Journal *of* Singularities
Volume 22 (2020), 114-133

Proc. of 15th International Workshop on Singularities, São Carlos, 2018
DOI: 10.5427/jsing.2020.22h

APPARENT CONTOURS OF STABLE MAPS OF SURFACES WITH BOUNDARY INTO THE PLANE

TAKAHIRO YAMAMOTO

Dedicated to Professor Takashi Nishimura on the occasion of his 60th birthday.

Abstract. Let M be a connected compact surface with boundary. A C^∞ map $M \to \mathbb{R}^2$ is admissible if it is non-singular on a neighborhood of the boundary. For a C^∞ stable map $f\colon M \to \mathbb{R}^2$, denote by $c(f)$ and $n(f)$, $i(f)$ the number of cusps and nodes, connected components of the set of singular points respectively. In this paper, we introduce the notion of admissibly homotopic among C^∞ maps $M \to \mathbb{R}^2$, and we will determine the minimal number $c + n$ for each admissibly homotopy class.

1. Introduction

Let M be a connected compact surface with boundary ∂ and P a surface without boundary. Denote by $C^\infty(M, P)$ the set of C^∞ maps $M \to P$ equipped with the Whitney C^∞ topology. A C^∞ map $f\colon M \to P$ is called a C^∞ *stable map*, (or *stable map* for short), if there exists a neighborhood $N(f) \subset C^\infty(M, P)$ of f such that every map $g \in N(f)$ is C^∞ right-left equivalent[1] to f. A C^∞ map $f\colon M \to P$ is stable if and only if f has fold, cusp and B_2 as its singularities, and $f|_{(S(f)\cup\partial)\setminus(C(f)\cup B(f))}$ is an immersion with normal crossings, where $C(f)$ and $B(f)$ denote the set of cusp points and B_2 points of f respectively, see Proposition 2.2 for details.

Note that if a C^∞ map $f\colon M \to P$ is stable, then $f|_\partial\colon \partial \to P$ is stable. Note also that a B_2 point is a fold point (or regular point) if we ignore the boundary (resp. we restrict f to boundary).

A C^∞ map $f\colon M \to P$ is called *admissible* if it is submersive on an open neighborhood of the boundary. Note that a C^∞ stable map $f\colon M \to P$ is admissible if and only if it has no B_2 points.

For a C^∞ stable map $f\colon M \to P$, denote by $c(f)$ and $n(f)$, $i(f)$ the numbers of cusps and nodes, connected components of the set singular points of f respectively.

Denote by M_k a connected compact surface with exactly k boundary components. A connected compact and orientable (or non-orientable) surface of genus g with exactly k boundary components is denoted by $\Sigma_{g,k}$ (resp. $N_{g,k}$). The 2-dimensional sphere and the plane are denoted by S^2 and $\mathbb{R}^2$ respectively.

For a C^∞ map $f\colon M \to P$, define the set of singular points of f as

$$S(f) = \{p \in M \mid \operatorname{rank} d_p f < 2\}.$$

We call $f(S(f))$ the *apparent contour* (or *contour* for short) of f and denote it by $\gamma(f)$. For a closed surface M, the apparent contour of a stable map $M \to P$ ($P = \mathbb{R}^2$, S^2) relates the topology of M as classical result of Thom [11] and a formula obtained by Pignoni [9] show.

2000 *Mathematics Subject Classification.* Primary 57R45; Secondary 57R35, 57R90, 58K65 .
Key words and phrases. stable map, cusps, nodes.

[1]Two maps f, $g \in C^\infty(M, P)$ are C^∞ *right-left equivalent* if there exist a diffeomorphism $\Phi\colon M \to M$ preserving the boundary and a diffeomorphism $\psi\colon P \to P$ such that $f \circ \Phi = \psi \circ g$.

Pignoni [9] introduced the notion of a minimal contour of a closed surface: The contour $\gamma(f)$ of a stable map $f\colon M \to \mathbb{R}^2$ is called a *minimal contour* of M if the number $c(f)+n(f)$ is the smallest among stable maps $g\colon M \to \mathbb{R}^2$ which satisfy $i(g)=1$. Then, Demoto [2] introduced the notion of a minimal contour of a C^∞ map $f_0\colon M \to P$ between surfaces and studied that of a C^∞ map $S^2 \to S^2$: Let $f_0\colon M \to P$ be a C^∞ map and $f\colon M \to P$ a C^∞ stable map which is homotopic to f_0 and satisfies $i(f)=1$. Call $\gamma(f)$ a *minimal contour* of f_0 if the number $c(f)+n(f)$ is the smallest among C^∞ stable maps $g\colon M \to P$ which are homotopic to f_0 and $i(g)=1$. Then, Kamenosono and the author [7] studied minimal contours of C^∞ maps $M \to S^2$ of closed surfaces M. Apparent contours of stable maps between surfaces were also studied in [15, 16, 3, 17]. Studying minimal contours of C^∞ maps make the very first step toward classifying generic C^∞ maps of surfaces up to right-left equivalence.

In this paper, we study minimal contour of C^∞ maps of surfaces with boundary. More precisely, for a surface M with boundary and a surface P without boundary, we introduce the notion of *admissibly homotopic* which is an equivalence relation among admissible C^∞ maps $M \to P$, and *admissible minimal contour* of an admissible C^∞ map $M \to P$. Then, we study admissible minimal contours of admissible C^∞ maps $M_1 \to \mathbb{R}^2$.

This paper is organized as follows. In §2, we prepare some notions and introduce the main-theorems (Theorems 2.3 and 2.5). In §3, we prepare some notions concerning stable maps $f\colon M_k \to \mathbb{R}^2$ ($k \geq 1$) and introduce the formula as an application of formulas obtained by Pignoni [9] and Imai [6]. In §4, we construct admissible stable maps $\Sigma_{g,1} \to \mathbb{R}^2$ ($g \geq 0$) and $N_{g,1} \to \mathbb{R}^2$ ($g \geq 1$) which are in the lists of Theorem 2.3 and 2.5 respectively. In §5, we show the contours of stable maps constructed in § 4 are admissible minimal contours. In §6, we pose a problem which concerns the apparent contours of stable fold maps $f\colon M_k \to \mathbb{R}^2$, where a stable map $f\colon M_k \to \mathbb{R}^2$ of a surface with boundary is called *fold map* if it has no cups as its singularities.

Throughout this paper, all surfaces are connected and smooth of class C^∞, and all maps are smooth of class C^∞ unless stated otherwise. The symbols r and $g \geq 0$ denote integers. For a topological space X, id_X denotes the identity map of X.

2. Main-Theorem

In this section, we introduce some notions and introduce the main-theorems (Theorems 2.3 and 2.5).

Let M_k be a compact and connected surface with exactly k boundary components $\partial_1 \cup \cdots \cup \partial_k$. Then, admissible C^∞ maps $f_0, f_1\colon M_k \to \mathbb{R}^2$ are said *admissibly homotopic* if there exists a C^∞ map $H\colon M_k \times [0,1] \to \mathbb{R}^2$ such that $H_t = H(\cdot, t)\colon M_k \to \mathbb{R}^2$ is an admissible C^∞ map for each $t \in [0,1]$, and $H_0 = f_0$ and $H_1 = f_1$.

Let $f\colon M_k \to \mathbb{R}^2$ be an admissible C^∞ map. Then, for each component ∂_j, orient the regular curve $f(\partial_j) \subset \mathbb{R}^2$ so that at each point, the inner of $f(M_k)$ is in the left hand side. Note that the definition of the orientation for $f(\partial_j) \subset \mathbb{R}^2$ is well-defined by virtue of the assumption that f is admissible. Then, call the rotation number of $f(\partial_j) \subset \mathbb{R}^2$ the *boundary rotation number* of ∂_j (or *rotaion number* of ∂_j for short) with respect to f and denote it by $W(f; \partial_j)$. If $k=1$, then call the rotation number of $f(\partial) \subset \mathbb{R}^2$ the *boundary rotation number* of f and denote it by $W(f)$. Furthermore, in the case that $M = \Sigma_g$ and $k=1$, define $s(f) = +1$ (or -1) if there exists a neighborhood of $N(\partial)$ of ∂ such that $f|_{N(\partial)}$ preserves (resp. reverses) the orientation of $N(\partial)$.

Proposition 2.1. (1) *Admissible stable maps $f_0, f_1\colon \Sigma_{g,1} \to \mathbb{R}^2$ are admissibly homotopic if and only if $W(f_0) = W(f_1)$ and $s(f_0) = s(f_1)$.*

(2) *Admissible stable maps* $f_0, f_1\colon N_{g,1} \to \mathbb{R}^2$ *are admissibly homotopic if and only if* $W(f_0) = W(f_1)$.

Proof. (1) If f_0 and f_1 are admissibly homotopic, then $s(f_0) = s(f_1)$ and regular curves $f_0(\partial)$ and $f_1(\partial)$ are regularly homotopic. It implies that $W(f_0) = W(f_1)$.

We consider the opposite direction. If $W(f_0) = W(f_1)$, then regular curves $f_0(\partial)$ and $f_1(\partial)$ with the canonical orientation are regularly homotopic. Thus, there exists a C^∞ map

$$H'\colon \partial \times [0,1] \to \mathbb{R}^2$$

so that $H'(\cdot, 0) = f_0|_\partial$ and $H'(\cdot, 1) = f_1|_\partial$. Then, we can extend H' to a C^∞ map

$$H''\colon N(\partial) \times [0,1] \to \mathbb{R}^2$$

on a neighborhood of ∂ so that $H''|_{\partial\times[0,1]} = H'$ and $H''_t = H''(\cdot, t)\colon N(\partial) \to \mathbb{R}^2$ is a submersion for any $t \in [0,1]$. Note that if $s(f_0) = s(f_1) = +1$ (or $s(f_0) = s(f_1) = -1$), then $H''_t = H''(\cdot, t)$ is an immersion which preserves (resp. reverses) orientation of a neighborhood of ∂ for each $t \in [0,1]$. On the other hand, we decompose $\Sigma_{g,1}$ into a simplicial complex. We also decompose $\Sigma_{g,1} \times [0,1]$ into a simplicial complex which is compatible with the simplicial decomposition of $\Sigma_{g,1}$. We define a map $H\colon \Sigma_{g,1} \times [0,1] \to \mathbb{R}^2$ by the following manner:

0-simplex: If a 0-simplex $\sigma =< a_0 >$ is in $N(\partial) \times [0,1]$ (or $\Sigma_{g,1} \times \{0\}$, $\Sigma_{g,1} \times \{1\}$), then we define $H(a_0) = H''(a_0)$ (resp. $H(a_0) = f_0(a_0)$, $H(a_0) = f_1(a_0)$). Otherwise, we define $H(a_0) = 0 \in \mathbb{R}^2$.

1-simplex: If a 1-simplex $\sigma =< a_0, a_1 >$ is in $N(\partial) \times [0,1]$, (or $\Sigma_{g,1} \times \{0\}$, $\Sigma_{g,1} \times \{1\}$), then $H|_\sigma$ is defined by $H|_\sigma = H''|_\sigma$ (resp. $H|_\sigma = f_0|_\sigma$, $H|_\sigma = f_1|_\sigma$). Otherwise, we define $H|_\sigma$ by $H(x) = \lambda_0 H(a_0) + \lambda_1 H(a_1)$, where $x = \lambda_0 a_0 + \lambda_1 a_1 \in \sigma$ with the property that $\lambda_i \in \mathbb{R}_{\geq 0}$ and $\lambda_0 + \lambda_1 = 1$.

2-simplex: If a 2-simplex $\sigma =< a_0, a_1, a_2 >$ is in $N(\partial) \times [0,1]$ (or $\Sigma_{g,1} \times \{0\}$, $\Sigma_{g,1} \times \{1\}$), then $H|_\sigma$ is defined by $H|_\sigma = H''|_\sigma$ (resp. $H|_\sigma = f_0|_\sigma$, $H|_\sigma = f_1|_\sigma$). Otherwise, we define $H|_\sigma$ by $H(x) = \lambda_0 H(a_0) + \lambda_1 H(a_1) + \lambda_2 H(a_2)$, where $x = \lambda_0 a_0 + \lambda_1 a_1 + \lambda_2 a_2 \in \sigma$ with the property that $\lambda_i \in \mathbb{R}_{\geq 0}$ $(i = 0, 1, 2)$, and $\lambda_0 + \lambda_1 + \lambda_2 = 1$.

3-simplex: If a 3-simplex $\sigma =< a_0, a_1, a_2, a_3 >$ is in $N(\partial) \times [0,1]$, then $H|_\sigma$ is defined by $H|_\sigma = H''|_\sigma$. Otherwise, we define $H|_\sigma$ by

$$H(x) = \lambda_0 H(a_0) + \lambda_1 H(a_1) + \lambda_2 H(a_2) + \lambda_3 H(a_3),$$

where $x = \lambda a_0 + \lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3 \in \sigma$ with the property that $a_i \in \mathbb{R}$, $a_i > 0$ $(i = 0, 1, 2, 3)$, and $a_0 + a_1 + a_2 + a_3 = 1$.

Then, by perturbing H slightly, if necessary, we obtain a desired C^∞ map $\Sigma_{g,1} \times [0,1] \to \mathbb{R}^2$. Namely, f_0 and f_1 are admissibly homotopic.

(2) The case of C^∞ maps $N_{g,1} \to \mathbb{R}^2$ is also proved by similar way of (1). We omit the proof here. □

C^∞ stable maps of compact and connected surfaces with boundary into surfaces without boundary are characterized by the following way.

Proposition 2.2 (Bluce and Giblin [1]). *Let M be a compact and connected surface possibly with boundary ∂ and P a surface without boundary. A C^∞ map $f\colon M \to P$ is C^∞ stable if and only if it satisfies the following conditions.*

(1) (*Local conditions*) *In the following, for $p \in \partial$, we use local coordinates (x, y) around p such that* Int$M$ *and ∂ correspond to the sets $\{y > 0\}$ and $\{y = 0\}$ respectively.*

(1a) *For $p \in \mathrm{Int}M$, the germ of f at p is right-left equivalent to one of the following:*

$$(x,y) \mapsto \begin{cases} (x,y), & p\text{: regular point,} \\ (x,y^2), & p\text{: fold point,} \\ (x,y^3+xy), & p\text{: cusp point.} \end{cases}$$

(1b) *For $p \in \partial$, the germ of f at p is right-left equivalent to one of the following:*

$$(x,y) \mapsto \begin{cases} (x,y) & p\text{: regular point of } f|_{N(\partial M)}, \\ (x,y^2+xy) & p\text{: } B_2 \text{ point.} \end{cases}$$

(2) (*Global conditions*) *For each $q \in f(S(f) \cup \partial)$, the multi-germ*

$$(f|_{S(f)\cup\partial}, f^{-1}(q) \cap (S(f) \cup \partial))$$

is right-left equivalent to one of the four multi-germs whose images are as depicted in Figure 1, where blue curves and gray curves represent $f(S(f))$ and $f(\partial)$ respectively: (1) represent immersion mono-germs $(\mathbb{R},0) \ni t \mapsto (t,0) \in (\mathbb{R}^2,0)$ which correspond to a single fold point or a single boundary point respectively, and (2) represents cusp mono-germ $(\mathbb{R},0) \ni t \mapsto (t^2,t^3) \in (\mathbb{R}^2,0)$ which correspond to a cusp point, (3) represents B_2 multi-germ which corresponds to a single point in $\partial \cap S(f)$, (4) represent normal crossings of two immersion germs, each of which corresponds to a fold point or a boundary point.

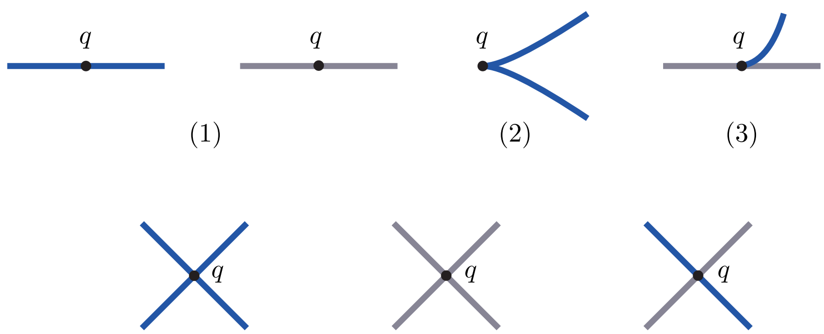

FIGURE 1. The images of multi-germs of $f|_{S(f)\cup S(f|_{\partial M})}$

Let $f_0\colon M_1 \to P$ be an admissible C^∞ map and $f\colon M_1 \to P$ an admissible C^∞ stable map which is admissibly homotopic to f_0. Call $\gamma(f)$ an *admissible minimal contour* of f_0 if the number $c(f)+n(f)$ is the smallest among stable maps $g\colon M_1 \to \mathbb{R}^2$ which are admissibly homotopic to f_0 and $i(g)=1$. Note that the number of connected components of the set of singular points is allowed to vary during admissible homotopy.

Theorem 2.3. *Let $g \geq 0$ be an integer and $f\colon \Sigma_{g,1} \to \mathbb{R}^2$ be a rotation number r admissible stable map. The contour $\gamma(f)$ is an admissible minimal contour if and only if the pair $(c(f),n(f))$ is one of the pairs below:*

$g=0$**:**

$$(c(f),n(f)) = \begin{cases} (r+1,0) & \text{if } r \geq 0, \\ (-r-1,-r-1) & \text{if } r \leq -1. \end{cases}$$

$g = 1$**:**

$$(c(f), n(f)) = \begin{cases} (r+3, 0) \ \textit{or} \ (r-1, 4) & \textit{if } r \geq 1, \\ (r+3, 0) & \textit{if } -2 \leq r \leq 0, \\ (-r-3, -r-3) & \textit{if } r \leq -3, \end{cases}$$

$g = 2$**:**

$$(c(f), n(f)) = \begin{cases} (r-3, 6) & \textit{if } r \geq 3, \\ (1, 5) & \textit{if } r = 2, \\ (r+1, 4) \ \textit{or} \ (r+5, 0) & \textit{if } -1 \leq r \leq 1, \\ (r+5, 0) & \textit{if } -4 \leq r \leq -2, \\ (-r-5, -r-5) & \textit{if } r \leq -5, \end{cases}$$

$g \geq 3$**:**

$$(c(f), n(f)) =$$

$$\begin{cases} (r-2g+1, 2g+2) & \textit{if } r \geq 2g-1, \\ (2, 6+2k) & \textit{if } r = 9-2g+4k, \ k = 0, \dots, g-3, \\ (1, 6+2k) & \textit{if } r = 8-2g+4k, \ k = 0, \dots, g-3, \\ (0, 6+2k) & \textit{if } r = 7-2g+4k, \ k = 0, \dots, g-3, \\ (1, 5+2k) & \textit{if } r = 6-2g+4k, \ k = 0, \dots, g-2, \\ (r+2g-3, 4) \ \textit{or} \ (r+2g+1, 0) & \textit{if } 3-2g \leq r \leq 5-2g, \\ (r+2g+1, 0) & \textit{if } -2g \leq r \leq 2-2g, \\ (-r-2g-1, -r-2g-1) & \textit{if } r \leq -1-2g. \end{cases}$$

Remark that the number $c+n$ of an admissible minimal contour of a C^∞ map $f_0 \colon \Sigma_{g,1} \to \mathbb{R}^2$ depend only on the boundary rotation number $W(f_0)$. It does not depend on the sign $s(f_0)$.

Corollary 2.4. *The number $c+n$ of an admissible minimal contour of a rotation number r admissible stable map $\Sigma_{g,1} \to \mathbb{R}^2$ is one of the items below:*

$$c+n = \begin{cases} r+3 & \textit{if } r \geq 2g-1, \\ (r+2g+5)/2 & \textit{if } 3-2g \leq r < 2g-1 \ \textit{and} \ r \equiv 3-2g \mod 4, \\ (r+2g+6)/2 & \textit{if } 2-2g \leq r < 2g-1 \ \textit{and} \ r \equiv 2-2g \ \textit{or} \ -2g \mod 4, \\ (r+2g+7)/2 & \textit{if } 1-2g \leq r < 2g-1 \ \textit{and} \ r \equiv 1-2g \mod 4, \\ r+2g+1 & \textit{if } -2g \leq r \leq 2-2g, \\ -2(r+1+2g) & \textit{if } r \leq -1-2g. \end{cases}$$

Theorem 2.5. *Let $g \geq 1$ be an integer and $h \colon N_{g,1} \to \mathbb{R}^2$ be a rotation number r admissible stable map. The contour $\gamma(h)$ is an admissible minimal contour if and only if the pair $(c(h), n(h))$ is one of the items below:*

$$(c(h), n(h)) = \begin{cases} (1, |g+r-4|/2) & \textit{if } r \geq 2-g \ \textit{and} \ r \equiv g \mod 2, \\ (0, |g+r-3|/2) & \textit{if } r \geq 1-g \ \textit{and} \ r \not\equiv g \mod 2, \\ (1, -(g+r)/2) & \textit{if } r \leq -g \ \textit{and} \ r \equiv g \mod 2, \\ (0, -(g+r+1)/2) & \textit{if } r \leq -1-g \ \textit{and} \ r \not\equiv g \mod 2. \end{cases}$$

3. Topological formula of apparent contour

In this section, we introduce topological formula of apparent contours of admissible stable maps $M \to \mathbb{R}^2$ of surfaces with boundary.

Let us recall some notions introduced by Pignoni [9]. Let M_k be a compact and connected surface with exactly k boundary components $\partial = \partial_1 \cup \cdots \cup \partial_k$ and $f\colon M_k \to \mathbb{R}^2$ an admissible stable map whose contour is non-empty. Then, for each component ∂_j, orient the regular curve $f(\partial_j) \subset \mathbb{R}^2$ so that at each point, the inner of $f(M_k)$ is in the left hand side. Note that the definition of the orientation for $f(\partial_j)$ is well-defined by virtue of the assumption that f is admissible. Let $S(f) = S_1 \cup \cdots \cup S_\ell$ be the decomposition of $S(f)$ into the connected components and set $\gamma_i = f(S_i)$ $(i = 1, \ldots, \ell)$. Note that $\gamma(f) = \gamma_1 \cup \cdots \cup \gamma_\ell$. For each γ_i, denote by U_i the unbounded component of $\mathbb{R}^2 \setminus \gamma_i$. Note that $\partial U_i \subset \gamma_i$.

Orient γ_i so that at each fold point image, the surface is "folded to the left hand side". More precisely, for a point $y \in \gamma_i$ which is not a cusp or a node, choose a normal vector v of γ_i at y such that $f^{-1}(y')$ contains more elements than $f^{-1}(y)$, where y' is a regular value of f close to y in the direction of v. Let τ be a tangent vector of γ_i at y such that the ordered pair (τ, v) is compatible with the given orientation of $\mathbb{R}^2$. It is easy to see that τ gives a well-defined orientation for γ_i.

Definition 3.1. A point $y \in \partial U_i \setminus \{\text{cusps, nodes}\}$ is said to be *positive* if the normal orientation v at y points toward U_i. Otherwise, it is said to be *negative*.

A component γ_i is said to be *positive* if all points of $\partial U_i \setminus \{\text{cusps, nodes}\}$ are positive; otherwise, γ_i is said to be *negative*. The numbers of positive and negative components are denoted by i^+ and i^- respectively.

By the geometrical condition of the surface $\Sigma_{g,1}$, we obtain the following lemma.

Lemma 3.2. *Let $f\colon \Sigma_{g,1} \to \mathbb{R}^2$ be an admissible stable map whose singular points set consists of one component. Then, the contour is a negative component.*

Definition 3.3. A point $y \in \partial U_i \setminus \{\text{cusps, nodes}\}$ is called an *admissible starting point* if y is a positive (or negative) point of a positive (resp. negative) component γ_i. Note that for each i, there always exists an admissible starting point on γ_i.

Definition 3.4. Let $y \in \gamma_i$ be an admissible starting point and $Q \in \gamma_i$ a node. Let $\alpha\colon [0,1] \to \gamma_i$ be a parameterization consistent with the orientation which is singular only when the image is a cusp such that $\alpha^{-1}(y) = \{0, 1\}$. Then, there are two numbers $0 < t_1 < t_2 < 1$ satisfying $\alpha(t_1) = \alpha(t_2) = Q$.

We say that Q is *positive* if the orientation of $\mathbb{R}^2$ at Q defined by the ordered pair $(\alpha'(t_1), \alpha'(t_2))$ coincides with that of $\mathbb{R}^2$ at Q; *negative*, otherwise.

The number of positive (or negative) nodes on γ_i is denoted by N_i^+ (resp. N_i^-). The definition of a positive (or negative) node on γ_i depends on the choice of an admissible starting point y. However, it is known that the algebraic number $N_i^+ - N_i^-$ does not depend on the choice of y, see [12] for details. Thus, the algebraic number $N^+ - N^- = \sum_{i=1}^{k}(N_i^+ - N_i^-)$ is well defined. Note that nodes arising from $\gamma_i \cap \gamma_j$ $(i \neq j)$ play no role in the computation.

Then, we have the following formula as an application of the formula of Pignoni [9] and Imai [6].

Proposition 3.5. *For an admissible stable map $f\colon M_k \to \mathbb{R}^2$, we have*

$$g = \varepsilon(M_k)\left((N^+ - N^-) + \frac{c(f)}{2} + (1 + i^+ - i^-) - \frac{1}{2}\sum_{j=1}^{k}(r_j + 1)\right) \tag{3.1}$$

where $\varepsilon(M_k)$ is equal to 1 if M_k is orientable or 2 if M_k is non-orientable, and r_j denotes the rotation number of $f|_{\partial_i}$.

Proof. To compute the Euler characteristic $\chi(M_k)$, apply a result of Levine [8]: For an admissible stable map $f\colon M_k \to \mathbb{R}^2$, we have

$$\chi(M_k) = \sum_{i=1}^{\ell} \tau(\gamma_i) + \frac{1}{2}\sum_{j=1}^{k} \tau(e_j),$$

where γ_i and e_j denote $f(S_i)$ and $f(\partial_j)$ respectively, and $\tau(\gamma_i)$ and $\tau(e_j)$ denote the double tangent turning number of γ_i and e_j with respect to the canonical orientation respectively. For an oriented closed curve α, the double tangent turning number $\tau(\alpha)$ is defined as the degree of the map $\alpha \to \mathbb{R}P^1$ assigning to each point on the curve its tangent line. This map is also defined at cusp points. If α has no cusps, then $\tau(\alpha) = 2r(\alpha)$ where $r(\alpha)$ denotes the normal degree of α. To compute $\tau(\alpha)$, apply a result of Quine [10]: For a closed plane curve α, we have

$$\tau(\alpha) = 2\eta(\alpha) + 2n^+ - 2n^- + c^+ - c^-,$$

where $\eta(\alpha) = \pm 1$ is defined according to the orientation of the curve α, c^+ (or c^-) denotes the number of positive (resp. negative) cusps of α, and n^+ (or n^-) the number of positive (resp. negative) nodes of α, see [10] for details. Comparing the definitions of the items in the Quine's formula with the ones introduced in this paper, we see: (a) the sign of the double points is the opposite of that defined by Quine; (b) when the contour is endowed with its canonical orientation, each cusp is negative. Thus,

$$\tau(\gamma_i) = 2\eta(\gamma_i) + 2N_i^- - 2N_i^+ - c_i,$$

where c_i denotes the number of cusps of γ_i. $\eta(\gamma_i) = +1$ if and only if γ_i is negative.

$$\sum_{i=1}^{k} \tau(\gamma_i) = 2i^- - 2i^+ + 2N^- - 2N^+ - c(f).$$

Each $f(\partial_j)$ is a closed curve with no cusp: $\tau(f(\partial_j)) = 2r_j$. Hence, by applying the formula of Levine to f, we obtain

$$\chi(M_k) = 2i^- - 2i^+ + 2N^- - 2N^+ - c(f) + \sum_{j=1}^{k} r_j. \tag{3.2}$$

Then, the result follows immediately. □

Corollary 3.6. *Let $f\colon \Sigma_{g,1} \to \mathbb{R}^2$ be an admissible stable map of rotation number r. Then, the number of cusps of f and the rotation number r never have the same parity.*

Lemma 3.7. *Let $f\colon \Sigma_{g,1} \to \mathbb{R}^2$ be an admissible stable map. If $\gamma(f)$ has a node, then it has at least one negative node.*

4. Admissible stable maps $M_1 \to \mathbb{R}^2$

In this section, we construct boundary rotation number $r \in \mathbb{Z}$ stable maps $f_{r,g}\colon \Sigma_{g,1} \to \mathbb{R}^2$ $(g \geq 0)$ and $h_{r,g}\colon N_{g,1} \to \mathbb{R}^2$ $(g \geq 1)$ whose singular points sets consist of one component and whose pairs (c, n) are in the lists of Theorems 2.3 and 2.5 respectively. Note that constructing such stable maps is a part of a proof of Theorem 2.3 (or Theorem 2.5).

Note that in Figures, boundary curves are drawn in gray and the image of boundary curves are also drawn in gray.

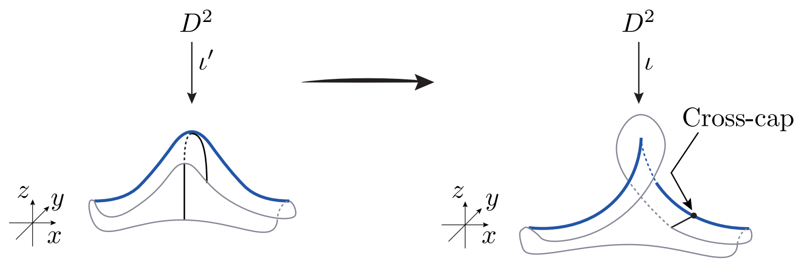

FIGURE 2. Modification I: By applying this modification, the rotation number increase by one.

4.1. **Admissible stable maps $\Sigma_{0,1} \to \mathbb{R}^2$.** For a boundary rotation number r' admissible stable map $f' \colon \Sigma_{g',1} \to \mathbb{R}^2$ whose singular points set consists of i' components and have c' cusps and n' nodes, by applying modifications I (or II, III) defined by Figure 2 (resp. Figures 3, 4), we obtain a boundary rotation number r admissible stable map $f \colon \Sigma_{g,1} \to \mathbb{R}^2$ whose singular points set consists of i components and has c cusps and n nodes. Note that a C^∞ map $\Sigma_{g,1} \to \mathbb{R}^2$ is locally defined by the projection $\mathbb{R}^3 \to \mathbb{R}^2$ into the xz-plane composed with a C^∞ map $\iota' \colon D^2 \to \mathbb{R}^3$ of the 2-dimensional disc. Figures 2, 3 and 4 represent modifications for a C^∞ map $\iota' \colon D^2 \to \mathbb{R}^3$. Note that the modified maps $\iota \colon D^2 \to \mathbb{R}^3$ in Figure 2 and 3, 4 have one cross-cap:

(1) Modification I (Figure 2):
$$(r, g, i, c, n) = (r' + 1, g', i', c' + 1, n')$$

(2) Modification II (Figures 3):
$$(r, g, i, c, n) = (r' - 1, g', i', c' + 1, n' + 1)$$

(3) Modification III (Figure 4):
$$(r, g, i, c, n) = (r' - 2, g' + 1, i', c', n')$$

Figure 5 define a rotation number -1 admissible stable map $f_{-1,0} \colon \Sigma_{0,1} \to \mathbb{R}^2$ whose triple (i, c, n) is equal to $(1, 0, 0)$. More precisely, $f_{-1,0}$ is defined by $f_{-1,0} = \pi_{xz} \circ \iota$.

By applying modification I inductively to $f_{-1,0}$, we obtain an admissible stable map
$$f_{r,0} \colon \Sigma_{0,1} \to \mathbb{R}^2$$
whose triple (i, c, n) is equal to $(1, r + 1, 0)$ for each integer $r \geq -1$.

By applying modification II inductively to $f_{-1,0}$, we obtain an admissible stable map
$$f_{r,0} \colon \Sigma_{0,1} \to \mathbb{R}^2$$
whose triple (i, c, n) is equal to $(1, -r - 1, -r - 1)$ for each integer $r \leq -1$.

4.2. **Admissible stable maps $\Sigma_{1,1} \to \mathbb{R}^2$.** For each integer $r' \leq 2$, by applying modification III to $f_{r',0}$, we obtain boundary rotation number $r \leq 0$ admissible stable maps $f_{r,1}$ whose triples (i, c, n) are one of the items below:
$$(i, c, n) = \begin{cases} (1, r + 3, 0) & \text{if } -2 \leq r \leq 0, \\ (1, -r - 3, -r - 3) & \text{if } r \leq -3. \end{cases}$$

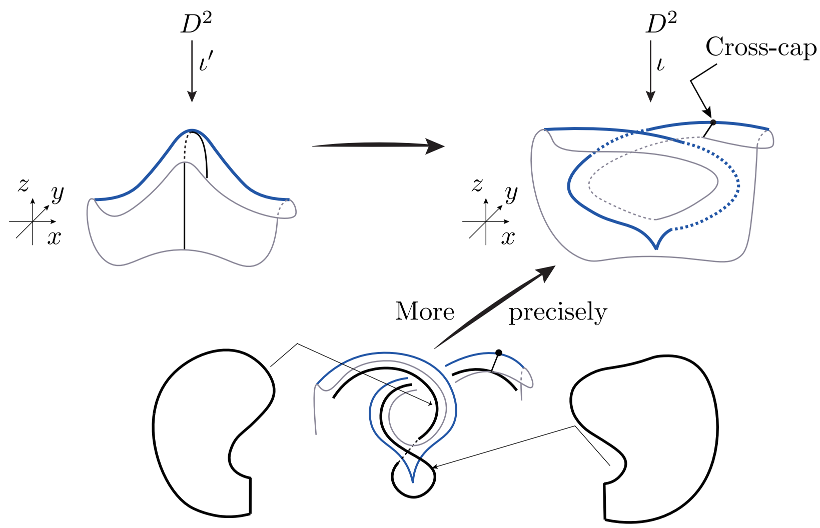

FIGURE 3. Modification II: By applying this modification, the rotation number decrease by one.

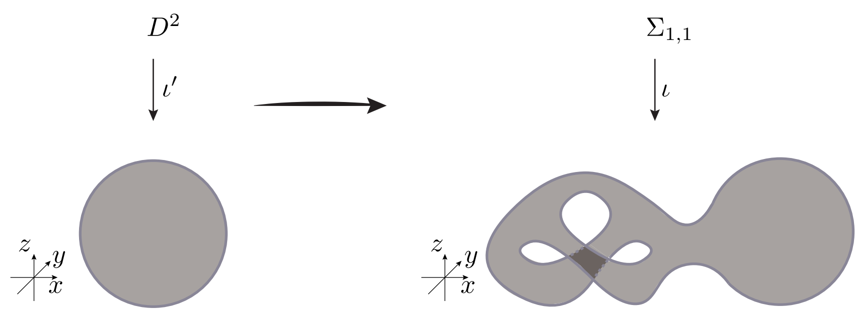

FIGURE 4. Modification III: By applying this modification, the rotation number decrease by two and the genus of the source surface increase by one.

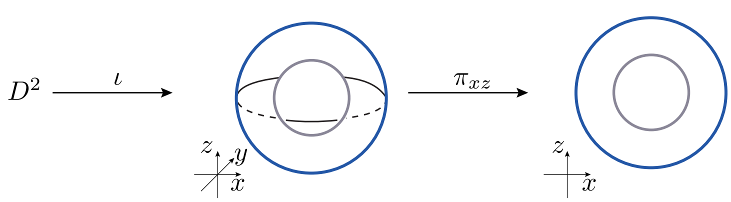

FIGURE 5. Admissible stable map $D^2 \to \mathbb{R}^2$ of rotation number -1.

Let us construct stable maps $f_{r,1}$ ($r \geq 1$). Figures 6 and 7 show degree one stable maps $f_1', f_2' \colon \Sigma_1 \to S^2$ obtained by Kamenosono and the author [7]. Note that the contours of these

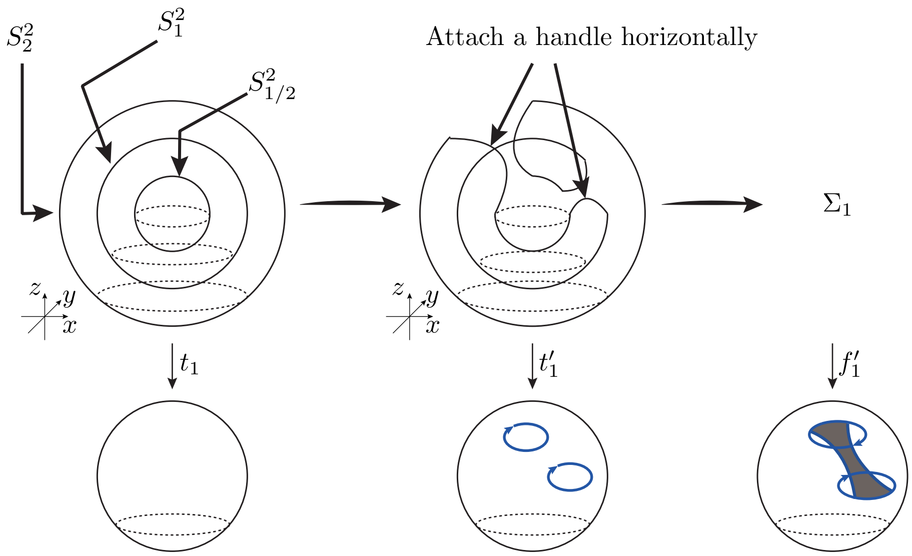

FIGURE 6. A degree one stable map $f_1\colon \Sigma_g \to S^2$: f'_1 is obtained by the following manner: (1) Define $S^2_r = \{(x, y, z) \in \mathbb{R}^3 \mid z^2 + y^2 + z^2 = r^2\}$ and put $M = S^2_{1/2} \cup S^2_1 \cup S^2_2$. Define $t_1\colon M \to S^2_1$ by $x \mapsto x/|x|$. (2) By attaching two handles vertically between $S^2_{1/2}$ and S^2_1, $S^2_{1/2}$ and S^2_2, we obtain a degree one stable map $t'_1\colon S^2 \to S^2$ whose triple is equal to $(2, 0, 0)$. (3) By attaching a handle horizontally as the Figure, we obtain a degree one stable map $f'_1\colon \Sigma_g \to S^2$ whose triple $(i, , c, n)$ is equal to $(1, 0, 4)$.

maps are minimal contours. Stable maps $f'_1, f'_2\colon \Sigma_1 \to S^2$ induce rotation number one admissible stable maps $f^1_{1,1}, f^2_{1,1}\Sigma_{1,1} \to \mathbb{R}^2$ whose contours are as depicted in right-hand side of Figures 8 and 9 respectively. By applying modification I inductively to $f^1_{1,1}$ and $f^2_{1,1}$, we obtain rotation number $r \geq 1$ admissible stable maps $f^1_{r,1}$, $f^2_{r,1}\colon \Sigma_{1,1} \to \mathbb{R}^2$ whose triples (i, c, n) are equal to $(1, r-1, 4)$, $(1, r+3, 0)$ respectively.

4.3. **Admissible stable maps $\Sigma_{2,1} \to \mathbb{R}^2$.** For each $r' \leq 0$ (or $r' = 1, 2, 3$), by applying modification III to $f_{r',1}$ (resp. $f^1_{r',1}$, $f^2_{r',1}$), we obtain boundary rotation number $r \leq -2$ (resp. $r = -1, 0, 1$) admissible stable maps $f_{r,2}$ (resp. $f^1_{-1,2}$, $f^1_{0,2}$, $f^1_{1,2}$, $f^2_{-1,2}$, $f^2_{0,2}$, $f^2_{1,2}$) whose triples (i, c, n) are one of the items below:

$$(i, c, n) = \begin{cases} (1, r+1, 4) \text{ or } (r+5, 0) & \text{if } -1 \leq r \leq 1, \\ (1, r+5, 0) & \text{if } -4 \leq r \leq -2, \\ (1, -r-5, -r-5) & \text{if } r \leq -5. \end{cases}$$

Let us construct rotation number $r \geq 2$ admissible stable maps $\Sigma_{2,1} \to \mathbb{R}^2$.

Proposition 4.1. *For each $g \geq 2$, there are rotation numbers $2g-2$ and $2g-1$ admissible stable maps $f_{2g-2,g}$ and $f_{2g-1,g}\colon \Sigma_{g,1} \to \mathbb{R}^2$ whose triples (i, c, n) are equal to $(1, 1, 2g+1)$ and $(1, 0, 2g+2)$ respectively.*

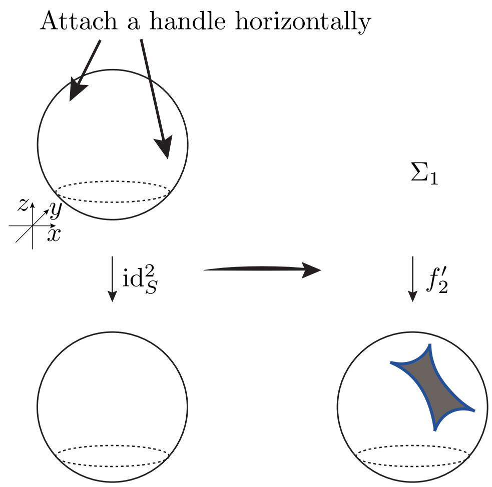

FIGURE 7. A degree one stable map $f_2' \colon \Sigma_g \to S^2$: f_2 is obtained by attaching a handle horizontally to the source sphere of the identity map on S^2.

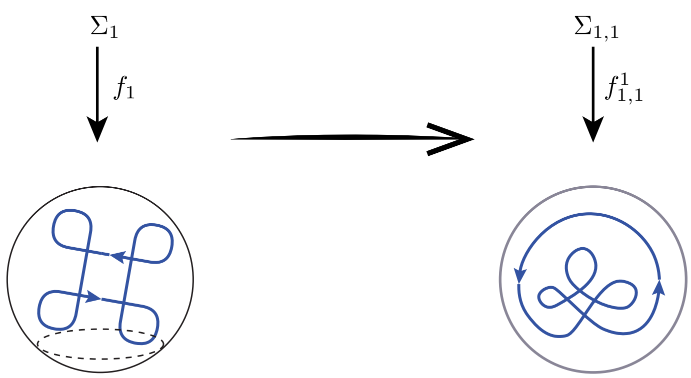

FIGURE 8. Admissible stable map $f_{1,1}^1 \colon \Sigma_{1,1} \to \mathbb{R}^2$.

Proof. Figures 10 and 11 define boundary rotation number two and three admissible stable maps $f_{2,2}$ and $f_{3,2} \colon \Sigma_{2,1} \to \mathbb{R}^2$ whose triples (i, c, n) are equal to $(1, 1, 5)$ and $(1, 0, 6)$ respectively. More precisely, to define $f_{2,2}$ (or $f_{3,2}$), we decompose $\Sigma_{2,1}$ into three pieces. Then, define inclusions of each pieces into $\mathbb{R}^3$ as depicted in Figure 10 (resp. Figure 11). Note that $\Sigma_{2,1}$ is restored by attaching the three pieces along bold curves and dotted lines which are labeled in Figure 10 (resp. Figure 11). An admissible stable map $f_{2,2}$ (resp. $f_{3,2}$) is defined by the projection π_{xz} composed with the inclusion.

We can construct such admissible stable maps $f_{2g-2,g}$ and $f_{2g-1,g}$ as well as the cases $f_{2,2}$ and $f_{3,2}$. □

By applying modification I inductively to $f_{3,2}$, we obtain a rotation number r admissible stable map $f_{r,2} \colon \Sigma_{2,1} \to \mathbb{R}^2$ whose triple (i, c, n) is equal to $(1, r-3, 6)$ for each $r \geq 3$.

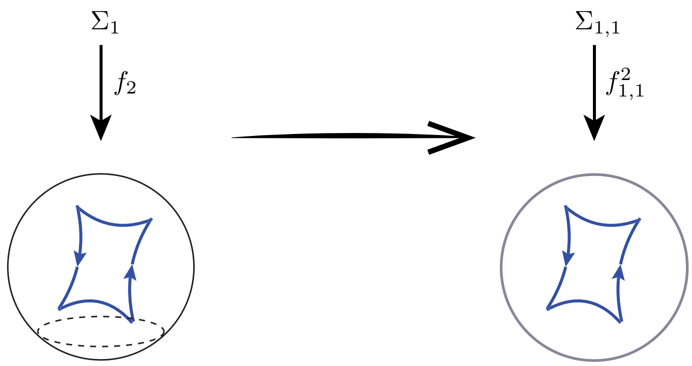

FIGURE 9. Admissible stable map $f^2_{1,1}\colon \Sigma_{1,1} \to \mathbb{R}^2$

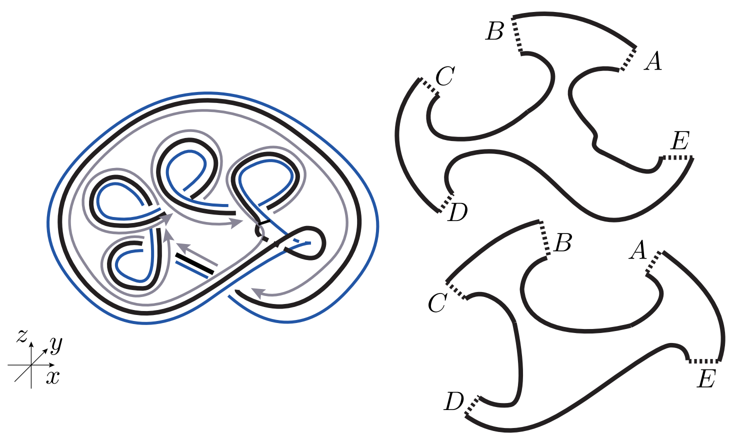

FIGURE 10. Admissible stable map $\Sigma_{2,1} \to \mathbb{R}^2$.

4.4. **Admissible stable maps $\Sigma_{g,1} \to \mathbb{R}^2$ ($g \geq 3$).** Let us consider the case $g = 3$. In this case, we already have admissible stable maps $f_{4,3}$ and $f_{5,3}$ whose triples (i, c, n) are equal to $(1, 1, 7)$ and $(1, 0, 8)$ respectively by Proposition 4.1.

By applying modification III to $f_{r',2}$ where $2 \leq r' \leq 5$ or $r' \leq -2$ (or $f^1_{r',2}$, $f^2_{r',2}$ where $-1 \leq r' \leq 1$), we obtain boundary rotation number $0 \leq r \leq 3$ or $r \leq -4$ (resp. $-3 \leq r \leq -1$) admissible stable maps $f_{r,3}$ (resp. $f^1_{r,3}$, $f^2_{r,3}$) whose triples (i, c, n) are one of the items below:

$$(i, c, n) = \begin{cases} (1, r-1, 6) & \text{if } 1 \leq r \leq 3, \\ (1, 1, 5) & \text{if } r = 0, \\ (1, r+3, 4) \text{ or } (r+7, 0) & \text{if } -3 \leq r \leq -1, \\ (1, r+7, 0) & \text{if } -6 \leq r \leq -4, \\ (1, -r-7, -r-7) & \text{if } r \leq -7. \end{cases}$$

Then, by applying modification I inductively to $f_{5,3}$, we obtain a boundary rotation number r admissible stable map $f_{r,3}\colon \Sigma_{3,1} \to \mathbb{R}^2$ whose triple (i, c, n) is equal to $(1, r-5, 8)$ for each $r \geq 5$.

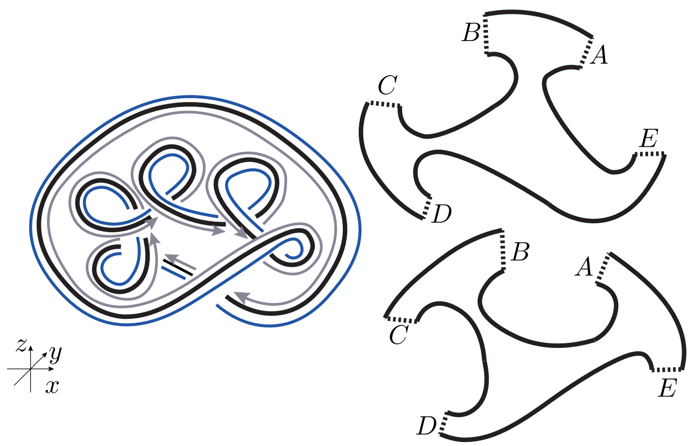

FIGURE 11. Admissible stable map $\Sigma_{2,1} \to \mathbb{R}^2$.

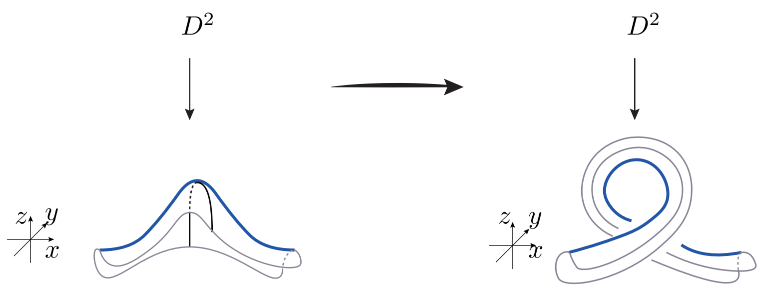

FIGURE 12. Modification IV: By applying this modification, the rotation number increases by two.

Similarly, for each $g \geq 4$ and $r \leq 2g-3$, we construct $f_{r,g}$ where $5-2g \leq r \leq 2g-3$ or $r \leq 2-2g$ (or $f^1_{r,g}$, $f^2_{r,g}$ where $3-2g \leq r \leq 5-2g$) by applying modification III to $f_{r'+2,g'-1}$ (resp. $f^1_{r'+2,g'-1}$, $f^2_{r'+2,g'-1}$ where $5-2g' \leq r' \leq 7-2g'$). Then, by applying modification I inductively to $f_{2g-1,g}$, we obtain an admissible stable map $f_{r,g}$ for each $r \geq 2g-1$. Note that we already have $f_{2g-2,g}$ in Proposition 4.1.

4.5. **Admissible stable maps** $N_{g,1} \to \mathbb{R}^2$**.** By applying modification IV (or V, VI) defined by Figure 12 (resp. Figures 13, 14) for a boundary rotation number r' admissible stable map $h\colon N_{g',1} \to \mathbb{R}^2$ whose singular points set consists of i' components and has c' cusps and n' nodes, we obtain a boundary rotation number r admissible stable map $h\colon N_{g,1} \to \mathbb{R}^2$ whose singular points set consists of i components and has c cusps and n nodes:

(4) Modification IV
$$(r, g, i, c, n) = (r'+2, g', i', c', n'+1)$$

(5) Modification V
$$(r, g, i, c, n) = (r'-2, g', i', c', n'+1)$$

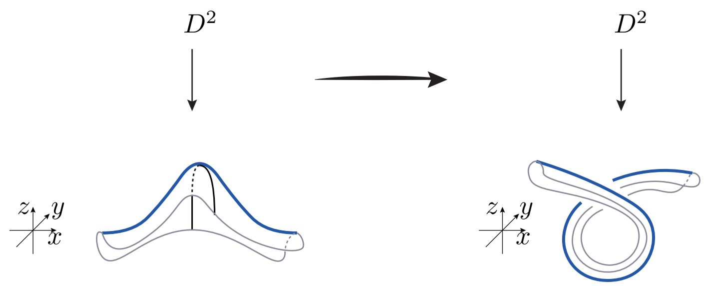

FIGURE 13. Modification V: By applying this modification, the rotation number increases by two.

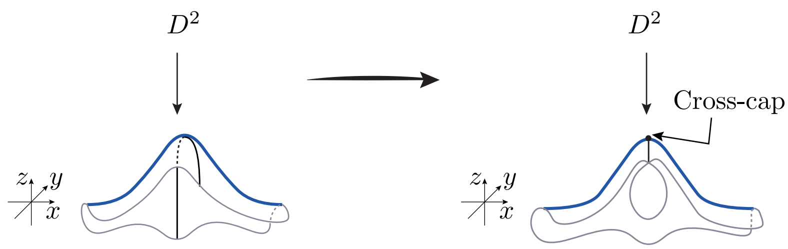

FIGURE 14. Modification VI: By applying this modification, the rotation number decreases by one.

(6) Modification VI

$$(r, g, i, c, n) = (r' - 1, g' + 1, i', c', n')$$

Note that the modified map $\iota' \colon D^2 \to \mathbb{R}^3$ have one cross-cap.

Furthermore, by applying modification III to a boundary rotation number r' admissible stable map $h' \colon N_{g',1} \to \mathbb{R}^2$, we obtain a boundary rotation number $r' - 2$ admissible stable map $h' \colon N_{g'+2,1} \to \mathbb{R}^2$.

Figure 15 defines C^∞ maps $\iota_i \colon N_{1,1} \to \mathbb{R}^3$ ($i = -2, -1, 2$ and 3). Then, the projection π_{xz} composed with ι_{-2}, ι_{-1}, ι_2 and ι_3 define boundary rotation number -2, -1, 2 and 3 admissible stable maps $h_{-2,1}$, $h_{-1,1}$, $h_{2,1}$ and $h_{3,1} \colon N_{1,1} \to \mathbb{R}^2$ whose triples (i, c, n) are equal to $(1, 0, 0)$, $(1, 1, 0)$, $(1, 0, 0)$ and $(1, 1, 0)$ respectively.

By applying modification IV to $h_{-2,1}$ and $h_{-1,1}$, we obtain boundary rotation number zero and one admissible stable maps $h_{0,1}$ and $h_{1,1} \colon N_{1,1} \to \mathbb{R}^2$ whose triples (i, c, n) are equal to $(1, 0, 1)$ and $(1, 1, 1)$ respectively.

By applying modification IV inductively to $h_{2,1}$ and $h_{3,1}$, we obtain a boundary rotation number $r \geq 2$ admissible stable map $h_{r,1} \colon N_{1,1} \to \mathbb{R}^2$ whose triple (i, c, n) is equal to $(1, 0, (r - 2)/2)$ if $r \geq 2$ is even, $(1, 1, (r - 3)/2)$ otherwise.

Similarly, by applying modification V inductively to $h_{-2,1}$ and $h_{-1,1}$, we obtain a boundary rotation number $r \leq -1$ admissible stable map $h_{r,1} \colon N_{1,1} \to \mathbb{R}^2$ whose triple (i, c, n) is equal to $(1, 0, (-r - 2)/2)$ if $r \leq -1$ is even, $(1, 1, (-r - 1)/2)$ otherwise.

Thus, we see that for each triple (i, c, n) in the list of Theorem 2.5 ($g = 1$), there exists an admissible stable map $N_{1,1} \to \mathbb{R}^2$ whose triple (i, c, n) is the triple.

Then, by applying modification III inductively to $h_{r',1} \colon N_{1,1} \to \mathbb{R}^2$, we obtain a boundary rotation number r admissible stable map $h_{r,g} \colon N_{g,1} \to \mathbb{R}^2$ whose triples (i, c, n) are in the list of Theorem 2.5 for each odd number $g \geq 1$ and each $r \in \mathbb{Z}$. Furthermore, by applying modification

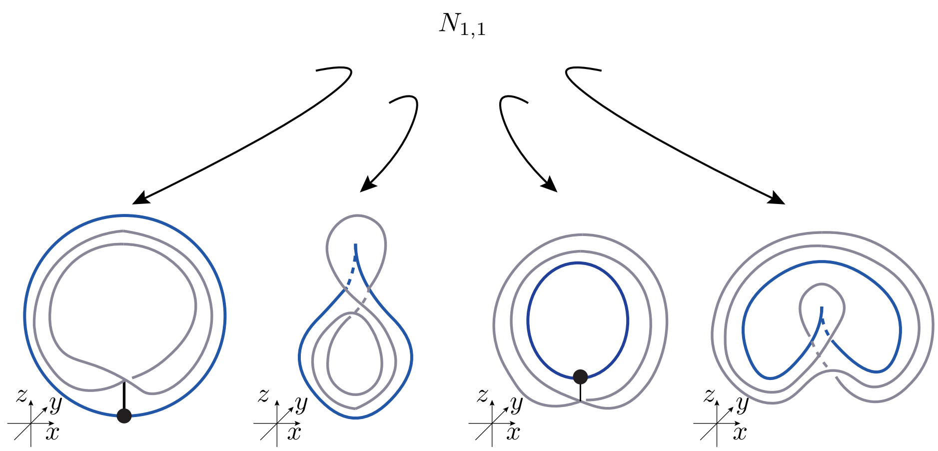

FIGURE 15. Admissible stable maps $N_{1,1} \to \mathbb{R}^2$ of rotation numbers $-2, -1, 2$ and 3, respectively

VI inductively to $h_{r',g'}\colon N_{g',1} \to \mathbb{R}^2$ with odd $g' \geq 1$, we obtain $h_{r,g}\colon N_{g,1} \to \mathbb{R}^2$ whose triples (i, c, n) are in the list of Theorem 2.5 for each even $g \geq 2$ and $r \in \mathbb{Z}$.

Thus, we see that for each (i, c, n) in the list of Theorem 2.5, there is a boundary rotation number r admissible stable map $h\colon N_{g,1} \to \mathbb{R}^2$ whose triple (i, c, n) is equal to the triple.

5. Proof of minimum of $c + n$ in Theorem 2.3

Let $g \in \mathbb{Z}_{\geq 0}$ and $r \in \mathbb{Z}$. To prove Theorem 2.3 we need the following Lemmas.

Lemma 5.1 (M. Yamamoto [14])**.** *Let $f\colon \Sigma_{g,1} \to \mathbb{R}^2$ be a rotation number r admissible stable map whose singular points set consists of one component. Then, $c(f) \geq |r+1| - 2g$ and $c(f) \not\equiv r$* mod 2.

Lemma 5.2. *Let $f\colon \Sigma_{g,1} \to \mathbb{R}^2$ be a rotation number r admissible stable map whose singular points set consists of one component.*

(1) *If f has no cusps, then $r \equiv 2g - 1 \mod 4$.*
(2) *If $r \equiv 2g + 1 \mod 4$, then $\gamma(f)$ has at least two cusps.*

Proof. (1) For such stable map $f\colon \Sigma_{g,1} \to \mathbb{R}^2$, $\Sigma_{g,1}$ is decomposed into three pieces as

$$\Sigma_{g,1} = \Sigma_{g-t,1} \sqcup N(S(f)) \sqcup \Sigma_{t,2}, \quad 0 \leq t \leq g,$$

where $N(S(f))$ denote a tubular neighborhood of $S(f)$. Note that $f_1; = f|_{\Sigma_{g-t,1}}$ and $f_2 := f|_{\Sigma_{t,2}}$ are immersions. Then, by applying a result of Heafliger:

For an immersed surface $M_k \subset \mathbb{R}^2$, the Euler-Poincare characteristic
$\chi(M_k)$ is equal to the normal degree of ∂M_k.

If $W(f_1) = k$, then we have $\chi(\Sigma_{g-t,1}) = k$ and $\chi(\Sigma_{t,2}) = k + r$. This shows that $2g = 1 + r + 4t$.

(2) Put $r = 2g + 1 + 4k$. Then, formula (3.1) implies the conclusion. □

Let us divide a proof into two cases $g = 0$ and $g \geq 1$.

5.1. $g = 0$. Lemma 5.1 shows that the contour $\gamma(f_{r,0})$ is an admissible minimal contour for each $r \geq 0$.

Let us consider the case $r \leq -1$. Let $f\colon \Sigma_{0,1} \to \mathbb{R}^2$ be an admissible stable map of rotation number r whose singular points set consists of one component. Then, Lemma 5.1 implies that $c(f) \geq -(r+1)$. In this case, (3.1) and Lemma 3.2 show that

$$\frac{r+1}{2} = (N^+ - N^-) + \frac{c(f)}{2}.$$

Then, we have

$$\frac{r+1}{2} = (N^+ - N^-) + \frac{c(f)}{2} \geq (N^+ - N^-) - \frac{r+1}{2}.$$

This implies that $(r+1) \geq (N^+ - N^-)$. Note that $(r+1)$ is negative. Thus, we have $N^- \geq -(r+1)$. Then,

$$c(f) + n(f) \geq \frac{c(f)}{2} + \frac{r+1}{2} + 2N^- \geq -2(r+1).$$

Thus, for such admissible stable maps, we have $c(f) + n(f) \geq -2(r+1)$. This shows that the contour $\gamma(f_{r,0})$ $(r \leq -1)$ is an admissible minimal contour.

5.2. $g \geq 1$. At first, let us consider the case $r \geq 2g-1$. Let $f\colon \Sigma_{g,1} \to \mathbb{R}^2$ be an admissible stable map of rotation number r whose singular points set consists of one component. Then the formula (3.1) and Lemma 3.2 show that

$$g + \frac{r+1}{2} = (N^+ - N^-) + \frac{c(f)}{2}. \tag{5.1}$$

If $\gamma(f)$ has no node, then $c(f) = 2g + r + 1$. If $\gamma(f)$ hsa a node, then Lemma 3.7 and Lemma 5.1 yeild that

$$c(f) + n(f) \geq \frac{c(f)}{2} + g + \frac{r+1}{2} + 2N^- \geq r + 3.$$

This shows that the contour $\gamma(f_{r,g})$ $(r \geq 2g-1)$ is an admissible minimal contour.

The case $-2g \leq r \leq 2g$ is also proved by using Lemmas 5.1, 5.2 and the similarly argument as the above case.

Then, let us consider the case $r \leq -2g-1$. Let $f\colon \Sigma_{g,1} \to \mathbb{R}^2$ be a rotation number r admissible stable map whose singular points set consists of one component. The formula (5.1) and Lemma 5.1 imply

$$g + \frac{r+1}{2} \geq (N^+ - N^-) + \frac{-r-1-2g}{2}.$$

Thus, we have

$$2g + r + 1 \geq (N^+ - N^-).$$

Note that $2g + r + 1$ is negative. Thus, $N^- \geq -(2g + r + 1)$. Then,

$$c(f) + n(f) \geq \frac{c(f)}{2} + g + \frac{r+1}{2} + 2N^- \geq -2(r + 2g + 1).$$

Therefore, the contour $\gamma(f_{r,g})$ $(r \leq -2g-1)$ is admissible minimal contour.

It completes the proof of Theorem 2.3.

6. Proof of minimum of $c+n$ in Theorem 2.5

Let $g \in \mathbb{Z}_{\geq 1}$ and $r \in \mathbb{Z}$. Proposition 3.5 yeilds the following lemma.

Lemma 6.1. *Let $h\colon N_{g,1} \to \mathbb{R}^2$ be a boundary rotation number r admissible stable map whose singular points set consists of one component. Then, the numbers $g+r$ and $c(h)$ never have the same parity. In particular, if $g+r$ is an even number, then h has at least one cusp.*

Proof. Let $h\colon N_{g,1} \to \mathbb{R}$ be a such stable map. Then, formula (3.1) induces the following modulo two equation

$$g \equiv c(h) - (r+1).$$

It implies the conclusion. □

We divide a proof into two cases $g=1$ and $g \geq 2$.

6.1. $g=1$**.** Lemma 6.1 shows that the contours $\gamma(h_{r,1})$ $(r=-2,-1,2,3)$ are admissible minimal contours.

At first, let us consider the case $r \geq 4$. Let $h\colon N_{1,1} \to \mathbb{R}^2$ be a boundary rotation number r admissible stable map whose singular points set consists of one component.

(i1) $i^+ = 1$. Then, the formula (3.1) implies $2(N^+ - N^-) + c(h) = r-2$. If $\gamma(h)$ has no nodes, then $c(h) = r-2$. If $\gamma(h)$ has a node, then

$$c(h) + n(h) = \frac{r-2+c(h)}{2} + 2N^- \geq \frac{r-2+c(h)}{2}.$$

This yeilds that if $r \geq 4$ is odd (or even), then

$$c(h) + n(h) \geq (r-1)/2$$

(resp. $c(h)+n(h) \geq (r-2)/2$).

(i2) $i^- = 1$. Then, the formula (3.1) implies $2(N^+ - N^-) + c(h) = r+2$. If $\gamma(h)$ has no nodes, then $c(h) = r+2$. If $\gamma(h)$ has a node, then

$$c(h) + n(h) = \frac{r+c(h)+2}{2} + 2N^- \geq \frac{r+c(h)+2}{2}.$$

This yields that if $r \geq 4$ is odd (or even), then

$$c(h) + n(h) \geq (r+3)/2$$

(resp. $c(h)+n(h) \geq (r+2)/2$).

(i1) and (i2) show that if $r \geq 4$ is odd (or even), then $c(h)+n(h) \geq (r-1)/2$ (resp. $c(h)+n(h) \geq (r-2)/2$). This implies that the contour $\gamma(h_{r,1})$ $(r \geq 4)$ is an admissible minimal contour.

Then, let us consider the case $r \leq -3$. Let $h\colon N_{1,1} \to \mathbb{R}^2$ be a boundary rotation number r admissible stable map whose singular points set consists of one component.

(i1) $i^+ = 1$. Then, the formula (3.1) induces $2(N^+ - N^-) = r - c(h) - 2$. Note that $r - c(h) - 2 \leq 0$. Thus, we have $N^- \geq -(r-c(h)-2)/2$. Then,

$$c(h) + n(h) = \frac{r+c(h)-2}{2} + 2N^- \geq \frac{3c(h)-r+2}{2}.$$

Lemma 6.1 yields that if $r \leq -3$ is odd (or even), then $c(h) + n(h) \geq (-r + 5)/2$ (resp. $c(h) + n(h) \geq (-r + 2)/2$).

(i2) $i^- = 1$. Then, the formula (3.1) induces $2(N^+ - N^-) = r - c(h) + 2$. If $\gamma(h)$ has no nodes, then $c(h) = r + 2$. If $\gamma(h)$ has a node, then $(N^+ - N^-) = (r - c(h) + 2)/2 \leq 0$. Thus, we have $N^- \geq -(r - c(h) + 2)/2$. Then,

$$c(h) + n(h) = \frac{r - c(h) + 2}{2} + 2N^- \geq \frac{3c(h) - r - 2}{2}.$$

Lemma 6.1 shows that if $r \leq -3$ be odd (or even), then $c(h) + n(h) \geq (-r + 1)/2$ (resp. $(-r - 2)/2$).

(i1) and (i2) show that $\gamma(h_{r,1})$ ($r \leq -3$) is an admissible minimal contour.

Formula (3.1) implies the following.

Lemma 6.2. *Let $h\colon N_{1,1} \to \mathbb{R}^2$ be a boundary rotation number 0 admissible stable map whose singular points set consists of one component. Then, $c(h) + n(h) \geq 1$.*

Therefore, $\gamma(h_{0,1})$ is an admissible minimal contour.

We can show that $\gamma(h_{1,1})$ is minimal as the above case.

Thus, we complete the proof of the Theorem 2.5 for $g = 1$.

6.2. $g \geq 2$. Lemma 6.1 shows that the contours $\gamma(h_{-g,g})$ and $\gamma(h_{-g-1,g})$ are admissible minimal contours.

At first, let us consider $r \geq -g + 1$. Let $h\colon N_{g,1} \to \mathbb{R}^2$ be a boundary rotation number r admissible stable map whose singular points set consists of one component.

(i1) $i^+ = 1$. Then, formula (3.1) shows that $2(N^+ - N^-) + c = g + r - 3$. If $\gamma(h)$ has no nodes, then $c(h) = g + r - 3$. If $\gamma(h)$ has a node, then

$$c(h) + n(h) = c(h) + \frac{g + r - c(h) - 3}{2} + 2N^- \geq \frac{g + r + c(h) - 3}{2}.$$

Lemma 6.1 shows that if $g + r$ is even (or odd), then $c(h) + n(h) \geq (g + r - 2)/2$ (resp. $c(h) + n(h) \geq (g + r - 3)/2$).

(i2) $i^- = 1$. Then, formula (3.1) shows that $2(N^+ - N^-) + c(h) = g + r + 1$. If $\gamma(h)$ has no nodes, then $c(h) = g + r + 1$. If $\gamma(h)$ has a node, then

$$c(h) + n(h) = c(h) + \frac{g + r - c(h) + 1}{2} + 2N^- \geq \frac{g + r + c(h) + 1}{2}.$$

Lemma 6.1 shows that if $g + r$ is even (or odd), then $c(h) + n(h) \geq (g + r + 2)/2$ (resp. $c(h) + n(h) \geq (g + r + 1)/2$).

(i1) and (i2) implies that the conturs $\gamma(h_{r,g})$ ($r \geq -g+1$) are an admissible minimal contours.

Then, let $r \leq -g - 2$. Let $h\colon N_{g,1} \to \mathbb{R}^2$ be a boundary rotation number r admissible stable map whose singular points set consists of one component.

(i1) $i^+ = 1$. Formula (3.1) shows that $2(N^+ - N^-) = g + r - c(h) - 3 \leq 0$. Thus, we have $N^- \geq -(g + r - c(h) - 3)/2$, Then,

$$c(h) + n(h) = c(h) + \frac{g + r - c(h) - 3}{2} + 2N^- \geq \frac{-g - r + c(h) + 3}{2}.$$

Lemma 6.1 shows that if $g + r$ is even (or odd), then $c(h) + n(h) \geq (-g - r + 4)/2$ (resp. $c(h) + n(h) \geq (-g - r + 3)/2$).

(i2) $i^- = 1$. Formula (3.1) shows that $2(N^+ - N^-) = g + r - c(h) + 1 \leq 0$. Thus, we have $N^- \geq -(g + r - c(h) + 1)/2$, Then,

$$c(h) + n(h) = c(h) + \frac{g + r - c(h) + 1}{2} + 2N^- \geq \frac{-g - r + 3c(h) - 1}{2}.$$

Lemma 6.1 shows that $g + r$ is even (or odd), then

$$c(h) + n(h) \geq (-g - r + 2)/2$$

(resp. $c(h) + n(h) \geq (-g - r - 1)/2$).

(i1) and (i2) implies that $\gamma(h_{r,g})$ $(r \geq -g - 2)$ is an admissible minimal contour.

It completes the proof of Theorem 2.5.

7. Problem

Let M be a compact connected surface with boundary and P a surface without boundary. A C^∞ map $f\colon M \to P$ is called a *fold map* if f has only fold points as its singularities.

Let $f\colon M \to \mathbb{R}^2$ be a boundary rotation number r admissible stable fold map. Then, call the contour $\gamma(f)$ an $\mathcal{F}$-(i,n)-*minimal contour* of boundary rotation number r maps $M \to \mathbb{R}^2$ if the pair $(i(f), n(f))$ is the smallest among rotation number r admissible stable fold maps $M \to \mathbb{R}^2$ with respect to the lexicographic order.

Problem 7.1. Let $M = \Sigma_{g,1}$ or $N_{g,1}$. Study an $\mathcal{F}$-(i,n)-minimal contour of boundary rotation number r maps $M \to \mathbb{R}^2$.

References

[1] J. W. Bruce, P. J. Giblin, *Projections of surfaces with boundary*, Proc. London Math. Soc. (3) 60 (1990), no. 2, 392–416.

[2] S. Demoto, *Stable maps between 2-spheres with a connected fold curve*. Hiroshima Math. J. **35** (2005), no. 1, 93–113. DOI: 10.32917/hmj/1150922487

[3] T. Fukuda and T. Yamamoto, *Apparent contours of stable maps into the sphere*, J. Sing. **3** (2011), 113–125. DOI: 10.5427/jsing.2011.3g

[4] M. Golubitsky and V. Guillemin, *Stable mappings and their singularities*, Grad. Texts in Math., Vol. 14, Springer, New York-Heidelberg, 1973.

[5] A. Haefliger, *Quelques remarques sur les applications différentiables d'une surface dans le plan*, (French) Ann. Inst. Fourier. Grenoble 10 1960 47–60. DOI: 10.5802/aif.97

[6] N. Imai, *Discriminant set of a stable map and the Euler characteristic of a surface with boundary* (in Japanese), Master Thesis, Hiroshima Univ., March 1999.

[7] A. Kamenosono and T. Yamamoto, *The minimal numbers of singularities of stable maps between surfaces*, Topology Appl. **156** (2009), pp. 2390-2405. DOI: 10.1016/j.topol.2009.06.010

[8] H. Levine, *Computing the Euler charactristic of a manifold with boundary*, Proc. Amer. Math. Soc. 123 (1995), no. 8, 2563–2567.

[9] R. Pignoni, *Projections of surfaces with a connected fold curve*, Topology Appl. **49** (1993), no. 1, 55–74. DOI: 10.1016/0166-8641(93)90129-2

[10] J. R. Quine, *Plűcker equations for curves.* Amer. Math. Monthly **88** (1981), no. 1, 21–29.

[11] R. Thom. *Les singularités des applications différentiables.* (French) Ann. Inst. Fourier (Grenoble) 6 (1955/56), 43–87.

[12] H. Whitney, *On regular families of curves*, Bull. Amer. Math. Soc. 47, (1941). 145–147. DOI: 10.1090/s0002-9904-1941-07395-7

[13] H. Whitney, *On singularities of mappings of euclidean spaces. I. Mappings of the plane into the plane.* Ann. of Math. (2) **62** (1955), 374–410. DOI: 10.2307/1970070

[14] M. Yamamoto, *Pseudo-immersions of oriented surfaces with one boundary component into the plane.* Proc. Roy. Soc. Edinburgh Sect. A **139** (2009), no. 6, 1327–1335.
[15] T. Yamamoto, *Apparent contours with minimal number of singularities*, Kyushu J. Math. **64**(2010), no. 1, 1–16. DOI: 10.2206/kyushujm.64.1
[16] T. Yamamoto, *Apparent contours of stable maps between closed surfaces*, Kodai Math. J. 40 (2017), no. 2, 358–378. DOI: 10.2996/kmj/1499846602
[17] T. Yamamoto, *Number of singularities of stable maps on surfaces*, Pacific J. Math., Vol. 280 (2016), No. 2, 489–510. DOI: 10.2140/pjm.2016.280.489

Takahiro Yamamoto, Department of Mathematics, Tokyo Gakugei University, Koganei, Tokyo, 184-8501, Japan

Email address: `yamush@u-gakugei.ac.jp`