M.Mustata, *Contact loci in arc spaces*, Compos. Math. 140-5, (2004), pp.1229-1244. DOI: 10.1112/s0010437x04000429

[6] T. de Fernex, *The space of arcs of an algebraic variety*, Algebraic Geometry, Salt Lake City 2015, Proc. Symp. Pure Math., 97-1, (2018). DOI: 10.1090/pspum/097.1/01672

[7] T. de Fernex, *Three-dimensional counter-examples to the Nash problem* , Compos. Math. 149-9, (2013), pp.1519-1534. DOI: 10.1112/s0010437x13007252

[8] T. de Fernex and R.Docampo, *Terminal valuations and the Nash problem*, Invent. Math. 203 (2016), pp.303-331. DOI: 10.1007/s00222-015-0597-5

[9] J. Fernandez de Bobadilla, M. Pe Pereira, *The Nash problem for surfaces*, Ann. Math. 176-3, (2012), pp.2003-2029. DOI: 10.4007/annals.2012.176.3.11

[10] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Math. 52, New York-Heidelberg: Springer-Verlag, (1977).

[11] S. Ishii and J. Kollár, *The Nash problem on arc families of singularities*, Duke Math. J. 120-3 (2003), pp.601-620. DOI: 10.1215/s0012-7094-03-12034-7

[12] J.M.Johnson and J. Kollár, *Arc spaces of cA-type singularities*, J. of Sing. 7 (2013), pp.238-252.

[13] Lê D.T. and M. Tosun, *Combinatorics of rational surface singularities*, Comm. Math. Helv. 79, (2004), pp.582-604.

[14] M. Leyton-Alvarez, H. Mourtada, M. Spivakovsky, *Newton non-degenerate mu-constant deformations admit simultaneous embedded resolutions*, (2020). arχiv: 2001.10316

[15] M. Lejeune-Jalabert, H. Mourtada and A. Reguera, *Jet schemes and minimal embedded desingularization of plane branches*, Rev. de la Real Acad. de Cienc. Exactas, Fis. Nat. Ser. A. Mat.107-1 (2013), pp.145-157. DOI: 10.1007/s13398-012-0091-5

[16] H. Mourtada, *Jet schemes of complex plane branches and equisingularity*, Ann. de l'Inst. Fourier, 61-6, (2011), pp.2313-2336. DOI: 10.5802/aif.2675
[17] H. Mourtada, *Jet schemes of rational double point surface singularities*, Valuation Theory in Interaction, EMS Ser. Congr. Rep., Eur. Math. Soc., (2014), pp.373-388. DOI: 10.4171/149-1/18
[18] H. Mourtada, *Jet schemes and minimal generating sequences of divisorial valuations in dimension* 2, Michigan Math. J. 66-1 (2017), pp.155-174. DOI: 10.1307/mmj/1488510030
[19] H. Mourtada, *Jet schemes of normal toric surfaces*, Bull. Soc. Math. France 145-2, (2017), pp.237-266. DOI: 10.24033/bsmf.2736
[20] H. Mourtada, C.Plénat, *Jet schemes and minimal toric embedded resolutions of rational double point singularities*, Comm. in Algebra, 46-3, (2018), pp.1314-1332. DOI: 10.1080/00927872.2017.1344695
[21] M.Mustaţă, *Jet schemes of locally complete intersection canonical singularities*, Invent. Math. 145-3, (2001), pp.397-424. With an appendix by D.Eisenbud and E. Frenkel. DOI: 10.1007/s002220100152
[22] J. F. Nash, *Arc structure of singularities*, Duke Math. J. 81-1, (1996), pp.31-38.
[23] M. Oka, *Non-degenerate complete intersection singularity*, Actualité Math. Hermann, Paris, 1997.
[24] C.Plénat, M.Spivakovsky, *The Nash problem and its solution: a survey*, J. Sing. 13, (2015), pp.229-244.
[25] J.Stevens, *On the classification of rational surface singularities*, J. of Sing. 7, (2013), pp.108-133.
[26] G.N. Tjurina, *Absolute isolation of rational singularities, and triple rational points*, (Russian) Funkc. Anal. Prilozen. 2-4, (1968), pp.70–81.
[27] A.N. Varchenko, *Zeta-function of monodromy and Newton's diagram*, Invent. Math. 37-3 (1976), pp.253-262. DOI: 10.1007/bf01390323

B. Karadeniz, Department of Mathematics, Gebze Technical University, Gebze 41400, Kocaeli, Turkey
Email address: busrakaradeniz@gtu.edu.tr

H. Mourtada, Université de Paris, Sorbonne Université, CNRS, Institut Mathématiques de Jussieu-Paris Rive Gauche, F-75013 Paris, France
Email address: hussein.mourtada@imj-prg.fr

C. Plénat, Aix Marseille Univ, CNRS, Centrale Marseille, I2M, CMI, Technopôle Château-Gombert, 39, rue F. Joliot Curie, 13453 Marseille Cedex 13
Email address: camille.plenat@univ-amu.fr

M. Tosun, Department of Mathematics, Galatasaray University, Ortaköy 34357, Istanbul, Turkey
Email address: mtosun@gsu.edu.tr

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TOPOLOGY OF COMPLEMENTS TO REAL AFFINE SPACE LINE ARRANGEMENTS

GOO ISHIKAWA AND MOTOKI OYAMA

ABSTRACT. It is shown that the diffeomorphism type of the complement to a real space line arrangement in any dimensional affine ambient space is determined only by the number of lines and the data on multiple points.

1. INTRODUCTION

Let $\mathscr{A} = \{\ell_1, \ell_2, \ldots, \ell_d\}$ be a real space line arrangement, or a configuration, consisting of affine d-lines in $\mathbb{R}^3$. The different lines $\ell_i, \ell_j (i \neq j)$ may intersect, so that the union $\cup_{i=1}^d \ell_i$ is an affine real algebraic curve of degree d in $\mathbb{R}^3$ possibly with multiple points. In this paper we determine the topological type of the complement $M(\mathscr{A}) := \mathbb{R}^3 \setminus (\cup_{i=1}^d \ell_i)$ of $\mathscr{A}$, which is an open 3-manifold. We observe that the topological type $M(\mathscr{A})$ is determined only by the number of lines and the data on multiple points of $\mathscr{A}$. Moreover we determine the diffeomorphism type of $M(\mathscr{A})$.

Set $D^n := \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$, the n-dimensional closed disk. The pair $(D^i \times D^j, D^i \times \partial(D^j))$ with $i + j = n,\ 0 \leq i, 0 \leq j$, is called an n-dimensional *handle of index* j (see [17][1] for instance).

Now take one D^3 and, for any non-negative integer g, attach to it g-number of 3-dimensional handles $(D_k^2 \times D_k^1, D_k^2 \times \partial(D_k^1))$ of index 1 $(1 \leq k \leq g)$, by an attaching embedding

$$\varphi : \bigsqcup_{k=1}^{g} (D_k^2 \times \partial(D_k^1)) \to \partial(D^3) = S^2$$

such that the obtained 3-manifold

$$B_g := D^3 \bigcup\nolimits_{\varphi} (\bigsqcup\nolimits_{k=1}^{g} (D_k^2 \times D_k^1))$$

is orientable. We call B_g the 3-*ball with trivial* g-*handles of index* 1 (Figure 1.)

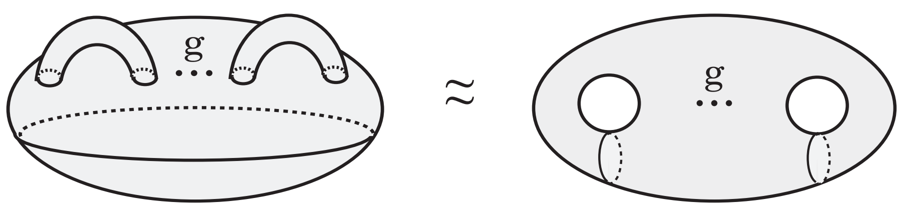

FIGURE 1. 3-ball with trivial g-handles of index 1.

Note that the topological type of B_g does not depend on the attaching map φ and is uniquely determined only by the number g. The boundary of B_g is the orientable closed surface Σ_g of genus g.

2010 *Mathematics Subject Classification.* Primary 57M25; Secondary 57R45, 57M15, 58K15.

Key words and phrases. trivial handle attachment, height function, space graph complement, open three-manifold.
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Let $\mathscr{A}$ be any d-line arrangement in $\mathbb{R}^3$. Let $t_i = t_i(\mathscr{A})$ denote the number of multiple points with multiplicity i, $i = 2,\dots,d$. The vector $(t_d, t_{d-1}, \dots, t_2)$ provides a degree of degeneration of the line arrangement $\mathscr{A}$. Set $g := d + \sum_{i=2}^{d}(i-1)t_i$. In this paper we show the following result:

Theorem 1.1. *The complement $M(\mathscr{A})$ is homeomorphic to the interior of* 3*-ball with trivial g-handles of index* 1.

Corollary 1.2. *$M(\mathscr{A})$ is homotopy equivalent to the bouquet $\bigvee_{k=1}^{g} S^1$.*

The above results are naturally generalised to any line arrangements in $\mathbb{R}^n (n \geq 3)$.

Let $\mathscr{A} = \{\ell_1, \ell_2, \dots, \ell_d\}$ be a line arrangement in $\mathbb{R}^n$ and set $M(\mathscr{A}) := \mathbb{R}^n \setminus (\cup_{i=1}^{d} \ell_i)$. Again let t_i denote the number of multiple points of $\mathscr{A}$ of multiplicity i, $i = 2,\dots,d$. Set $g := d + \sum_{i=2}^{d}(i-1)t_i$. Then we have

Theorem 1.3. *$M(\mathscr{A})$ is homeomorphic to the interior of n-ball B_g with trivially attached g-handles of index $n-2$.*

Thus we see that the topology of complements of real space line arrangements is completely determined by the combinational data, the *intersection poset* in particular. Recall that the intersection poset $P = P(\mathscr{A})$ is the partially ordered set which consists of all multiple points, the lines themselves $\ell_1, \ell_2, \dots, \ell_d$ and $T = \mathbb{R}^n$ as elements, endowed with the inclusion order. Then the number t_i is recovered as the number of minimal points x such that $\#\{y \in P \mid x < y, y \neq T\} = i$ and d as the number of maximal points of $P \setminus \{T\}$.

Corollary 1.4. *$M(\mathscr{A})$ is homotopy equivalent to the bouquet $\bigvee_{k=1}^{g} S^{n-2}$.*

In particular $M(\mathscr{A})$ is a minimal space, i.e. it is homotopy equivalent to a *CW* complex such that the number of i-cells is equal to its i-th Betti number for all $i \geq 0$.

Even for semi-algebraic open subsets in $\mathbb{R}^n$, homotopical equivalence does not imply topological equivalence in general. However we see this is the case for complements of real affine line arrangements, as a result of Theorem 1.3 and Corollary 1.4.

By the uniqueness of smoothing of corners, and by careful arguments at all steps of the proof of Theorem 1.3, we see that Theorem 1.3 can be proved in differentiable category.

Theorem 1.5. *$M(\mathscr{A})$ is diffeomorphic to the interior of n-ball B_g with trivially attached g-handles of index $n-2$.*

Note that the relative classification problem of line arrangements $(\mathbb{R}^n, \cup_{i=1}^{d} \ell_i)$ is classical but far from being solved ([6] for instance). Moreover there is a big difference in differentiable category and topological category. In fact even the local classification near multiple points of high multiplicity i, $i \geq n+2$ has moduli in differentiable category while it has no moduli in topological category. The classification of complements turns to be easier and simpler as we observe in this paper.

The real line arrangements on the plane $\mathbb{R}^2$ is one of classical and interesting subjects to study. It is known or easy to show that the number of connected components of the complement to a real plane line arrangement is given exactly by $1+g$ using the number $g = d + \sum_{i=2}^{d}(i-1)t_i$. This can be derived from Corollary 1.4 by just setting $n = 2$. For example, it can be shown from known combinatorial results for line arrangements on projective plane (see [4] for instance). In fact we prove it using our method in the process of the proof of Theorem 1.3. Therefore Theorem 1.3 and Corollary 1.4 are regarded as a natural generalisation of the classical fact.

Though our object in this paper is the class of real affine line arrangements, it is natural to consider also real projective line arrangements consisting of projective lines in the projective space $\mathbb{R}P^n$, or corresponding real linear plane arrangements consisting of 2-dimensional linear subspaces in $\mathbb{R}^{n+1}$. However the topology of complements in both cases are not determined, in general, by the intersection posets, which are defined similarly to the affine case. In fact there exists an example of pairwise transversal linear plane arrangements $\mathscr{B}$ and $\mathscr{B}'$ in $\mathbb{R}^4$ with $d = 4$ such that the complements $M(\mathscr{B})$ and $M(\mathscr{B}')$ have non-isomorphic cohomology algebras and therefore they are not homotopy equivalent, so, not homeomorphic to each other ([19], Theorem 2.1).

A linear plane arrangement in $\mathbb{R}^4$ is pairwise transverse if and only if the corresponding projective line arrangement is non-singular (without multiple points) in $\mathbb{R}P^3$. Non-singular line arrangements in $\mathbb{R}P^3$, which are called skew line configurations, are studied in details (see [6, 13, 15, 16] for instance). Moreover, the topology of non-singular real algebraic curves in $\mathbb{R}P^3$ is studied, related to Hilbert's 16th problem, by many authors (see [8] for instance). Also refer to the surveys on the study of real algebraic varieties ([5, 14]).

It is natural to consider also complex line arrangements in $\mathbb{C}^n = \mathbb{R}^{2n}$. The topology of complex subspace arrangements in $\mathbb{C}^n$, in particular, homotopy types of them is studied in detail (see [10, 19] for instance). Then it is known that the intersection poset turns to have more information in complex cases than in real cases. Refer to [12, 20], for instance, on the theory on the homotopy types of complements for general subspace arrangements.

In §2, we define the notion of trivial handle attachments clearly. In §3, we show Theorem 1.3 and Theorem 1.5 in parallel, using an idea of stratified Morse theory ([3]) in a simple situation. We then realize a difference of topological features between the complements to line arrangements and to knots, links, tangles or general space graphs (Remark 3.8). In the last section, related to our results, we discuss briefly the topology of real projective line arrangements and real linear plane arrangements.

The authors thank Professor Masahiko Yoshinaga for his valuable suggestion to turn authors' attention to real space line arrangements. They thank also an anonymous referee for his/her valuable comments.

2. Trivial handle attachments

First we introduce the local model of trivial handle attachments.

Let $j < n$. Let $S^j \subset \mathbb{R}^n$ be the sphere defined by $x_1^2 + \cdots + x_j^2 + x_n^2 = 1, x_{j+1} = 0, \dots, x_{n-1} = 0$, and $\partial(D^j) = S^{j-1} = S^j \cap \{x_n = 0\}$. Let $e_\ell \in \mathbb{R}^n$ be the vector defined by $(e_\ell)_i = \delta_{\ell i}$. Then define an embedding $\widetilde{\Phi} : D^{n-j} \times S^j \to \mathbb{R}^n$ by

$$\widetilde{\Phi}(t_1, \dots, t_{n-j-1}, t_{n-j}, x) := x + t_1 e_{n-1} + \cdots + t_{n-j-1} e_{j+1} + t_{n-j} x,$$

which gives a tubular neighbourhood of S^j in $\mathbb{R}^n$. Set

$$\varphi_{\mathrm{st}} := \widetilde{\Phi}|_{D^{n-j} \times \partial(D^j)} : D^{n-j} \times S^{j-1} \to \mathbb{R}^{n-1} \subset \mathbb{R}^n,$$

which gives a tubular neighbourhood of S^{j-1} in $\mathbb{R}^{n-1} = \{x_n = 0\}$. We call φ_{st} the *standard* attaching map of the handle of index j. Note that the embedding φ_{st} extends to the *standard* handle $\Phi : D^{n-j} \times D^j \to \mathbb{R}^n$, which is defined by

$$\Phi(t_1, \dots, t_{n-j-1}, t_{n-j}, x_1, \dots, x_j) := \widetilde{\Phi}\left(t_1, \dots, t_{n-j-1}, t_{n-j}, x_1, \dots, x_j, 0, \dots, 0, \sqrt{1 - \textstyle\sum_{i=1}^j x_i^2}\right),$$

attached to $\{x_n \le 0\}$ along φ_{st}.

Let M be a topological (resp. differentiable) n-manifold with a *connected* boundary ∂M.

Let $p \in \partial M$. A coordinate neighbourhood (U, ψ), $\psi : U \to \psi(U) \subset \mathbb{R}^{n-1} \times \mathbb{R}$ around p in M is called *adapted* if $\psi : U \to \mathbb{R}^n$ is a homeomorphism of U and $\psi(U) \cap \{x_n \le 0\}$ which maps $U \cap \partial M$ into $\mathbb{R}^{n-1} = \{x_n = 0\}$.

Now we consider an attaching of several handles of index j to M along ∂M. We call a handle attaching map $\varphi : \bigsqcup_{k=1}^{\ell} (D_k^{n-j} \times \partial(D_k^j)) \to \partial M$ *trivial* if there exist disjoint adapted coordinate neighbourhoods

$(U_1,\psi_1),\ldots,(U_\ell,\psi_\ell)$ on M such that $\varphi(D_k^{n-j}\times\partial(D_k^j))\subset U_k$ and $\psi_k\circ\varphi: D_k^{n-j}\times\partial(D_k^j)\to\mathbb{R}^{n-1}\times\mathbb{R}$ is the standard attachment for $k=1,\ldots,\ell$. (Figure 2)

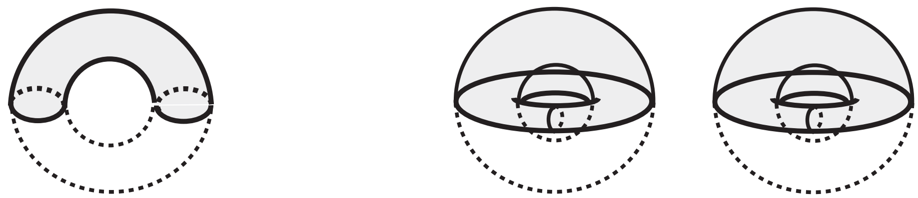

FIGURE 2. Trivial handle attachments: the cases $n=3, j=1, \ell=1$ and $n=4$, j = $2, \ell=2$.

Then $M\cup_\varphi\left(\bigsqcup_{k=1}^{\ell}(D_k^{n-j}\times D_k^j)\right)$ is called the manifold obtained from M by attaching standard handles and the topological type of M does not depend on the attaching map φ but depends only on j and ℓ. Moreover if M is a differentiable manifold, then the diffeomorphism type of the attached manifold is uniquely determined by the smoothing or straightening of corners (see Proposition 2.6.2 of [17] for instance). Note that the diffeomorphism type of the interior does not change by the smoothing.

Note that, if φ is a trivial handle attaching map, then $\varphi|_{0\times\partial(D_k^j)}: 0\times\partial(D_k^j)\to\partial M$ is unknotted and $\varphi|_{\bigsqcup_{k=1}^{\ell}(0\times\partial(D_k^j))}:\bigsqcup_{k=1}^{\ell}(0\times\partial(D_k^j))\to\partial M$ is unlinked (see Figure 4). Therefore we can slide the trivial attachment mapping $\bigsqcup_{k=1}^{\ell}(D_k^{n-j}\times\partial(D_k^j))$ to an embedding into a disjoint union to an arbitrarily small neighbourhoods of any disjoint ℓ number points on ∂M up to isotopy (cf. Homogeneity Lemma [9]).

Remark 2.1. The assumption that ∂M is connected is essential. For example, let

$$M=\{x\in\mathbb{R}^n\mid -1\le x_n\le 1\}.$$

Then we have at least two non-homeomorphic spaces by different attachments of two trivial handles of index 1 (Figure 3).

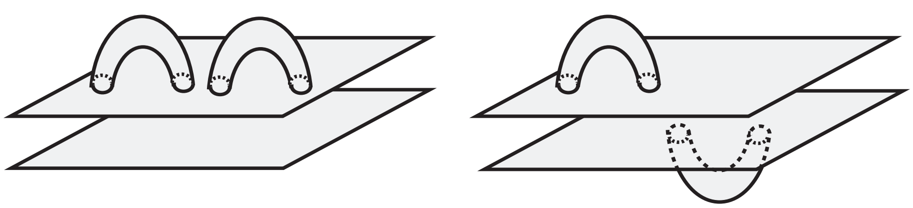

FIGURE 3. Non-homeomorphic attachments of trivial handles $n=3, j=1, \ell=2$.

We see that iterative trivial attachments gives a homeomorphic (resp. differentiable) manifold to the manifold obtained by the simultaneous trivial attachments.

Lemma 2.2. *Let M' be a topological (resp. differentiable) n-manifold with connected boundary $\partial M'$. Suppose M' is homeomorphic (diffeomorphic) to a space $M_1 := M\cup_\varphi\left(\bigsqcup_{k=1}^{\ell}(D_k^{n-j}\times D_k^j)\right)$ obtained, from a topological (differentiable) manifold M with connected boundary, by attaching k number of trivial* *handles of index j. Then the space $M_2 := M' \cup_{\varphi'} \left(\bigsqcup_{k=\ell+1}^{\ell+m} (D_k^{n-j} \times D_k^j) \right)$ obtained from M' by attaching m number of trivial handles of index j is homeomorphic (diffeomorphic) to the space*

$$M_3 := M \cup_{\varphi''} \left(\bigsqcup_{k=1}^{\ell+m} (D_k^{n-j} \times D_k^j) \right)$$

obtained from M by attaching $\ell + m$ number of trivial handles of index j.

See Figure 4 for the case $j = 1$.

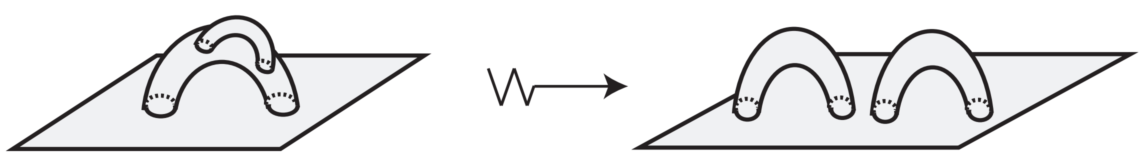

FIGURE 4. Sliding of trivial handle attachments.

Proof of Lemma 2.2. Let $f : M_1 \to M'$ be a homeomorphism (resp. a diffeomorphism). Then

$$f\big(\bigsqcup_{k=1}^{\ell} (D_k^{n-j} \times D_k^j) \big)$$

is not contained in $\partial M'$. Then we slide, up to isotopy, the attaching map $\varphi' : \bigsqcup_{k=\ell+1}^{\ell+m} (D_k^{n-j} \times \partial D_k^j) \to \partial M'$ to $\varphi''' : \bigsqcup_{k=\ell+1}^{\ell+m} (D_k^{n-j} \times \partial D_k^j) \to \partial M'$ such that

$$f\left(\varphi \left(\bigsqcup_{k=1}^{\ell} \left(D_k^{n-j} \times \partial D_k^j \right) \right) \right) \cap \varphi''' \left(\bigsqcup_{k=\ell+1}^{\ell+m} \left(D_k^{n-j} \times \partial D_k^j \right) \right) = \emptyset.$$

Consider $\varphi'' := \varphi \bigsqcup f^{-1} \circ \varphi''' : \bigsqcup_{k=1}^{\ell+m} (D_k^{n-j} \times \partial D_k^j) \to \partial M$. Then M_2 is homeomorphic (resp. diffeomorphic) to M_3. $\square$

3. AFFINE LINE ARRANGEMENTS

Let $n \geq 2$.

We consider line arrangements in $\mathbb{R}^n$ or more generally consider a subset X in $\mathbb{R}^n$ which is a union of finite number of closed line segments and half lines. Then X may be regarded as a finite graph (with compact and non-compact edges) embedded as a closed set in $\mathbb{R}^n$ (Figure 5). Here we admit vertices of valency 1.

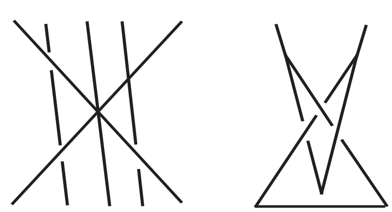

FIGURE 5. A line arrangement and a space graph

Take a unit vector $v \in S^{n-1} \subset \mathbb{R}^n$ and define the height function $h : \mathbb{R}^n \to \mathbb{R}$ by $h(x) := x \cdot v$ using Euclidean inner product. Choose v so that

(i) v is neither perpendicular to any line segments nor half lines in X.

(ii) For each c, the hyperplane $h(x) = c$ of level c contains at most one vertex of X.

Note that there exists a union Σ of finite number of great hyperspheres such that any unit vector in $S^{n-1} \setminus \Sigma$ satisfies the conditions (i) and (ii).

After a rotation of $\mathbb{R}^n$, we may suppose $h(x) = x_n$. We write $x = (x', x_n)$, where $x' = (x_1, \dots, x_{n-1})$. Set $M = \mathbb{R}^n \setminus X$ and, for any $c \in \mathbb{R}$,

$$M_{\leq c} := \{x \in M \mid x_n \leq c\}, \quad M_{<c} := \{x \in M \mid x_n < c\}.$$

Let $V \subset X$ be the set of vertices of X. Set $V = \{u_1, u_2, \dots, u_r\}, c_i = h(u_i)$ and $C = h(V) = \{c_1, c_2, \dots, c_r\}$ with $c_1 < c_2 < \cdots < c_r$.

Though the following lemma is clear intuitively, we give a proof to make sure.

Lemma 3.1. *The topological (resp. diffeomorphism) type of $M_{\leq c}$ is constant on $c_i < c < c_{i+1}$ and the topological (diffeomorphism) type of $M_{<c}$ is constant on $c_i < c \leq c_{i+1}$, $i = 0, 1, \dots, r$, with $c_0 = -\infty, c_{r+1} = \infty$. Here $M_{<\infty}$ means M itself.*

Proof: First we treat the case $i < r$. Take a sufficiently large $R > 0$ such that

$$\{x \in X \mid c_i < x_n < c_{i+1}, \|x'\| > R/2\} = \emptyset.$$

Consider the cylinder

$$C := \{x \in \mathbb{R}^n \mid c_i < x_n < c_{i+1}, \|x'\| \leq R\}.$$

Then $\mathscr{C} := \{\mathrm{Int}\, C \setminus X, X \cap C, \partial C\}$ is a Whitney stratification of C. The function $h : C \to (c_i, c_{i+1})$ is proper and the restriction of h to each stratum is a submersion. Now we follow the standard method (the proof of Thom's first isotopy lemma [11, 7]) to show differentiable triviality of mappings. Note that the flow used in the proof of isotopy lemma is differentiable in each stratum. For any $\varepsilon > 0$, take a vector field η over (c_i, c_{i+1}) such that $\eta = 0$ on $(c_i, c_i + \varepsilon/2)$ and $\eta = \partial/\partial y$ on $(c_i + \varepsilon, c_{i+1})$, where y is the coordinate on $\mathbb{R}$. Then η lifts to a controlled vector field ξ over C such that ξ tangents to each stratum. We extend $\xi|_{\partial c}$ to $\{x \in \mathbb{R}^n \mid c_i < x_n < c_{i+1}, \|x'\| \geq R\}$ via the retraction $x = (x', x_n) \mapsto (\frac{1}{\|x'\|} R x', x_n)$ and to $\{x \in \mathbb{R}^n \mid x_n < c_i + \varepsilon/2\}$ by letting it 0, and we have an integrable vector field ξ on $\{x \in \mathbb{R}^n \mid x_n < c_{i+1}\}$. By integrating ξ, we have a homeomorphism of $M_{\leq c}$ and $M_{\leq c'}$ for any $c, c' \in (c_i, c_{i+1})$ and a diffeomorphism of $M_{<c}$ and $M_{<c'}$ for any $c, c' \in (c_i, c_{i+1}]$. Note that the differentiable flow of the vector field may not be defined through $x_n = c_{i+1}$ but it gives a diffeomorphism of $M_{<c}$ and $M_{<c_{i+1}}$.

Second we treat the case $i = r$. Consider the quadratic cone $\|x'\|^2 - R x_n^2 = 0$ in $\mathbb{R}^n$. Supposing $c_{r+1} > 0$ after a translation along x_n-axis in necessary, and taking R sufficiently large, we have that $X \cap \{x \in \mathbb{R}^n \mid c_{r+1} < x_n\}$ lies inside of the cone $\|x'\|^2 - R x_n^2 < 0$. Now set

$$D := \{x \in \mathbb{R}^n \mid c_{r+1} < x_n, \|x'\|^2 - R x_n^2 \leq 0\},$$

and consider the proper map $h : D \to (c_{r+1}, \infty)$ with the Whitney stratification

$$\mathscr{D} := \{\mathrm{Int}\, D \setminus X, X \cap D, \partial D\}.$$

For any $\varepsilon > 0$, take a (non-complete) vector field η over (c_{r+1}, ∞) such that $\eta = 0$ on $(c_{r+1}, c_{r+1} + \varepsilon/2)$ and $\eta = (1 + y^2)\partial/\partial y$ on (c_{r+1}, ∞). We lift η to a controlled vector filed ξ over D and then over $\mathbb{R}^n$. Then, using the integration of ξ, we have a diffeomorphism of $M_{\leq c}$ and $M_{\leq c'}$ for any $c, c' \in (c_i, c_{i+1})$, and a diffeomorphism of $M_{<c}$ and $M_{<c'}$ for any $c, c' \in (c_i, c_{i+1}]$. In particular we have that $M_{<c}$ for $c_{r+1} < c$ is diffeomorphic to M itself. $\square$

Remark 3.2. The topological (resp. diffeomorphism) type of $M_{\leq c}$ (resp. $h^{-1}(c) \setminus X$) is not necessarily constant at $c = c_{i+1}$.

We observe the topological change of $M_{<c}$ when c moves across a critical value c_i as follows:

Lemma 3.3. *Let u be a vertex of X and let $c = h(u)$. Let $s = s(u)$ denote the number of edges of X which are adjacent to u from above with respect to h.*

Then, for a sufficiently small $\varepsilon > 0$, the open set $M_{<c+\varepsilon}$ is diffeomorphic to the interior of

$$M_{\leq c-\varepsilon} \bigcup_{\varphi} \Big(\bigsqcup_{i=1}^{s-1} (D_i^2 \times D_i^{n-2}) \Big),$$

obtained by an attaching map

$$\varphi : \bigsqcup_{i=1}^{s-1} D^2 \times \partial(D^{n-2}) \longrightarrow h^{-1}(c-\varepsilon) \setminus X \ = \partial(M_{\leq c-\varepsilon}) \subset M_{\leq c-\varepsilon},$$

of $(s-1)$ number of trivial handles of index $n-2$, provided $s \geq 1$.

In particular $M_{<c+\varepsilon}$ is diffeomorphic to $M_{<c-\varepsilon}$ if $s = 1$.

If $s = 0$ then $M_{<c+\varepsilon}$ is diffeomorphic to the interior of $M_{\leq c-\varepsilon} \bigcup_{\varphi} (D^1 \times D^{n-1})$ obtained by an attaching map $\varphi : D^1 \times \partial(D^{n-1}) \to h^{-1}(c-\varepsilon) \setminus X$ of a (not necessarily trivial) handle of index $n-1$. (See Figure 6.)

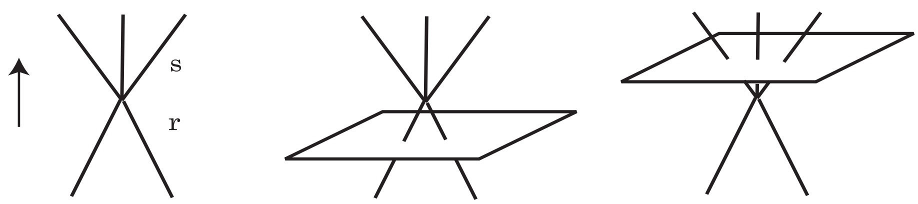

FIGURE 6. Topological bifurcations.

Remark 3.4. In the case $s = 0$, the handle attachment is not necessarily trivial since the core of the attachment does not necessarily bounds a disk. (See Figure 13.)

Remark 3.5. Note that if $r = r(u)$ denotes the number of edges of X which are adjacent to p from *below* with respect to h, then the intersection $X \cap h^{-1}(c-\varepsilon)$ consists of r-points in the hyperplane $h^{-1}(c-\varepsilon)$ and thus $h^{-1}(c-\varepsilon) \setminus X$ is a punctured hyperplane by r-points.

Remark 3.6. Note that locally in a neighbourhood of each vertex u of X, the topological equivalence class of the germ of a generic height function $h : (\mathbb{R}^n, X, u) \to (\mathbb{R}, c)$ is determined only by s and r, the numbers of branches. This can be shown by using Thom's isotopy lemma ([7]).

Proof of Lemma 3.3. For sufficiently small $0 < \varepsilon < \varepsilon'$, $M_{<c-\varepsilon} \setminus M_{\leq c-\varepsilon'}$ is a space

$$\{x \in \mathbb{R}^n \mid c - \varepsilon' < h(x) < c - \varepsilon\}$$

deleted r-half-lines. We may suppose the intersection $X \cap h^{-1}(c-\varepsilon)$ lies on a line, up to a diffeomorphism of $M_{\leq c-\varepsilon}$. We delete r-small tubular neighbourhoods of the half-lines from the half space, then still we have a diffeomorphic space to $M_{<c-\varepsilon} \setminus M_{\leq c-\varepsilon'}$. Then we connect the r-holes by boring a sequence of canals without changing the diffeomorphism type of complements. See Figures 7 and 8. The boring a canal means, in general dimension, to delete $D^1 \times D^{n-1}$ along the line segment connecting the holes.

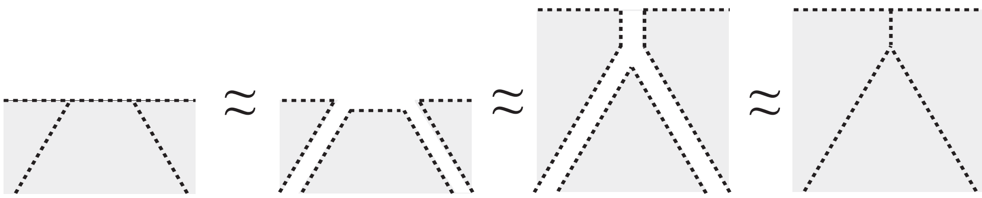

FIGURE 7. No topological changes of complements occur when $s = 1$.

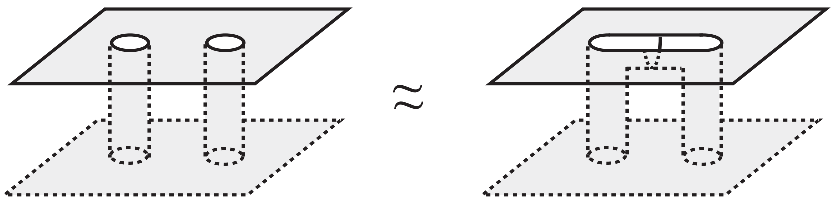

FIGURE 8. Boring a canal does not change the topology of ground.

First let $s = 1$. Then the resulting space is diffeomorphic to $M_{<c+\varepsilon} \setminus M_{\leq c-\varepsilon'}$. The diffeomorphism is taken to be the identity on $M_{\leq c-\varepsilon'}$ and it extends to a diffeomorphism between $M_{<c-\varepsilon}$ and $M_{<c+\varepsilon}$. This shows Lemma 3.3 in the case $s = 1$.

Next we teat the case $s = 2, r = 0$. The topological change from $M_{c-\varepsilon}$ to $M_{c+\varepsilon}$ is give by digging a tunnel, which is, equivalently, given by a handle attaching of index $n-2$. In fact, we examine the topological change of the complement to

$$\sqcup = \{(0, x_{n-1}, x_n) \in \mathbb{R}^n \mid (-2 \leq x_{n-1} \leq 2, x_n = 0) \text{ or } (x_{n-1} = -2, x_n \geq 0) \text{ or } (x_{n-1} = 2, x_n \geq 0)\},$$

in $\mathbb{R}^n$ when x_n goes across $x_n = c = 0$. Take the closed tube T of radius 1 of $\sqcup$. Then for the complement $M = \mathbb{R}^n \setminus T$, $M_{<\varepsilon}$ is diffeomorphic to the interior of the half space $\{x_n \leq 0\}$ attached the handle

$$H = \{x \in \mathbb{R}^n \mid -1 \leq x_{n-1} \leq 1,\ \frac{1}{2} \leq x_1^2 + \cdots + x_{n-2}^2 + x_n^2 \leq 2,\ x_n \geq 0\}.$$

along

$$H \cap \{x_n \leq 0\} = \{x \in \mathbb{R}^n \mid -1 \leq x_{n-1} \leq 1,\ \frac{1}{2} \leq x_1^2 + \cdots + x_{n-2}^2 \leq 2\}.$$

The pair $(H, H \cap \{x_n \leq 0\})$ is diffeomorphic to the pair $(D^2 \times D^{n-2}, D^2 \times \partial D^{n-2})$, where the core $(0 \times D^{n-2}, \partial D^{n-2})$ corresponds to

$$\{x_1^2 + \cdots + x_{n-2}^2 + x_n^2 = 1, x_{n-1} = 0, x_n \geq 0\} \quad \text{and} \quad \{x_1^2 + \cdots + x_{n-2}^2 = 1, x_{n-1} = 0, x_n = 0\}.$$

Note that the latter bounds an $n-1$-dimensional disk $\{x_1^2 + \cdots + x_{n-2}^2 \leq 1, x_{n-1} = 0, x_n = 0\}$, which does not touch the boundary $\partial M_{<\varepsilon}$. See Figures 9 and 10.

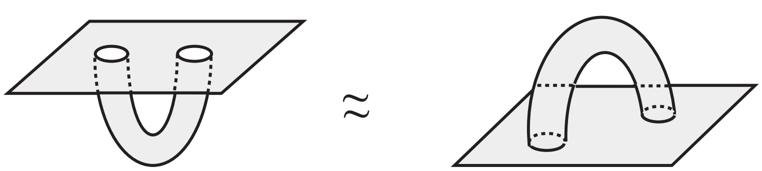

FIGURE 9. Digging a tunnel is same as bridging for the topology of ground.

The same argument works for any r. See Figure 10 for the case $s = 2, r = 2$. Note that complements to "X" and "H" are diffeomorphic. See Figures 10, 11 and 12.

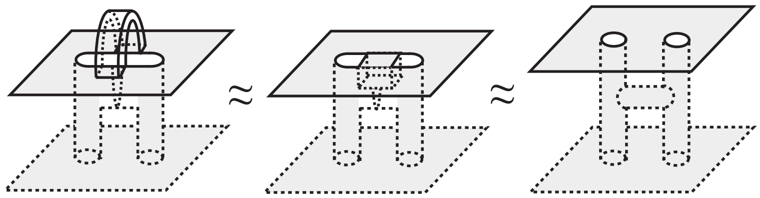

FIGURE 10. The case $s = 2, r = 2$.

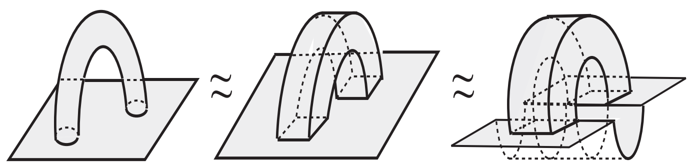

FIGURE 11. Trivial handle attachment and topological bifurcation.

In general, for any $s \geq 2$, the topological change is obtained by attaching trivial $s-1$ handles of index $n-2$. See Figure 12.

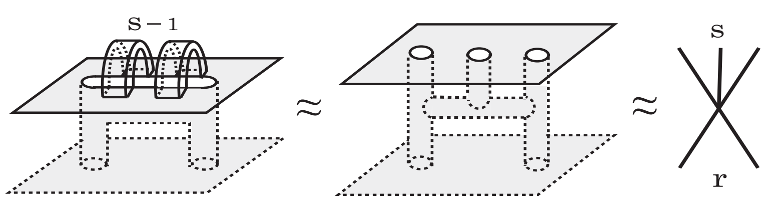

FIGURE 12. The case $s = 3, r = 2$.

In the case $s = 0$, contrarily to above, the change of diffeomorphism type is obtained by an attaching not necessarily trivial handle. See Figure 13.

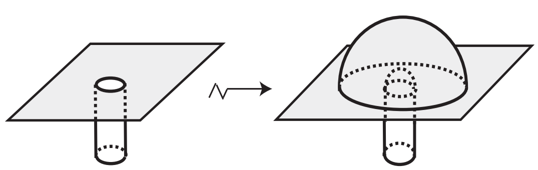

FIGURE 13. Topological change in the case $s = 0$.

When $n = 2$, the topological bifurcation occurs just as putting $s-1$ number of disjoint open disks. Thus we have Lemma 3.3. □

First let us apply Lemma 3.1 and Lemma 3.3 to the case $n = 2$.

For a $c \in \mathbb{R}$ of sufficiently large $|c|$, supposing a generic height function is given by $h = x_2$ as above. Then $M_{\leq c}$ (resp. $M_{<c}$) is diffeomorphic to the half plane $\{x_2 \leq c\}$ (resp. $\{x_n < c\}$ deleted d number of half lines. The number of connected components is equal to $1+d$. By passing a multiple point of multiplicity i, then by Lemma 3.3, we see that the number of connected components of $M_{\leq c}$ (resp. $M_{<c}$) increases exactly by $(i-1)$. Thus, after passing all multiple points, the number of connected components of $M_{<c}$, which is homeomorphic to $M(\mathscr{A})$, is given by $1+d+\sum_{i=2}^{d}(i-1)t_i$.

Proof of Theorem 1.5. For a $c \in \mathbb{R}$ with $c \ll 0$, the space $M_{\leq c}$ (resp. $M_{<c}$) is diffeomorphic to the half space $\{x_n \leq c\}$ (resp. $\{x_n < c\}$ deleted d number of half lines. By passing a multiple point of multiplicity i, for a sufficiently large c, the space $M_{\leq c}$ is obtained by attaching $i-1$ number of trivial handles of index $n-2$, by Lemma 3.3. After passing all multiple points, the space $M_{\leq c}$ is diffeomorphic to the space obtained by attaching $\sum_{i=2}^{d}(i-1)t_i$ number of trivial handles of index $n-2$ to the half space deleted d number of half lines. Then $M_{<c}$ is diffeomorphic to the interior of B_g with $g = d+\sum_{i=2}^{d}(i-1)t_i$. By Lemma 3.1, for $c \in \mathbb{R}$ with $0 \ll c$, $M_{<c}$ is diffeomorphic to $M(\mathscr{A})$. Hence we have Theorem 1.5. $\square$

Proofs of Theorem 1.3 and Theorem 1.1. Theorem 1.3 follows from Theorem 1.5 and Theorem 1.1 follows from Theorem 1.3 by setting $n = 3$. $\square$

Remark 3.7. Let X be a subset of $\mathbb{R}^n$ which is a union of finite number of closed line segments and half lines. Then similarly to the proof of Theorem 1.1 using Lemma 3.3, we see that, if there exists a height function $h : \mathbb{R}^n \to \mathbb{R}$ satisfying (i)(ii) such that $h|_X : X \to \mathbb{R}$ has no local maximum, then the complement $\mathbb{R}^n \setminus X$ is diffeomorphic to the interior of n-ball with trivially attached g-handles of index $n-2$, for some g. If $X \subset \mathbb{R}^n$ is compact, then any height function has a maximum, so non-trivial attachments may occur.

Remark 3.8. The knot complements have more information than line arrangement complements. For example, it is known that, for knots $K, K' \subset S^3$, if $S^3 \setminus K$ and $S^3 \setminus K'$ are homeomorphic, then the pairs (S^3, K) and (S^3, K') are homeomorphic ([2]). Taking account of it, consider $(\mathbb{R}^3, X)$ for a line arrangement $\mathscr{A} = \{\ell_1, \ldots, \ell_d\}$ in $\mathbb{R}^3$ and $X := \bigcup_{i=1}^{d} \ell_i \subset \mathbb{R}^3$ and its one-point compactification $(S^3, \overline{X})$. Then the complement $S^3 \setminus \overline{X}$ is homeomorphic to $M(\mathscr{A})$ and to B_g, which depends only on the number

$$g = d + \sum_{i=1}^{d}(i-1)t_i,$$

while g does not determine the topological type of the pair $(S^3, \overline{X})$ in general.

4. Projective line and linear plane arrangements

Let $\widetilde{\mathscr{A}} = \{\widetilde{\ell}_1, \ldots, \widetilde{\ell}_2, \ldots, \widetilde{\ell}_d\}$ be a real projective line arrangement in the projective space $\mathbb{R}P^n$ and let $\mathscr{B} = \{L_1, L_2, \ldots, L_d\}$ be the real linear plane arrangement in $\mathbb{R}^{n+2}$ corresponding to $\widetilde{\mathscr{A}}$. Then the complement $M(\mathscr{B})$ of $\mathscr{B}$ is homeomorphic to the link complement $S^n \cap M(\mathscr{B})$ times $\mathbb{R}_{>0}$, where S^n is a sphere in $\mathbb{R}^{n+1}$ centred at the origin. Moreover $S^n \cap M(\mathscr{B})$ is a double cover of $M(\widetilde{\mathscr{A}})$ for the corresponding projective line arrangement $\widetilde{\mathscr{A}}$ in $\mathbb{R}P^n$.

Take a projective hyperplane $H \subset \mathbb{R}P^n$ such that H intersects transversely to all lines $\widetilde{\ell}_i, 1 \leq i \leq d$, and that H does not pass through any multiple point of $\widetilde{\mathscr{A}}$. Then identify $\mathbb{R}P^n \setminus H$ with the affine space $\mathbb{R}^n$ and the affine line arrangement $\mathscr{A}$ obtained by setting $\ell_i := \widetilde{\ell}_i \setminus H \subset \mathbb{R}^n$. Take a ball

$$D^n = \{x \in \mathbb{R}^n \mid \|x\| \leq r\} \subset \mathbb{R}^n$$

for a sufficiently large radius r such that interior of D^n contains all multiple points of $\mathscr{A}$ and the boundary $\partial(D^n) = S^{n-1}$ intersects transversally to all lines $\ell_i, 1 \leq i \leq d$. Then the closure $\overline{U}$ of $U := \mathbb{R}P^n \setminus D^n$

is regarded as a tubular neighbourhood of H in $\mathbb{R}P^n$. The closure $\overline{U}$ is homeomorphic to the space $(S^{n-1}\times[-1,1])/\sim$, where $(x,t)\sim(-x,-t)$. Let $a_1,\dots,a_{2d}$ be disjoint $2d$ points in S^{n-1}.

Let $W_k^{n-1}\subset S^{n-1}$ be a sufficiently small open disk neighbourhood of $a_k,(1\le k\le 2d)$. Set

$$N := S^{n-1}\setminus W_k^{n-1} \quad \text{and} \quad \widetilde{N} := (N\times[-1,1])/\sim \ (\subset\ (S^{n-1}\times[-1,1])/\sim).$$

Then $\widetilde{N}$ is an n-dimensional manifold with boundary N, which is doubly covered by a "punctured shell" $N\times[-1,1]$ (see Figure 14).

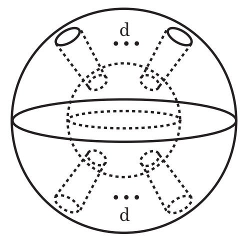

FIGURE 14. Punctured shell.

Thus we observe

Proposition 4.1. *The intersection $U\cap M(\widetilde{\mathscr{A}})$ is homeomorphic to the interior of $\widetilde{N}$. The complement $M(\widetilde{\mathscr{A}})\subset\mathbb{R}P^n$ is homeomorphic to the interior of $B_g\bigcup_\varphi\widetilde{N}$ for an attaching embedding $\varphi: N\to\partial(B_g)$. The homeomorphism class of $M(\widetilde{\mathscr{A}})$ is determined by the isotopy class of the embedding φ. The embedding φ is determined by the intersection of $M(\mathscr{A})$ and a hypersphere of sufficiently large radius in $\mathbb{R}^n$.*

Proof: We see that the intersection of $M(\mathscr{A})$ and a hypersphere of sufficiently large radius in $\mathbb{R}^n$ is homeomorphic to the sphere deleted $2d$-points. Then we have Proposition 4.1 by Theorem 1.3. □

REFERENCES

[1] R.E. Gompf, A.I. Stipsicz, *4-manifolds and Kirby calculus,* Graduate Studies in Math. **20**, Amer. Math. Soc., (1999). DOI: 10.1090/gsm/020

[2] C. McA. Gordon, J. Luecke, *Knots are determined by their complements,* Bull. Amer. Math. Soc., **20-1** (1989), 83–87. DOI: 10.1090/s0273-0979-1989-15706-6

[3] M. Goresky, R. MacPherson, *Stratified Morse theory,* Springer-Verlag, Berlin, Heidelberg, New York, (1988).

[4] B. Grünbaum, *Arrangements and Spreads,* Regional Conference Series in Math. **10**, Amer. Math. Soc. (1972).

[5] D.A. Gudkov, *The topology of real projective algebraic varieties,* Uspekhi Mat. Nauk, **29-4** (1974), 3–79; transl. Russ. Math. Surveys, **29-4** (1974), 1–79. DOI: 10.1070/rm1974v029n04abeh001288

[6] J. W. P. Hirschfeld, *A configuration of lines in three dimensions,* Proc. Edinburgh Math. Soc., **18-2** (1972), 105–123.

[7] J.N. Mather, *Stratifications and mappings,* Dynamical Systems, Academic Press, (1973), pp. 195–232.

[8] G. Mikhalkin, S. Orevkov, *Real algebraic knots and links of low degree,* J. Knot Theory Ramifications **25** (2016), 34 pages. DOI: 10.1142/S0218216516420104

[9] J.W. Milnor, *Topology from the differentiable view point,* Princeton Univ. Press (1997).

[10] P. Orlik, H. Terao, *Arrangements of Hyperplanes,* Grundlehren Math. Wiss. **300**, Springer-Verlag, New York, (1992).

[11] R.Thom, *Ensembles et morphismes stratifies,* Bull. Amer. Math. Soc., **75** (1969), 240–284. DOI: 10.1090/s0002-9904-1969-12138-5

[12] V.A. Vasiliev *Topology of plane arrangements and their complements,* Uspekhi Mat. Nauk, **56-2** (2001) 167–203; transl. Russ. Math. Surv., **56-2** (2001), 365–401. DOI: 10.1070/rm2001v056n02abeh000384

[13] O.Ya. Viro, *Topological problems on lines and points of three-dimensional space,* Dokl. Akad. Nauk SSSR **284-5** (1985), 1049–1052; transl. Soviet Math. Dokl. **32-2** (1985), 528–531.

[14] O.Ya. Viro, *Progress in the topology of real algebraic varieties over the last six years,* Uspekhi Mat. Nauk, **41-3** (1986), 45–67; transl. Russ. Math. Surveys, **41-3** (1986), 55–82. DOI: 10.1070/rm1986v041n03abeh003317

[15] O.Ya. Viro, Yu.V. Drobotukhina, *Configurations of skew-lines,* Algebra i Analiz **1-4** (1989), 222–246; transl. Leningrad Math. J. **1-4**(1990), 1027–1050.
[16] J. Viro, O. Viro, *Configurations of skew lines.* arχiv: math/0611374 [math.GT]
[17] C.T.C. Wall, *Differential Topology,* Cambridge studies in advanced math., **156**, Cambridge Univ. Press, (2016).
[18] M. Yoshinaga, *Minimality of hyperplane arrangements and basis of local system cohomology,* Singularities in geometry and topology, IRMA Lect. Math. Theor. Phys., **20**, Eur. Math. Soc., Zürich, (2012), pp. 345–362. DOI: 10.4171/118-1/16
[19] G.M. Ziegler, *On the difference between real and complex arrangements,* Math. Z. **212-1** (1993), 1–11.
[20] G.M. Ziegler, R.T. Zivaljevi, *Homotopy types of subspace arrangements via diagrams of spaces,* Math. Ann., **295**(1993), 527–548. DOI: 10.1007/bf01444901

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCES, HOKKAIDO UNIVERSITY, SAPPORO 060-0810, JAPAN
Email address: ishikawa@math.sci.hokudai.ac.jp

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCES, HOKKAIDO UNIVERSITY, SAPPORO 060-0810, JAPAN
Email address: oyama.motoki@gmail.com

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