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DERIVED KZ EQUATIONS

VADIM SCHECHTMAN AND ALEXANDER VARCHENKO

To the 30th anniversary of [SV]

Abstract. In this note we strengthen the results in our previous work by presenting their derived version. Namely, we define a "derived Knizhnik - Zamolodchikov connection" and identify it with a "derived Gauss - Manin connection".

1. Introduction

1.0. Brief review of the paper. The main result of [SV] provided a realization of Knizhnik-Zamolodchikov equations arising in physics as equations on horizontal sections for a Gauss-Manin connection.

More explicitly, without going into details to be given below, the KZ connection acts on a space of functions depending on $\mathbf{z} \in B$ where B is a domain in $\mathbb{C}^n$ with values in a homology group $H_0(\mathfrak{n}, M)$ where $\mathfrak{n}$ is a certain Lie algebra, and M a (maybe infinite-dimensional) $\mathfrak{n}$-module. In other words the KZ connection acts on the trivial vector bundle over B with a fiber $H_0(\mathfrak{n}, M)$, this vector bundle to be denoted $\mathcal{H}_0(\mathfrak{n}, M)$.

All homology spaces $H_i(\mathfrak{n}, M)$ are Λ-graded

$$H_i(\mathfrak{n}, M) = \oplus_{\lambda\in\Lambda} H_i(\mathfrak{n}, M)_\lambda$$

where Λ is certain lattice. For a given λ only a finite number of spaces $H_i(\mathfrak{n}, M)_\lambda$, $0 \le i \le N$, are different from 0. Let us pick λ.

On the other hand one has introduced in *op. cit.* a fibration (a smooth surjective map)

$$p_\lambda : X_\lambda \longrightarrow B$$

and a $\mathcal{D}$-module $\mathcal{L}_\lambda$ over X_λ, and a finite group Σ_λ (a product of symmetric groups) which acts on X_λ and $\mathcal{L}_\lambda$.

One has constructed an isomorphism of the bundle $\mathcal{H}_0(\mathfrak{n}, M)_\lambda$ equipped with the KZ connection with the bundle $(R^N p_{\lambda *}\mathcal{L}_\lambda)^{\Sigma_\lambda}$ equipped with the GM connection.

In fact in [SV] for all $0 \le i \le N$ there were established isomorphisms

$$\beta_{i,\lambda} :\ \mathcal{H}_i(\mathfrak{n}, M)_\lambda \xrightarrow{\sim} (R^{N-i} p_{\lambda *}\mathcal{L}_\lambda)^{\Sigma_\lambda} \tag{1.0.1}$$

of vector bundles over B. However the question of identification of a connection on $\mathcal{H}_i(\mathfrak{n}, M)_\lambda$ corresponding to the GM connection on $(R^{N-i} p_{\lambda *}\mathcal{L}_\lambda)^{\Sigma_\lambda}$ was left open for $i > 0$, although a natural candidate has been given.

In the present note we establish this remaining point. To do this we start from the remark that by its very definition in *op. cit.* isomorphisms (1.0.1) are induced by a map of complexes

$$\eta_\lambda = (\eta_{\lambda,i}) :\ \mathcal{C}_\bullet(\mathfrak{n}, M)_\lambda \longrightarrow \Omega^{N-\bullet}_{X_\lambda/B}(\mathcal{L}_\lambda)^{\Sigma_\lambda} \tag{1.0.2}$$

where $\mathcal{C}_\bullet(\mathfrak{n}, M)_\lambda$ is the λ-homogeneous part of the Chevalley chain complex, and $\Omega^\bullet_{X_\lambda/B}(\mathcal{L}_\lambda)$ is a certain complex of differential form on X_λ, the relative de Rham complex of $\mathcal{L}_\lambda$.

A naive expectation would be that:

(a) *for the KZ part*:
each term $\mathcal{C}_i(\mathfrak{n}, M)_\lambda$ comes equipped with an integrable connection, these connections are compatible with differentials and thus induce a connection on the cohomology $\mathcal{H}_i(\mathfrak{n}, M)_\lambda$;
(b) *for the GM part*:
similarly, each term $\Omega^j_{X_\lambda/B}(\mathcal{L}_\lambda)$ comes equipped with an integrable Σ_λ-equivariant connection, these connections are compatible with differentials and thus induce a connection on the cohomology $R^j p_{\lambda *}\mathcal{L}_\lambda$;
(c) the map η_λ is compatible with the connections in (a), (b), and therefore the isomorphisms $\beta_{i,\lambda}$ (1.0.1) identify two connections.
In reality, (a) is literally true (and easy); this is present in [SV].
Point (b) is more delicate: there is *no* natural connection on the complex $\Omega^\bullet_{X_\lambda/B}(\mathcal{L}_\lambda)$. Happily, to define a connection on the cohomology a weaker structure is sufficient:
(b') there exists a filtered complex such that the term E_1 of the corresponding spectral sequence (recalled in Appendix) coincides with the de Rham complex of the GM connection on $R^\bullet p_{\lambda *}\mathcal{L}_\lambda$.
This filtered complex is described below: it is a generalization of the Katz - Oda construction for the GM connection, [KO].
Accordingly, (c) should be replaced by
(c') the map η_λ may be extended to a map of filtered complexes which, after passing to E_1-terms, induces a map from the de Rham complex of the KZ connection to the de Rham complex of the GM connection.

Now we will describe some details of what was said above.

1.1. Knizhnik - Zamolodchikov connection. Let $\mathfrak{g}$ be a complex Lie algebra equipped with an element
$$\Omega \in \mathfrak{g} \otimes \mathfrak{g}$$
having the following property:

1.1.1. *Let M_1, M_2 be arbitrary $\mathfrak{g}$-modules. The actions of Ω and $\mathfrak{g}$ on $M_1 \otimes M_2$ commute.*

1.1.2. Example. Let $\mathfrak{g}$ be finite-dimensional, equipped with a non-degenerate invariant symmetric bilinear form $(,)$. Denote
$$\Omega = \sum_i x_i \otimes x^i \in \mathfrak{g} \otimes \mathfrak{g}$$
where $\{x_i\} \subset \mathfrak{g}$ is any $\mathbb{C}$-base, and $\{x^i\}$ is the dual base, i.e. $(x_i, x^i) = \delta_{ij}$. This element ("the Casimir") does not depend on a choice of a base and satisfies 1.1.1.
Let $M_1, \ldots, M_n$ be $\mathfrak{g}$-modules, $n \geq 1$. Denote $M = M_1 \otimes \ldots \otimes M_n$.

For a smooth affine complex[1] algebraic variety U, $\Omega^\bullet(U)$ will denote the space of global sections for its *algebraic* de Rham complex $\Omega^\bullet_U$. Thus $\Omega^0_U = \mathcal{O}_U$ is the sheaf of functions, etc.
If M is a vector space, we denote
$$\Omega^\bullet(U; M) := \Omega^\bullet(U) \otimes M.$$
Let $n \geq 1$ be an integer. Let $M_1, \ldots, M_n$ be $\mathfrak{g}$-modules; set $M = M_1 \otimes \ldots \otimes M_n$. For each $i \neq j$ we have an operator
$$\Omega_{ij} : M \longrightarrow M$$

[1] In what follows the base field $\mathbb{C}$ of complex numbers may be replaced by any field of characteristics 0.

acting as Ω on $M_i \otimes M_j$ and as identity on the other factors.

Denote

$$U_n = \{\mathbf{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n |\ z_i \neq z_j \text{ for all } i \neq j\} \tag{1.1.1}$$

Thus $U_1 = \mathbb{C}$.

The KZ connection is an operator

$$\nabla_{KZ} : \Omega^0(U_n; M) = \mathcal{O}(U_n) \otimes M \longrightarrow \Omega^1(U_n; M)$$

given by

$$\nabla_{KZ} = d_{DR} + \Omega_{KZ} := d_{DR} - \frac{1}{\kappa} \sum_{i<j} \frac{\Omega_{ij}(dz_i - dz_j)}{z_i - z_j} \tag{1.1.2}$$

where d_{DR} is the de Rham differential. Here $\kappa \in \mathbb{C}^*$ is a complex parameter.

Thus $\nabla_{KZ} = d_{DR}$ if $n = 1$.

This connection is integrable: if we define, starting from ∇_{KZ}, operators

$$\nabla_{KZ} :\ \Omega^i(U_n; M) \longrightarrow \Omega^{i+1}(U_n; M)$$

for all i in the usual way then $\nabla_{KZ}^2 = 0$ (this amounts to the classical YB equation for the differential form Ω_{KZ}).

In other words, ∇_{KZ} is an integrable connection (i.e. it defines a structure of a $\mathcal{D}_{U_n}$-module) on the trivial bundle $\mathcal{M}$ over U_n with fiber M.

1.2. The Chevalley complex and the derived KZ. Let $\mathfrak{n} \subset \mathfrak{g}$ be a Lie subalgebra.

We will be interested in Chevalley chain complexes

$$C_\bullet(\mathfrak{n}, M) :\ \ldots \longrightarrow \Lambda^2\mathfrak{n} \otimes M \longrightarrow \mathfrak{n} \otimes M \longrightarrow M \longrightarrow 0$$

where $d(g \otimes x) = gx$,

$$d(g_1 \wedge g_2 \otimes x) = g_1 \otimes g_2 x - g_2 \otimes g_1 x - [g_1, g_2] \otimes x,$$

etc., cf. [SV], (5.4.2).

Let $\mathcal{C}_\bullet(\mathfrak{n}, M)$ denote the trivial vector bundle over U_n with a fiber $C_\bullet(\mathfrak{n}, M)$, so it is a complex of vector bundles.

We define the *derived KZ connection* as an integrable connection on $\mathcal{C}_\bullet(\mathfrak{n}, M)$ given by the same formula as above,

$$\nabla_{KZ} = d_{DR} + \Omega_{KZ} := d_{DR} - \frac{1}{\kappa} \sum_{i<j} \frac{\Omega_{ij}(dz_i - dz_j)}{z_i - z_j} \tag{1.2.1}$$

where now the operators

$$\Omega_{ij} :\ C_l(\mathfrak{n}, M) = \Lambda^l\mathfrak{n} \otimes M \longrightarrow C_l(\mathfrak{n}, M)$$

are acting through the factor M.

Whence we get the corresponding de Rham complex

$$\Omega^\bullet_{KZ}(U_n, C_\bullet(\mathfrak{n}, M)) = DR(\mathcal{C}_\bullet(\mathfrak{n}, M), \nabla_{KZ})(U_n). \tag{1.2.2}$$

It is a double complex: the commutation of the Chevalley differential with ∇_{KZ} follows from 1.1.1.

We call it the *KZ-Chevalley complex.*

In fact this complex appears *avant la lettre* already in [SV] 7.2.3.

1.3. Derived Gauss - Manin connection. Let $N \geq 0$ be an integer. Consider the affine space $\mathbb{C}^{n+N}$ with coordinates $z_1, \dots, z_n, t_1, \dots, t_N$, and inside it an open subspace

$$U_{n,N} = \{z_i \neq z_j,\ z_i \neq t_a,\ t_a \neq t_b\}.$$

We have an obvious projection

$$p : U_{n,N} \longrightarrow U_n.$$

The de Rham algebra $\Omega^\bullet(U_{n,N})$ is the total complex of a bicomplex

$$\Omega^\bullet(U_{n,N}) = \operatorname{Tot} \Omega^{\bullet\bullet}(U_{n,N})$$

where $\Omega^{pq}(U_{n,N})$ is the space of forms containing p differentials dz_m and q differentials dt_i, the full de Rham differential being the sum

$$d_{DR} = d_z + d_t.$$

The *relative* de Rham complex is by definition

$$\Omega^\bullet(U_{n,N}/U_n) = (\Omega^{0\bullet}(U_{n,N}), d_t);$$

one has a projection

$$p:\ \Omega^\bullet(U_{n,N}) \longrightarrow \Omega^\bullet(U_{n,N}/U_n)$$

Let $\mathcal{L}$ be a $\mathcal{D}_{U_{n,N}}$-module, i.e. a quasicoherent $\mathcal{O}_{U_{n,N}}$-module equipped with an integrable connection

$$\nabla :\ \mathcal{L} \longrightarrow \Omega^1_{U_{n,N}} \otimes \mathcal{L};$$

its de Rham complex is

$$DR(\mathcal{L}) :\ 0 \longrightarrow \mathcal{L} \xrightarrow{\nabla} \Omega^1_{U_{n,N}} \otimes \mathcal{L} \xrightarrow{\nabla} \Omega^2_{U_{n,N}} \otimes \mathcal{L} \longrightarrow \dots$$

By definition, *the de Rham complex of the derived Gauss - Manin connection* on the direct image $Rp_*\mathcal{L}$, to be denoted $DR(Rp_*\mathcal{L})$, is the same complex $DR(\mathcal{L})$ equipped with a decreasing filtration

$$F^0_z DR(\mathcal{L}) = DR(\mathcal{L}) \supset F^1_z DR(\mathcal{L}) \supset \dots \tag{1.3.1}$$

where $F^i_z DR(\mathcal{L})$ is the subcomplex containing $\geq i$ differentials dz_a.

Note that the utmost left column of F^0/F^1 is the relative de Rham complex representing $Rp_*\mathcal{L}$ whose cohomology are the sheaves $R^ip_*\mathcal{L}$. These sheaves carry the usual GM connections ∇^i.

The complexes $E^i(DR(Rp_*\mathcal{L}), F^\bullet_z)$ defined in the Appendix, A2.1 (the components of the E_1 term of the spectral sequence for our filtered complex) are nothing else but the de Rham complexes of $R^ip_*\mathcal{L}$:

$$E^i(DR(Rp_*\mathcal{L}), F^\bullet_z) \overset{\sim}{=} DR(R^ip_*\mathcal{L}, \nabla^i).$$

This isomorphism justifies the above definition.

1.3.1. Remark. For the case of a trivial connection on $\mathcal{O}_{U_{n,N}}$ the above construction is nothing else but the Katz - Oda definition of the usual GM connection, [KO].

1.4. Coulomb $\mathcal{D}$-modules. Let V be a finite-dimensional complex vector space equipped with a symmetric bilinear form $(,)$. Let

$$\mu = (\mu_1, \dots, \mu_n) \in V^n,\ \alpha = (\alpha_1, \dots, \alpha_N) \in V^N,$$

and $\kappa \in \mathbb{C}^*$.

We associate to these data a $\mathcal{D}$-module $\mathcal{L}(\mu,\alpha)$, to be called *a Coulomb* [2] *$\mathcal{D}$-module*, over $U_{n,N}$: by definition it is the structure sheaf $\mathcal{O}_{U_{n,N}}$ equipped with a connection

$$\nabla(\mu,\alpha) = d_{DR} + \frac{1}{\kappa}\omega(\mu,\alpha)$$

where

$$\omega(\mu,\alpha) = \sum_{i<j}(\mu_i,\mu_j)d\ln(z_i - z_j) -$$
$$-\sum_{i,a}(\mu_i,\alpha_a)d\ln(z_i - t_a) + \sum_{a<b}(\alpha_a,\alpha_b)d\ln(t_a - t_b).$$

1.5. On the other hand we can associate with the data (V,μ,α) above a Lie algebra $\mathfrak{g} = \mathfrak{g}(\alpha)$ ("a Kac-Moody algebra without Serre relations") and a collection of "contragradient Verma" $\mathfrak{g}$-modules $M(\mu_1)^c,\ldots,M(\mu_n)^c$.

For example if V is one-dimensional and $\alpha_1 = \ldots = \alpha_N$ then $\mathfrak{g} = \mathfrak{sl}_2$.

Let

$$M = M(\mu_1)^c\otimes\ldots\otimes M(\mu_n)^c,$$

and consider the total complex of the de Rham complex (1.2.2) $\operatorname{Tot}\Omega^\bullet(U_n, C_\bullet(\mathfrak{n}, M))$. It is Λ-graded where $\Lambda = \sum_i \mathbb{Z}\alpha_i \subset V$, and it carries a decreasing filtration

$$F_z^\bullet \operatorname{Tot}\Omega^\bullet(U_n, C_\bullet(\mathfrak{n}, M))$$

where

$$F_z^\bullet \operatorname{Tot}\Omega^\bullet(U_n, C_\bullet(\mathfrak{n}, M)) \subset \operatorname{Tot}\Omega^\bullet(U_n, C_\bullet(\mathfrak{n}, M))$$

is the subcomplex of differential forms containing $\geq i$ differentials dz_a.

Let $\lambda = \sum_i \alpha_i \in \Lambda$. Our main result defines a map from the λ-homogeneous component of this filtered complex to the filtered complex $(DR(Rp_*\mathcal{L}(\mu,\alpha), F_z^\bullet)$.

For details see Theorem 3.8 and Corollary 3.9.

1.6. This paper studies the dependence on parameters of integrals of closed holomorphic forms over cycles. The study of such functions is the classical topic of singularity theory.

Plan of the paper

In the next §2 we discuss in detail the case $\mathfrak{g} = \mathfrak{sl}_2$. The general case is discussed in §3. In the Appendix we recall some standard homological algebra of filtered complexes.

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§2. The case $\mathfrak{g} = \mathfrak{sl}_2$

2.0. Setup. We consider the Lie algebra $\mathfrak{g} = \mathfrak{sl}_2$ with standard generators e, f, h; let $\mathfrak{n} := \mathbb{C}f \subset \mathfrak{g}$ (resp. $\mathfrak{n}_+ := \mathbb{C}e$) be the lower (resp. upper) triangular subalgebra. We will identify $\mathfrak{n}_+$ with $\mathfrak{n}^*$, with e being dual to f.

[2] "Loi fondamentale de l'Élictricité. La force répulsive des deux petits globes électrisés de la même nature d'électricité, est en raison inverse du carré de la distance du centre de deux globes." Charles-Augustin de Coulomb, *Premier Mémoire sur l'Électricité et le Magnétisme*, 1785.

The Casimir element is

$$\Omega = \frac{1}{2}h \otimes h + e \otimes f + f \otimes e.$$

2.0.1. Invariance lemma. *Let M_1, M_2 be arbitrary $\mathfrak{g}$-modules. The actions of Ω and $\mathfrak{g}$ on $M_1 \otimes M_2$ commute.*

Proof: exercise for the reader.

2.1. Chevalley complex.

If M is a $\mathfrak{g}$-module, $C_\bullet(\mathfrak{n}^*, M^*) = C_\bullet(\mathfrak{n}_+, M^*)$ will denote the Chevalley chain complex

$$0 \longrightarrow \mathfrak{n}^* \otimes M^* \xrightarrow{d^*} M^* \longrightarrow 0 \tag{2.1.1}$$

living in degrees $-1, 0$. Here the action of $\mathfrak{n}^*$ on the dual space M^* is given by

$$(f^*\alpha)(x) = \alpha(ex),\ x \in M,\ \alpha \in M^* \tag{2.1.1a}$$

where $f^* \in \mathfrak{n}^*$ is defined by $f^*(f) = 1$.

Next, $C^\bullet(\mathfrak{n}, M^c)$ will denote the dual complex

$$0 \longrightarrow M \xrightarrow{d} \mathfrak{n} \otimes M \longrightarrow 0,\ d(x) = f \otimes ex \tag{2.1.2}$$

living in degrees $0, 1$.

For $m \in \mathbb{C}$, $M(m)$ will denote the Verma module with a vacuum vector v such that $hv = mv$, $ev = 0$. It is $\mathbb{N}$-graded:

$$M(m) = \oplus_{k\geq 0} M(m)_k$$

where $M(m)_k = \mathbb{C}f^k v$.

Fix a natural $n \geq 1$ and an n-tuple $\mathbf{m} = (m_1, \ldots, m_n) \in \mathbb{C}^n$, and consider the tensor product

$$M(\mathbf{m}) = M(m_1) \otimes \ldots \otimes M(m_n)$$

The above grading on each $M(m_i)$ gives rise to an $\mathbb{N}$-grading on $M(\mathbf{m})$:

$$M(\mathbf{m}) = \oplus_{k\geq 0} M(\mathbf{m})_k$$

where

$$M(\mathbf{m})_k = \oplus_{k_1+\ldots+k_n=k} M(m_1)_{k_1} \otimes \ldots \otimes M(m_n)_{k_n}.$$

For a multi-index $a = (k_1, \ldots, k_n)$ we denote

$$|a| = \sum_{i=1}^{n} a_i, \tag{2.1.3}$$

and

$$f^a v := f^{a_1}v_1 \otimes \ldots \otimes f^{a_n}v_n \in M(\mathbf{m})_{|a|}. \tag{2.1.4}$$

The Chevalley complex acquires a grading as well:

$$C^\bullet(\mathfrak{n}, M(\mathbf{m})^c) = \oplus_{k\geq 0} C^\bullet(\mathfrak{n}, M(\mathbf{m})^c)_k,$$

with

$$C^\bullet(\mathfrak{n}, M(\mathbf{m})^c)_k : 0 \longrightarrow M(\mathbf{m})_k \longrightarrow \mathfrak{n} \otimes M(\mathbf{m})_{k-1} \longrightarrow 0. \tag{2.1.5}$$

2.2. Logarithmic forms. Recall that $\kappa \in \mathbb{C}^*$ is fixed.

We fix an integer $N \geq 0$ and consider the space $U_{n,N}$ (see 1.4 above).

We are going to define certain logarithmic forms on this space. For a function u we denote

$$d\ln u := \frac{du}{u}$$

The symmetric group Σ_N acts on forms from $\Omega^\bullet(U_{n,N})$ by permuting variables $t_1, \dots, t_N$.

For a differential form w we define by Alt w the skew-symmetrization of w with respect to the Σ_N-action,

$$\mathrm{Alt}\; w(t_1, \dots, t_N) = \sum_{\sigma \in \Sigma_N} (-1)^\sigma w(t_{\sigma(1)}, \dots, t_{\sigma(N)}).$$

All forms appearing in our constructions are skew-symmetric. They are given by the following formulas. For $a = (a_1, \dots, a_n) \in \mathbb{N}^n$, $|a| := \sum a_i = N$, we define

$$w_a = \frac{1}{a_1! \dots a_n!} \mathrm{Alt}\; u_a,$$

where

$$\begin{aligned} u_a \quad = \quad & d\ln(t_1 - z_1) \wedge \dots \wedge d\ln(t_{a_1} - z_1) + \\ & + d\ln(t_{a_1+1} - z_2) \wedge \dots \wedge d\ln(t_{a_1+a_2} - z_2) + \\ & \dots + \, d\ln(t_{a_1+\dots+a_{n-1}+1} - z_n) \wedge \dots \wedge d\ln(t_N - z_n). \end{aligned}$$

Similarly, for $b = (b_1, \dots, b_n)$, $|b| = N - 1$, we define

$$w_b = \frac{1}{b_1! \dots b_n!} \mathrm{Alt}\; u_b,$$

where

$$\begin{aligned} u_b \quad = \quad & -\kappa \Big(\; d\ln(t_2 - z_1) \wedge \dots \wedge d\ln(t_{b_1+1} - z_1) + \\ & + d\ln(t_{b_1+2} - z_2) \wedge \dots \wedge d\ln(t_{b_1+b_2+1} - z_2) + \\ & \dots + \, d\ln(t_{b_1+\dots+b_{n-1}+2} - z_n) \wedge \dots \wedge d\ln(t_N - z_n) \Big). \end{aligned}$$

In this formula we start from the variable t_2 and have the factor $-\kappa$ in front of the exterior product.

For example if $N = 2$, $a = (2, 0)$, $b = (1, 0)$, then

$$\begin{aligned} w_a \quad &= \quad d\ln(t_1 - z_1) \wedge d\ln(t_2 - z_1) \\ w_b \quad &= \quad -\kappa(d\ln(t_2 - z_1) + d\ln(t_1 - z_2)). \end{aligned}$$

2.3. Coulomb $\mathcal{D}$-module. Define a "Coulomb interaction" closed 1-form

$$\begin{aligned} \omega_{\mathbf{m}} := \sum_{1 \le s < u \le n} \frac{m_s m_u}{2} \, d\ln(z_s - z_u) + \sum_{1 \le i < j \le N} 2 \, d\ln(t_i - t_j) - \\ - \sum_{i=1}^{N} \sum_{s=1}^{n} m_s \, d\ln(t_i - z_s) \in \Omega^1(U_{n,N}) \end{aligned} \tag{2.3.1}$$

Define a differential $\nabla_{\mathbf{m}}$ on the graded space $\Omega^\bullet(U_{n,N})$

$$\nabla_{\mathbf{m}} := d_{DR} + \frac{1}{\kappa}\omega_{\mathbf{m}} : \; \Omega^i(U_{n,N}) \longrightarrow \Omega^{i+1}(U_{n,N})$$

Note that $\nabla_{\mathbf{m}}^2 = 0$ since $d_{DR}\omega_{\mathbf{m}} = 0$.

We will denote by $\Omega^\bullet_{\mathbf{m}}(U_{n,N})$ the space $\Omega^\bullet(U_{n,N})$ equipped with the differential $\nabla_{\mathbf{m}}$.

This is nothing else but the complex of global sections for the de Rham complex $DR(\mathcal{L}(\mathbf{m}, N))$ of the *Coulomb $\mathcal{D}$-module* $\mathcal{L}(\mathbf{m}) = \mathcal{L}(\mathbf{m}, N)$ over $U_{n,N}$ which is by definition the structure sheaf $\mathcal{O}_{U_{n,N}}$ equipped with a connection $\nabla_{\mathbf{m}} := d_{DR} + \frac{1}{\kappa}\omega_{\mathbf{m}}$.

2.4. Coulomb - KZ - Chevalley complex and a canonical N-cocycle in it. Recall a Chevalley complex $C^\bullet(\mathfrak{n}, M(\mathbf{m})^c)_N$.

Consider a double complex which as a bigraded vector space is a tensor product

$$C^{\bullet\bullet}_{\mathbf{m},N} := \{C^{pq}_{\mathbf{m},N}\}$$

where

$$C^{pq}_{\mathbf{m},N} := \Omega^p(U_{n,N}) \otimes C^q(\mathfrak{n}, M(\mathbf{m})^c)_N$$

Note that the components are nontrivial only if $0 \le q \le 1$.

By definition it is equipped with two differentials:

— the horizontal one is a KZ - Coulomb differential

$$\nabla_{\mathrm{KZ,Coul}} = d_{DR} + \frac{1}{\kappa}\omega_{\mathbf{m}} - \frac{1}{\kappa}\omega_{\mathrm{KZ}}$$

where

$$\omega_{\mathrm{KZ}} := \sum_{1\le i<j\le n} \Omega_{ij} d\ln(z_i - z_j) \tag{2.4.1}$$

It acts on the index p:

$$\nabla_{\mathrm{KZ,Coul}} : \Omega^p(U_{n,N}) \otimes C^q(\mathfrak{n}, M(\mathbf{m})^c)_N \longrightarrow \Omega^{p+1}(U_{n,N}) \otimes C^q(\mathfrak{n}, M(\mathbf{m})^c)_N$$

— the vertical one is the Chevalley differential d_{Ch} acting on the second factor.

We will be interested in the associated total complex

$$C^{\bullet}_{\mathbf{m},N} := \mathrm{Tot}\, C^{\bullet\bullet}_{\mathbf{m},N},$$

to be called the *Coulomb - KZ - Chevalley complex.*

Recall the notations (2.1.4).

Define elements

$$\begin{aligned}
\mathcal{I}_0 &:= \sum_{|a|=N} w_a \otimes f^{(a)} v \in C^{N0}_{\mathbf{m},N}\\
\mathcal{I}_1 &:= \sum_{|b|=N-1} w_b \otimes (f \otimes f^{(b)} v) \in C^{N-1,1}_{\mathbf{m},N}\\
\mathcal{I} &:= \mathcal{I}_0 + \mathcal{I}_1 \in C^{N}_{\mathbf{m},N}
\end{aligned}$$

2.5. Theorem. *$\mathcal{I}$ is a cocycle in $C^\bullet_{\mathbf{m},N}$ of total degree N. In components:*

$$\nabla_{\mathrm{KZ,Coul}} \mathcal{I}_0 = 0 \tag{2.5.1}$$

$$d_{\mathrm{Ch}}\mathcal{I}_0 + \nabla_{\mathrm{KZ,Coul}}\mathcal{I}_1 = 0. \tag{2.5.2}$$

Proof. We deduce Theorem 2.5 from the two main results in [SV]. The first of them is [SV, Theorem 6.16.2] on the relation between the Lie algebra differential and the de Rham differential. The second is [SV, Theorem 7.2.5″] on the relation between the KZ equations and the Gauss-Manin connection.

Since all forms w_a are closed the equation (2.5.1) may be rewritten as

$$\frac{1}{\kappa}\Big(\omega_{\mathbf{m}} - \omega_{KZ}\Big)\mathcal{I}_0 = 0.$$

This equation is the statement of [SV, Theorem 7.2.5"] applied to the $\mathfrak{sl}_2$ case.

Equation (2.5.2) may be rewritten as

$$\frac{1}{\kappa}\Big(\omega_{\mathbf{m}} - \omega_{KZ}\Big)\mathcal{I}_1 + d_{Ch}\mathcal{I}_0 = 0$$

and it can be split into two equations.

One of these equations follows from [SV, Theorem 7.2.5"] applied to the situation with $N-1$ of t-variables instead of the N variables $t_1, \dots, t_N$, and the other equation follows from [SV, Theorem 6.16.2].

More precisely, consider the splitting

$$\frac{1}{\kappa}\omega_{\mathbf{m}}\mathcal{I}_1 = \mathcal{P}_1 + \mathcal{P}_2,$$

where $\mathcal{P}_1, \mathcal{P}_2$ are defined as follows. We have

$$\mathcal{I}_1 = \sum_{|b|=N-1}\sum_{\sigma\in S_N}(-1)^\sigma u_b(t_{\sigma(2)},\dots,t_{\sigma(N)})\otimes\left(f\otimes f^{(b)}v\right)$$

and $\omega_{\mathbf{m}}$ is the sum of 1-forms, $\omega_{\mathbf{m}} = \sum_\alpha \omega_\alpha$, see (2.3.1). We say that a summand

$$(-1)^\sigma\omega_\alpha\wedge u_b(t_{\sigma(2)},\dots,t_{\sigma(k)})\otimes\left(f\otimes f^{(b)}v\right)$$

belongs to $\mathcal{P}_1$ if ω_α does not have the variable $t_{\sigma(1)}$, otherwise it belongs to $\mathcal{P}_2$.

2.5.1. Lemma. *We have*

$$\mathcal{P}_1 - \frac{1}{\kappa}\omega_{\mathrm{KZ}}\mathcal{I}_1 = 0, \tag{2.5.1.1}$$

$$\mathcal{P}_2 + d_{\mathrm{Ch}}\mathcal{I}_0 = 0. \tag{2.5.1.2}$$

Proof of the Lemma. Equation (2.5.1.2) follows from [SV], Theorem 6.16.2. Equation (2.5.1.1) follows from [SV], Theorem 7.2.5$''$. □

This implies (2.5.2) and achieves the proof of 2.5. □

2.6. Interpretation of the cocycle $\mathcal{I}$ as a map $\eta : DR(\mathbf{KZ}) \longrightarrow DR(\mathbf{GM})$. Note that the Coulomb de Rham complex $\Omega^\bullet_{\mathbf{m}}(U_{n,N})$ is a dg-module over the de Rham algebra $\Omega^\bullet(U_{n,N})$ which in turn is a $\Omega^\bullet(U_n)$-algebra due to the projection $p : U_{n,N} \longrightarrow U_n$.

Consider the trivial vector bundle $\mathcal{M}(\mathbf{m})$ over U_n with a fiber $M(\mathbf{m})$; it carries the integrable KZ connection

$$\nabla_{\mathrm{KZ}} = d_z - \frac{1}{\kappa}\omega_{\mathrm{KZ}} \tag{2.6.1}$$

which makes of it a $\mathcal{D}_{U_n}$-module. The space of global sections of its de Rham complex will be

$$DR(\mathcal{M}(\mathbf{m}))(U_n) = \Omega^\bullet(U_n)\otimes_{\mathbb{C}} M(\mathbf{m}).$$

As usual this object is $\mathbb{N}$-graded.

Next we can pass to Chevalley chains and consider a complex of vector bundles

$$\mathcal{C}_\bullet(\mathfrak{n}^*, M(\mathbf{m})^*_N) = U_n\times C_\bullet(\mathfrak{n}^*, \mathcal{M}(\mathbf{m})^*_N),$$

whose dual will be

$$\mathcal{C}^\bullet(\mathfrak{n}, M(\mathbf{m})^c_N) = U_n\times C^\bullet(\mathfrak{n}, \mathcal{M}(\mathbf{m})^c_N).$$

Both complexes carry KZ connections induced by (2.6.1); therefore we may consider their de Rham complexes which are $\Omega^\bullet_{U_n}$-modules.

Our main hero, the KZ - Coulomb - Chevalley complex may be rewritten in a form

$$C^{\bullet\bullet}_{\mathbf{m},N} = DR(\mathcal{C}^\bullet(\mathfrak{n}, M(\mathbf{m})^c_N)(U_n)\otimes_{\Omega^\bullet(U_n)}\Omega^\bullet_{\mathbf{m}}(U_{n,N})$$

By linear algebra, to give a 0-cocycle

$$Z\in \mathrm{Tot}(A^\bullet\otimes B^\bullet)^0$$

in the total complex of a tensor product of two complexes $A^\bullet\otimes B^\bullet$ is equivalent to giving a map of complexes

$$\eta(Z):\ A^{\bullet *}\longrightarrow B^\bullet.$$

Therefore our cocycle $\mathcal{I}$ gives rise to a map between two complexes

$$\eta = \eta(\mathcal{I}) : DR(\mathcal{C}_\bullet(\mathfrak{n}^*, M(\mathbf{m})^*_N)(U_n) \longrightarrow \Omega^\bullet_{\mathbf{m}}(U_{n,N})[N] \tag{2.6.2}$$

Both complexes are filtered:

namely, we define

$$F^i DR(\mathcal{C}_\bullet(\mathfrak{n}^*, M(\mathbf{m})^*_N)(U_n) \subset DR(\mathcal{C}_\bullet(\mathfrak{n}^*, M(\mathbf{m})^*_N)(U_n)$$

to be the subcomplex of forms of degree $\geq i$, and

$$F^i_z \Omega^\bullet_{\mathbf{m}}(U_{n,N}) \subset \Omega^\bullet_{\mathbf{m}}(U_{n,N})$$

to be the subcomplex of forms containing $\geq i$ differentials dz_a.

2.6.1. Key fact. *The map η is compatible with the filtrations.* □

As a corollary , the induced map of E_1-terms of the corresponding spectral sequences gives rise to maps between the de Rham complexes

$$\eta^i :\ DR(H_i(\mathfrak{n}^*, \mathcal{M}(\mathbf{m})^*_N), \nabla_{\mathrm{KZ}}) \longrightarrow DR(Rp_*^{N-i}\mathcal{L}(\mathbf{m}, N), \nabla_{\mathrm{GM}}),$$

$0 \leq i \leq 1$, cf 1.3.

By construction these maps land in the subsheaves of anti-invariants

$$\eta^i :\ DR(H_i(\mathfrak{n}^*, \mathcal{M}(\mathbf{m})^*_N), \nabla_{\mathrm{KZ}}) \longrightarrow DR(Rp_*^{N-i}\mathcal{L}(\mathbf{m}, N)^{\Sigma_N,-}, \nabla_{\mathrm{GM}}),$$

Let us sum up our results.

2.7. Theorem. *The map* (2.6.2) *is a morphism of filtered complexes. The induced map of E_1 terms for the corresponding spectral sequences is a pair of morphisms*

$$\eta^i :\ DR(H_i(\mathfrak{n}^*, \mathcal{M}(\mathbf{m})^*_N), \nabla_{\mathrm{KZ}}) \longrightarrow DR(Rp_*^{N-i}\mathcal{L}(\mathbf{m}, N)^{\Sigma_N,-}, \nabla_{\mathrm{GM}}),$$

$0 \leq i \leq 1$.

Here η^i is a map from the de Rham complex of $H_i(\mathfrak{n}^*, \mathcal{M}(\mathbf{m})^*_N)$ equipped with the KZ connection to the de Rham complex of $Rp_*^{N-i}\mathcal{L}(\mathbf{m}, N)^{\Sigma_N,-}$ equipped with the Gauss-Manin connection, or, which is the same, a morphism of lisse $\mathcal{D}$-modules over U_n:

$$\eta^i :\ (H_i(\mathfrak{n}^*, \mathcal{M}(\mathbf{m})^*_N), \nabla_{\mathrm{KZ}}) \longrightarrow (Rp_*^{N-i}\mathcal{L}(\mathbf{m}, N)^{\Sigma_N,-}, \nabla_{\mathrm{GM}}).$$

These maps are isomorphisms for generic κ.

2.8. Corollary: integral solutions for higher KZ. Let us return to the notations of 2.1.

Consider the complex $C^\bullet(\mathfrak{n}, M(\mathbf{m})^c)_N$, see (2.1.3)

$$0 \longrightarrow M(\mathbf{m})_N \longrightarrow \mathfrak{n} \otimes M(\mathbf{m})_{N-1} \longrightarrow 0$$

which we denote here for brevity

$$C^\bullet :\ 0 \longrightarrow C^0 \xrightarrow{d} C^1 \longrightarrow 0,$$

and the dual complex

$$C^{*\bullet} :\ 0 \longrightarrow C^{1*} \xrightarrow{d^*} C^{0*} \longrightarrow 0$$

In this subsection we consider the analytic version of our varieties and $\mathcal{D}$-modules.

For any $\mathbf{z} = (z_1, \ldots, z_n) \in U_n$ we denote by $F(\mathbf{z})$ the fiber

$$F(\mathbf{z}) := p^{-1}(\mathbf{z}) = \{(t_1, \ldots, t_N) \in \mathbb{C}^N |\ t_i \neq t_j; t_i \neq z_a\} \subset \mathbb{C}^N.$$

We will deal with the analytic Coulomb $\mathcal{D}$-module $\mathcal{L}^{\mathrm{an}}(\mathbf{m})$ over $U_{n,N}$. Consider its de Rham complex

$$\Omega^{\mathrm{an}\bullet}_{\mathbf{m}} := DR(\mathcal{L}^{\mathrm{an}}(\mathbf{m})).$$

For each $\mathbf{z} \in U_n$ let $\Omega^\bullet_{\mathbf{m}}(\mathbf{z})$ denote the restriction of $\Omega^{\mathrm{an}\bullet}_{\mathbf{m}}$ to the fiber $F(\mathbf{z})$; inside it we have the skew-symmetric part

$$\Omega^\bullet_{\mathbf{m}}(\mathbf{z})^{\Sigma_N,-} \subset \Omega^\bullet_{\mathbf{m}}(\mathbf{z})$$

Next, inside $\Omega^\bullet_{\mathbf{m}}(\mathbf{z})^{\Sigma_N,-}$ consider the finite-dimensional Aomoto subcomplex of differential forms with logarithmic singularities along all hyperplanes $t_i = t_j$ and $t_i = z_a$; let us denote this subcomplex

$$A^\bullet(\mathbf{z}) : \ 0 \longrightarrow A^{N-1}(\mathbf{z}) \overset{d_A(\mathbf{z})}{\longrightarrow} A^N(\mathbf{z}) \longrightarrow 0,$$

the differential $d_A(\mathbf{z})$ being the multiplication by the one-form

$$\frac{1}{\kappa}\omega_{\mathbf{m}}(\mathbf{z}) = \frac{1}{\kappa}\Bigg(\sum_{1\le i<j\le N} 2\, d\ln(t_i - t_j) - \sum_{i=1}^{N}\sum_{s=1}^{n} m_s\, d\ln(t_i - z_s)\Bigg) \in \Omega^1(F(\mathbf{z}))$$

cf. (2.3.1). This subcomplex will have only two nontrivial components living in degrees $N-1$ and N.

We denote by

$$W^i(\mathbf{z}) := H^i(A^\bullet(\mathbf{z})), \ i = N-1, N,$$

its cohomology.

Global maps η^i

The space

$$C^0 = M(\mathbf{m})_N$$

admits a base $\{f^a v,\ |a| = N\}$; let us denote by $\{f^{a\vee}\}$ the dual base of C^{0*}.

Similarly, the space

$$C^1 = \mathfrak{n} \otimes M(\mathbf{m})_{N-1}$$

admits a base $\{f \otimes f^b v,\ |b| = N-1\}$; let us denote $\{f^\vee \otimes f^{b\vee}\}$ the dual base of C^{1*}.

Define two maps

$$\eta^i : \ C^{i*} \longrightarrow \Omega^{N-i}(U_{n,N}), \ i = 0, 1,$$

by

$$\eta^0(f^{a\vee}) = w_a,$$

and

$$\eta^1(f^\vee \otimes f^{b\vee}) = w_b.$$

Denote

$$\mathcal{A}^{N-i} := \eta^i(C^{i*}) \subset \Omega^{N-i}(U_{n,N}).$$

Let $\mathbf{z} \in U_n$. The restriction to the fiber $F(\mathbf{z})$ induces maps

$$\mathcal{A}^{N-i} \longrightarrow A^{N-i}(\mathbf{z});$$

composing them with the maps η_i we get maps

$$\eta^i(\mathbf{z}) : \ C^{i*} \longrightarrow A^{N-i}(\mathbf{z})$$

According to [SV] these maps are isomorphisms; moreover, they induce an isomorphism of complexes

$$\eta^\bullet(\mathbf{z}) : C^{*\bullet} \overset{\sim}{\longrightarrow} A^\bullet(\mathbf{z})[N]$$

where on the left we have the Chevalley differential whereas on the right we have the twisted de Rham differential in the de Rham complex of the fiber.

Chains of the Betti realization

For each $\mathbf{z}$ let $\mathcal{L}_{\mathbf{m}}(\mathbf{z})$ denote the restriction of $\mathcal{L}(\mathbf{m})$ to $F(\mathbf{z})$; let

$$\mathcal{L}_{\mathbf{m}}(\mathbf{z})^{\vee} = \mathcal{H}om(\mathcal{L}_{\mathbf{m}}(\mathbf{z}), \mathcal{O}^{\mathrm{an}}_{F(\mathbf{z})})$$

be the dual D-module. Let

$$\mathcal{L}_{\mathbf{m}}(\mathbf{z})^{\vee\mathrm{hor}} \subset \mathcal{L}_{\mathbf{m}}(\mathbf{z})^{\vee}$$

be the subsheaf of horizontal sections; it is a locally constant sheaf over $F(z)$.

Let $C_{\bullet}(F(\mathbf{z}), \mathcal{L}_{\mathbf{m}}(\mathbf{z})^{\vee\mathrm{hor}})$ denote the complex

$$0 \longrightarrow C_{2N}(F(\mathbf{z}), \mathcal{L}_{\mathbf{m}}(\mathbf{z})^{\vee\mathrm{hor}}) \longrightarrow \ldots \longrightarrow C_0(F(\mathbf{z}), \mathcal{L}_{\mathbf{m}}(\mathbf{z})^{\vee\mathrm{hor}}) \longrightarrow 0$$

of finite singular chains with coefficients in $\mathcal{L}_{\mathbf{m}}(\mathbf{z})^{\vee\mathrm{hor}}$. We will be dealing with subspaces of i-cycles

$$Z_i(F(\mathbf{z}), \mathcal{L}_{\mathbf{m}}(\mathbf{z})^{\vee\mathrm{hor}}) \subset C_i(F(\mathbf{z}), \mathcal{L}_{\mathbf{m}}(\mathbf{z})^{\vee\mathrm{hor}})$$

and with homology spaces $H_i(F(\mathbf{z}), \mathcal{L}_{\mathbf{m}}(\mathbf{z})^{\vee\mathrm{hor}})$.

2.8.1. GM connection: Betti realization. When $\mathbf{z}$ varies, these complexes form a complex of (infinite-dimensional) vector bundles over U_n, denoted by $\mathcal{C}_{\bullet}(F(\mathbf{z}), \mathcal{L}_{\mathbf{m}}(\mathbf{z})^{\vee\mathrm{hor}})$. Each term $\mathcal{C}_i(F(\mathbf{z}), \mathcal{L}_{\mathbf{m}}(\mathbf{z})^{\vee\mathrm{hor}})$ carries a flat connection.

Indeed, given $\mathbf{z}_0$ and a finite singular chain

$$\gamma(\mathbf{z}_0) \in C_i(F(\mathbf{z}_0), \mathcal{L}_{\mathbf{m}}(\mathbf{z}_0)^{\vee\mathrm{hor}}),$$

we can move $\mathbf{z}$ in a small neighbourhood $V \ni \mathbf{z}_0$ such that nothing changes topologically; this provides a parallel transport of $\gamma(\mathbf{z}_0)$ over V, i.e. a flat family of chains

$$\{\gamma(\mathbf{z}) \in C_i(F(\mathbf{z}), \mathcal{L}_{\mathbf{m}}(\mathbf{z})^{\vee\mathrm{hor}})\}_{\mathbf{z}\in V}. \tag{2.8.1.1}$$

These connections are obviously compatible with boundary, i.e. we get a flat connection on the complex $\mathcal{C}_{\bullet}(F(\mathbf{z}), \mathcal{L}_{\mathbf{m}}(\mathbf{z})^{\vee\mathrm{hor}})$. This is the Betti incarnation of the derived GM connection.

It induces flat connections on the bundles of cycles $\mathcal{Z}_i(F(\mathbf{z}), \mathcal{L}_{\mathbf{m}}(\mathbf{z})^{\vee\mathrm{hor}})$ and on the homology $\mathcal{H}_i(F(\mathbf{z}), \mathcal{L}_{\mathbf{m}}(\mathbf{z})^{\vee\mathrm{hor}})$.

□

We can integrate i-forms against i-chains, i.e. we have pairings

$$\int : \; C_i(F(\mathbf{z}), \mathcal{L}_{\mathbf{m}}(\mathbf{z})^{\vee\mathrm{hor}}) \otimes \Omega^i_{\mathbf{m}}(\mathbf{z}) \longrightarrow \mathbb{C}.$$

Let

$$\{\gamma_i(\mathbf{z}) \in Z_i(F(\mathbf{z}), \mathcal{L}_{\mathbf{m}}(\mathbf{z})^{\vee\mathrm{hor}})\}_{\mathbf{z}\in V} \tag{2.8.1}$$

be a flat family of cycles over a small open $V \subset U_n$ whose classes in $H_i(F(\mathbf{z}), \mathcal{L}_{\mathbf{m}}(\mathbf{z})^{\vee\mathrm{hor}})$ form a flat section of the GM connection.

Let $i = N$. For each $x \in C^{0*}$ and $\mathbf{z} \in V$ we get a number

$$\int_{\gamma_N(\mathbf{z})} \eta^0(\mathbf{z})(x) \in \mathbb{C};$$

it is linear with respect to x, so we've got an element

$$\int_{\gamma_N(\mathbf{z})} \eta^0(\mathbf{z})(\bullet) \in (C^{0*})^* = C^0 = M(\mathbf{m})_N. \tag{2.8.2}$$

Note that if $x = d^*y$ then $\eta^0(\mathbf{z})(x)$ is a coboundary in $\Omega^{\mathrm{an}}_{\mathbf{m}}(\mathbf{z})$, so the integral is zero since $\gamma_N(\mathbf{z})$ is a cycle. This means that (2.8.2) belongs to the subspace of "singular vectors"

$$(\mathrm{Coker}(d^*))^* = \mathrm{Ker}\, d = \mathrm{Ker}(e : M(\mathbf{m})_N \longrightarrow M(\mathbf{m})_{N-1}).$$

Similarly if $i = N-1$ then for each $x \in C^{1*}$ and $\mathbf{z} \in V$ we get a number

$$\int_{\gamma_{N-1}(\mathbf{z})} \eta^1(\mathbf{z})(x) \in \mathbb{C};$$

which is linear with respect to x, so we've got an element

$$\int_{\gamma_{N-1}(\mathbf{z})} \eta^1(\mathbf{z})(\bullet) \in (C^{1*})^* = C^1 = M(\mathbf{m})_{N-1}.$$

Its image in

$$(\mathrm{Ker}(d^*))^* = C^1/dC^0 = \mathrm{Coker}(e : M(\mathbf{m})_N \longrightarrow M(\mathbf{m})_{N-1})$$

depends only on the homology class

$$\overline{\gamma_{N-1}(\mathbf{z})} \in H_i(F(\mathbf{z}), \mathcal{L}_{\mathbf{m}}(\mathbf{z})^{\vee \mathrm{hor}}).$$

2.8.1. Theorem. (a) *For any local flat family of N-cycles*

$$\{\gamma_N(\mathbf{z}) \in Z_N(F(\mathbf{z}), \mathcal{L}_{\mathbf{m}}(\mathbf{z})^{\vee hor})\}_{z \in V},\ V \subset U_n,$$

the linear map $\int_{\gamma(\mathbf{z})}$ defines a solution of the KZ equations with values in $Ker(d)$, i.e. in the weight component of

$$\mathrm{Ker}\, e:\ M(\mathbf{m}) \longrightarrow M(\mathbf{m})$$

of weight $\sum_{s=1}^n m_i - 2N$. For generic κ any solution of the KZ equations in this space is given by a suitable family $\gamma_N(\mathbf{z})$.

(b) *For any local flat family of $(N-1)$-cycles*

$$\{\gamma_{N-1}(\mathbf{z}) \in Z_{N-1}(F(\mathbf{z}), \mathcal{L}_{\mathbf{m}}(\mathbf{z})^{\vee hor})\}_{z \in V},\ V \subset U_n,$$

the linear map $\int_{\gamma(\mathbf{z})}$ defines a solution of the KZ equations with values in $\mathrm{Coker}(d)$*, i.e. in the weight component of $M(\mathbf{m})/eM(\mathbf{m})$ of weight $\sum_{s=1}^n m_i - 2(N-1)$. For generic κ any solution of the KZ equations in this space is given by a suitable family $\gamma_{N-1}(\mathbf{z})$.*

Proof. Part (a) is proved in [SV], whereas part (b) is new and follows from Theorem 2.7 □

See [CV], where the dimensions of the spaces $\mathrm{Ker}\, d$ and $\mathrm{Coker}\, d$ are calculated for nonnegative integers $m_1, \ldots, m_n$.

2.9. Exotic (dual) KZ equations.

Let $N = 1, n = 2$. Let us look up more attentively at the KZ - Coulomb part of our cocycle. So we have a 2-dimensional subspace

$$M_1 \subset M(m_1) \otimes M(m_2)$$

with a base $\{fv_1 \otimes v_2,\ v_1 \otimes fv_2\}$ whose elements we will write as columns.

The Casimir Ω acts on this subspace by the matrix

$$\Omega = \begin{pmatrix} (m_1-2)m_2/2 & m_2 \\ m_1 & m_1(m_2-2)/2 \end{pmatrix}$$

Consider a double Coulomb - KZ complex $\Omega^{\bullet\bullet}(M)$: as a graded space

$$\Omega^{\bullet\bullet}(M_1) := \Omega^{\bullet\bullet}(U_{2,1}) \otimes M_1$$

where Ω^{ij} are differential forms in z, t, of degree i (resp. j) with respect to z (resp. to t).

The first (horizontal) differential is a KZ connection

$$d' = \nabla_{KZ} = d_z - \frac{1}{\kappa}\frac{\Omega(dz_1 - dz_2)}{z_1 - z_2} = d_z + A_1 dz_1 + A_2 dz_2$$

where d_z means de Rham with respect to z, whereas the second (vertical) differential

$$d'' = d_t$$

(de Rham with respect to t)

The identity $\nabla^2_{KZ} = 0$ means that the KZ connection is integrable.

In coordinates:

$$\partial_{z_2} A_1 - \partial_{z_1} A_2 - [A_1, A_2] = 0,$$

in our case $[A_1, A_2] = 0$.

Now we will descibe the relevant part of the cocycle $\mathcal{I}$ from Theorem 2.5.

Consider a form

$$\omega^{01} = I = \begin{pmatrix} (t-z_1)^{-1}\Phi dt \\ (t-z_2)^{-1}\Phi dt \end{pmatrix} = \begin{pmatrix} I_1 \\ I_2 \end{pmatrix} \in \Omega^{01}(M_1)$$

2.9.1. Claim. *We have*

$$d''\omega^{01} = 0,$$

(obvious), and

$$d'\omega^{01} = d''\omega^{10} \tag{2.9.1}$$

where

$$\omega^{10} = J_1 dz_1 + J_2 dz_2 \in \Omega^{10}(M_1),$$

with

$$J_1 = \begin{pmatrix} -(t-z_1)^{-1}\Phi \\ 0 \end{pmatrix},\ J_2 = \begin{pmatrix} 0 \\ -(t-z_2)^{-1}\Phi \end{pmatrix}.$$

2.9.2. Claim. *We have*

$$d'\omega^{10} = 0,$$

in coordinates

$$-\partial_{z_2} J_1 + \partial_{z_1} J_2 - \frac{1}{\kappa}\frac{\Omega J_2}{z_1 - z_2} + \frac{1}{\kappa}\frac{\Omega J_1}{z_1 - z_2} = 0. \tag{2.9.2}$$

The last differential equation is called the *dual KZ equation*: it is a system of two linear differential equations on two functions (nonzero coordinates of vectors J_1, J_2).

The equation does not depend on t, whereas our vectors J_1, J_2 do. For all t the couple

$$(J_1(t,z), J_2(t,z))$$

is a solution of (2.9.2).

§3. Kac-Moody case

3.1. Kac-Moody algebras without Serre relations. We start with the data from [SV], 6.1. Let $\mathfrak{h}$ be a finite-dimensional vector space equipped with a non-degenerate symmetric bilinear form $(,)$.

We fix a finite set of non-zero covectors $\{\alpha_1, \dots, \alpha_r\} \subset \mathfrak{h}^*$ whose elements are called simple roots; let $B = (b_{ij})$ where $b_{ij} = (\alpha_i, \alpha_j)$ (this is "the symmetrized Cartan matrix").

We denote by

$$h_i = b(\alpha_i)$$

where $b : \mathfrak{h}^* \xrightarrow{\sim} \mathfrak{h}$ is the isomorphism induced by $(,)$.

We define $\mathfrak{g} = \mathfrak{g}(B)$ as a Lie algebra with generators $e_i, f_i, 1 \le i \le r$, and $\mathfrak{h}$ and relations

$$[e_i, f_j] = \delta_{ij} h_i,$$

$$[h, e_i] = \alpha_i(h) e_i,\ [h, f_i] = -\alpha_i(h) f_i,$$

$$[h, h'] = 0,\ h, h' \in \mathfrak{h}$$

We denote by $\mathfrak{n} = \mathfrak{n}_- \subset \mathfrak{g}$ (resp. by $\mathfrak{n}_+$) the Lie subalgebra generated by all elements f_i (resp. e_i); it is a free Lie algebra with these generators.

We have the triangular decomposition

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$$

Root lattice

Let

$$\Lambda = \sum_i \mathbb{Z}\alpha_i \subset \mathfrak{h}^*$$

denote the abelian subgroup generated by α_i.

We will use the notations for "positive" and "negative" submonoids:

$$\Lambda_{\geq 0} := \sum_{i=1}^{r} \mathbb{Z}_{\geq 0}\alpha_i \subset \Lambda,\ \Lambda_{>0} := \Lambda_{\geq 0} \setminus \{\mathbf{0}\};$$

$$\Lambda_{\leq 0} := -\Lambda_{\leq 0}, \Lambda_{<0} := \Lambda_{\leq 0} \setminus \{\mathbf{0}\}$$

Principal gradation

Our algebra $\mathfrak{g}$ is Λ-graded:

$$\mathfrak{g} = \oplus_{\lambda \in \Lambda}\ \mathfrak{g}_\lambda$$

where

$$\mathfrak{g}_\lambda = \{x \in \mathfrak{g}|\ [h, x] = \lambda(h)x \text{ for all } h \in \mathfrak{h}\}$$

with

$$\mathfrak{h} = \mathfrak{g}_{\mathbf{0}},$$

$$\mathfrak{n} := \mathfrak{n}_- = \oplus_{\lambda \in \Lambda_{<0}}\ \mathfrak{g}_\lambda = \oplus_{\lambda \in \Lambda_{<0}}\ \mathfrak{n}_\lambda$$

$$\mathfrak{n}_+ = \oplus_{\lambda \in \Lambda_{>0}}\ \mathfrak{g}_\lambda = \oplus_{\lambda \in \Lambda_{>0}}\ \mathfrak{n}_\lambda$$

Verma modules

For $\mu \in \mathfrak{h}^*$ $M(\mu)$ will denote a $\mathfrak{g}$-module with one generator $v = v_\mu$ and relations

$$hv_\mu = \mu(h)e_\mu,\ e_i v_\mu = 0.$$

It is $(\mu + \Lambda_{\leq 0})$-graded:

$$M(\mu) = \oplus_{\lambda \in \mu + \Lambda_{\leq 0}} M(\mu)_\lambda$$

where

$$M(\mu)_\lambda = \{x \in M(\mu)|\ hx = \lambda(h)x\}$$

A map

$$U\mathfrak{n} \longrightarrow M(\mu),\ x \mapsto xv_\mu$$

is an isomorphism of vector spaces.

Notation: duals for Λ-graded spaces

In the sequel we will be dealing with various Λ-graded spaces $V = \oplus_{\lambda\in\Lambda} V_\lambda$ with finite-dimensional components V_λ. In that case V_λ^* will denote the restricted dual:

$$V^* = \oplus_{\lambda\in\Lambda} V_\lambda^*.$$

Double

The Borel Lie subalgebra

$$\mathfrak{b} := \mathfrak{n} \oplus \mathfrak{h} \subset \mathfrak{g}$$

carries a structure of a *Lie bialgebra* (see [D]) described in [SV], 6.14.1. This means in particular that we have a cobracket map

$$\mathfrak{b} \longrightarrow \mathfrak{b} \wedge \mathfrak{b}$$

which gives, after the passage to duals, a Lie algebra structure on the space $\mathfrak{b}^*$. The projection $\mathfrak{b} \longrightarrow \mathfrak{n}$ induces an embedding $\mathfrak{n}^* \hookrightarrow \mathfrak{b}^*$, and the subspace $\mathfrak{n}^*$ is a Lie subalgebra of $\mathfrak{b}^*$.

This allows one to define its *Drinfeld double* $\tilde{\mathfrak{g}} = D(\mathfrak{b})$; it is a Lie algebra which as a vector space is

$$\tilde{\mathfrak{g}} = \mathfrak{b} \oplus \mathfrak{b}^* = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{h}^* \oplus \mathfrak{n}^*.$$

If M is a Verma module, one introduces a structure of a $\mathfrak{b}^*$-module on M^* which, together with an obvious structure of a a $\mathfrak{b}$-module gives rise to a $D(\mathfrak{b})$-module structure on M^*, see [SV], 6.16.

3.2. Chevalley complexes. We fix $n \geq 1$ and an n-tuple of weights

$$\mu = (\mu_1, \dots, \mu_n) \in \mathfrak{h}^{*n}$$

Consider

$$M(\mu) = M(\mu_1) \otimes \dots \otimes M(\mu_n)$$

The Λ-gradations on each $M(\mu_i)$ gives rise to a Λ-gradation on their tensor product $M(\mu)$.

Each $M(\mu_i)^*$ is a $\tilde{\mathfrak{g}}$-module, whence the tensor product

$$M(\mu)^* = M(\mu_1)^* \otimes \dots \otimes M(\mu_n)^*$$

is a $\tilde{\mathfrak{g}}$-module as well. In particular due to the inclusions of Lie algebras

$$\mathfrak{n}^* \subset \mathfrak{b}^* \subset D(\mathfrak{b}^*) = \tilde{\mathfrak{g}}$$

$M(\mu)^*$ is a $\mathfrak{n}^*$-module.

We will be interested in Chevalley homology complexes:

$$C_\bullet(\mathfrak{n}^*, M(\mu)^*) : \ \dots \longrightarrow \Lambda^2\mathfrak{n}^* \otimes M(\mu)^* \longrightarrow \mathfrak{n}^* \otimes M(\mu)^* \longrightarrow M(\mu)^* \longrightarrow 0 \qquad (3.2.1)$$

They are analogues of (2.1.1).

They carry a Λ-grading induced by gradings on $\mathfrak{n}$ and $M(\mu)$:

$$C_\bullet(\mathfrak{n}^*, M(\mu)^*) = \oplus_{\lambda\in\Lambda} C_\bullet(\mathfrak{n}^*, M(\mu)^*)_\lambda$$

where we denote by $C_\bullet(\mathfrak{n}^*, M(\mu)^*)_\lambda$ the subcomplex of weight $|\mu| + \lambda$, $|\mu| := \sum_{a=1}^n \mu_a$.

3.3. The Casimir element and KZ equation.

We have an invariant Casimir element

$$\Omega \in \tilde{\mathfrak{g}} \hat{\otimes} \tilde{\mathfrak{g}}^* \qquad (3.3.1)$$

defined in [SV], 7.2. Namely,

$$\Omega := \sum_{\lambda\in\Lambda_{<0}} \Omega_\lambda + \Omega_0 + \sum_{\lambda\in\Lambda_{>0}} \Omega_\lambda \in$$
$$\in \mathfrak{n}\otimes\mathfrak{n}^* \oplus \mathfrak{h}\otimes\mathfrak{h}^* \oplus \mathfrak{h}^*\otimes\mathfrak{h} \oplus \mathfrak{n}^*\otimes\mathfrak{n}$$

where

$$\Omega_0 = \frac{1}{2}(\Omega_{\mathfrak{h}} + \Omega_{\mathfrak{h}^*})$$

and $\Omega_\lambda \in \mathfrak{n}_\lambda\otimes\mathfrak{n}^*_\lambda$ for $\lambda < 0$ (resp. $\in \mathfrak{n}^*_\lambda\otimes\mathfrak{n}_\lambda$ for $\lambda > 0$) are canonical elements.

Recall the space U_n from 1.1.

Let $\mathcal{C}_\bullet(\mathfrak{n}^*, M(\mu)^*) = C_\bullet(\mathfrak{n}^*, \mathcal{M}(\mu)^*)$ be the trivial vector bundle over U_n with a fiber $C_\bullet(\mathfrak{n}^*, M(\mu)^*)$; it is a Λ-graded complex of vector bundles:

$$C_\bullet(\mathfrak{n}^*, \mathcal{M}(\mu)^*) = \oplus_{\lambda\in\Lambda} C_\bullet(\mathfrak{n}^*, \mathcal{M}(\mu)^*)_\lambda$$

For

$$\lambda = -\sum_{i=1}^r k_i\alpha_i \in \Lambda_{\leq 0}$$

with $\sum_{i=1}^r k_i = N$ the complex $\mathcal{C}_\bullet(\mathfrak{n}^*, M(\mu)^*)_\lambda$ lives in degrees $[-N, 0]$.

The invariant Casimir element allows one to define the KZ connection on each $\mathcal{C}_\bullet(\mathfrak{n}^*, M(\mu)^*)_\lambda$, see 1.2.

3.4. A Coulomb $\mathcal{D}$-module and its de Rham complex. Pick

$$\lambda = -\sum_{i=1}^r k_i\alpha_i \in \Lambda_{\leq 0};$$

let $N = \sum k_i$.

Consider the space $\mathbb{C}^{n,N} = \mathbb{C}^{n+N}$ with coordinates $z_1, \ldots, z_n, t_1, \ldots, t_N$ and a subspace

$$U_{n,N} = \{(\mathbf{z}, \mathbf{t}) \in \mathbb{C}^{n,N} |\ z_i \neq z_j, t_i \neq t_j, z_i \neq t_j\}$$

We have a projection

$$p = p_{n,N} :\ U_{n,N} \longrightarrow U_n$$

We shall use a notation $[k] = \{1, \ldots, k\}$.

Pick a map of sets

$$\pi : [N] \longrightarrow [r]$$

such that

$$|\pi^{-1}(i)| = k_i,\ i \in [r].$$

We will denote by

$$\Sigma_\pi \overset{\sim}{=} \Sigma_{k_1} \times \ldots \times \Sigma_{k_r}$$

a subgroup of the symmetric group respecting all the fibers $\pi^{-1}(i)$.

A Coulomb $\mathcal{D}$-module $\mathcal{L}(\mu, \lambda)$

By definition $\mathcal{L}(\mu, \lambda)$ is a $\mathcal{D}_{U_{n,N}}$-module which is $\mathcal{O}_{U_{n,N}}$ equipped with an integrable connection

$$\nabla_{\mu,\lambda} = d_{DR} + \frac{1}{\kappa}\omega_{\mu,\lambda}$$

where $\omega_{\mu,\lambda}$ is a closed differential 1-form

$$\omega_{\mu,\lambda} = \sum_{1\leq i<j\leq n} (\mu_i, \mu_j)\frac{dz_i - dz_j}{z_i - z_j} - \sum_{i\in[n],k\in[N]} (\mu_i, \alpha_{\pi(k)})\frac{dz_i - dt_k}{z_i - t_k}$$

$$+\sum_{1\leq k<l\leq N}(\alpha_{\pi(k)},\alpha_{\pi(l)})\frac{dt_k-dt_l}{t_k-t_l} \tag{3.4.1}$$

It gives rise to the de Rham complex

$$\Omega^\bullet_{\mu,\lambda}(U_{n,N}):=DR(\mathcal{L}(\mu,\lambda))(U_{n,N})=(\Omega^\bullet(U_{n,N}),\nabla_{\mu,\lambda})$$

We will be interested in the subcomplex of Σ_π-skew-invariants

$$\Omega^\bullet_{\mu,\lambda}(U_{n,N})^{\Sigma_\pi,-}\subset\Omega^\bullet_{\mu,\lambda}(U_{n,N})$$

3.5. Relative de Rham complexes and derived Gauss-Manin.

(a) The de Rham complex $\Omega^\bullet(U_{n,N})$ is the total complex of a bicomplex

$$\Omega^\bullet(U_{n,N})=\operatorname{Tot}\Omega^{\bullet\bullet}(U_{n,N})$$

where $\Omega^{pq}(U_{n,N})$ is the space of forms containing p differentials dz_m and q differentials dt_i, the full de Rham differential being the sum

$$d_{DR}=d_z+d_t.$$

The *relative* de Rham complex is by definition

$$\Omega^\bullet(U_{n,N}/U_n)=(\Omega^{0\bullet}(U_{n,N}),d_t);$$

one has a projection

$$p:\ \Omega^\bullet(U_{n,N})\longrightarrow\Omega^\bullet(U_{n,N}/U_n)$$

(b) *Coulomb twisting*

Similarly the form

$$\omega_{\mu,\lambda}=\omega_{\mu,\lambda,z}+\omega_{\mu,\lambda,t}$$

with

$$\omega_{\mu,\lambda,t}=\sum_{i\in[n],k\in[N]}(\mu_i,\alpha_{\pi(k)})\frac{dt_k}{z_i-t_k}+\sum_{1\leq k<l\leq N}(\alpha_{\pi(k)},\alpha_{\pi(l)})\frac{dt_k-dt_l}{t_k-t_l}$$

which is d_t-closed.

We define

$$\Omega^\bullet_{\mu,\lambda}(U_{n,N}/U_n):=(\Omega^\bullet(U_{n,N}/U_n),d_t+\frac{1}{\kappa}\omega_{\mu,\lambda,t})$$

We have an epimorphism of complexes

$$p:\ \Omega^\bullet_{\mu,\lambda}(U_{n,N})\longrightarrow\Omega^\bullet_{\mu,\lambda}(U_{n,N}/U_n) \tag{3.5.1}$$

(c) *Derived Gauss - Manin connection.*

The complex $\Omega^\bullet_\mu(U_{n,N})$ is the total complex of a double complex

$$\Omega^{\bullet\bullet}_{\mu,\lambda}(U_{n,N}):=(\Omega^{\bullet\bullet}(U_{n,N}),\nabla_{\mu,\lambda,z}+\nabla_{\mu,\lambda,t})$$

where

$$\nabla_{\mu,\lambda,z}=d_z+\frac{1}{\kappa}\omega_{\mu,\lambda,z},\ \nabla_{\mu,\lambda,t}=d_z+\frac{1}{\kappa}\omega_{\mu,\lambda,t}.$$

We shall write the differential $\nabla_{\mu,z}$ (resp. $\nabla_{\mu,t}$) horizontally (resp. vertically).

The map p (3.5.1) is nothing but the projection to the utmost left vertical component.

We can identify $\Omega^{\bullet\bullet}_{\mu,\lambda}(U_{n,N})$ with the de Rham complex of the connection $\nabla_{\mu,\lambda,z}$ on the complex $\Omega^\bullet_{\mu,\lambda}(U_{n,N}/U_n)$:

$$\nabla_{\mu,\lambda,z}:\ \Omega^\bullet_{\mu,\lambda}(U_{n,N}/U_n)=\Omega^{0\bullet}_{\mu,\lambda}(U_{n,N})\longrightarrow\Omega^\bullet_{\mu,\lambda}(U_{n,N}/U_n)\otimes\Omega^1(U_n)=\Omega^{1\bullet}_{\mu,\lambda}(U_{n,N})$$

This is the *derived GM connection* on the complex $Rp_*\mathcal{L}(\mu,\lambda)(U_n)$.

3.7. A map η and its lifting $\tilde{\eta}$.
In [SV] a map of complexes

$$\eta : C_\bullet(\mathfrak{n}^*, \mathcal{M}(\mu)^*)_\lambda \longrightarrow \Omega^\bullet_{\mu,\lambda}(U_{n,N}/U_n)^{\Sigma_\pi,-}[N] \tag{3.7.1}$$

has been defined, see *op. cit.* (7.2.4).

Here we consider both complexes appearing in (3.7.1) as cohomological complexes concentrated in degrees $[-N, 0]$.

For each $\mathbf{z} \in U_n$ consider a fiber

$$U_\mathbf{z} = U_{n,N;\mathbf{z}} := p^{-1}(\mathbf{z})$$

We can compose η with the restriction map

$$r_\mathbf{z} : \Omega^\bullet_{\mu,\lambda}(U_{n,N}/U_n) \longrightarrow \Omega^\bullet_{\mu,\lambda}(U_{n,N;\mathbf{z}})$$

to get

$$\eta_\mathbf{z} = r_\mathbf{z} \circ \eta : C_\bullet(\mathfrak{n}_-, \mathcal{M}(\mu)^*)_\lambda \longrightarrow \Omega^\bullet_{\mu,\lambda}(U_{n,N;\mathbf{z}})^{\Sigma_\pi,-}[N]$$

A remarkable feature of the mappings $\eta_\mathbf{z}$ is the following:
for generic values of κ the maps $\eta_\mathbf{z}$ are quasi-isomorphisms for all $\mathbf{z} \in U_n$.
Here "generic" means $\kappa \in \mathbb{C}\backslash$ (an explicitly given discrete countable subset).

Main result

We start with a definition of a map of *graded $\mathcal{O}(U_n)$-modules*

$$\tilde{\eta} : C_\bullet(\mathfrak{n}^*, \mathcal{M}(\mu)^*)_\lambda \longrightarrow \Omega^\bullet_{\mu,\lambda}(U_{n,N})^{\Sigma_\pi,-}[N] \tag{3.7.2}$$

which lifts η, i.e. such that

$$\eta = p \circ \tilde{\eta}.$$

Here is a picture:

$$\begin{array}{ccc} & & \Omega^\bullet_{\mu,\lambda}(U_{n,N})^{\Sigma_\pi,-}[N] \\ & \tilde{\eta} \nearrow & \downarrow p \\ C_\bullet(\mathfrak{n}^*, \mathcal{M}(\mu)^*)_\lambda & \xrightarrow{\eta} & \Omega^\bullet_{\mu,\lambda}(U_{n,N}/U_n)^{\Sigma_\pi,-}[N] \end{array}$$

The definition of $\tilde{\eta}$ is a modification of that of η. Namely, for a monomial

$$x \in C_i(\mathfrak{n}^*, \mathcal{M}(\mu)^*)_\lambda$$

the corresponding differential form

$$\eta(x) \in \Omega^{N-i}_{\mu,\lambda}(U_{n,N}/U_n)$$

contains fractions of the form $dt_i/(t_i - z_p)$. To obtain $\tilde{\eta}(x)$ we replace all these fractions by $d\ln(t_i - z_p)$. That's it.

Compare the definition of forms u_a, u_b in 2.2 for $\mathfrak{g} = \mathfrak{sl}_2$.

The map $\tilde{\eta}$ induces a map of $\Omega^\bullet(U_{n,N})$-modules

$$\tilde{\eta} : \Omega^\bullet(U_{n,N}) \otimes_{\Omega^\bullet(U_n)} C_\bullet(\mathfrak{n}^*, \mathcal{M}(\mu)^*)_\lambda \longrightarrow \Omega^\bullet_{\mu,\lambda}(U_{n,N})^{\Sigma_\pi,-}[N] \tag{3.7.3}$$

The space on the left is the underlying space of the De Rham complex for the derived KZ; it carries the KZ differential ∇_{KZ}.

3.8. Theorem. (a) *The map $\tilde{\eta}$* (3.7.3) *commutes with the differentials on both sides.*

(b) *Both sides of* (3.7.3) *carry natural decreasing filtrations in z direction, and the map $\tilde{\eta}$ respects these filtrations.*

In other words, we've got a map of filtered complexes

$$\begin{array}{ccc} DR(\text{derived KZ}) & \xrightarrow{\tilde{\eta}} & DR(\text{derived GM}) \\ \| & & \| \\ \operatorname{Tot}\Omega^{\bullet}_{KZ}(U_n, C_{\bullet}(\mathfrak{n}^*, \mathcal{M}(\mu)^*)_{\lambda}) & \xrightarrow{\tilde{\eta}} & \operatorname{Tot}(\Omega^{\bullet\bullet}_{\mu,\lambda})(U_{n,N})^{\Sigma_{\pi},-}[N] \end{array} \tag{3.8.1}$$

The proof is similar to that of Section 2.

3.9. Corollary. *The map $\tilde{\eta}$ induces maps of $\mathcal{D}_{U_n}$-modules (which are isomorphisms for generic κ)*

$$\eta_i : \ (\mathcal{H}_i(\mathfrak{n}^*, \mathcal{M}(\mu)^*)_{\lambda}, \nabla_{KZ}) \longrightarrow (R^{N-i}p_*\mathcal{L}(\mu,\lambda)^{\Sigma_{\pi},-}, \nabla_{GM})$$

for all $0 \le i \le N$.

For $i = 0$ such a mapping has been constructed in [SV].

As in Section 2, we get from this an integral representation for the solutions.

Appendix

We recall here some standard constructions from homological algebra.

A.1. Bicomplexes. A a *bicomplex* in an abelian category $\mathcal{C}$ is a collection of objects $A^{\bullet\bullet} = \{A^{pq}, \ p, q \in \mathbb{Z}\}$, and arrows

$$d_h^{pq} : \ A^{pq} \longrightarrow A^{p+1,q}, \ d_v^{pq} : \ A^{pq} \longrightarrow A^{p,q+1}$$

such that

$$d_h^2 = d_v^2 = 0, \ d_h d_v = d_v d_h$$

One associates to it a simple complex, to be denoted

$$A^{\bullet} = \operatorname{Tot} A^{\bullet\bullet}$$

with components

$$A^i = \oplus_{p+q=i} A^{pq}$$

and a differential $d : A^i \longrightarrow A^{i+1}$ with components

$$d^{pq} = d_h^{pq} + (-1)^p d_v^{pq} : \ A^{pq} \longrightarrow A^{p+1,q} \oplus A^{p,q+1}.$$

A.2. Filtered complexes. Let $A^{\bullet}$ be a simple complex. Consider a decreasing filtration by subcomplexes on it:

$$F^0 A^{\bullet} = A^{\bullet} \supset F^1 A^{\bullet} \supset \dots .$$

We associate to it a collection of complexes

$$E(A^{\bullet}, F)^i : \ 0 \longrightarrow H^i(F^0A^{\bullet}/F^1A^{\bullet}) \longrightarrow H^{i+1}(F^1A^{\bullet}/F^2A^{\bullet}) \longrightarrow \dots, \tag{A.2.1}$$

$i \ge 0$, where a differential

$$H^{i+p}(F^pA^{\bullet}/F^{p+1}A^{\bullet}) \longrightarrow H^{i+p+1}(F^{p+1}A^{\bullet}/F^{p+2}A^{\bullet})$$

is the boundary map for the short exact sequence

$$0 \longrightarrow F^{p+1}A^{\bullet}/F^{p+2}A^{\bullet} \longrightarrow F^pA^{\bullet}/F^{p+2}A^{\bullet} \longrightarrow F^pA^{\bullet}/F^{p+1}A^{\bullet} \longrightarrow 0$$

(This is nothing else but the E_1 term of the spectral sequences for $(A^{\bullet}, F^{\bullet})$.)

A.3. Example. Suppose that $A^{\bullet} = \operatorname{Tot} A^{\bullet\bullet}$ with $A^{pq} = 0$ for $p < 0$, and a filtration is defined by

$$F^iA^j = \oplus_{p \ge i, p+q=j} A^{pq}.$$

Then a p-th graded piece

$$F^pA^{\bullet}/F^{p+1}A^{\bullet} = \{A^{pq}, \ q \in \mathbb{Z}\},$$

and the differential induced by d on it coincides with the vertical differential d_v.

It follows that a complex $E(A^\bullet, F^\bullet)^i$ is identified with

$$0 \longrightarrow H^i_v(A^{\bullet 0}) \longrightarrow H^i_v(A^{\bullet 1}) \longrightarrow \dots,$$

with a differential induced by d_h.

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Vadim Schechtman, Institut de Mathématiques de Toulouse, Université Paul Sabatier, 118 route de Narbonne, 31062 Toulouse, France

Alexander Varchenko, Department of Mathematics, University of North Carolina at Chapel Hill, Chapel Hill, NC 27599-3250, USA; Faculty of Mathematics and Mechanics, Lomonosov Moscow State University, Leninskiye Gory 1, 119991 Moscow GSP-1, Russia; Moscow Center of Fundamental and Applied Mathematics Leninskiye Gory 1, 119991 Moscow GSP-1, Russia