

EULER CHARACTERISTICS OF LINEAR SYMPLECTIC QUOTIENTS AND $O(2)$ -SPACES

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ABSTRACT. We give explicit computations of the Γ -Euler characteristic for several families of orbit space definable translation groupoids. These include the translation groupoids associated to finite-dimensional linear representations of the circle and real and unitary representations of the real 2×2 orthogonal group. In the case of translation groupoids associated to linear symplectic quotients of representations of an arbitrary compact Lie group G , we show that unlike the other cases, the Γ -Euler characteristic depends only on the group and not on the representation.

1. INTRODUCTION

The Euler characteristic χ is classically defined in the context of triangulable topological spaces as a homotopy invariant. It can be generalized to definable spaces, a generalization of semialgebraic spaces, sacrificing homotopy invariance but preserving finite additivity. The Euler characteristic has been generalized to many contexts, including the introduction of many Euler characteristics for orbifolds, spaces locally modeled on the quotient of a manifold by a finite group. These various orbifold Euler characteristics can be indexed by a finitely generated group Γ , resulting in the Γ -Euler characteristics χ_Γ . Orbifolds can be represented by certain classes of groupoids; when the groupoid corresponds to a global action of a group G on a topological space X , it is called a translation groupoid and denoted $G \ltimes X$. In this case, the object space of the groupoid is given by X , the arrow space is $G \times X$, with (g, x) representing an arrow from x to gx .

In the recent paper [8], the first and third author gave the definition of the Γ -Euler characteristics χ_Γ of sufficiently well-behaved topological groupoids with definable orbit spaces. This extended the work of [13], which generalized a large collection of orbifold Euler characteristics to translation groupoids by compact Lie groups that may have infinite isotropy and hence do not present orbifolds.

For each finitely presented group Γ , the Γ -Euler characteristic χ_Γ is an invariant of *orbit space definable groupoids*, roughly topological groupoids whose orbit spaces and their partitions into (weak) orbit types can be made definable. In the case of a translation groupoid $G \ltimes X$ given by the definable action of a compact Lie group G on a definable space X , which is the primary focus of this paper, the groupoid is orbit space definable if the usual orbit space $G \backslash X$ is definable, and the partition of $G \backslash X$ into orbits with isomorphic isotropy groups is a finite partition into definable sets. In this context, the Γ -Euler characteristic χ_Γ is defined as an integral over the orbit space $G \backslash X$ using the Euler characteristic as a finitely additive measure [29]. When Γ is trivial, the integrand is 1 yielding the usual Euler characteristic of $G \backslash X$. For general Γ , the

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integrand is $\chi(G_x \backslash \text{Hom}(\Gamma, G_x))$, the Euler characteristic of the space of conjugacy classes of group homomorphisms from Γ to an isotropy group G_x ; see Equation (2.1). When $\Gamma = \mathbb{Z}^\ell$, $\chi_{\mathbb{Z}^\ell}$ recovers the *orbifold Euler characteristic of order ℓ* defined in [13]; in particular, $\chi_{\mathbb{Z}}$ is equal to the *orbifold Euler characteristic*. This is proven in [8, Theorem 5.4] and follows from the fact that χ_Γ can also be computed as the integral over the space of conjugacy classes of homomorphisms $\Gamma \rightarrow G$, of the Euler characteristic of $C_G(\phi) \backslash X^{(\phi)}$, where $X^{(\phi)}$ denotes the points fixed by the image of $\phi \in \text{Hom}(\Gamma, G)$ and $C_G(\phi)$ denotes the centralizer of the image of ϕ . The Γ -Euler characteristics χ_Γ are invariant under Morita equivalence of topological groupoids, multiplicative over Cartesian products of groupoids, and additive over disjoint unions of orbit spaces in the appropriate sense. With additional mild hypotheses on the orbit space definable groupoids, the Γ -Euler characteristics can be realized as the usual (finitely additive) Euler characteristics of definable topological spaces, the Γ -inertia spaces. For a translation groupoid $G \times X$, the Γ -inertia space is the orbit space of the Γ -loop space $\{(\phi, x) \in \text{Hom}(\Gamma, G) \times X : \phi(\gamma) \in G_x \forall x \in \Gamma\}$ by the G -action given by $g(\phi, x) = (g\phi g^{-1}, gx)$. When $\Gamma = \mathbb{Z}$, the \mathbb{Z} -loop space can be identified with the collection of isotropy groups attached to the corresponding point in X , and when $\Gamma = \mathbb{Z}^\ell$, the \mathbb{Z}^ℓ -loop space can be identified with the collection of commuting ℓ -tuples of elements of isotropy groups. For a general orbit space definable groupoid \mathbf{G} , the Γ -inertia space is the orbit space of the *inertia groupoid* $\Lambda_\Gamma \mathbf{G}$, whose object space is the space $\text{Hom}(\Gamma, \mathbf{G})$ of groupoid homomorphisms $\Gamma \rightarrow \mathbf{G}$, which can be understood as the space of homomorphisms from Γ into the isotropy groups of \mathbf{G} , and whose arrows are given by the action of \mathbf{G} on $\text{Hom}(\Gamma, \mathbf{G})$ by conjugation. The precise definitions of χ_Γ and orbit space definable groupoids are recalled in Section 2; see [8] for more details.

In this paper, we give explicit computations of χ_Γ for several collections of translation groupoids. In each case, the translation groupoid is given by a linear action of a compact Lie group G on a finite-dimensional vector space V or a G -invariant definable subset X of V , and hence is orbit space definable. We consider the explicit cases where $G = \mathbb{S}^1$ is the circle or $G = \text{O}(2)$ is the real 2×2 orthogonal group in detail, and also consider the linear symplectic quotient associated to an arbitrary compact Lie group. A key tool is the use of circle actions on the orbit space of the groupoid to simplify these computations. To this end, we generalize the localization formula of [19] for the Euler characteristic of a compact Riemannian manifold on which a torus acts smoothly to affine definable spaces with definable torus actions; see Theorem 3.1.

These computations serve several purposes. First, they give an indication of the degree to which $\chi_\Gamma(G \times X)$ of a translation groupoid $G \times X$ depends on the group G and its action on X . In some of the simplest examples one might initially compute, $\chi_\Gamma(G \times X) = 0$; however, the cases considered here indicate that $\chi_\Gamma(G \times X)$ generally depends heavily on both G and its action on X ; see Remarks 4.5 and 5.6. In addition, these computations offer counterexamples to properties one might expect from χ_Γ ; see e.g. Remark 4.4.

Our computations also indicate general properties of the Γ -Euler characteristics χ_Γ . A notable example is the case of linear symplectic quotients, the (singular) symplectic quotients associated to unitary representations of compact Lie groups. As is recalled at the end of Section 6, linear symplectic quotients are local models for symplectic quotients of manifolds, whose topology is an active area of investigation [6]. Explicit computations of χ_Γ for translation groupoids defining linear symplectic quotients corresponding to representations of \mathbb{S}^1 and $\text{O}(2)$ suggested that the χ_Γ of these groupoids have a particularly simple form in general. This led to the proof of the main result of this paper, Theorem 6.1, which demonstrates that χ_Γ of a linear symplectic quotient by a compact Lie group G depends only on the group and not on the representation, and is therefore equal to χ_Γ of the group G as a groupoid with one unit. This differs significantly from many of the other cases considered in this paper, for which χ_Γ depends both on the group and the action.

The outline of this paper is as follows. In Section 2, we recall the background needed to define χ_Γ focusing on the case of translation groupoids by linear group actions. Section 3 presents the localization formula for the Euler characteristic in the presence of torus actions as well as specializations that allow us to avoid explicit descriptions of some of the orbit types in the orbit space $G \backslash X$ in the computation of $\chi_\Gamma(G \times X)$. In Section 4 we compute $\chi_\Gamma(\mathbb{S}^1 \times V)$ where V is a finite-dimensional representation of the circle as well as $\chi_\Gamma(\mathbb{S}^1 \times X)$ for an \mathbb{S}^1 -invariant subspace of such a V ; Section 5 gives similar computations for a representation of the orthogonal group $O(2)$. In Section 6, we prove Theorem 6.1, which determines the Γ -Euler characteristic of the linear symplectic quotient associated to a representation of an arbitrary compact Lie group G .

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2. BACKGROUND

In this section, we briefly summarize the background and definitions that we will need. The reader is referred to [8, Sec. 2–3] for more details.

Recall [28, Ch. 1, Def. (2.1)] that an *o-minimal structure* on \mathbb{R} is a sequence $(\mathcal{S}_n)_{n \in \mathbb{N}}$ where each \mathcal{S}_n is a collection of subsets of \mathbb{R}^n that is closed under unions and set differences, with the following properties: if $A \in \mathcal{S}_n$, then $A \times \mathbb{R}, \mathbb{R} \times A \in \mathcal{S}_{n+1}$; each \mathcal{S}_n contains the diagonal $\{(x, \dots, x) : x \in \mathbb{R}\} \subset \mathbb{R}^n$; the projection of $A \in \mathcal{S}_{n+1}$ to \mathcal{S}_n as the first n coordinates is in \mathcal{S}_n ; \mathcal{S}_1 is the set of finite unions of points and open intervals; and \mathcal{S}_2 contains $\{(x, y) \in \mathbb{R}^2 : x < y\}$. An example of an o-minimal structure on \mathbb{R} is that of the semialgebraic sets, that is, the finite unions of solution sets to finite systems of polynomial equations and inequalities. Throughout, fix an o-minimal structure $(\mathcal{S}_n)_{n \in \mathbb{N}}$ on \mathbb{R} such that each \mathcal{S}_n contains the semialgebraic sets in \mathbb{R}^n ; the o-minimal structure of semialgebraic sets will be sufficient for much of this paper, but we state some of the results of Section 3 more generally. We say that a subset $A \subseteq \mathbb{R}^n$ that is an element of \mathcal{S}_n is *definable*, and a function $f: A \rightarrow \mathbb{R}^m$ is a *definable function* if its graph is a definable subset of \mathbb{R}^{m+n} . An *affine definable space* is a pair (X, ι_X) such that X is a topological space and $\iota_X: X \rightarrow \mathbb{R}^n$ is a topological embedding, where \mathbb{R}^n has the standard topology, such that $\iota_X(X)$ is a definable set. A subset $A \subset X$ is *definable* if $\iota_X(A)$ is a definable subset of \mathbb{R}^n , and a morphism of affine definable spaces (X, ι_X) and (Y, ι_Y) is a continuous function $f: X \rightarrow Y$ such that $\iota_Y \circ f \circ \iota_X^{-1}$ is a definable function on $\iota_X(X)$. By [28, Ch. 3, (2.11)], every definable subset $A \subseteq \mathbb{R}^n$ admits a finite decomposition into *cells* [28, Ch. 3, Def. (2.3)], the Euler characteristic of a d -dimensional cell is defined to be $(-1)^d$, and the *Euler characteristic* $\chi(A)$ of A is defined to be the sum of the Euler characteristics of its cells [28, Ch. 4, Sec. 2]. If (X, ι_X) is an affine definable space and $A \subseteq X$ is definable, then the Euler characteristic $\chi(A)$ of A is the Euler characteristic of $\iota_X(A)$. By [2, Thm. 2.2], the Euler characteristic $\chi(A)$ depends only on the underlying topological space A .

For a groupoid $\mathbf{G}_1 \rightrightarrows \mathbf{G}_0$, where \mathbf{G}_0 is the *space of objects* and \mathbf{G}_1 is the *space of arrows*, we denote the structure functions as source $s: \mathbf{G}_1 \rightarrow \mathbf{G}_0$, target $t: \mathbf{G}_1 \rightarrow \mathbf{G}_0$, multiplication $m: \mathbf{G}_1 \times_t \mathbf{G}_1 \rightarrow \mathbf{G}_1$, unit $u: \mathbf{G}_0 \rightarrow \mathbf{G}_1$, and inverse $i: \mathbf{G}_1 \rightarrow \mathbf{G}_1$. For $x \in \mathbf{G}_0$, we let \mathbf{G}_x^x denote the isotropy group, $\mathbf{G}_x^x := (s, t)^{-1}(x, x)$, and $\mathbf{G}x$ the orbit, $\mathbf{G}x := \{y \in \mathbf{G}_0 : \exists g \in (s, t)^{-1}(x, y)\}$.

A *topological groupoid* is a groupoid such that \mathbf{G}_0 and \mathbf{G}_1 are topological spaces and the structure functions are continuous; we will also assume that \mathbf{G}_0 and \mathbf{G}_1 are Hausdorff and paracompact and s is an open map. We let $|\mathbf{G}|$ denote the orbit space and $\pi: \mathbf{G}_0 \rightarrow |\mathbf{G}|$ the orbit map, which is a quotient map as s is open. A topological groupoid is *orbit space definable* if it is equipped with a map $\iota_{|\mathbf{G}|}: |\mathbf{G}| \rightarrow \mathbb{R}^n$ such that $(|\mathbf{G}|, \iota_{|\mathbf{G}|})$ is an affine definable space, \mathbf{G}_x^x is a compact Lie group for each $x \in \mathbf{G}_0$, and the partition of $|\mathbf{G}|$ into sets of orbits of points with isomorphic isotropy groups is a finite partition into definable subsets. Following [26, Def. 5.6], we refer to the elements of this partition of $|\mathbf{G}|$ as *weak orbit types*; i.e., the orbits $\mathbf{G}x$ and $\mathbf{G}y$ of respective points $x, y \in \mathbf{G}_0$ are in the same weak orbit type if $\mathbf{G}_x^x \simeq \mathbf{G}_y^y$ as compact Lie groups.

Let us describe an important class of examples that will be the focus of much of this paper; we summarize the relevant definitions and results here and refer the reader to [8, Sec. 2.2] and [4, 24] for more details and references. Let G be a compact Lie group. Then G has a faithful representation and hence is isomorphic to a compact linear algebraic group. This identifies G with a *semialgebraic group*, a semialgebraic subset of some Euclidean space that is a topological group whose multiplication and inverse maps are definable functions in the o-minimal structure of semialgebraic sets. It is moreover a *semialgebraic linear group*, i.e., is semialgebraically isomorphic to a semialgebraic subgroup of $\mathrm{GL}_m(\mathbb{R})$. The corresponding semialgebraic linear group is unique and hence gives G the unique structure of an *affine definable topological group* with respect to any o-minimal structure that contains the semialgebraic sets. That is, with respect to the above embedding of G into $\mathrm{GL}_m(\mathbb{R}) \subset \mathbb{R}^{m^2}$, the multiplication and inverse maps are morphisms of the affine definable spaces $G \times G \rightarrow G$ and $G \rightarrow G$, respectively.

An *affine definable G -space* X is then an affine definable space (X, ι_X) where X is a topological G -space and the action map $G \times X \rightarrow X$ is a morphism of affine definable spaces; we then say that the G -action is a *definable G -action*. It follows that X admits a *definable proper quotient*, i.e., $G \backslash X$ admits a unique structure, up to definable homeomorphism, as an affine definable space with respect to which the orbit map $X \rightarrow G \backslash X$ is a morphism of affine definable spaces. If the o-minimal structure is that of semialgebraic sets and $X \subseteq \mathbb{R}^n$ is semialgebraic, i.e., ι_X is the identity, X is called a *semialgebraic G -set*. In this case, the translation groupoid $G \ltimes X$ is an orbit space definable groupoid by [8, Cor. 3.6]. In particular, if V is a finite-dimensional linear representation of G and X is a semialgebraic subset of V , then $G \ltimes X$ is an orbit space definable groupoid.

If \mathbf{G} is an orbit space definable groupoid and Γ is a finitely presented discrete group, the Γ -Euler characteristic of \mathbf{G} [8, Def. 3.8] is

$$(2.1) \quad \chi_\Gamma(\mathbf{G}) = \int_{|\mathbf{G}|} \chi(\mathbf{G}_x^x \backslash \mathrm{Hom}(\Gamma, \mathbf{G}_x^x)) d\chi(\mathbf{G}x),$$

where $\mathrm{Hom}(\Gamma, \mathbf{G}_x^x)$ is the space of group homomorphisms from Γ to the isotropy group \mathbf{G}_x^x with the compact-open topology, the integral is that with respect to the Euler characteristic [29], see also, e.g., [5], and the action of \mathbf{G}_x^x on $\mathrm{Hom}(\Gamma, \mathbf{G}_x^x)$ is by pointwise conjugation. Because the integrand $\chi(\mathbf{G}_x^x \backslash \mathrm{Hom}(\Gamma, \mathbf{G}_x^x))$ depends only on the isomorphism class of \mathbf{G}_x^x , this can be reformulated as follows. Let

$$|\mathbf{G}| = |\mathbf{G}|_1 \sqcup |\mathbf{G}|_2 \sqcup \cdots \sqcup |\mathbf{G}|_r$$

denote the partition of the orbit space into weak orbit types, i.e., orbits of points with isomorphic isotropy groups. For each i , let G_i denote the isotropy group of a choice of point whose orbit is in $|\mathbf{G}|_i$. Then

$$(2.2) \quad \chi_\Gamma(\mathbf{G}) = \sum_{i=1}^r \chi(|\mathbf{G}|_i) \chi(G_i \backslash \mathrm{Hom}(\Gamma, G_i)).$$

In the same way, if

$$|\mathbf{G}| = X_1 \sqcup X_2 \sqcup \cdots \sqcup X_s$$

is a partition of $|\mathbf{G}|$ into definable subsets that are each contained in a weak orbit type and G_i denotes a choice of isotropy group of a point whose orbit is in X_i , then

$$(2.3) \quad \chi_\Gamma(\mathbf{G}) = \sum_{i=1}^s \chi(X_i) \chi(G_i \backslash \text{Hom}(\Gamma, G_i)).$$

Finally, let us fix some notation that is used throughout this paper. If G is a compact Lie group, X is a G -space, and $H \leq G$, we let (H) denote the G -conjugacy class of H in G and define $X^H := \{x \in X : hx = x \ \forall h \in H\}$, the set of points fixed by H ; $X_H := \{x \in X : G_x = H\}$, points with isotropy group H ; and $X_{(H)} := \{x \in X : G_x \in (H)\}$, points with isotropy group conjugate to H . The set $X_{(H)}$ is called the *orbit type* associated to the conjugacy class (H) . Because $X_{(H)}$ is G -invariant, we will use the term “partition into orbit types” to describe the partition of X as well as the induced partition of the orbit space $G \backslash X$. Note that if $G \ltimes X$ is an orbit space definable groupoid, then an orbit type of $|G \ltimes X| = G \backslash X$ is obviously contained in a weak orbit type.

For a topological space X , we let $\text{Homeo}(X)$ denote the group of homeomorphisms $X \rightarrow X$. If $\rho: G \rightarrow \text{GL}(V)$ is a linear representation of G , $g \in G$, and $W \subseteq V$ is a $\rho(g)$ -invariant subspace, we use the notation $\rho(g)|_W$ to denote the restriction of $\rho(g)$ to W . If W is $\rho(g)$ -invariant for all $g \in G$, we use $\rho|_W$ to denote the resulting restricted representation $G \rightarrow \text{GL}(W)$ defined by $g \mapsto \rho(g)|_W$. If X is a $\rho(g)$ -invariant subset of V , we also use $\rho(g)|_X$ to denote the restriction of $\rho(g)$ to X and, if X is $\rho(g)$ -invariant for all G , $\rho|_X$ to denote the map $G \rightarrow \text{Homeo}(X)$ given by $g \mapsto \rho(g)|_X$. For a positive integer n , we let $\underline{n} = \{1, 2, \dots, n\}$, $R(n)$ denote the group of n th roots of unity, D_{2n} denote the dihedral group with $2n$ elements, and \mathbb{F}_n denote the free group with n generators.

3. LOCALIZATION TO THE FIXED POINTS OF A TORUS ACTION

An important tool we will use throughout this paper is the simplification of Euler characteristic computations in the presence of circle actions. This is a consequence of the following localization theorem for the Euler characteristic, which is well-known in the context of torus actions on manifolds; see [19, Remark on p. 64] and [9, Thm. 9.3]. In this section, we will prove this result and then elaborate consequences for the computation of χ_Γ in specific circumstances.

Theorem 3.1. *Let $T = (\mathbb{S}^1)^\ell$ be a torus and let X be an affine definable space that admits a definable T -action. Then $\chi(X) = \chi(X^T)$. In particular, $\chi(X) = 0$ if $X^T = \emptyset$.*

Proof. As T is compact, X admits a definably proper quotient $\pi: X \rightarrow T \backslash X$ so that $T \backslash X$ is an affine definable space and π is a morphism of affine definable spaces; see [8, pp.2348–9]. By the trivialization theorem [28, Ch. 9 (1.7)], there is a partition $T \backslash X = Y_1 \sqcup \dots \sqcup Y_r$ of $T \backslash X$ into definable subsets such that $\pi^{-1}(Y_i)$ is definably homeomorphic to $F_i \times Y_i$ for a definable set F_i that is itself definably homeomorphic to $\pi^{-1}(x)$ for each $x \in Y_i$. Reindex the Y_i if necessary so that F_i is a single point for $i \leq k$ and not a singleton for $i > k$, where we may have $k = 0$ or $k = r$. As the action map $T \times X \rightarrow X$ induces for each $x \in X$ a definable homeomorphism of $T_x \backslash T$ onto the orbit $\pi^{-1}(x)$ of x in X , it follows that each F_i is definably homeomorphic to $T_i \backslash T$ where T_i is the isotropy group of a point $x \in \pi^{-1}(Y_i)$. Therefore, $T_i = T$ for $i \leq k$ and $T_i \lneq T$ for $i > k$; that is, $X^T = \bigsqcup_{i=1}^k \pi^{-1}(Y_i)$. If $i > k$, then as $F_i \simeq T_i \backslash T$ is a positive-dimensional compact connected abelian Lie group and hence a nontrivial torus, we have by the multiplicativity of χ [28, Ch. 4, Cor. (2.11)] that $\chi(\pi^{-1}(Y_i)) = \chi(F_i \times Y_i) = 0$. It follows that

$$\chi(X) = \sum_{i=1}^r \chi(\pi^{-1}(Y_i)) = \sum_{i=1}^k \chi(\pi^{-1}(Y_i)) = \chi(X^T). \quad \square$$

Combining Theorem 3.1 with Equation (2.3) yields the following.

Corollary 3.2. *Let \mathbf{G} be an orbit space definable groupoid, let*

$$|\mathbf{G}| = X_1 \sqcup X_2 \sqcup \cdots \sqcup X_s$$

be a partition of $|\mathbf{G}|$ into definable sets such that each X_i is contained in a weak orbit type, and let G_i be the isotropy group of a point whose orbit is in X_i . Let T_i be a (possibly trivial) torus for each i , suppose that each X_i admits a definable T_i -action, and let $I = \{i : X_i^{T_i} \neq \emptyset\}$. Then

$$(3.1) \quad \chi_\Gamma(\mathbf{G}) = \sum_{i \in I} \chi(X_i^{T_i}) \chi(G_i \backslash \text{Hom}(\Gamma, G_i)).$$

In the sequel, we will primarily apply Theorem 3.1 in situations where $T = \mathbb{S}^1$, and it will be convenient to state a few specializations to this case. As the only proper closed subgroups of \mathbb{S}^1 are finite, we have the following.

Corollary 3.3. *Let X be an affine definable space that admits a definable \mathbb{S}^1 -action such that the isotropy group of each $x \in X$ is finite. Then $\chi(X) = 0$.*

We will need the following specific application of Corollary 3.3.

Corollary 3.4. *Let G be a compact Lie group and $\rho : G \rightarrow \text{GL}(V)$ a finite-dimensional linear G -representation. Suppose $X \subseteq V$ is a G -invariant definable subset, H is the isotropy group of a point in X , and $Z \subset X_{(H)}$ is G -invariant and definable. Let $Y = Z \cap X_H$. Suppose further that $T \leq C_{\text{Homeo}(Y)}(\rho(N_G(H))|_Y)$ is a subgroup of the centralizer of $\rho(N_G(H))|_Y$ in $\text{Homeo}(Y)$ such that $T \simeq \mathbb{S}^1$, T acts on Y with finite isotropy, and the intersection of T and $\rho(N_G(H))|_Y$ in $\text{Homeo}(Y)$ is finite. Then*

$$\chi_\Gamma(G \ltimes X) = \chi_\Gamma(G \ltimes (X \setminus Z)).$$

Proof. The orbit type in $|G \ltimes X|$ associated to H is given by $G \backslash X_{(H)}$, which is homeomorphic to $N_G(H) \backslash X_H$ by [25, Thm. 4.3.10]. Restricting this homeomorphism to $G \backslash Z$ yields a homeomorphism to $N_G(H) \backslash Y$. As T commutes with $\rho(N_G(H))|_Y$, the T -action descends to a definable action on $N_G(H) \backslash Y$, and the fact that T has finite intersection with $\rho(N_G(H))|_Y$ in $\text{Homeo}(Y)$ implies that the T -action on $N_G(H) \backslash Y$ has finite isotropy. \square

Continuing to let V denote a linear G -representation, let $D = \{e^{\sqrt{-1}\theta} I \in \text{GL}(V) : \theta \in \mathbb{R}\} \simeq \mathbb{S}^1$ denote the subgroup of $\text{GL}(V)$ that acts on V as scalar multiplication by $e^{\sqrt{-1}\theta}$. Because D is in the center of $\text{GL}(V)$, for each isotropy group H of the G -action on V , we have that V_H , $V_{(H)}$, and V^H are D -invariant. Then as D fixes no points in V except the origin, we have the following.

Corollary 3.5. *Let G be a compact Lie group, $\rho : G \rightarrow \text{GL}(V)$ a finite-dimensional linear G -representation, and let $D = \{e^{\sqrt{-1}\theta} I \in \text{GL}(V) : \theta \in \mathbb{R}\} \simeq \mathbb{S}^1$ denote the subgroup of $\text{GL}(V)$ that acts on V as scalar multiplication by $e^{\sqrt{-1}\theta}$. Let $X \subseteq V$ be G -invariant, D -invariant, and definable; let*

$$G \backslash X = (G \backslash X)_1 \sqcup (G \backslash X)_2 \sqcup \cdots \sqcup (G \backslash X)_s$$

denote the partition of $G \backslash X$ into orbit types; and for each i , let H_i denote the isotropy group of a choice of point with orbit in $(G \backslash X)_i$. For a fixed j , suppose $0 \notin X_{H_j}$ and the intersection of $D|_{X_{H_j}}$ with $\rho(N_G(H_j))|_{X_{H_j}}$ in $\text{Homeo}(X_{H_j})$ is finite. Then for any finitely presented Γ , $(G \backslash X)_j$ does not contribute to the Γ -Euler characteristic of $G \ltimes X$, i.e.,

$$\chi_\Gamma(G \ltimes X) = \sum_{\substack{i=1 \\ i \neq j}}^s \chi((G \backslash X)_i) \chi(G_i \backslash \text{Hom}(\Gamma, G_i)).$$

4. COMPUTATIONS FOR \mathbb{S}^1 -REPRESENTATIONS

In this section, we compute the Γ -Euler characteristics of translation groupoids associated to linear representations of the circle. We additionally consider some cases of invariant subsets of representations.

Let V be a finite-dimensional hermitian vector space, let $U(V)$ denote the group of unitary transformations $V \rightarrow V$, and let $\rho: \mathbb{S}^1 \rightarrow U(V)$ be a unitary \mathbb{S}^1 -representation. Choosing a basis with respect to which the action is diagonal, let (x_1, \dots, x_n) be coordinates for V with respect to this basis, and then the action of \mathbb{S}^1 is described by a weight vector $(a_1, \dots, a_n) \in \mathbb{Z}^n$. Specifically, the action is given by

$$z(x_1, \dots, x_n) = (z^{a_1} x_1, \dots, z^{a_n} x_n), \quad z \in \mathbb{S}^1, \quad (x_1, \dots, x_n) \in V.$$

The isotropy group of a point $(x_1, \dots, x_n) \in V$ is determined by the coordinates x_i that are nonzero, and the possible isotropy groups are either \mathbb{S}^1 or the finite group $R(m)$ of m th roots of unity isomorphic to $\mathbb{Z}/m\mathbb{Z}$, $m \geq 1$.

Let $\underline{n} = \{1, 2, \dots, n\}$. We partition V by defining for $I \subseteq \underline{n}$ the set

$$V_I = \{(x_1, \dots, x_n) \in V : x_i \neq 0 \forall i \in I \text{ and } x_i = 0 : \forall i \notin I\}.$$

We also define $\mathbf{a}_I = \{a_i : i \in I\}$ to be the set of weights corresponding to elements of I and set $\mathbf{a}_\emptyset = \{0\}$. Then each V_I is a subset of an orbit type in V so that $\{V_I : I \subseteq \underline{n}\}$ is a partition subordinate to the partition of V into orbit types. Specifically, the isotropy group of V_I is given by the group $R(\gcd(\mathbf{a}_I))$ of $\gcd(\mathbf{a}_I)$ th roots of unity, where in the case $\mathbf{a}_I = \{0\}$, we use the common convention that $\gcd(\{0\}) = 0$ and adopt the convention that $R(0) = \mathbb{S}^1$. Note that the closure V_I^{cl} of V_I is the coordinate subspace of V given by

$$V_I^{\text{cl}} = \{(x_1, \dots, x_n) \in V : x_i = 0 : \forall i \notin I\}.$$

Note further that $V^{\mathbb{S}^1} = \bigsqcup_{I \subseteq \underline{n} : \mathbf{a}_I = \{0\}} V_I$.

Theorem 4.1. *Let $\rho: \mathbb{S}^1 \rightarrow U(V)$ be a unitary \mathbb{S}^1 -representation with weight vector $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{Z}^n$ and let Γ be a finitely presented discrete group. Then*

$$\chi_\Gamma(\mathbb{S}^1 \ltimes V) = \chi(\text{Hom}(\Gamma, \mathbb{S}^1)) - \sum_{\substack{i=1 \\ a_i \neq 0}}^n \chi(\text{Hom}(\Gamma, \mathbb{Z}/a_i\mathbb{Z})).$$

Proof. Consider the partition of $\mathbb{S}^1 \backslash V$ into the sets $\{\mathbb{S}^1 \backslash V_I : I \subseteq \underline{n}\}$. Each element $\mathbb{S}^1 \backslash V_I$ is contained in the weak orbit type of the group $R(\gcd(\mathbf{a}_I))$. In particular,

$$\mathbb{S}^1 \backslash V^{\mathbb{S}^1} = V^{\mathbb{S}^1} = \bigsqcup_{I \subseteq \underline{n} : \mathbf{a}_I = \{0\}} V_I,$$

and each $\mathbb{S}^1 \backslash V_I$ with $\mathbf{a}_I \neq \{0\}$ is contained in the weak orbit type of the finite group $R(\gcd(\mathbf{a}_I))$. As \mathbb{S}^1 is abelian so that conjugation is trivial, we can rewrite Equation (2.3) in this context as

$$(4.1) \quad \begin{aligned} \chi_\Gamma(\mathbb{S}^1 \ltimes V) &= \chi(\text{Hom}(\Gamma, \mathbb{S}^1)) \chi(\mathbb{S}^1 \backslash V^{\mathbb{S}^1}) \\ &+ \sum_{I \subseteq \underline{n} : \mathbf{a}_I \neq \{0\}} \chi(\text{Hom}(\Gamma, \mathbb{Z}/\gcd(\mathbf{a}_I)\mathbb{Z})) \chi(\mathbb{S}^1 \backslash V_I). \end{aligned}$$

The set $\mathbb{S}^1 \backslash V^{\mathbb{S}^1} = V^{\mathbb{S}^1}$ is a complex subspace of V , hence a single even-dimensional cell with Euler characteristic 1, so that the first term in Equation (4.1) reduces to $\chi(\text{Hom}(\Gamma, \mathbb{S}^1))$.

For each i such that $a_i \neq 0$, $V_{\{i\}} \simeq \mathbb{C} \setminus \{0\}$, and $z \in \mathbb{S}^1$ acts on $V_{\{i\}}$ as multiplication by z^{a_i} . It follows that the orbit space $\mathbb{S}^1 \backslash V_{\{i\}}$ is an open interval with isotropy group $R(|a_i|) \simeq \mathbb{Z}/a_i\mathbb{Z}$, and the corresponding term in (4.1) is $-\chi(\text{Hom}(\Gamma, \mathbb{Z}/a_i\mathbb{Z}))$.

To complete the proof, we claim that for each I that contains at least two elements and such that $\mathbf{a}_I \neq \{0\}$, we have $\chi(\mathbb{S}^1 \setminus V_I) = 0$. Let $I = \{i_1, \dots, i_k\}$ with $k \geq 2$ and at least one $a_{i_j} \neq 0$ so that $\mathbf{a}_I \neq \{0\}$. If the a_{i_1}, \dots, a_{i_k} do not all coincide, then the circle action on V_I by scalar multiplication, i.e., the restriction of the action with weight vector $(1, \dots, 1)$ on the vector space V_I^{cl} , commutes with the \mathbb{S}^1 -action on V_I and has finite intersection with $\rho(\mathbb{S}^1)|_{V_I}$. If $a_{i_1} = a_{i_2} = \dots = a_{i_k}$, then the same holds for the circle action with weight vector $(1, \dots, 1, -1)$ on V_I^{cl} . In either case, $\chi(\mathbb{S}^1 \setminus V_I) = 0$ by Corollary 3.4. \square

Remark 4.2. The only real irreducible representation of \mathbb{S}^1 that does not admit a unitary structure is the trivial 1-dimensional representation. Hence, if V is a finite-dimensional real linear representation of \mathbb{S}^1 then $V = W \oplus V^{\mathbb{S}^1}$ where W admits a unitary structure and $V^{\mathbb{S}^1}$ is a real vector space. Say W has weight vector $(a_1, \dots, a_n) \in \mathbb{Z}^n$ and note that each $a_i \neq 0$ by construction. As groupoids, $\mathbb{S}^1 \times V \simeq (\mathbb{S}^1 \times W) \times V^{\mathbb{S}^1}$ where $V^{\mathbb{S}^1}$ is the base groupoid consisting only of units. Then by the multiplicativity of χ_Γ [8, Lem. 4.17], we have that the Γ -Euler characteristic $\chi_\Gamma(\mathbb{S}^1 \times V)$ is given by

$$(-1)^d \chi_\Gamma(\mathbb{S}^1 \times W) + (-1)^d \chi(\text{Hom}(\Gamma, \mathbb{S}^1)) + (-1)^{d+1} \sum_{i=1}^n \chi(\text{Hom}(\Gamma, \mathbb{Z}/a_i \mathbb{Z})),$$

where d is the real dimension of $V^{\mathbb{S}^1}$.

Recall that \mathbb{F}_ℓ denotes the free group with ℓ generators. When $\Gamma = \mathbb{Z}^\ell$ or \mathbb{F}_ℓ , we have $\text{Hom}(\Gamma, \mathbb{S}^1) \simeq (\mathbb{S}^1)^\ell$ and $\text{Hom}(\Gamma, \mathbb{Z}/a_i \mathbb{Z}) \simeq (\mathbb{Z}/a_i \mathbb{Z})^\ell$, implying the following.

Corollary 4.3. *Let $\rho: \mathbb{S}^1 \rightarrow \text{U}(V)$ be a unitary \mathbb{S}^1 -representation with weight vector $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{Z}^n$. Then*

$$\chi_{\mathbb{Z}^\ell}(\mathbb{S}^1 \times V) = \chi_{\mathbb{F}_\ell}(\mathbb{S}^1 \times V) = - \sum_{i=1}^n |a_i|^\ell.$$

If V is a real representation so that $V = W \oplus V^{\mathbb{S}^1}$ where W is unitary with weight vector $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{Z}^n$ and each $a_i \neq 0$, then

$$\chi_{\mathbb{Z}^\ell}(\mathbb{S}^1 \times V) = \chi_{\mathbb{F}_\ell}(\mathbb{S}^1 \times V) = (-1)^{d+1} \sum_{i=1}^n |a_i|^\ell,$$

where d is the real dimension of $V^{\mathbb{S}^1}$.

Remark 4.4. From Corollary 4.3, we see that when $\Gamma = \mathbb{Z}^\ell$ or \mathbb{F}_ℓ , if V_1 and V_2 are two unitary representations of \mathbb{S}^1 , then

$$\chi_\Gamma(\mathbb{S}^1 \times (V_1 \oplus V_2)) = \chi_\Gamma(\mathbb{S}^1 \times V_1) + \chi_\Gamma(\mathbb{S}^1 \times V_2).$$

By Theorem 4.1 and Remark 4.2, this identity does not hold for Γ such that $\chi(\text{Hom}(\Gamma, \mathbb{S}^1)) \neq 0$, nor for $\Gamma = \mathbb{Z}^\ell$ or \mathbb{F}_ℓ when the V_i are real representations. We will see in Section 5 that this also does not hold for representations of $\text{O}(2)$, even when restricting to unitary representations; see Theorem 5.1.

Remark 4.5. Regarding the question of how much the Γ -Euler characteristics χ_Γ depend on the representation, let us observe that if V_1 and V_2 are unitary representations of the circle such that $\chi_\Gamma(\mathbb{S}^1 \times V_1) = \chi_\Gamma(\mathbb{S}^1 \times V_2)$ for $\Gamma = \mathbb{Z}^\ell$ with $\ell = 1, \dots, n$, then the absolute values of the weights of V_1 and V_2 coincide up to permuting weights. This follows from Corollary 4.3 and the facts that the power sums generate the elementary symmetric polynomials and the elementary symmetric polynomials separate orbits of points in \mathbb{C}^n . If the V_i are real representations, then the same holds, and the sign of any $\chi_{\mathbb{Z}^\ell}(\mathbb{S}^1 \times V_i)$ determines the parity of the real dimensions of the $V_i^{\mathbb{S}^1}$. Of course, if the real dimensions of the $V_i^{\mathbb{S}^1}$ have the same parity and the absolute values of the

weights are the same for both, then $\chi_\Gamma(\mathbb{S}^1 \times V_1) = \chi_\Gamma(\mathbb{S}^1 \times V_2)$ for any Γ by Theorem 4.1 and Remark 4.2.

A slight modification of the proof of Theorem 4.1 also yields the following.

Corollary 4.6. *Let $\rho: \mathbb{S}^1 \rightarrow \mathrm{U}(V)$ be a unitary \mathbb{S}^1 -representation with weight vector $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{Z}^n$. Suppose $X \subset V$ is an \mathbb{S}^1 -invariant definable subset with the following property: For each $I = \{i_1, \dots, i_k\}$ with $k \geq 2$ and $\mathbf{a}_I \neq \{0\}$ such that $V_I \cap X \neq \emptyset$, if \mathbf{a}_I has more than one element, then $X \cap V_I$ is invariant under the circle action by scalar multiplication, and if \mathbf{a}_I is a singleton, then $X \cap V_I$ is invariant under a circle action with weight $(\pm 1, \dots, \pm 1)$ where each sign occurs at least once. Then*

$$\chi_\Gamma(\mathbb{S}^1 \times X) = \chi(\mathrm{Hom}(\Gamma, \mathbb{S}^1))\chi(X \cap V^{\mathbb{S}^1}) + \sum_{\substack{i=1 \\ a_i \neq 0}}^n \chi(G \setminus (X \cap V_{\{i\}}))\chi(\mathrm{Hom}(\Gamma, \mathbb{Z}/a_i\mathbb{Z})).$$

Note that the hypotheses of Corollary 4.6 are satisfied for any X defined by restrictions on the moduli of the coordinates of elements of V . As an example, let

$$S = \{(x_1, \dots, x_n) \in V : |x_1|^2 + \dots + |x_n|^2 = 1\} \simeq \mathbb{S}^{2n-1}$$

denote the unit sphere in V . Then $S \cap V^{\mathbb{S}^1}$ is empty or an odd-dimensional sphere so that $\chi(S \cap V^{\mathbb{S}^1}) = 0$, and if $a_i \neq 0$, then $G \setminus (S \cap V_{\{i\}})$ is a point so that $\chi(G \setminus (S \cap V_{\{i\}})) = 1$, yielding

$$\chi_\Gamma(\mathbb{S}^1 \times S) = \sum_{\substack{i=1 \\ a_i \neq 0}}^n \chi(\mathrm{Hom}(\Gamma, \mathbb{Z}/a_i\mathbb{Z})).$$

Similarly, if $B = \{(x_1, \dots, x_n) \in V : |x_1|^2 + \dots + |x_n|^2 \leq 1\}$ is the unit ball in V , then $B \cap V^{\mathbb{S}^1}$ is a point or closed ball so that $\chi(B \cap V^{\mathbb{S}^1}) = 1$, and $G \setminus (B \cap V_{\{i\}})$ is a half-open interval with $\chi(G \setminus (B \cap V_{\{i\}})) = 0$, yielding

$$\chi_\Gamma(\mathbb{S}^1 \times B) = \chi(\mathrm{Hom}(\Gamma, \mathbb{S}^1)).$$

Another important case of an \mathbb{S}^1 -invariant subset of a unitary \mathbb{S}^1 -representation V is the zero fiber of the moment map, which will be considered in Section 6 in the general setting of a unitary representation of an arbitrary compact Lie group.

5. COMPUTATIONS FOR O(2)-REPRESENTATIONS

Throughout this section, $G = \mathrm{O}(2) = \mathrm{O}(2, \mathbb{R})$, which consists of $\mathrm{SO}(2)$, the rotations

$$r_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

and $\mathrm{O}(2) \setminus \mathrm{SO}(2)$, the reflections

$$s_\theta = \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix},$$

for $\theta \in \mathbb{R}$. We briefly recall the representation theory of G and refer the reader to [23, Theorem 7.2.1] and [18, Section 11.2] for more information.

The irreducible unitary representations of $\mathrm{O}(2)$ are those induced from the 1-dimensional irreducible representations of $\mathrm{SO}(2) \simeq \mathbb{S}^1$. For $a \in \mathbb{Z}$, let ϵ_a denote the 1-dimensional circle representation given by multiplication by z^a for $z \in \mathbb{S}^1$ (i.e., the representation with weight vector (a) in the language of Section 4). Let τ_a denote the induced representation of $\mathrm{O}(2)$. Then τ_0 splits into the trivial 1-dimensional representation and the 1-dimensional representation \det with kernel $\mathrm{SO}(2)$ whose action is multiplication by the determinant ± 1 . For $a \neq 0$, τ_a is irreducible and isomorphic to τ_{-a} . We will therefore only consider τ_α for $\alpha > 0$, using the

notation α to emphasize that the integer must be positive. Then $\{\det, \tau_\alpha : \alpha > 0\}$ is a complete list of unitary irreducible representations of $O(2)$. Expressed as matrices, $\det: r_\theta \mapsto (1)$ and $\det: s_\theta \mapsto (-1)$, while for $\alpha > 0$,

$$\tau_\alpha: r_\theta \mapsto \begin{pmatrix} e^{\sqrt{-1}\alpha\theta} & 0 \\ 0 & e^{-\sqrt{-1}\alpha\theta} \end{pmatrix}, \quad \tau_\alpha: s_\theta \mapsto \begin{pmatrix} 0 & e^{\sqrt{-1}\alpha\theta} \\ e^{-\sqrt{-1}\alpha\theta} & 0 \end{pmatrix}.$$

In either \det or τ_α , the isotropy group of the origin is of course $O(2)$. The isotropy group of any nonzero point of \det is $SO(2)$. In τ_α with coordinates (x_1, x_2) , the kernel is the group $R(\alpha) \simeq \mathbb{Z}/\alpha\mathbb{Z}$ of α th roots of unity in $SO(2)$. If $|x_1| = |x_2|$, then $x_1 = e^{\sqrt{-1}\alpha\theta}x_2$ has solutions corresponding to α distinct reflections so that (x_1, x_2) has isotropy isomorphic to $D_{2\alpha}$, the dihedral group with 2α elements.

5.1. Computations for $O(2)$ -representations with general Γ . An arbitrary unitary representation V of $O(2)$ such that $V^{O(2)} = \{0\}$ is of the form

$$(5.1) \quad V = \left(\bigoplus_{i=1}^n \tau_{\alpha_i} \right) \oplus d \det$$

where each $\alpha_i > 0$, $d \geq 0$, and $\dim V = 2n + d$. Note that the underlying action of $SO(2) \simeq \mathbb{S}^1$ has weight vector $(\alpha_1, -\alpha_1, \dots, \alpha_n, -\alpha_n, \overbrace{0, \dots, 0}^d)$.

Let $(x_{i,1}, x_{i,2})$ be complex coordinates for τ_{α_i} and let y_j be the complex coordinate for the j th \det factor. Then the complex coordinates for V are given by

$$(x_{1,1}, x_{1,2}, x_{2,1}, \dots, x_{n,1}, x_{n,2}, y_1, \dots, y_d) \in \mathbb{C}^{2n+d}.$$

It will be convenient to partition V into subsets of orbit types as follows:

- The origin, which has isotropy $O(2)$.
- For each $i \in \underline{n}$, the set \mathbf{X}_i of points $\mathbf{v} \in V$ such that exactly one of $x_{i,1}$ and $x_{i,2}$ are nonzero and all other coordinates are zero. These points have isotropy group $R(\alpha_i)$ isomorphic to $\mathbb{Z}/\alpha_i\mathbb{Z}$.
- For each nonempty $I \subseteq \underline{n}$, the set \mathbf{X}_I^* of points $\mathbf{v} \in V$ with $x_{i,1}, x_{i,2} \neq 0$ for $i \in I$ and all other coordinates zero such that $|x_{i,1}| = |x_{i,2}|$ for all i and there is a common solution $\theta \in [0, 2\pi)$ to $e^{\sqrt{-1}\alpha_i\theta}x_{i,2} = x_{i,1}$ for $i \in I$. It follows that there are $\gcd\{\alpha_i : i \in I\}$ such solutions so that these points have isotropy isomorphic to $D_{2 \gcd\{\alpha_i : i \in I\}}$.
- For each $I \subseteq \underline{n}$ with at least two elements, the set \mathbf{X}_I^\times of points $\mathbf{v} \in V$ such that either all of the $x_{i,1} \neq 0$ for $i \in I$ or all of the $x_{i,2} \neq 0$ for $i \in I$, with all other coordinates zero. These points have isotropy isomorphic to $\mathbb{Z}/\gcd\{\alpha_i : i \in I\}\mathbb{Z}$.
- For each nonempty $I \subseteq \underline{n}$, the set \mathbf{X}_I of points $\mathbf{v} \in V$ such that at least one of $x_{i,1}$ and $x_{i,2}$ are nonzero for each $i \in I$; $x_{i,1} = x_{i,2} = 0$ for $i \notin I$; $y_j = 0$ for all j ; $\mathbf{v} \notin \mathbf{X}_I^*$; $\mathbf{v} \notin \mathbf{X}_I^\times$; and, if I has one element i , both $x_{i,1}, x_{i,2} \neq 0$ so that $\mathbf{v} \notin \mathbf{X}_i$. These points have isotropy isomorphic to $\mathbb{Z}/\gcd\{\alpha_i : i \in I\}\mathbb{Z}$.
- For each nonempty $J \subseteq \underline{d}$, the set \mathbf{Y}_J of points $\mathbf{v} \in V$ such that $y_j \neq 0$ for $j \in J$ with all other coordinates zero. These points have isotropy $SO(2)$.
- For each nonempty $I \subseteq \underline{n}$ and nonempty $J \subseteq \underline{d}$, the set $\mathbf{X}_{I,J}$ of points $\mathbf{v} \in V$ such that at least one of $x_{i,1}$ and $x_{i,2}$ are nonzero for each $i \in I$, $y_j \neq 0$ for $j \in J$, and all other coordinates are zero. These points have isotropy isomorphic to $\mathbb{Z}/\gcd\{\alpha_i : i \in I\}\mathbb{Z}$.

Note that if $x_{i,1} \neq 0$ and $x_{i,2} = 0$ for a point $\mathbf{v} \in V$, then the action of $O(2) \setminus SO(2)$ maps \mathbf{v} to a point satisfying $x_{i,1} = 0$ and $x_{i,2} \neq 0$. Otherwise, the $O(2)$ -action preserves the zero and nonzero coordinates of \mathbf{v} . If $\mathbf{v} \in \mathbf{X}_I^*$ for some $I \subseteq \underline{n}$, then the action of $O(2)$ preserves the number of common solutions $\theta \in [0, 2\pi)$ to $e^{\sqrt{-1}\alpha_i\theta}x_{i,2} = x_{i,1}$ for $i \in I$. Therefore, each of the

sets $\mathbf{X}_i, \mathbf{X}_i^*, \mathbf{X}_I^\times, \mathbf{X}_I, \mathbf{Y}_J$, and $\mathbf{X}_{I,J}$ defined above are $O(2)$ -invariant. Note that points in $\mathbf{X}_I^*, \mathbf{X}_I^\times$, and \mathbf{X}_I have at least two nonzero coordinates.

Theorem 5.1. *Let $\rho: O(2) \rightarrow U(V)$ be a unitary $O(2)$ -representation with $V^{O(2)} = \{0\}$ of the form (5.1) and let Γ be a finitely presented discrete group. Then*

$$(5.2) \quad \chi_\Gamma(O(2) \ltimes V) = \chi(O(2) \backslash \text{Hom}(\Gamma, O(2))) - \sum_{i=1}^n \chi(\text{Hom}(\Gamma, \mathbb{Z}/\alpha_i \mathbb{Z})).$$

Proof. The orbit space of the origin is a single point with isotropy $O(2)$, yielding the term $\chi(O(2) \backslash \text{Hom}(\Gamma, O(2)))$. For $i \in \underline{n}$, the set $O(2) \backslash \mathbf{X}_i$ is homeomorphic to an open interval; specifically, each point in \mathbf{X}_i contains a unique point in its orbit such that $x_{i,1}$ is positive real and $x_{i,2} = 0$. Therefore, each \mathbf{X}_i contributes $-\chi(\text{Hom}(\Gamma, \mathbb{Z}/\alpha_i \mathbb{Z}))$ to $\chi_\Gamma(O(2) \ltimes V)$.

For a set \mathbf{X}_I^\times corresponding to $I \subseteq \underline{n}$ with $|I| \geq 2$ elements such that the α_i with $i \in I$ all coincide, \mathbf{X}_I^\times is homeomorphic to $(\mathbb{C} \setminus \{0\})^{|I|} \sqcup (\mathbb{C} \setminus \{0\})^{|I|}$, and $O(2) \backslash \mathbf{X}_I^\times$ is homeomorphic to $(0, 1) \times (\mathbb{C} \setminus \{0\})^{|I|-1}$. To see this, note that the orbit of each $\mathbf{v} \in \mathbf{X}_I^\times$ contains a unique point such that $x_{i,1} \neq 0$ for each $i \in I$ and, if i_1 is the smallest element of I , then $x_{i_1,1}$ is positive real; for $i \neq i_1$, the $x_{i,1}$ can take any value in $\mathbb{C} \setminus \{0\}$. As $|I| - 1 \geq 1$, it follows that $\chi(O(2) \backslash \mathbf{X}_I^\times) = \chi((0, 1) \times (\mathbb{C} \setminus \{0\})^{|I|-1}) = 0$.

For any set Z of the form $\mathbf{X}_I^*, \mathbf{X}_I, \mathbf{Y}_J$, or $\mathbf{X}_{I,J}$, or of the form \mathbf{X}_I^\times where the α_i for $i \in I$ do not all coincide, the circle action by scalar multiplication on V preserves Z and has finite intersection with $O(2)$ in $\text{Homeo}(Z)$. Hence the corresponding contribution to $\chi_\Gamma(O(2) \ltimes V)$ is zero by Corollary 3.5, completing the proof. \square

Remark 5.2. The hypothesis in Theorem 5.1 that $V^{O(2)} = \{0\}$ can easily be removed by noting that, for an arbitrary unitary $O(2)$ -representation W , $W = W^{O(2)} \times V$ with V as in (5.1), where $W^{O(2)}$ is a complex vector space so that $\chi(W^{O(2)}) = 1$. Hence, as in Remark 4.2, $\chi_\Gamma(O(2) \ltimes V) = \chi_\Gamma(O(2) \ltimes W)$.

Continuing to let V denote a unitary $O(2)$ -representation such that $V^{O(2)} = \{0\}$ of the form (5.1), let $V_\mathbb{R} \subset V$ denote the set of real points in V . That is,

$$V_\mathbb{R} = \{(x_{1,1}, x_{1,2}, x_{2,1}, \dots, x_{n,1}, x_{n,2}, y_1, \dots, y_d) \in V : \overline{x_{i,1}} = x_{i,2} \forall i \text{ and } y_j \in \mathbb{R} \forall j\}.$$

Then $V_\mathbb{R}$ is a real representation of $O(2)$. For example, the defining representation of $O(2)$ is given by $(\tau_1)_\mathbb{R}$.

Theorem 5.3. *Let $\rho: O(2) \rightarrow U(V)$ be a unitary $O(2)$ -representation with $V^{O(2)} = \{0\}$ of the form (5.1) and let Γ be a finitely presented discrete group. Then*

$$(5.3) \quad \begin{aligned} \chi_\Gamma(O(2) \ltimes V_\mathbb{R}) &= \chi(O(2) \backslash \text{Hom}(\Gamma, O(2))) + \frac{(-1)^d - 1}{2} \chi(\text{Hom}(\Gamma, \mathbb{S}^1)) \\ &\quad - \sum_{\emptyset \neq I \subseteq \underline{n}} (-2)^{|I|-1} \chi(D_{2 \gcd\{\alpha_I\}} \backslash \text{Hom}(\Gamma, D_{2 \gcd\{\alpha_I\}})) \\ &\quad - \sum_{I=\{i_1, i_2\} \subseteq \underline{n}} \chi(D_{2 \gcd\{\alpha_I\}} \backslash \text{Hom}(\Gamma, D_{2 \gcd\{\alpha_I\}})) \\ &\quad + \frac{1 - (-1)^d}{2} \sum_{i=1}^n \chi(\text{Hom}(\Gamma, \mathbb{Z}/\alpha_i \mathbb{Z})), \end{aligned}$$

where we use the shorthand $\{\alpha_I\}$ to denote $\{\alpha_i : i \in I\}$.

Proof. The orbit space of the origin is again a point with isotropy $O(2)$, yielding the term $\chi(O(2) \backslash \text{Hom}(\Gamma, O(2)))$. The sets \mathbf{X}_i and \mathbf{X}_I^\times do not intersect $V_\mathbb{R}$, so we consider the sets \mathbf{X}_I^* for nonempty $I \subseteq \underline{n}$ with $|I|$ elements. Let $I = \{i_1, \dots, i_k\}$ and then the $O(2)$ -orbit

of $(x_{i_1,1}, x_{i_1,2}) \in (\tau_{i_1})_{\mathbb{R}}$ contains a unique element with $x_{i_1,1} = x_{i_1,2}$ positive real, which has isotropy $D_{2\alpha_{i_1}}$. In order to be an element of \mathbf{X}_I^* , $(x_{i_2,1}, x_{i_2,2}) \in (\tau_{i_2})_{\mathbb{R}}$ must be contained in one of $\alpha_{i_1}/\gcd(\alpha_{i_1}, \alpha_{i_2})$ lines through the origin (excluding the origin), and the action of $D_{2\alpha_{i_1}}$ identifies these lines. Hence the $D_{2\alpha_{i_1}}$ -orbit of $(x_{i_2,1}, x_{i_2,2})$ contains a unique element with $x_{i_2,1}$ on one of two rays (excluding the origin), the positive real axis and the set of points with argument $\pi \gcd(\alpha_{i_1}, \alpha_{i_2})/\alpha_{i_1}$; such points $(x_{i_1,1}, x_{i_1,2}, x_{i_2,1}, x_{i_2,2}) \in (\tau_{i_1} \oplus \tau_{i_2})_{\mathbb{R}}$ have isotropy $D_{2 \gcd(\alpha_{i_1}, \alpha_{i_2})}$. In the same way, the $D_{2 \gcd(\alpha_{i_1}, \alpha_{i_2})}$ -orbit of $(x_{i_3,1}, x_{i_3,2})$ contains a unique element on one of two rays excluding the origin. Continuing by induction, the orbit space of \mathbf{X}_I^* is homeomorphic to $(0, 1) \times ((0, 1) \sqcup (0, 1))^{|I|-1}$. It therefore has Euler characteristic $-(-2)^{|I|-1}$ and isotropy $D_{2 \gcd\{\alpha_i: i \in I\}}$, yielding the first sum in Equation (5.3).

The set \mathbf{X}_I only intersects $V_{\mathbb{R}}$ if I has at least two elements, as any real point $(x_{i_1,1}, x_{i_1,2})$ satisfies $|x_{i_1,1}| = |x_{i_1,2}|$ and hence is an element of \mathbf{X}_I^* . First assume $I = \{i_1, i_2\}$ has two elements, and then as in the case of \mathbf{X}_I^* , there is a unique point in the $O(2)$ -orbit of $(x_{i_1,1}, x_{i_1,2}) \in (\tau_{i_1})_{\mathbb{R}}$ such that $x_{i_1,1} = x_{i_1,2}$ is positive real; this point has isotropy $D_{2\alpha_{i_1}}$. Then $x_{i_2,1}$ must lie in the complement of the $\alpha_{i_1}/\gcd(\alpha_{i_1}, \alpha_{i_2})$ lines through the origin fixed by a reflection in $D_{2\alpha_{i_1}}$. This separates \mathbb{C} into $2\alpha_{i_1}/\gcd(\alpha_{i_1}, \alpha_{i_2})$ open 2-cells, and there is a unique point in the $D_{2\alpha_{i_1}}$ -orbit of $(x_{i_2,1}, x_{i_2,2}) \in (\tau_{i_2})_{\mathbb{R}}$ contained in the 2-cell bounded by the positive real axis and the ray of points with argument $\pi \gcd(\alpha_{i_1}, \alpha_{i_2})/\alpha_{i_1}$. It follows that the $O(2)$ -orbit space of \mathbf{X}_I is homeomorphic to $(0, 1) \times \mathbb{C}$ and hence $\chi(O(2) \backslash \mathbf{X}_I) = -1$ with isotropy $D_{2 \gcd(\alpha_{i_1}, \alpha_{i_2})}$. Now suppose I has at least three elements, let i_1 be the smallest, and then there must be at least one other element i_2 such that $(x_{i_2,1}, x_{i_2,2}) \in (\tau_{i_2})_{\mathbb{R}}$ is not fixed by any of the reflections in $D_{2\alpha_{i_1}}$ (otherwise, the point is in \mathbf{X}_I^*). Choose a third element $i_3 \in I$, and then $x_{i_3,1}$ is unconstrained in $\mathbb{C} \setminus \{0\}$. Arguing as in the case I has two elements, there is a unique point in the $O(2)$ -orbit of $(x_{i_1,1}, x_{i_1,2}, x_{i_2,1}, x_{i_2,2}, x_{i_3,1}, x_{i_3,2}) \in (\tau_{i_1} \oplus \tau_{i_2} \oplus \tau_{i_3})_{\mathbb{R}}$ such that $x_{i_1,1}$ is positive real, $x_{i_2,1}$ is contained in a single open 2-cell, and $x_{i_3,1}$ is an element of $(\mathbb{Z}/\gcd(\alpha_{i_1}, \alpha_{i_2}, \alpha_{i_3})\mathbb{Z}) \backslash (\mathbb{C} \setminus \{0\})$ with the cyclic group acting as rotations. It follows that the $O(2)$ -orbit space of \mathbf{X}_I is homeomorphic to $(0, 1) \times \mathbb{C} \times (\mathbb{C} \setminus \{0\}) \times \cdots$ and hence has vanishing Euler characteristic. Note that if I has more than three elements, the remaining factors contributed by each will each be homeomorphic $\mathbb{C} \setminus \{0\}$ by the same argument as in the case of x_{i_3} . It follows that the \mathbf{X}_I contribute the second sum in Equation (5.3).

Now let $J \subseteq \underline{d}$ be nonempty with $|J|$ elements, and then \mathbf{Y}_J is homeomorphic to $(\mathbb{R} \setminus \{0\})^{|J|}$. The action of $O(2)$ is by scalar multiplication by -1 , so the orbit space $O(2) \backslash \mathbf{Y}_J$ is homeomorphic to $\mathbb{R} \times (\mathbb{R} \setminus \{0\})^{|J|-1}$ and has Euler characteristic $-(-2)^{|J|-1}$. These points have abelian isotropy \mathbb{S}^1 so that the \mathbf{Y}_J contribute

$$\begin{aligned} - \sum_{\emptyset \neq J \subseteq \underline{d}} (-2)^{|J|-1} \chi(\text{Hom}(\Gamma, \mathbb{S}^1)) &= -\chi(\text{Hom}(\Gamma, \mathbb{S}^1)) \sum_{|J|=1}^d \binom{d}{|J|} (-2)^{|J|-1} \\ &= \frac{(-1)^d - 1}{2} \chi(\text{Hom}(\Gamma, \mathbb{S}^1)) \end{aligned}$$

to $\chi_{\Gamma}(O(2) \times V_{\mathbb{R}})$.

It remains only to consider the $\mathbf{X}_{I,J}$ for nonempty sets $I \subset \underline{n}$ and $J \subset \underline{d}$. First assume $I = \{i\}$ is a singleton and J is arbitrary with $|J| > 0$ elements, and let j_1 be the smallest element of J . The $O(2)$ -orbit of $(x_{i,1}, x_{i,2}, y_{j_1})$ contains a unique point with $x_{i,1}$ positive real and y_{j_1} positive, and the isotropy group $\mathbb{Z}/\alpha_i\mathbb{Z}$ of this point acts trivially on the remaining coordinates y_j for $j \in J$. It follows that the orbit space $O(2) \backslash \mathbf{X}_{I,J}$ of $\mathbf{X}_{I,J}$ is homeomorphic to $\mathbb{R}^2 \times (\mathbb{R} \setminus \{0\})^{|J|-1}$ and hence has Euler characteristic $(-2)^{|J|-1}$, contributing $(-2)^{|J|-1} \chi(\text{Hom}(\Gamma, \mathbb{Z}/\alpha_i\mathbb{Z}))$ to $\chi_{\Gamma}(O(2) \times V_{\mathbb{R}})$. If $I = \{i_1, i_2, \dots\}$ has at least two elements, then choosing an $O(2)$ -orbit representative such that $x_{i_1,1}$ is positive real and $y_{j_1} > 0$, $x_{i_2,1}$ can take

any nonzero value in $\mathbb{C} \setminus \{0\}$, and the $\mathbb{Z}/\alpha_{i_1}\mathbb{Z}$ -action on $\mathbb{C} \setminus \{0\}$ results in a $(\mathbb{Z}/\alpha_{i_1}\mathbb{Z}) \backslash (\mathbb{C} \setminus \{0\})$ factor in the orbit space homeomorphic to $\mathbb{C} \setminus \{0\}$. It follows that $O(2) \backslash \mathbf{X}_{I,J}$ has vanishing Euler characteristic. Hence, the contribution to $\chi_\Gamma(O(2) \times V_{\mathbb{R}})$ of the sets $\mathbf{X}_{I,J}$ is given by

$$\begin{aligned} & \sum_{i=1}^n \sum_{\emptyset \neq J \subseteq \underline{d}} (-2)^{|J|-1} \chi(\text{Hom}(\Gamma, \mathbb{Z}/\alpha_i\mathbb{Z})) \\ &= \sum_{i=1}^n \chi(\text{Hom}(\Gamma, \mathbb{Z}/\alpha_i\mathbb{Z})) \sum_{|J|=1}^d \binom{d}{|J|} (-2)^{|J|-1} \\ &= \sum_{i=1}^n \frac{1 - (-1)^d}{2} \chi(\text{Hom}(\Gamma, \mathbb{Z}/\alpha_i\mathbb{Z})). \quad \square \end{aligned}$$

5.2. Computations for O(2)-representations with free and free abelian Γ . To specialize the computations of Section 5.1 to the groups $\Gamma = \mathbb{Z}^\ell$ and \mathbb{F}_ℓ , we need only compute $\chi(H \backslash \text{Hom}(\Gamma, H))$ for the isotropy groups H that appear in Equations (5.2) and (5.3). For abelian H , this computation is trivial and was considered in Section 4; we have

$$\chi(\mathbb{S}^1 \backslash \text{Hom}(\Gamma, \mathbb{S}^1)) = \chi(\text{Hom}(\Gamma, \mathbb{S}^1)) = 0$$

and

$$\chi((\mathbb{Z}/\alpha\mathbb{Z}) \backslash \text{Hom}(\Gamma, (\mathbb{Z}/\alpha\mathbb{Z}))) = \chi(\text{Hom}(\Gamma, (\mathbb{Z}/\alpha\mathbb{Z}))) = \alpha^\ell$$

for both $\Gamma = \mathbb{Z}^\ell$ and $\Gamma = \mathbb{F}_\ell$. We now consider the remaining cases.

By choosing a generating set for \mathbb{F}_ℓ , $\text{Hom}(\mathbb{F}_\ell, H)$ can be identified with the set of ordered ℓ -tuples of elements of H , and the action of \mathbb{F}_ℓ on $\text{Hom}(\mathbb{F}_\ell, H)$ corresponds to simultaneous conjugation of an ℓ -tuple. The partition of the set of ℓ -tuples of elements of $O(2)$ into $O(2)$ -orbit types under simultaneous conjugation is as follows. Note that the center of $O(2)$ is given by $\{r_0, r_\pi\}$, the centralizer of any other rotation is $SO(2)$, and the centralizer of a reflection s_θ is $\{r_0, r_\pi, s_\theta, r_\pi s_\theta = s_{\theta+\pi}\}$; all s_θ are conjugate, and r_θ is conjugate only to $r_{-\theta}$.

- (i) ℓ -tuples $(g_1, \dots, g_\ell) \in O(2)$ such that each $g_i \in \{r_0, r_\pi\}$. These ℓ -tuples are fixed by $O(2)$.
- (ii) ℓ -tuples $(g_1, \dots, g_\ell) \in O(2)$ such that each $g_i \in SO(2)$ and at least one $g_i \notin \{r_0, r_\pi\}$. The isotropy group of these ℓ -tuples is $SO(2)$.
- (iii) ℓ -tuples $(g_1, \dots, g_\ell) \in O(2)$ in which a single reflection s_θ occurs at least once and all g_i commute with s_θ . The isotropy group of any such ℓ -tuple is $\{r_0, r_\pi, s_\theta, s_{\theta+\pi}\}$.
- (iv) ℓ -tuples $(g_1, \dots, g_\ell) \in O(2)$ fixed only by the center $\{r_0, r_\pi\}$ of $O(2)$. These ℓ -tuples contain at least one reflection s_θ and at least one g_i that does not commute with s_θ . These ℓ -tuples have isotropy $\{r_0, r_\pi\}$.

The ℓ -tuples of types (i), (ii), and (iii) are commuting ℓ -tuples and hence represent elements of $\text{Hom}(\mathbb{Z}^\ell, O(2))$, while the ℓ -tuples of type (iv) do not commute.

Lemma 5.4. *For any $\ell \geq 1$,*

$$(5.4) \quad \chi(O(2) \backslash \text{Hom}(\mathbb{Z}^\ell, O(2))) = 2^{2^\ell - 1},$$

and for $\ell \geq 2$, we have

$$(5.5) \quad \chi(O(2) \backslash \text{Hom}(\mathbb{F}_\ell, O(2))) = 2^{\ell-2}(2^\ell + 1).$$

Proof. We consider the orbits of ℓ -tuples of each type listed above and compute

$$\chi(O(2) \backslash \text{Hom}(\Gamma, O(2)))$$

for the Γ under consideration.

As the center of $O(2)$ has two elements, there are 2^ℓ ℓ -tuples of type (i), and the $O(2)$ -orbit of each is a singleton. The space of ℓ -tuples of types (i) and (ii) combined form the space $SO(2)^\ell$,

and those of type (i) yield 2^ℓ discrete points in $\mathrm{SO}(2)^\ell$. Hence, the ℓ -tuples of type (ii) consist of the complement of these points, a space with Euler characteristic -2^ℓ . Each type (ii) ℓ -tuple is centralized by $\mathrm{SO}(2)$ and hence has an orbit with two elements; it follows that the space of type (ii) tuples forms a 2-to-1 cover of its orbit space, which therefore has Euler characteristic $-2^{\ell-1}$. The space of orbits of ℓ -tuples of type (iii) is discrete; although there are infinitely many choices for s_θ , they are all conjugate, so we can for instance choose the unique representative of each orbit such that the first reflection to occur in the ℓ -tuple is s_0 . To count the resulting orbits of ℓ -tuples, let r be the first position for which g_r is a reflection. There are 2 choices for each g_i with $i < r$ and 4 choices for each g_i with $i > r$, yielding $\sum_{r=1}^{\ell} 2^{r-1} 4^{\ell-r} = 2^{\ell-1}(2^\ell - 1)$ orbits. Because $\mathrm{Hom}(\mathbb{Z}^\ell, \mathrm{O}(2))$ is identified with the space of commuting ℓ -tuples, i.e., those of types (i), (ii), and (iii), we have

$$\chi(\mathrm{O}(2) \setminus \mathrm{Hom}(\mathbb{Z}^\ell, \mathrm{O}(2))) = 2^\ell - 2^{\ell-1} + 2^{\ell-1}(2^\ell - 1) = 2^{2\ell-1},$$

establishing Equation (5.4).

If $\ell = 1$, then there are no ℓ -tuples of type (iv), as any singleton containing a reflection is type (iii). We claim by induction on ℓ that if $\ell \geq 2$, then the space of $\mathrm{O}(2)$ -orbits of ℓ -tuples of type (iv) has Euler characteristic $-2^{\ell-2}(2^\ell - 1)$. First consider $\ell = 2$. A type (iv) tuple must contain a reflection, and there is a unique element of each orbit such that the first reflection that occurs is s_0 and the other element is either a rotation r_θ or a reflection s_θ with $\theta \in (0, \pi)$. The orbit space is therefore three intervals: the tuples of the form (s_0, s_θ) , those of the form (s_0, r_θ) , and those of the form (r_θ, s_0) , so that the Euler characteristic is $-3 = -2^{\ell-2}(2^\ell - 1)$.

Now, let X be the space of ℓ -tuples of type (iv) for some $\ell \geq 2$, and assume

$$\chi(\mathrm{O}(2) \setminus X) = -2^{\ell-2}(2^\ell - 1).$$

If $(g_1, \dots, g_{\ell+1})$ is a type (iv) $(\ell + 1)$ -tuple, then (g_1, \dots, g_ℓ) must be type (ii), (iii), or (iv); we first consider the case that (g_1, \dots, g_ℓ) is type (iv). Choosing a representative of each orbit in $\mathrm{O}(2) \setminus X$ and concatenating any $g_{\ell+1} \in \mathrm{O}(2)$ to this representative yields a representative of the orbit of a type (iv) $(\ell + 1)$ -tuple. Therefore, the set of orbits of type (iv) $(\ell + 1)$ -tuples such that (g_1, \dots, g_ℓ) is as well type (iv) is homeomorphic to $(\mathrm{O}(2) \setminus X) \times \mathrm{O}(2)$ and has zero Euler characteristic.

As seen above, the space of orbits of type (ii) ℓ -tuples has Euler characteristic $-2^{\ell-1}$, and each is centralized by $\mathrm{SO}(2)$. Choose a representative (g_1, \dots, g_ℓ) of each orbit, and then we may concatenate any reflection $g_{\ell+1} = s_\theta$ to yield a type (iv) $(\ell + 1)$ -tuple. Conjugating by $\mathrm{SO}(2)$, there is a unique representative of the resulting $(\ell + 1)$ -tuple such that $g_{\ell+1} = s_0$. It follows that the space of orbits of type (iv) $(\ell + 1)$ -tuples whose first ℓ coordinates are type (ii) has Euler characteristic $-2^{\ell-1}$.

Similarly, the space consisting of orbits of type (iii) ℓ -tuples has Euler characteristic $2^{\ell-1}(2^\ell - 1)$. We can choose a unique representative of each orbit as above such that the first reflection to occur is s_0 , and the centralizer of this ℓ -tuple is $\{r_0, r_\pi, s_0, s_\pi\}$. We may form a type (iv) $(\ell + 1)$ -tuple by concatenating any element of $\mathrm{O}(2)$ that is not in this centralizer, and then we may conjugate by s_0 to choose a unique element of the orbit of the resulting $(\ell + 1)$ -tuple such that $g_{\ell+1} = s_\theta$ or r_θ with $\theta \in (0, \pi)$. It follows that the Euler characteristic of the space of $(\ell + 1)$ -tuples formed in this way is $-2^\ell(2^\ell - 1)$. Combining the above computations, the space of orbits of type (iv) $(\ell + 1)$ -tuples has Euler characteristic

$$-2^\ell(2^\ell - 1) - 2^{\ell-1} = -2^{\ell-1}(2^{\ell+1} - 1),$$

completing the induction.

With this, Equation (5.5) is obtained by adding $-2^{\ell-2}(2^\ell - 1)$, the Euler characteristic of the space of orbits of type (iv) ℓ -tuples, to Equation (5.4). \square

Combining Lemma 5.4 with Theorem 5.1 yields the following.

Corollary 5.5. *Let $\rho: O(2) \rightarrow U(V)$ be a unitary $O(2)$ -representation of the form (5.1) such that $V^{O(2)} = \{0\}$. For any $\ell \geq 1$,*

$$\chi_{\mathbb{Z}^\ell}(O(2) \ltimes V) = 2^{2^\ell - 1} - \sum_{i=1}^n \alpha_i^\ell,$$

and for $\ell \geq 2$,

$$\chi_{\mathbb{F}_\ell}(O(2) \ltimes V) = 2^{\ell-2}(2^\ell + 1) - \sum_{i=1}^n \alpha_i^\ell.$$

Remark 5.6. As in Remark 4.5, observe that the $\chi_{\mathbb{Z}^\ell}(O(2) \ltimes V)$ for $\ell = 1, \dots, n$ determine the α_i , and the same holds for the $\chi_{\mathbb{F}_\ell}(O(2) \ltimes V)$. Hence, among unitary $O(2)$ -representations of the same dimension, finitely many of the $\chi_{\mathbb{Z}^\ell}(O(2) \ltimes V)$ or $\chi_{\mathbb{F}_\ell}(O(2) \ltimes V)$ determine the representation.

Remark 5.7. With V as in Theorem 5.1, note that $O(2) \ltimes V$ fits into a short exact sequence of Lie groupoids

$$\mathbf{1}_V \longrightarrow \mathbb{S}^1 \ltimes V \xrightarrow{\nu} O(2) \ltimes V \xrightarrow{\kappa} \mathbb{Z}/2\mathbb{Z} \ltimes (\mathbb{S}^1 \setminus V) \longrightarrow \mathbf{1}_V$$

where $\mathbf{1}_V$ denotes the base groupoid $V \rightrightarrows V$ over V in which all arrows are units, ν is the embedding of the connected component containing the units, and κ is the quotient map. The

\mathbb{S}^1 -action has weight vector $(\alpha_1, -\alpha_1, \dots, \alpha_n, -\alpha_n, \overbrace{0, \dots, 0}^d)$ as noted after Equation (5.1). The nontrivial element of $\mathbb{Z}/2\mathbb{Z}$ acts on $\mathbb{S}^1 \setminus V$ by

$$\mathbb{S}^1(x_{1,1}, x_{1,2}, \dots, x_{n,1}, x_{n,2}, y_1, \dots, y_d) \mapsto \mathbb{S}^1(x_{1,2}, x_{1,1}, \dots, x_{n,2}, x_{n,1}, -y_1, \dots, -y_d).$$

The action of \mathbb{S}^1 on V by scalar multiplication fixes only the origin and commutes with the $O(2)$ -action so that by Corollary 3.2,

$$\chi_\Gamma(\mathbb{Z}/2\mathbb{Z} \ltimes (\mathbb{S}^1 \setminus V)) = \chi(\text{Hom}(\Gamma, \mathbb{Z}/2\mathbb{Z}))$$

for each finitely presented Γ . In particular, for each positive integer ℓ , it follows that

$$\chi_{\mathbb{Z}^\ell}(\mathbb{Z}/2\mathbb{Z} \ltimes (\mathbb{S}^1 \setminus V)) = \chi_{\mathbb{F}_\ell}(\mathbb{Z}/2\mathbb{Z} \ltimes (\mathbb{S}^1 \setminus V)) = 2^\ell.$$

We have $\chi_\Gamma(\mathbf{1}_V) = 1$ for all Γ , and by Corollary 4.3,

$$\chi_{\mathbb{Z}^\ell}(\mathbb{S}^1 \ltimes V) = \chi_{\mathbb{F}_\ell}(\mathbb{S}^1 \ltimes V) = -2 \sum_{i=1}^n \alpha_i^\ell.$$

In [8, Theorem 5.13], it was demonstrated that χ_Γ is multiplicative over short exact sequences $1 \rightarrow \mathbf{B} \rightarrow \mathbf{G} \rightarrow \mathbf{H} \rightarrow 1$ where \mathbf{B} is a bundle of compact Lie groups, \mathbf{H} is a translation groupoid, and the isotropy groups in \mathbf{G} are abelian; [8, Example 14] then demonstrated that there is no simple relationship between the χ_Γ when the assumption that the \mathbf{G}_x are abelian was dropped. Comparing the above χ_Γ to Corollary 5.5 further indicates that multiplicativity of χ_Γ over short exact sequences only holds in specific circumstances. In particular, $\chi_{\mathbb{F}_\ell}$ and $\chi_{\mathbb{Z}^\ell}$ coincide for $\mathbf{1}_V$, $\mathbb{S}^1 \ltimes V$, and $\mathbb{Z}/2\mathbb{Z} \ltimes (\mathbb{S}^1 \setminus V)$ yet do not for $O(2) \ltimes V$.

We now consider $\chi(D_{2m} \setminus \text{Hom}(\mathbb{Z}^\ell, D_{2m}))$. The center of D_{2m} is $\{r_0, r_\pi\}$ if m is even and trivial if m is odd. If m is even, there are two conjugacy classes of reflections; if m is odd, all reflections are conjugate. For any m , the D_{2m} -conjugacy class of any rotation r_θ with $\theta \neq 0, \pi$ is the pair $r_{\pm\theta}$.

Lemma 5.8. *Let m be any positive integer and let $P(m) = 2$ if m is even and 1 if m is odd. For any $\ell \geq 1$,*

$$(5.6) \quad \chi(D_{2m} \setminus \text{Hom}(\mathbb{Z}^\ell, D_{2m})) = \frac{m^\ell + P(m)^\ell(2^{\ell+1} - 1)}{2},$$

and for $\ell \geq 2$, we have

$$(5.7) \quad \chi(D_{2m} \setminus \text{Hom}(\mathbb{F}_\ell, D_{2m})) = \frac{(2P(m))^\ell + P(m)m^{\ell-1}(2^\ell - 1) + m^\ell}{2}.$$

Proof. First assume m is odd, in which case $r_\pi \notin D_{2m}$. Then there is only one type (i) ℓ -tuple in D_{2m}^ℓ , and its orbit is a singleton. There are $m^\ell - 1$ type (ii) ℓ -tuples, each with an orbit of size 2, yielding $(m^\ell - 1)/2$ orbits. There are $m(2^\ell - 1)$ type (iii) ℓ -tuples that contain a single reflection and the identity, each with an orbit of size m , yielding $2^\ell - 1$ orbits. Hence the orbits of commuting ℓ -tuples consist of

$$1 + \frac{m^\ell - 1}{2} + (2^\ell - 1) = \frac{m^\ell + 2^{\ell+1} - 1}{2}$$

points, yielding Equation (5.6) in this case. Subtracting the numbers of type (i), (ii), and (iii) ℓ -tuples from the total number $(2m)^\ell$ of ℓ -tuples yields $(m^\ell - m)(2^\ell - 1)$ type (iv) ℓ -tuples, each with an orbit of size $2m$, hence $(m^{\ell-1} - 1)(2^\ell - 1)/2$ orbits of type (iv) ℓ -tuples. Then $\chi(D_{2m} \setminus \text{Hom}(\mathbb{F}_\ell, D_{2m}))$ for m odd is given by

$$\frac{m^\ell + 2^{\ell+1} - 1}{2} + \frac{(m^{\ell-1} - 1)(2^\ell - 1)}{2} = \frac{2^\ell + m^{\ell-1}(2^\ell - 1) + m^\ell}{2},$$

corresponding to Equation (5.7) in this case.

Now assume m is even. There are 2^ℓ type (i) ℓ -tuples, each with trivial orbit, and $m^\ell - 2^\ell$ type (ii) ℓ -tuples partitioned into orbits of size 2. For a given reflection s_θ , there are $4^\ell - 2^\ell$ tuples consisting of $\{r_0, r_\pi, s_\theta, s_{-\theta}\}$ that are not of type (ii), and $m/2$ choices for s_θ , yielding $m(4^\ell - 2^\ell)/2$ type (iii) ℓ -tuples in orbits of size $m/2$. Hence,

$$\chi(D_{2m} \setminus \text{Hom}(\mathbb{Z}^\ell, D_{2m})) = 2^\ell + \frac{m^\ell - 2^\ell}{2} + (4^\ell - 2^\ell) = \frac{m^\ell + 2^\ell(2^{\ell+1} - 1)}{2},$$

completing the proof of Equation (5.6).

We again count the ℓ -tuples of type (iv) by subtracting those of type (i), (ii), and (iii) from $(2m)^\ell$, yielding

$$(2m)^\ell - 2^\ell - (m^\ell - 2^\ell) - \frac{m(4^\ell - 2^\ell)}{2} = \frac{(2^\ell - 1)(2m^\ell - m2^\ell)}{2}$$

ℓ -tuples of type (iv) in orbits of size m . Adding the resulting number of orbits to

$$\chi(D_{2m} \setminus \text{Hom}(\mathbb{Z}^\ell, D_{2m}))$$

yields $\chi(D_{2m} \setminus \text{Hom}(\mathbb{F}_\ell, D_{2m}))$ and completes the proof of Equation (5.7). \square

Combining Lemmas 5.4 and 5.8 with Theorem 5.3 yields the following.

Corollary 5.9. *Let $\rho: \text{O}(2) \rightarrow \text{U}(V)$ be a unitary $\text{O}(2)$ -representation of the form (5.1) with $V^{\text{O}(2)} = \{0\}$ and let $V_{\mathbb{R}}$ denote the set of real points in V . Let $P(m) = 2$ if m is even and 1 if m is odd. For any $\ell \geq 1$, using the shorthand $\{\alpha_I\}$ to denote $\{\alpha_i : i \in I\}$,*

$$\begin{aligned} \chi_{\mathbb{Z}^\ell}(\text{O}(2) \times V_{\mathbb{R}}) &= 2^{2\ell-1} \\ &+ \sum_{\emptyset \neq I \subseteq \underline{n}} (-2)^{|I|-2} (\gcd\{\alpha_I\}^\ell + P(\gcd\{\alpha_I\})^\ell (2^{\ell+1} - 1)) \\ &- \frac{1}{2} \sum_{I=\{i_1, i_2\} \subseteq \underline{n}} (\gcd\{\alpha_I\}^\ell + P(\gcd\{\alpha_I\})^\ell (2^{\ell+1} - 1)) + \frac{1 - (-1)^d}{2} \sum_{i=1}^n \alpha_i^\ell, \end{aligned}$$

and for $\ell \geq 2$,

$$\begin{aligned} \chi_{\mathbb{F}_\ell}(\mathrm{O}(2) \ltimes V_{\mathbb{R}}) &= 2^{\ell-2}(2^\ell + 1) + \frac{1 - (-1)^\ell}{2} \sum_{i=1}^n \alpha_i \\ &+ \sum_{\emptyset \neq I \subseteq \underline{n}} (-2)^{|I|-2} ((2P(\mathrm{gcd}\{\alpha_I\}))^\ell + P(\mathrm{gcd}\{\alpha_I\}) \mathrm{gcd}\{\alpha_I\}^{\ell-1} (2^\ell - 1) + \mathrm{gcd}\{\alpha_I\}^\ell) \\ &- \frac{1}{2} \sum_{I=\{i_1, i_2\} \subseteq \underline{n}} ((2P(\mathrm{gcd}\{\alpha_I\}))^\ell + P(\mathrm{gcd}\{\alpha_I\}) \mathrm{gcd}\{\alpha_I\}^{\ell-1} (2^\ell - 1) + \mathrm{gcd}\{\alpha_I\}^\ell). \end{aligned}$$

6. COMPUTATIONS FOR LINEAR SYMPLECTIC QUOTIENTS

In this section, we compute $\chi_\Gamma(\mathbf{G})$ in the case that \mathbf{G} is the translation groupoid representing the real linear symplectic quotient associated to a unitary representation of an arbitrary compact Lie group G . In contrast to many of the other cases considered in this paper, we demonstrate that $\chi_\Gamma(\mathbf{G})$ only depends on the group G and not on the representation.

Let G be a compact Lie group and $\rho: G \rightarrow \mathrm{GL}(V)$ a finite-dimensional linear G -representation. Equip V with a G -invariant hermitian inner product, and then the imaginary part of the hermitian product is a symplectic form on the underlying real vector space of V . The G -action on V is Hamiltonian, admitting a unique homogeneous quadratic moment map $\mu: V \rightarrow \mathfrak{g}^*$ where \mathfrak{g} denotes the Lie algebra of G and \mathfrak{g}^* its dual. Specifically, identifying \mathfrak{g}^* with \mathbb{R}^ℓ by choosing a basis, the component functions of μ have bi-degree $(1, 1)$, i.e., are sums of terms of the form $x_i \bar{x}_j$ where (x_1, \dots, x_n) are complex coordinates for V . The set $\mu^{-1}(0)$, called the *shell*, is G -invariant and real algebraic, hence definable. The *real linear symplectic quotient at level 0* is the quotient $G \backslash \mu^{-1}(0)$. Note that if G is finite, the moment map is zero so that $\mu^{-1}(0) = V$ and the symplectic quotient is the usual quotient $G \backslash V$. See [1, 27, 16] for more information.

Theorem 6.1. *Let G be a compact Lie group, $\rho: G \rightarrow \mathrm{GL}(V)$ a finite-dimensional linear G -representation, and Γ a finitely presented discrete group. Equip V with a G -invariant hermitian inner product and let $\mu: V \rightarrow \mathfrak{g}^*$ denote the homogeneous quadratic moment map so that the real symplectic quotient at level 0 is the orbit space $|G \times \mu^{-1}(0)|$ of the groupoid $G \times \mu^{-1}(0)$. Then*

$$(6.1) \quad \chi_\Gamma(G \times \mu^{-1}(0)) = \chi(G \backslash \mathrm{Hom}(\Gamma, G)) = \chi_\Gamma(G \times \{pt\}).$$

Proof. Let

$$\mu^{-1}(0) = X_1 \sqcup X_2 \sqcup \dots \sqcup X_s$$

denote the partition of $\mu^{-1}(0)$ into orbit types, let H_i be the isotropy group of some point in X_i for each i , and let $Y_i = (X_i)_{H_i} = X_{H_i}$ denote the points in X with isotropy equal to H_i . Order the X_i so that $H_1 = G$, i.e., $X_1 = Y_1 = \mu^{-1}(0)^G$ is the element of the partition that contains the origin. As in Section 3, let $D = \{e^{\sqrt{-1}\theta} I \in \mathrm{GL}(V) : \theta \in \mathbb{R}\} \simeq \mathbb{S}^1$. Recall that as D commutes with the action of G , each Y_i is D -invariant, and $D|_{Y_i}$ denotes the subgroup of $\mathrm{Homeo}(Y_i)$ given by the restriction of D . Fixing $j > 1$, we claim that $D|_{Y_j} \cap \rho(N_G(H_j))|_{Y_j}$ is finite; this is of course clear if G is finite.

Suppose for contradiction that $D|_{Y_j} \cap \rho(N_G(H_j))|_{Y_j}$ is infinite. As $D|_{Y_j} \cap \rho(N_G(H_j))|_{Y_j}$ is a closed subgroup of $D|_{Y_j} \simeq \mathbb{S}^1$, it follows that $D|_{Y_j} \cap \rho(N_G(H_j))|_{Y_j} = D|_{Y_j}$ so that

$$D|_{Y_j} \leq \rho(N_G(H_j))|_{Y_j}.$$

Let W be the complex linear subspace of V spanned by Y_j , and then $W \subseteq V^{H_j}$. By linearity, it follows that $D|_W \subseteq \rho(N_G(H_j))|_W$. Let \mathfrak{n}_j denote the Lie algebra of $N_G(H_j)$ and let $\mu_W: W \rightarrow \mathfrak{n}_j^*$ denote the moment map for the $N_G(H_j)$ -action on W . Choose a basis for \mathfrak{n}_j^* that contains an element dual to the Lie algebra of the subgroup $D|_W \leq N_G(H_j)$ and let $\mu_W^D: W \rightarrow \mathbb{R}$ denote the component of μ_W corresponding to this element. Note that μ_W is the restriction to W of the

moment map of the $N_G(H_j)$ -action on V so that for $w \in W$, $\mu(w) = 0$ implies $\mu_W(w) = 0$. In complex coordinates (w_1, \dots, w_k) for W , the moment map of $D|_W$ is given by a scalar multiple of $|w_1|^2 + |w_2|^2 + \dots + |w_k|^2$, see [11, Ex. 2.1], [14, Sec. 2], or [7, pp. 5]. Therefore, Y_j consists only of the origin, a contradiction. Hence, for each $j > 1$, D acts on X_j with finite isotropy. By Corollary 3.5, it follows that

$$\chi_\Gamma(G \ltimes \mu^{-1}(0)) = \chi(G \setminus X_1) \chi(G \setminus \text{Hom}(\Gamma, G)).$$

As G acts trivially on V^G , we have $\mu|_{V^G} = 0$ so that $V^G \subseteq \mu^{-1}(0)$. Therefore,

$$X_1 = \mu^{-1}(0)^G = V^G$$

is a complex vector space and hence has Euler characteristic 1, completing the proof. \square

Remark 6.2. If $\rho: G \rightarrow \text{GL}(V)$ is a finite-dimensional linear representation of a compact Lie group G , then the proof of Theorem 6.1 applies as well with $\mu^{-1}(0)$ replaced by other G - and D -invariant definable subsets $X \subseteq V$. Letting H_i , $i = 1, \dots, s$ be a choice of isotropy group from each conjugacy class, we require that if $N_G(H_i)$ contains a subgroup isomorphic to \mathbb{S}^1 that acts on V^{H_i} as scalar multiplication (up to a finite kernel), then $X \cap V_{H_i} = \emptyset$. If this holds, then Equation (6.1) does as well with $\mu^{-1}(0)$ replaced with X . When $G = \mathbb{S}^1$ acts on $V \simeq \mathbb{C}^n$ with weight vector (a_1, \dots, a_n) , a subset $X \subseteq V$ defined by an equation of the form

$$\sum_{i=1}^n b_i |x_i|^{p_i} = 0$$

with each $p_i \in \mathbb{Z}$ positive satisfies this hypothesis provided that the b_i are nonzero and, if $a_{i_1} = a_{i_2} = \dots = a_{i_k}$ for some subset $\{i_1, \dots, i_k\} \subseteq \underline{n}$, then the signs of the b_{i_j} , $j = 1, \dots, k$, coincide. The hypothesis about the signs of the b_i can as well be removed using the argument of Theorem 4.1, replacing D with a circle acting in the coordinates $(x_{i_1}, \dots, x_{i_k})$ with positive and negative weights so that its intersection with the G -action is finite.

Let us illustrate Theorem 6.1 with the following.

Example 6.3. Let $G = \mathbb{S}^1$ and $V \simeq \mathbb{C}^3$ with weight vector $(-6, 2, 3)$. In coordinates (x_1, x_2, x_3) for V , a generating set of the real invariants of the action is given by $m_1 = |x_1|^2$, $m_2 = |x_2|^2$, and $m_3 = |x_3|^2$, along with the real and imaginary parts of $p_1 = x_1 x_3^2$, $p_2 = x_1 x_2^3$, and $p_3 = x_2^3 x_3^{-2}$. We have

$$\mu^{-1}(0) = \{(x_1, x_2, x_3) \in \mathbb{C}^3 : 6m_1 = 2m_2 + 3m_3\}$$

so that m_1 is redundant on the shell. The symplectic quotient can be identified with the image of the Hilbert embedding $H: \mu^{-1}(0) \rightarrow \mathbb{R}^8$ given by

$$H(x_1, x_2, x_3) = (m_2, m_3, \text{Re}(p_1), \text{Im}(p_1), \text{Re}(p_2), \text{Im}(p_2), \text{Re}(p_3), \text{Im}(p_3)).$$

On the shell, the invariants satisfy relations generated by the following, which we express in complex coordinates for brevity:

$$\begin{aligned} 2m_2 m_3^2 + 3m_3^3 - 6p_1 \bar{p}_1, & \quad 6p_1 \bar{p}_2 - 2m_2 \bar{p}_3 - 3m_3 \bar{p}_3, & \quad m_3^3 p_1 - p_2 \bar{p}_3 \\ 6\bar{p}_1 p_2 - 2m_2 p_3 - 3m_3 p_3, & \quad m_3^2 \bar{p}_2 - \bar{p}_1 p_3, & \quad m_3^2 p_2 - p_1 p_3, \\ 2m_2^4 + 3m_2^3 m_3 - 6p_2 \bar{p}_2, & \quad m_2^3 \bar{p}_1 - \bar{p}_2 p_3, & \\ 27m_3^5 - 24m_2^2 p_1 \bar{p}_1 + 36m_2 m_3 p_1 \bar{p}_1 - 54m_3^2 p_1 \bar{p}_1 + 8p_3 \bar{p}_3. & & \end{aligned}$$

These relations define the Zariski closure of the image of the Hilbert embedding in \mathbb{R}^8 . The Jacobian of the relations has rank 8 at points where all coordinates x_i , and hence all invariants, are nonzero.

Define the sets

$$\begin{aligned} X_1 &= \{0\}, & X_2 &= \{(x_1, x_2, 0) \in \mu^{-1}(0) : x_1, x_2 \neq 0\}, \\ X_3 &= \{(x_1, 0, x_3) \in \mu^{-1}(0) : x_1, x_3 \neq 0\}, \\ X_4 &= \{(x_1, x_2, x_3) \in \mu^{-1}(0) : x_1, x_2, x_3 \neq 0\}. \end{aligned}$$

Points in X_2 have isotropy $R(2)$ and satisfy $m_3 = p_1 = p_3 = 0$. These points are singular, as the Jacobian of the relations at these points has rank 5. The image under H of X_2 consists of points $(m_2, 0, 0, 0, \operatorname{Re}(p_2), \operatorname{Im}(p_2), 0, 0) \in \mathbb{R}^8$, such that $m_2 \geq 0$ and $3(\operatorname{Re}(p_2))^2 + \operatorname{Im}(p_2)^2 = m_2^4$; hence $\mathbb{S}^1 \backslash X_2$ is homeomorphic to $\mathbb{C} \setminus \{0\}$.

Points in X_3 , have isotropy $R(3)$ and satisfy $m_2 = p_2 = p_3 = 0$. These points are as well singular, as the Jacobian of the relations at these points has rank 6. The image under H of X_3 consists of points $(0, m_3, \operatorname{Re}(p_1), \operatorname{Im}(p_1), 0, 0, 0, 0) \in \mathbb{R}^8$ with $m_1 \geq 0$ and

$$2(\operatorname{Re}(p_2))^2 + \operatorname{Im}(p_2)^2 = m_1^3,$$

as well homeomorphic to $\mathbb{C} \setminus \{0\}$. Points in X_4 have trivial isotropy, and their image under H is the collection of nonsingular points in $\mathbb{S}^1 \backslash \mu^{-1}(0)$, homeomorphic to $(\mathbb{C} \setminus \{0\})^2$. Hence, in this case, one can verify from this description that $\chi_\Gamma(\mathbb{S}^1 \times \mu^{-1}(0)) = \chi(\mathbb{S}^1 \backslash \operatorname{Hom}(\Gamma, \mathbb{S}^1))$ for each finitely presented Γ .

Many low-dimensional examples of linear symplectic quotients by a positive-dimensional compact Lie group G admit homeomorphisms to symplectic orbifolds that preserve the decomposition into orbit types and many other structures; see [10, 20, 7]. By the results of [16], this phenomenon does not occur for larger representations; see also [17]. However, because the Γ -Euler characteristics depend on the isotropy groups, they distinguish between the G -symplectic quotient and the symplectic orbifold in many cases. We illustrate this with the following example; see [16, Prop. 5.3 and 5.5].

Example 6.4. Let $G = \operatorname{SU}(2)$, let R_d denote the $(d+1)$ -dimensional irreducible representation of $\operatorname{SU}(2)$ on binary forms of degree d , and let μ_{R_d} denote the moment map of R_d . Then by [16, Prop. 5.3 and 5.5], the symplectic quotient corresponding to R_3 is homeomorphic to the quotient of $W_3 = \mathbb{C}$ by $\langle i \rangle \simeq \mathbb{Z}/4\mathbb{Z}$ acting as scalar multiplication, and the symplectic quotient corresponding to R_4 is homeomorphic to the quotient of $W_4 = \mathbb{C}^2$ by the symmetric group \mathcal{S}_3 acting diagonally on two copies of the standard representation on $\mathbb{C} \simeq \mathbb{R}^2$. Each of these homeomorphisms is shown to in fact be a graded regular symplectomorphism, see the reference for the definition, and hence preserves the partition into orbit types. Therefore, in the case of R_3 , the orbit space of $\operatorname{SU}(2) \times \mu_{R_3}^{-1}(0)$ is partitioned into the origin and a space homeomorphic to $\mathbb{C} \setminus \{0\}$, while in the case of R_4 , $|\operatorname{SU}(2) \times \mu_{R_4}^{-1}(0)|$ is partitioned into the origin, a space homeomorphic to $\mathbb{C} \setminus \{0\}$, and a space finitely covered by \mathbb{C}^2 with three complex 1-dimensional subspaces removed. Except for the origin, each of these orbit types has vanishing Euler characteristic so that for any finitely presented Γ , we confirm the result of Theorem 6.1 that

$$\chi_\Gamma(\operatorname{SU}(2) \times \mu_{R_3}^{-1}(0)) = \chi_\Gamma(\operatorname{SU}(2) \times \mu_{R_4}^{-1}(0)) = \chi(\operatorname{SU}(2) \backslash \operatorname{Hom}(\Gamma, \operatorname{SU}(2))).$$

In the same way, this description and Theorem 6.1 yield

$$\chi_\Gamma(\langle i \rangle \times W_3) = \chi((\mathbb{Z}/4\mathbb{Z}) \backslash \operatorname{Hom}(\Gamma, \mathbb{Z}/4\mathbb{Z})),$$

and

$$\chi_\Gamma(\mathcal{S}_3 \times W_4) = \chi(\mathcal{S}_3 \backslash \operatorname{Hom}(\Gamma, \mathcal{S}_3)).$$

It is clear that the χ_Γ of the orbifolds differ from those of the corresponding $\operatorname{SU}(2)$ -symplectic quotients; when $\Gamma = \mathbb{Z}$, we have $\chi(\operatorname{SU}(2) \backslash \operatorname{Hom}(\mathbb{Z}, \operatorname{SU}(2))) = 1$, $\chi(\mathbb{Z}/4\mathbb{Z} \backslash \operatorname{Hom}(\mathbb{Z}, \mathbb{Z}/4\mathbb{Z})) = 4$, and $\chi(\mathcal{S}_3 \backslash \operatorname{Hom}(\mathbb{Z}, \mathcal{S}_3)) = 3$.

Example 6.5. For an example with $G = \mathrm{SU}(2)$ that is not graded regularly symplectomorphic to an orbifold, let $V = 2R_2$ using the notation of Example 6.4; see the proof of [16, Prop. 5.1] and [3, Sec. 2, 6.2]. The kernel of the action is $\{\pm 1\}$, and V can be identified with $4\mathbb{R}^3$ where $\mathrm{SU}(2)/\{\pm 1\} \simeq \mathrm{SO}(3)$ acts as the defining representation on each copy of \mathbb{R}^3 . Using coordinates $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4)$ for V where each $\mathbf{x}_i \in \mathbb{R}^3$, the invariants are given by $u_{ij} = \langle \mathbf{x}_i, \mathbf{x}_j \rangle$ for $i \leq j$. The shell is given by

$$\mu_{2R_2}^{-1}(0) = \{(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) \in 4\mathbb{R}^4 : \mathbf{x}_1 \wedge \mathbf{x}_2 + \mathbf{x}_3 \wedge \mathbf{x}_4 = \mathbf{0}\}.$$

The Hilbert embedding $H: \mu_{2R_2}^{-1}(0) \rightarrow \mathbb{R}^{10}$ is given by

$$H(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) = (u_{11}, u_{12}, u_{13}, u_{14}, u_{22}, u_{23}, u_{24}, u_{33}, u_{34}, u_{44}),$$

and the Zariski closure of the image of H is defined by the 9 relations

$$\begin{aligned} u_{14}u_{23} - u_{13}u_{24} + u_{34}^2 - u_{33}u_{44}, & \quad u_{14}u_{22} - u_{12}u_{24} + u_{24}u_{34} - u_{23}u_{44}, \\ u_{13}u_{22} - u_{12}u_{23} + u_{24}u_{33} - u_{23}u_{34}, & \quad u_{12}u_{14} - u_{11}u_{24} + u_{14}u_{34} - u_{13}u_{44}, \\ u_{12}u_{13} - u_{11}u_{23} + u_{14}u_{33} - u_{13}u_{34}, & \quad u_{12}^2 - u_{11}u_{22} - u_{34}^2 + u_{33}u_{44}, \\ u_{24}^2u_{33} - 2u_{23}u_{24}u_{34} + u_{22}u_{34}^2 + u_{23}^2u_{44} - u_{22}u_{33}u_{44}, & \\ u_{14}u_{24}u_{33} - 2u_{13}u_{24}u_{34} + u_{12}u_{34}^2 + u_{34}^3 + u_{13}u_{23}u_{44} - u_{12}u_{33}u_{44} - u_{33}u_{34}u_{44}, & \\ u_{14}^2u_{33} - 2u_{13}u_{14}u_{34} + u_{11}u_{34}^2 + u_{13}^2u_{44} - u_{11}u_{33}u_{44}. & \end{aligned}$$

The orbit types are given by $X_1 = \{\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}\}$ with isotropy $\mathrm{SU}(2)$, the set X_2 of $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4)$ such that the \mathbf{x}_i are parallel and not all $\mathbf{0}$ fixed by subgroups isomorphic to \mathbb{S}^1 , and the remainder X_3 of the shell with isotropy $\{\pm 1\}$. The rank of the Jacobian of the relations is 9 generically, but 3 on X_2 so that X_2 consists of singular points.

Example 6.6. Let $G = \mathbb{S}^1$, let $\rho: \mathbb{S}^1 \rightarrow \mathrm{U}(V)$ be a unitary \mathbb{S}^1 -representation, and let μ_V denote the corresponding moment map. Using the notation of Section 4, let (x_1, \dots, x_n) be coordinates for V with respect to a basis on which the action is diagonal, and let $(a_1, \dots, a_n) \in \mathbb{Z}^n$ be the corresponding weight vector.

Then the shell $\mu_V^{-1}(0)$ is the set of points such that $\sum_{i=1}^n a_i |x_i|^2 = 0$, which satisfies the hypotheses of Corollary 4.6. The set of fixed points $\mu^{-1}(0)^{\mathbb{S}^1} = V^{\mathbb{S}^1}$ is a complex vector space with Euler characteristic 1, and for each i such that $a_i \neq 0$, we have $\mu^{-1}(0) \cap V_{\{i\}} = \emptyset$. Hence Corollary 4.6 yields Theorem 6.1 in this case.

Example 6.7. Let $G = \mathrm{O}(2)$, let $\rho: \mathrm{O}(2) \rightarrow \mathrm{U}(V)$ be a unitary $\mathrm{O}(2)$ -representation with $V^{\mathrm{O}(2)} = \{0\}$ as in (5.1), and let μ_V denote the corresponding moment map. Using the description in the proof of Theorem 5.1, one may similarly describe a direct computation of

$$\chi_\Gamma(\mathrm{O}(2) \times \mu_V^{-1}(0)).$$

In the coordinates used in Theorem 5.1, the shell is given by

$$\mu_V^{-1}(0) = \left\{ \mathbf{v} \in V : \sum_{i=1}^n \alpha_i (|x_{i,1}| - |x_{i,2}|) = 0 \right\}.$$

The sets \mathbf{X}_i and \mathbf{X}_I^\times do not intersect the shell, and the sets \mathbf{X}_I^* and \mathbf{Y}_J are completely contained in the shell so that the argument of Theorem 5.1 applies. Similarly, the intersections of the sets \mathbf{X}_I and $\mathbf{X}_{I,J}$ with the shell are invariant under the action of the circle by scalar multiplication so that the same argument demonstrates that these sets have orbit spaces with vanishing Euler characteristic. One may also describe the \mathbf{X}_I^* , \mathbf{Y}_J , \mathbf{X}_I , and $\mathbf{X}_{I,J}$ explicitly; there are several cases to consider, but the orbit space of each admits a description with either $\mathbb{C} \setminus \{0\}$ or \mathbb{S}^1 as a factor, and hence has vanishing Euler characteristic.

To conclude, let us briefly recall how real linear symplectic quotients form local models for symplectic quotients of manifolds; see [15, Sec. 6] and [27, Sec. 2]. Suppose (M, ω) is a symplectic manifold equipped with the Hamiltonian action of a compact Lie group G with moment map $\mu_M: M \rightarrow \mathfrak{g}^*$. If $x \in M$ such that $\mu_M(x) = 0$, then by the local normal form theorem for the moment map [12, 21], a G -invariant neighborhood U of the G -orbit of x in M is G -equivariantly symplectomorphic to $Y = G \times_{G_x} (\mathfrak{m}^* \times V)$ where \mathfrak{m} is a complement to the Lie algebra \mathfrak{g}_x of G_x in \mathfrak{g} as G_x -representations and V is the *symplectic slice* at x , a subspace of the slice representation at x for the G -action on M . Then V can be equipped with a compatible complex structure with respect to which it is a unitary G_x -representation as above. Let $\mu_V: V \rightarrow \mathfrak{g}_x^*$ denote the moment map for V and $\mu_Y: Y \rightarrow \mathfrak{g}^*$ the moment map for Y . Using the results of [27, pp.384–7], the embedding of $\mu_V^{-1}(0)$ into $\mu_Y^{-1}(0)$ induces a weak equivalence $G_x \times \mu_V^{-1}(0) \rightarrow G \times \mu_Y^{-1}(0)$, see [8, Sec. 2.1]. It follows that the translation groupoid $G_x \times \mu_V^{-1}(0)$ presenting the real linear symplectic quotient at level 0 associated to V is Morita equivalent as a topological groupoid to the translation groupoid $G \times (U \cap \mu_M^{-1}(0))$ presenting a neighborhood of the orbit of x in the symplectic quotient $\mu_M^{-1}(0)/G$. In particular, $\chi_\Gamma(G \times (U \cap \mu_M^{-1}(0))) = \chi_\Gamma(G_x \times \mu_V^{-1}(0))$, and the latter can be computed using Theorem 6.1. See [22, Def. 58, Rem. 59] for the definition of Morita equivalence and [8, Cor. 4.15].

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