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FOLIATIONS ON $\mathbb{P}^2$ ADMITTING A PRIMITIVE MODEL

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Abstract. Given a foliation $\mathcal{F}$ on $\mathbb{P}^2_{\mathbb{C}}$, by fixing a line $L \subset \mathbb{P}^2_{\mathbb{C}}$, the polar pencil of $\mathcal{F}$ with axis L is the set of all polar curves of $\mathcal{F}$ with respect to points $l \in L$. In this work we study foliations $\mathcal{F}$ which admit a polar pencil whose generic element is reducible. To such an $\mathcal{F}$ is associated a primitive model, which is a foliation $\widetilde{\mathcal{F}}$ whose polar pencil, besides having irreducible generic element, is such that its fibers are contained in those of the polar pencil of $\mathcal{F}$. This work focuses on relating geometric properties of a foliation $\mathcal{F}$ with those of its primitive model $\widetilde{\mathcal{F}}$.

1. Introduction

This work deals with reducibility properties of the pencil of algebraic curves

$$\mathcal{P}:\ \{\alpha P(x,y) + \beta Q(x,y) = 0;\ (\alpha:\beta) \in \mathbb{P}^1\},$$

where $(x,y) \in \mathbb{C}^2$ and $P(x,y)$ and $Q(x,y)$ are polynomials in $\mathbb{C}[x,y]$. More specifically, we want to give conditions that identify when the generic element of this pencil is reducible. One situation is obvious: if the generators P and Q have a common irreducible factor, then this will be a factor for all elements in this pencil. Thus we can suppose P and Q relatively prime. In this case, Stein's factorization Theorem (see [3]) asserts that the generic element of $\mathcal{P}$ is reducible if and only if there are polynomials $\widetilde{P}(x,y)$ and $\widetilde{Q}(x,y)$ and a rational function $r : \mathbb{P}^1 \to \mathbb{P}^1$ of degree greater than one such that

$$\frac{P(x,y)}{Q(x,y)} = r\left(\frac{\widetilde{P}(x,y)}{\widetilde{Q}(x,y)}\right).$$

To this situation we associate two foliations on the projective plane $\mathbb{P}^2$: a foliation $\mathcal{F}$ induced in affine coordinates $(x,y) \in \mathbb{C}^2$ by the polynomial vector field

$$\mathbf{v} \ = \ P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y},$$

and a second foliation $\widetilde{\mathcal{F}}$ induced in the same affine coordinates by the vector field

$$\widetilde{\mathbf{v}} \ = \ \widetilde{P}(x,y)\frac{\partial}{\partial x} + \widetilde{Q}(x,y)\frac{\partial}{\partial y}.$$

We call $\mathcal{F}$ a *non-primitive* foliation and, if the generic element of the pencil

$$\widetilde{\mathcal{P}}:\ \{\alpha \widetilde{P}(x,y) + \beta \widetilde{Q}(x,y) = 0;\ (\alpha:\beta) \in \mathbb{P}^1\}$$

is irreducible, we say that $\widetilde{\mathcal{F}}$ is a *primitive* foliation, which is a *primitive model* for $\mathcal{F}$. Our idea is to study this configuration by relating geometric properties of $\mathcal{F}$ and $\widetilde{\mathcal{F}}$. As a byproduct, we

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will obtain information about problem of the reducibility of the generic element of the pencil $\mathcal{P}$ itself.

After presenting basic facts about foliations on $\mathbb{P}^2$ in section 2, we develop in section 3 the concept of primitive and non-primitive foliations. We prove that a non-primitive foliation and its primitive model have the same singularities in the affine plane $\mathbb{C}^2$ and, in Proposition 2, we establish a relation between their Milnor numbers. A consequence of this fact is that a foliation having only non-degenerate singularities is primitive. This, in its turn, implies that the generic foliation in the space of foliations of degree d on $\mathbb{P}^2$ is non-primitive.

We finally dedicate section 4 to the study of the singularities of $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ that lie over the line at infinity L_∞. Proposition 3 asserts that a non-primitive foliation always has singularities in L_∞. We also consider the case where L_∞ is invariant by $\mathcal{F}$ and the sum of its Milnor numbers over L_∞ is minimal, equal to the degree of the foliation plus one. By Proposition 5, this occurs if and only if all singularities of $\mathcal{F}$ in L_∞ are either non-degenerate or saddle nodes having L_∞ as a weak separatrix. Proposition 7 says that, when both a non-primitive foliation $\mathcal{F}$ and its primitive model $\widetilde{\mathcal{F}}$ leave L_∞ invariant, then the sum of the Milnor numbers at L_∞ is minimal for $\mathcal{F}$ if and only if it is minimal for $\widetilde{\widetilde{\mathcal{F}}}$. This apparently contrasts to what happens to Milnor numbers of singularities on the affine plane $\mathbb{C}^2$: the passage from the primitive model $\widetilde{\mathcal{F}}$ to the non-primitive $\mathcal{F}$ "degenerates" these singularities, in the sense that their Milnor numbers increase, as shown in Proposition 2.

2. Preliminaries

A foliation $\mathcal{F}$ of degree $d \geq 0$ in $\mathbb{P}^2 = \mathbb{P}^2_{\mathbb{C}}$ is induced in homogeneous coordinates $(X : Y : Z) \in \mathbb{P}^2$ by a 1-form

$$\omega \;=\; A(X,Y,Z)dX + B(X,Y,Z)dY + C(X,Y,Z)dZ, \tag{1}$$

where A, B and C are homogeneous polynomials of degree $d+1$ satisfying the Euler condition

$$XA(X,Y,Z) + YB(X,Y,Z) + ZC(X,Y,Z) = 0. \tag{2}$$

This means that we have a foliation of dimension two on $\mathbb{C}^3$ which contains in its leaves the lines through the origin, so that the foliation goes down to a foliation of dimension one on $\mathbb{P}^2$. The singular set of $\mathcal{F}$, denoted by $\mathrm{Sing}(\mathcal{F})$, is the set of common zeroes of A, B and C. We suppose, throughout this text, that $\mathrm{Sing}(\mathcal{F})$ has codimension two, which amounts to requiring that A, B and C have no common factor. In the affine plane $Z = 1$ with affine coordinates $x = X/Z$ and $y = Y/Z$ the foliation $\mathcal{F}$ is induced by the 1-form

$$\omega \;=\; A(x,y,1)dx + B(x,y,1)dy.$$

The foliation $\mathcal{F}$ is also given by the integral curves of the dual vector field of ω:

$$\mathbf{v} \;=\; P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}.$$

Here $P(x,y) = -B(x,y,1)$ and $Q(x,y) = A(x,y,1)$. We have two situations: if the line at infinity $L_\infty : \{Z = 0\}$ is invariant by $\mathcal{F}$ then Z divides A and B. Furthermore, for $k > 1$, Z^k is not a common factor for A and B, since otherwise Z would be a common factor of A, B and C by the Euler condition. This implies that $\max\{\deg P, \deg Q\} = d$. On the other hand, if the line at infinity is not invariant by $\mathcal{F}$, then Z is not a factor of both A and B, thus $P(x,y) = -B(x,y,1)$ as well as $Q(x,y) = A(x,y,1)$ have degree $d+1$. The Euler condition written in affine coordinates reads

$$xA(x,y,1) + yB(x,y,1) + C(x,y,1) = xQ(x,y) - yP(x,y) + C(x,y,1) = 0.$$

The terms of degree $d+2$ in the above relation give the equation

$$xQ_{d+1}(x,y) - yP_{d+1}(x,y) = 0,$$

where P_{d+1} and Q_{d+1} stand for the homogeneous part of degree $d+1$ of P and Q, respectively. Thus, there is a homogenous polynomial $G(x,y)$ of degree d such that $P_{d+1}(x,y) = xG(x,y)$ and $Q_{d+1}(x,y) = yG(x,y)$. We conclude that, when L_∞ is not invariant, $\mathcal{F}$ is induced by a vector field of the type

$$\mathbf{v} \ = \ (xG(x,y) + \hat{P}(x,y))\frac{\partial}{\partial x} + (yG(x,y) + \hat{Q}(x,y))\frac{\partial}{\partial y}, \tag{3}$$

where $\hat{P}$ and $\hat{Q}$ comprise the terms of degree d and lower of P and Q.

Reciprocally, let $\mathcal{F}$ be a foliation induced in affine coordinates (x,y) by a polynomial vector field of the form

$$\mathbf{v} \ = \ (xG(x,y) + \hat{P}(x,y))\frac{\partial}{\partial x} + (yG(x,y) + \hat{Q}(x,y))\frac{\partial}{\partial y}\,,$$

where G, when non-zero, is a homogeneous polynomial of degree d, while $\hat{P}$ and $\hat{Q}$ are either polynomials of degree d, when $G=0$, or of degree d or lower, when $G \neq 0$. Then $\mathcal{F}$ is a foliation of degree d and L_∞ is $\mathcal{F}$-invariant if and only if $G=0$.

Let now $\mathcal{F}$ be a germ of foliation at $p=(0,0) \in \mathbb{C}^2$, which is induced in local coordinates (x,y) by a vector field

$$\mathbf{v} \ = \ P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y},$$

where $P, Q \in \mathcal{O}_p$ are relatively prime germs of analytic functions. The *Milnor number* of $\mathcal{F}$ at p is defined as

$$\mu_p(\mathcal{F}) \ = \ \dim_{\mathbb{C}} \frac{\mathcal{O}_p}{(P,Q)},$$

where $(P,Q) \subset \mathcal{O}_p$ refers to the ideal generated by P and Q. Evidently, $\mu_p(\mathcal{F})$ is a non-negative integer, which is non-zero if and only p is a singularity for $\mathcal{F}$ (see [1] for more details).

Suppose now that the germ of foliation $\mathcal{F}$ has a smooth separatrix S, that is, a germ of holomorphic invariant curve passing through $p=(0,0)$. If we take local coordinates such that $S=\{y=0\}$ then $\mathcal{F}$ will be induced by a vector field of the form

$$\mathbf{v} \ = \ P(x,y)\frac{\partial}{\partial x} + y\overline{Q}(x,y)\frac{\partial}{\partial y},$$

which, restricted to S, is the vector field $\mathbf{v}_{|S} \ = \ P(x,0)\partial/\partial x$. We define the *relative Milnor number* of $\mathcal{F}$ with respect to S as the order of $\mathbf{v}_{|S}$ at $x=0$, that is

$$\mu_p(\mathcal{F},S) \ = \ \dim_{\mathbb{C}} \frac{\mathcal{O}_p}{(P,y)} \ = \ \mathrm{order}_{x=0}\mathbf{v}_{|S} \ = \ \mathrm{order}_{x=0}P(x,0).$$

It comes straight from the definition that $\mu_p(\mathcal{F},S) \leq \mu_p(\mathcal{F})$. We also remark that, when p is a regular point for $\mathcal{F}$, both numbers are zero.

Now, if S is a germ of a smooth analytic curve at p, non-invariant by $\mathcal{F}$, we take local coordinates (x,y) such that $p=(0,0)$ and $S : \{y=0\}$, so that $Q(x,0) \neq 0$. The *order of tangency* between $\mathcal{F}$ and S at p is the following number:

$$\tau_p(\mathcal{F},S) \ = \ \mathrm{order}_{x=0}Q(x,0).$$

The invariants $\mu_p(\mathcal{F})$, $\mu_p(\mathcal{F},S)$ and $\tau_p(\mathcal{F},S)$ are independent of the local coordinates and of the local expression of a vector field representing $\mathcal{F}$.

Next we state some global results about these invariants which will be used in the sequel. Let $\mathcal{F}$ be a foliation of degree d on $\mathbb{P}^2$. First of all, given a line $L \subset \mathbb{P}^2$ non-invariant by $\mathcal{F}$, then

$$\sum_{p\in\mathbb{P}^2} \tau_p(\mathcal{F}, L) \;=\; d.$$

In fact, we can take a system of affine coordinates $(x, y) \in \mathbb{C}^2$ for which that L has equation $y = 0$ and such that $\mathcal{F}$ and L are not tangent at $q = L \cap L_\infty$, that is $\tau_q(\mathcal{F}, L) = 0$. Here L_∞ denotes the line at infinity. We can also suppose that L_∞ is not $\mathcal{F}$-invariant, so that $\mathcal{F}$ is induced by a polynomial vector field as in (3). Simple calculations show that the fact that $\tau_q(\mathcal{F}, L) = 0$ is equivalent to the degree of $\hat{Q}$ in (3) being d. Furthermore, since L is not invariant by $\mathcal{F}$, the variable y does not divide $\hat{Q}$, so that $\hat{Q}(x, 0)$ actually has degree d. The result follows by noticing that at each point $p = (x_0, 0) \in L$, the order of tangency $\tau_p(\mathcal{F}, L)$ is the multiplicity of x_0 as a root of $\hat{Q}(x, 0)$.

Now, if $L \subset \mathbb{P}^2$ is an $\mathcal{F}$-invariant line it holds

$$\sum_{p\in L} \mu_p(\mathcal{F}, L) \;=\; d + 1. \tag{4}$$

To see this, it suffices to take an affine coordinate system $(x, y) \in \mathbb{C}^2$ such that L_∞ is not invariant by $\mathcal{F}$, L has equation $y = 0$ and $q = L\cap L_\infty$ is a regular point for $\mathcal{F}$, so that $\mu_q(\mathcal{F}, L) = 0$. Thus, supposing that $\mathcal{F}$ is induced by a vector field as in (3), for a point $p = (x_0, 0) \in L$, we have that $\mu_p(\mathcal{F}, L)$ is the order of x_0 as a root of $P(x, 0) = xG(x, 0) + \hat{P}(x, 0)$. The result follows from the fact that, since $q \notin \mathrm{Sing}(\mathcal{F})$, this polynomial has degree $d + 1$.

Finally, the sum of Milnor numbers of $\mathcal{F}$ on $\mathbb{P}^2$ gives a Bézout type theorem for $\mathcal{F}$, which reads

$$\sum_{p\in\mathbb{P}^2} \mu_p(\mathcal{F}) \;=\; d^2 + d + 1, \tag{5}$$

where d is the degree of $\mathcal{F}$. To see this we suppose that $\mathcal{F}$ is induced in affine coordinates $(x, y) \in \mathbb{C}^2$ by the polynomial vector field $\mathbf{v} \;=\; P(x, y)\partial/\partial x + Q(x, y)\partial/\partial y$. By an appropriate choice of the line at infinity L_∞ we may suppose that it does not contain any of the singularities of $\mathcal{F}$. This also implies that L_∞ is not invariant by $\mathcal{F}$, so that P and Q have degree $d+1$. Bézout's Theorem for the projective curves defined by P and Q give that the sum of their intersection numbers is $(d + 1)^2 = d^2 + 2d + 1$. The sum corresponding to points contained in the affine plane $\mathbb{C}^2$ equals the sum of the Milnor numbers of singularities of $\mathcal{F}$. The result is achieved by noticing that the two curves have d points of intersection over L_∞, with multiplicities counted.

3. Primitive models of foliations

Let $\mathcal{F}$ be a foliation on $\mathbb{P}^2$. Given a point $l \in \mathbb{P}^2$, the *polar curve* of $\mathcal{F}$ with *center* at $l \in \mathbb{P}^2$ is the closure of the set of points $p \in \mathbb{P}^2 \setminus \mathrm{Sing}(\mathcal{F})$ such that $T_p^{\mathbb{P}}\mathcal{F}$ passes through l:

$$P_l^{\mathcal{F}} \;=\; \overline{\{p \in \mathbb{P}^2 \setminus \mathrm{Sing}(\mathcal{F});\; l \in T_p^{\mathbb{P}}\mathcal{F}\}}.$$

Here $T_p^{\mathbb{P}}\mathcal{F}$ is the line through p with direction $T_p\mathcal{F}$. When $\mathcal{F}$ is induced in affine coordinates $(X : Y : Z) \in \mathbb{P}^2$ by a polynomial 1-form

$$\omega \;=\; A(X, Y, Z)dX + B(X, Y, Z)dY + C(X, Y, Z)dZ$$

as in (1), the polar curve with center $l = (\alpha : \beta : \gamma)$ has equation

$$\alpha A(X, Y, Z) + \beta B(X, Y, Z) + \gamma C(X, Y, Z) = 0.$$

It follows that if $\mathcal{F}$ has degree $d \geq 1$ then $P_l^{\mathcal{F}}$ is a curve of degree $d + 1$. Furthermore $P_l^{\mathcal{F}}$ contains all singularities of $\mathcal{F}$ as well as the point l. This object was studied in [2] and [4].

As the point $l \in \mathbb{P}^2$ moves, the curves $P_l^{\mathcal{F}}$ form a linear system of dimension two, the *polar net* of $\mathcal{F}$. If we fix a line $L \subset \mathbb{P}^2$ and take all polar curves of $\mathcal{F}$ whose centers lie in L we have the *polar pencil* of $\mathcal{F}$ with *axis* L. It is the set of curves

$$\alpha A(X,Y,Z) + \beta B(X,Y,Z) + \gamma C(X,Y,Z) = 0 \quad , \ (\alpha : \beta : \gamma) \in L \, ,$$

and will be denoted by $\mathcal{P}(\mathcal{F}, L)$.

Proposition 1. *Let $L \subset \mathbb{P}^2$ be an $\mathcal{F}$-invariant line. Then L is a fixed component of $\mathcal{P}(\mathcal{F}, L)$ with multiplicity one. Reciprocally, the only fixed component admitted in $\mathcal{P}(\mathcal{F}, L)$ is the line L, in which case it is $\mathcal{F}$-invariant and of multiplicity one. In particular, if L is not invariant by $\mathcal{F}$ then $\mathcal{P}(\mathcal{F}, L)$ has no fixed components.*

Proof. Suppose first that L is $\mathcal{F}$-invariant and fix $l \in L$. Then, the $\mathcal{F}$-invariance of L gives that $l \in T_p^{\mathbb{P}}\mathcal{F}$ for every $p \in L \setminus \text{Sing}(\mathcal{F})$. Thus, $L \subset P_l^{\mathcal{F}}$. Since $l \in L$ is arbitrary, we have $L \subset \mathcal{P}(\mathcal{F}, L)$. In what concerns its multiplicity, putting $L : \{Z = 0\}$ in the above system of homogeneous coordinates, we have

$$\mathcal{P}(\mathcal{F}, L) = \{\alpha A(X,Y,Z) + \beta B(X,Y,Z) = 0; \ (\alpha : \beta) \in \mathbb{P}^1\}.$$

Thus, if L were a fixed element of the pencil with multiplicity $k > 1$, then Z^k would be a divisor of both A and B, and the Euler condition (2) would imply that Z^{k-1} would be a divisor of C and we would find a component of codimension one in $\text{Sing}(\mathcal{F})$, which is not allowed. For the converse, we first remark that if $\mathcal{P}(\mathcal{F}, L)$ has a line L' in its base, then $L' = L$. Actually, if $p \in L' \setminus \text{Sing}(\mathcal{F})$ then $l \in T_p^{\mathbb{P}}\mathcal{F}$ for every $l \in L$. But, if $L' \neq L$ and if $p \notin L$, then $T_p^{\mathbb{P}}\mathcal{F}$ intersects L in only one point. Thus, the only possibility left is that $L' = L$. Then for a fixed $l \in L$ and for every $p \in L \setminus \text{Sing}(\mathcal{F})$ we have $l \in T_p^{\mathbb{P}}\mathcal{F}$. This means that $T_p^{\mathbb{P}}\mathcal{F} = L$ for every $p \in L \setminus \text{Sing}(\mathcal{F})$, which gives the $\mathcal{F}$-invariance of L. By the first part of the proof, L has multiplicity one. Finally, an irreducible fixed component of $\mathcal{P}(\mathcal{F}, L)$ of degree greater than one with equation $F(X,Y,Z) = 0$ would mean that F is a factor of both A and B and thus, by the Euler condition, it would be a factor of C, giving rise to a codimension one component in $\text{Sing}(\mathcal{F})$, which is impossible. $\square$

Let $\mathcal{F}$ be a foliation in $\mathbb{P}^2$ as before. Its *modified polar pencil* with axis at the line $L \subset \mathbb{P}^2$, denoted by $\mathcal{P}^*(\mathcal{F}, L)$, is the pencil obtained from $\mathcal{P}(\mathcal{F}, L)$ in the following way:

$$\mathcal{P}^*(\mathcal{F}, L) \ = \ \begin{cases} \mathcal{P}(\mathcal{F}, L) - L & \text{if } L \text{ is } \mathcal{F}\text{-invariant} \\ \mathcal{P}(\mathcal{F}, L) & \text{if } L \text{ is not } \mathcal{F}\text{-invariant} \end{cases}$$

Evidently $\mathcal{P}^*(\mathcal{F}, L)$ is free of fixed components.

We now choose an affine system of coordinates $(x, y) \in \mathbb{C}^2$ such that L is the line at infinity by making $L : \{Z = 0\}$, $x = X/Z$ and $y = Y/Z$, where $\mathcal{F}$ is induced by the vector field

$$\mathbf{v} \ = \ P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}.$$

In the coordinates (x, y), both $\mathcal{P}(\mathcal{F}, L)$ and $\mathcal{P}^*(\mathcal{F}, L)$ are given by

$$\{\alpha P(x,y) + \beta Q(x,y) = 0; \ (\alpha : \beta) \in \mathbb{P}^1\}.$$

By means of Bertini's Theorem concerning linear systems whose generic element is reducible, it is proved in [4] that the generic element of the polar net of a foliation on $\mathbb{P}^2$ is irreducible. However, it comes out that the polar net of a foliation might contain a pencil whose generic element is reducible. Evidently, if L is a line invariant by $\mathcal{F}$, then L belongs to all elements of the polar pencil having L as an axis, that is L is a fixed element of the polar pencil. By removing

L from the pencil, we can again ask if its generic element is reducible. Taking affine coordinates $(x, y) \in \mathbb{C}^2$ such that $L = L_\infty$ is the line at infinity then the polar pencil becomes

$$\{\alpha P(x,y) + \beta Q(x,y) = 0;\ (\alpha : \beta) \in \mathbb{P}^1\}.$$

We remark that now there are no elements of codimension one in the pencil, since the fact that $\mathrm{Sing}(\mathcal{F})$ has codimension 2 implies that P and Q have no common factor. We can then apply Stein's factorization Theorem (see [3]): the generic element of the pencil $\{\alpha P(x,y) + \beta Q(x,y) = 0\ ,\ (\alpha : \beta) \in \mathbb{P}^1\}$ is reducible if and only if there are polynomials $\widetilde{P}(x,y)$ and $\widetilde{Q}(x,y)$ and a rational function $r : \mathbb{P}^1 \to \mathbb{P}^1$ of degree greater than one such that

$$\frac{P(x,y)}{Q(x,y)} = r\left(\frac{\widetilde{P}(x,y)}{\widetilde{Q}(x,y)}\right).$$

This means that the pencil induced by P and Q "factors" through the one induced by $\widetilde{P}$ and $\widetilde{Q}$. We can ask once again if the generic element of the pencil $\{\alpha\widetilde{P}(x,y) + \beta\widetilde{Q}(x,y) = 0;\ (\alpha : \beta) \in \mathbb{P}^1\}$ is reducible. If true, we can repeat the process above, until we reach a situation where $\tilde{d}$ is minimal and the generic element of $\{\alpha\widetilde{P}(x,y) + \beta\widetilde{Q}(x,y) = 0;\ (\alpha : \beta) \in \mathbb{P}^1\}$ is irreducible.

We say that a foliation $\mathcal{F}$ on $\mathbb{P}^2$ is *primitive* if for every line $L \subset \mathbb{P}^2$ the modified polar pencil of $\mathcal{F}$ with axis L has irreducible generic element. If for some line $L \subset \mathbb{P}^2$ the modified polar pencil of $\mathcal{F}$ with respect to L has reducible generic element, we say that $\mathcal{F}$ is *non-primitive* (with respect to L). In this case, taking affine coordinates $(x, y) \in \mathbb{C}^2$ with respect to which L is the line at infinity, and a polynomial vector field

$$\mathbf{v} \ = \ P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y} \tag{6}$$

that induces $\mathcal{F}$, we find polynomials $\widetilde{P}(x,y)$ and $\widetilde{Q}(x,y)$ and a rational function $r : \mathbb{P}^1 \to \mathbb{P}^1$ of degree $m = \deg(r) \geq 2$ such that $P/Q = r(\widetilde{P}/\widetilde{Q})$ and so that the pencil $\mathcal{P}(\widetilde{P}, \widetilde{Q})$ has irreducible generic element. Notice that, putting $t = z/w$, we write $r(t) = r(z/w) = S(z,w)/T(z,w)$, where S and T are homogeneous polynomials of degree m, so that

$$\begin{cases} P(x,y) = S(\widetilde{P}(x,y), \widetilde{Q}(x,y)) \\ Q(x,y) = T(\widetilde{P}(x,y), \widetilde{Q}(x,y)). \end{cases} \tag{7}$$

We now define a foliation $\widetilde{\mathcal{F}}$ on $\mathbb{P}^2$ induced, in the same system of affine coordinates (x, y), by the vector field

$$\widetilde{\mathbf{v}} \ = \ \widetilde{P}(x,y)\frac{\partial}{\partial x} + \widetilde{Q}(x,y)\frac{\partial}{\partial y}.$$

Since $\mathcal{P}(\widetilde{P}, \widetilde{Q})$ has irreducible generic element, $\widetilde{P}$ and $\widetilde{Q}$ are relatively prime, so $\mathrm{Sing}(\widetilde{\mathcal{F}})$ has codimension two. $\widetilde{\mathcal{F}}$ is said to be a *primitive model* for $\mathcal{F}$. The number $m = \deg(r)$ will be called *degree of ramification* of $\mathcal{F}$. We remark that the property of being a non-primitive foliation and that of being the primitive model of a foliation involves fixing an affine plane with coordinates $(x, y) \in \mathbb{C}^2$ and a line at infinity $L_\infty \subset \mathbb{P}^2$. The degree of the vector field (6) inducing $\mathcal{F}$ is called *the affine degree* of $\mathcal{F}$, and is denoted by $\deg_a(\mathcal{F})$. If $\mathcal{F}$ is a non-primitive foliation admitting a primitive model $\widetilde{\mathcal{F}}$, we evidently have

$$\deg_a(\mathcal{F}) \ = \ m \deg_a(\widetilde{\mathcal{F}}),$$

where m is the degree of ramification.

Fix an affine plane in $\mathbb{P}^2$ with coordinates $(x,y) \in \mathbb{C}^2$. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be foliations on $\mathbb{P}^2$ induced, respectively, by polynomial vector fields

$$\mathbf{v}_1 \ = \ P_1(x,y)\frac{\partial}{\partial x} + Q_1(x,y)\frac{\partial}{\partial y} \quad \text{and} \quad \mathbf{v}_2 \ = \ P_2(x,y)\frac{\partial}{\partial x} + Q_2(x,y)\frac{\partial}{\partial y}.$$

Definition 1. The foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ are said to be *linearly equivalent* if there exist $a,b,c,d \in \mathbb{C}$ such that $ad - bc \neq 0$ and

$$\begin{cases} P_1(x,y) \ = \ aP_2(x,y) + bQ_2(x,y) \\ Q_1(x,y) \ = \ cP_2(x,y) + dQ_2(x,y). \end{cases}$$

The notion of linear equivalence defines equivalence classes in the space of foliations on $\mathbb{P}^2$. From the expression (3) it is easy to see that, in such an equivalence class, all foliations have the same degree d and leave L_∞ invariant, with the possible exception of one, which has degree $d-1$ and for which L_∞ is not invariant. Nevertheless, the affine degree is the same for all foliation in a class of linear equivalence. Therefore, a foliation of degree d for which L_∞ is not invariant is always linear equivalent to a foliation of degree $d+1$ which leaves L_∞ invariant. Evidently, two primitive models for the same foliation are linearly equivalent. On the other hand, two non-primitive foliations which are linearly equivalent have the same class of primitive models.

In the next two examples we introduce two classes of foliation which will appear in Theorem 1 below.

Example 1. We say that a foliation $\mathcal{F}$ on $\mathbb{P}^2$ is *homogeneous* with center at $l \in \mathbb{P}^2$ if $\mathcal{F}$ is induced in affine coordinates $(x,y) \in \mathbb{C}^2$ for which $l = (0,0)$ by a polynomial vector field

$$\mathbf{v} \ = \ P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}$$

such that $P(x,y)$ and $Q(x,y)$ are homogeneous polynomials of the same degree. One outstanding property of a foliation $\mathcal{F}$ which is homogeneous with center at $l \in \mathbb{P}^2$ is that its polar curve with center at l is $\mathcal{F}$-invariant and consists of $d+1$ lines passing through l, with multiplicities counted. If $\mathcal{F}$ is a homogeneous foliation centered at $l = (0,0)$ as above, then the line at infinity is invariant by $\mathcal{F}$ and $d = \deg(\mathcal{F}) = \deg_a(\mathcal{F})$. The only singularity in $\mathbb{C}^2$ is $l = (0,0)$, which has Milnor number $\mu_l(\mathcal{F}) = d^2$. Observe that this, along with expression (5), implies that

$$\sum_{p \in L_\infty} \mu_p(\mathcal{F}) = d + 1.$$

All the singularities of $\mathcal{F}$ on the line at infinity L_∞ are at the intersection of L_∞ and one of the invariant lines L which form the polar curve with center l. If L has multiplicity one as a component of $P_l^{\mathcal{F}}$, then $p = L \cap L_\infty$ is a non-degenerate singularity, meaning that the linear part of any vector field which induces $\mathcal{F}$ near p has two non-zero eigenvalues. On the other hand, if this multiplicity is $k > 1$, then $p = L \cap L_\infty$ is a saddle-node whose weak separatrix is contained in L_∞. We finally observe that any curve in the polar pencil of $\mathcal{F}$ with axis at L_∞ consists of $d+1$ lines passing through $(0,0) \in \mathbb{C}^2$.

Example 2. Let $\mathcal{F}$ be a foliation on $\mathbb{P}^2$. We say that $\mathcal{F}$ is a foliation *in one variable* if in some affine coordinate system $(x,y) \in \mathbb{C}^2$ it is induced by a polynomial vector field of the kind

$$P(x)\frac{\partial}{\partial x} + Q(x)\frac{\partial}{\partial y},$$

where P and Q are polynomials depending only on the variable x. Since $\mathrm{Sing}(\mathcal{F})$ has codimension two, P and Q are without common factors, which results that $\mathcal{F}$ has no singularities in the affine plane $\mathbb{C}^2$. It is easy to see that the line at infinity is $\mathcal{F}$-invariant, for its non-invariance would imply, from expression (3), that the higher order terms of P and Q would depend on both x and y. Thus, $d = \deg(\mathcal{F}) = \deg_a(\mathcal{F})$. We also remark that, if x_0 is a root of $Q(x)$, then the line $x = x_0$ is $\mathcal{F}$-invariant. These invariant lines all meet L_∞ at a singularity p. If $\deg(P) < \deg(Q)$, then this is the only singularity of $\mathcal{F}$. If $\deg(P) \geq \deg(Q)$ there is still another singularity on L_∞. For a foliation in one variable as above, any element of the polar pencil with axis at L_∞ consists of $d+1$ vertical lines, with multiplicities counted.

Theorem 1. *Let $\mathcal{F}$ be a non-primitive foliation on $\mathbb{P}^2$ which admits a primitive model of affine degree one. Then either $\mathcal{F}$ is a homogeneous foliation or it is a foliation in one variable.*

Proof. Let $\widetilde{\mathcal{F}}$ be a primitive model for $\mathcal{F}$, induced in affine coordinates $(x, y) \in \mathbb{C}^2$ by the polynomial vector field $\widetilde{P}(x, y)\partial/\partial x + \widetilde{Q}(x, y)\partial/\partial y$.

1st case: Either $\widetilde{P}$ or $\widetilde{Q}$ is a constant. Then, by means of a linear equivalence, we may suppose that $\widetilde{\mathcal{F}}$ is induced by a vector field of the form $(ax + by)\partial/\partial x + \partial/\partial y$, where $a \neq 0$ or $b \neq 0$. If $a = 0$, evidently $\widetilde{\mathcal{F}}$ is a foliation in one variable. If $a \neq 0$, by applying the affine change of coordinates $(u, v) = (ax + by, y)$, we arrive to the same conclusion.

2nd case: Both $\widetilde{P}$ and $\widetilde{Q}$ have degree one. Let us put $\widetilde{P} = ax + by + e$ and $\widetilde{Q} = cx + dy + f$. We first consider the situation where $\widetilde{P}$ and $\widetilde{Q}$ have no common root in the affine plane $\mathbb{C}^2$. This means that $ax + by$ is a multiple of $cx + dy$ by a non-zero constant. Thus, by linear equivalence, we can suppose that $\widetilde{P} = ax + by$ and $\widetilde{Q} = 1$ and we come to the first case, where $\mathcal{F}$ is a foliation in one variable. We then suppose that $\widetilde{P}$ and $\widetilde{Q}$ have a common root in $\mathbb{C}^2$. By an affine change of coordinates, we can suppose that this root is $(0, 0)$, which makes $\widetilde{P} = ax + by$ and $\widetilde{Q} = cx + dy$. If $r(t)$ is the rational map such that $P/Q = r(\widetilde{P}/\widetilde{Q})$, writing $t = z/w$, we have $r(z/w) = F(z, w)/G(z, w)$, where F and G are homogeneous polynomials of degree equal to the degree of r. We finally conclude that

$$P(x, y) = F(ax + by, cx + dy) \qquad \text{and} \qquad Q(x, y) = G(ax + by, cx + dy)$$

which says that $\mathcal{F}$ is a homogeneous foliation. □

If $\mathcal{F}$ is a non-primitive foliation with primitive model $\widetilde{\mathcal{F}}$ then, in the affine plane $\mathbb{C}^2$, the singular points for $\mathcal{F}$ and for $\widetilde{\mathcal{F}}$ are the same. In fact, with the notation of (7), we know that $P(x, y) = S(\widetilde{P}(x, y), \widetilde{Q}(x, y))$ and $Q(x, y) = T(\widetilde{P}(x, y), \widetilde{Q}(x, y))$. Evidently, the common zeroes of $\widetilde{P}$ and $\widetilde{Q}$ are zeroes of both P and Q, which gives $\mathrm{Sing}(\widetilde{\mathcal{F}})_{|\mathbb{C}^2} \subset \mathrm{Sing}(\mathcal{F})_{|\mathbb{C}^2}$. Reciprocally, the existence of a point (x_0, y_0) in $\mathbb{C}^2$ which is singular for $\mathcal{F}$ but not for $\widetilde{\mathcal{F}}$ would imply the existence of a common factor for $S(z, w)$ and $T(z, w)$. Thus we actually have $\mathrm{Sing}(\widetilde{\mathcal{F}})_{|\mathbb{C}^2} = \mathrm{Sing}(\mathcal{F})_{|\mathbb{C}^2}$.

Proposition 2. *Let $\mathcal{F}$ be a non-primitive foliation having $\widetilde{\mathcal{F}}$ as primitive model and m as the degree of ramification. If $p \in \mathbb{C}^2$ then*

$$\mu_p(\mathcal{F}) = m^2 \mu_p(\widetilde{\mathcal{F}}).$$

Proof. We keep the notation of (7). We consider the following maps from $\mathbb{C}^2$ to $\mathbb{C}^2$:

$$\begin{cases} \Phi(x, y) = (P(x, y), Q(x, y)), \\ \widetilde{\Phi}(x, y) = (\widetilde{P}(x, y), \widetilde{Q}(x, y)), \\ H(z, w) = (S(u, v), T(u, v)). \end{cases}$$

We have $\Phi = H \circ \widetilde{\Phi}$. We first remark that the Milnor number of the vector field $P\partial/\partial x + Q\partial/\partial y$ at a singularity p is the number of pre-images of $\Phi = (P, Q)$ lying near p of any point q sufficiently near $(0, 0) \in \mathbb{C}^2$. The result follows by noticing that, since S and T are homogeneous of degree

m and without common factors, the Milnor number of $S\partial/\partial u + T\partial/\partial v$ at $(0,0)$ is m^2 (see [1], section 2). $\square$

Corollary 1. *If $\mathcal{F}$ is a foliation having three non-aligned singularities each of them having the property that its Milnor number is not divisible by some m^2, where $m \in \mathbb{Z}$ and $m \geq 2$. Then $\mathcal{F}$ is a primitive foliation. In particular, if $\mathcal{F}$ has three non-aligned non-degenerate singularities, then $\mathcal{F}$ is primitive.*

Corollary 2. *Let $\mathcal{F}$ be a foliation having only non-degenerate singularities. Then $\mathcal{F}$ is primitive.*

Proof. Since all singularities of $\mathcal{F}$ have Milnor number 1, the above corollary implies that all singularities of $\mathcal{F}$ would lie in L_∞ if $\mathcal{F}$ were non-primitive. Summing up their Milnor numbers we have $\sum_{p\in L_\infty} \mu_p(\mathcal{F}) = d^2+d+1$, where d is the degree of $\mathcal{F}$. If L_∞ were $\mathcal{F}$-invariant, we would have $\sum_{p\in L_\infty} \mu_p(\mathcal{F}, L_\infty) = d+1$, which leads to a contradiction since $\mu_p(\mathcal{F}) = \mu_p(\mathcal{F}, L_\infty) = 1$ for a non-degenerate singularity. If L_∞ were non-invariant, then $\sum_{p\in L_\infty} \tau_p(\mathcal{F}, L_\infty) = d$, which is a contradiction since, when $p \in \mathrm{Sing}(\mathcal{F})$ is non-degenerate, it holds $\tau_p(\mathcal{F}, L_\infty) = \mu_p(\mathcal{F}) = 1$. $\square$

Corollary 3. *Let $\mathcal{F}ol(d)$ be the space of foliations of degree d in $\mathbb{P}^2$. Then the set of primitive foliations contain a non-empty Zariski open set.*

4. The study of the singularities on L_∞

We have seen in the previous section that a non-primitive foliation $\mathcal{F}$ and its primitive model $\widetilde{\mathcal{F}}$ have the same singularities in the affine plane $\mathbb{C}^2$, and its Milnor numbers are related by Proposition 2. The objective of this section is to explore the consequences of this fact to the singularities of $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ that lie over L_∞.

Let us consider a non-primitive foliation $\mathcal{F}$ of degree d_0 having a primitive model $\widetilde{\mathcal{F}}$ of degree $\tilde{d}_0$. We denote the affine degrees of $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ respectively by d and $\tilde{d}$. By summing up Milnor numbers we get

$$\begin{aligned}
\sum_{\mathbb{P}^2} \mu_p(\mathcal{F}) &= \sum_{\mathbb{C}^2} \mu_p(\mathcal{F}) + \sum_{L_\infty} \mu_p(\mathcal{F}) \\
&= m^2 \sum_{\mathbb{C}^2} \mu_p(\widetilde{\mathcal{F}}) + \sum_{L_\infty} \mu_p(\mathcal{F}) \\
&= m^2 \left(\sum_{\mathbb{P}^2} \mu_p(\widetilde{\mathcal{F}}) - \sum_{L_\infty} \mu_p(\widetilde{\mathcal{F}}) \right) + \sum_{L_\infty} \mu_p(\mathcal{F})
\end{aligned}$$

thus, using (5), we obtain

$$\begin{aligned}
\sum_{L_\infty} \mu_p(\mathcal{F}) - m^2 \sum_{L_\infty} \mu_p(\widetilde{\mathcal{F}}) &= \sum_{\mathbb{P}^2} \mu_p(\mathcal{F}) - m^2 \sum_{\mathbb{P}^2} \mu_p(\widetilde{\mathcal{F}}) \\
&= (d_0^2 + d_0 + 1) - m^2(\tilde{d}_0^{\,2} + \tilde{d}_0 + 1). \qquad (8)
\end{aligned}$$

The values of d_0 and $\tilde{d}_0$ in terms of the affine degrees d and $\tilde{d}$ depend only on the fact of L_∞ being $\mathcal{F}$-invariant or not. We consider three cases:

<u>1st case</u>: L_∞ is $\widetilde{\mathcal{F}}$-invariant but not $\mathcal{F}$-invariant.

We have $d_0 = d - 1$ and $\tilde{d}_0 = \tilde{d}$ and, putting this in equation (8),

$$\begin{aligned}\sum_{L_\infty} \mu_p(\mathcal{F}) - m^2 \sum_{L_\infty} \mu_p(\widetilde{\mathcal{F}}) &= (d^2 - d + 1) - m^2(\tilde{d}^2 + \tilde{d} + 1) \\ &= ((m\tilde{d})^2 - m\tilde{d} + 1) - m^2(\tilde{d}^2 + \tilde{d} + 1) \\ &= -m^2\tilde{d} - m\tilde{d} - m^2 + 1. \qquad (9)\end{aligned}$$

This allows us to conclude the following:

Proposition 3. *Let $\mathcal{F}$ be a non-primitive foliation. Then $\mathcal{F}$ has some singularity in L_∞.*

Proof. If L_∞ is $\mathcal{F}$ invariant then formula (4) implies that it must contain some singularity. Suppose now that L_∞ is not invariant by $\mathcal{F}$. By linear equivalence, we can suppose that $\widetilde{\mathcal{F}}$ leaves L_∞ invariant. If $\mathrm{Sing}(\mathcal{F}) \cap L_\infty = \emptyset$ then $\sum_{L_\infty} \mu_p(\mathcal{F}) = 0$. The above formula gives

$$-m^2 \sum_{L_\infty} \mu_p(\widetilde{\mathcal{F}}) = -m^2\tilde{d} - m\tilde{d} - m^2 + 1.$$

Thus, m would be a divisor of the right side of the equation, which is absurd. □

Suppose now that L_∞ is $\widetilde{\mathcal{F}}$-invariant and that $\sum_{L_\infty} \mu_p(\widetilde{\mathcal{F}}) = \tilde{d}_0 + 1 = \tilde{d} + 1$. In this case, expression (9) reads

$$\sum_{L_\infty} \mu_p(\mathcal{F}) - m^2(\tilde{d} + 1) = -m^2\tilde{d} - m\tilde{d} - m^2 + 1,$$

which implies

$$\sum_{L_\infty} \mu_p(\mathcal{F}) = -m\tilde{d} + 1.$$

This is a contradiction, since the right side is negative. We get the following conclusion:

Proposition 4. *Let $\mathcal{F}$ be a non-primitive foliation having a primitive model $\widetilde{\mathcal{F}}$ leaving L_∞ invariant. Suppose that $\sum_{L_\infty} \mu_p(\widetilde{\mathcal{F}}) = \tilde{d} + 1$, where $\tilde{d} = \deg(\widetilde{\mathcal{F}})$. Then L_∞ is $\mathcal{F}$-invariant. In particular, if all singularities of $\widetilde{\mathcal{F}}$ in L_∞ are non-degenerate, then L_∞ is $\mathcal{F}$-invariant.*

In the situation of the Proposition 4, relation (4) reads $\sum_{L_\infty} \mu_p(\widetilde{\mathcal{F}}, L_\infty) = \tilde{d} + 1$. Thus, the hypothesis $\sum_{L_\infty} \mu_p(\widetilde{\mathcal{F}}) = \tilde{d} + 1$ is a condition of minimality on the Milnor numbers of $\widetilde{\mathcal{F}}$ over L_∞, as explained in the next result:

Proposition 5. *Let $\mathcal{F}$ be a germ of foliation having a singularity at $p \in \mathbb{C}^2$ and let L be a germ of smooth separatrix at p. Then $\mu_p(\mathcal{F}, L) \leq \mu_p(\mathcal{F})$. Furthermore, equality occurs if and only if one of the two alternatives holds:*

(i) p is a non-degenerate singularity of $\mathcal{F}$;

(ii) p is a saddle-node having L as its weak separatrix.

Proof. Suppose that $\mathcal{F}$ is induced at p by a local vector field $P\partial/\partial x + Q\partial/\partial y$, where $P, Q \in \mathcal{O}_p$. Let us denote $\mu_p(P, Q) := \mu_p(\mathcal{F})$. For a vector field $P\partial/\partial x + Q_1Q_2\partial/\partial y$, where $Q_1, Q_2 \in \mathcal{O}_p$, we have $\mu_p(P, Q_1Q_2) = \mu_p(P, Q_1) + \mu_p(P, Q_2)$ (see [1]). Let us suppose that the separatrix L has equation $y = 0$, so that $\mathcal{F}$ is induced by a vector field of the form $P\partial/\partial x + yQ_1\partial/\partial y$ for some $Q_1 \in \mathcal{O}_p$. Thus

$$\mu_p(\mathcal{F}) = \mu_p(P, yQ_1) = \mu_p(P, y) + \mu_p(P, Q_1) = \mu_p(\mathcal{F}, L) + \mu_p(P, Q_1),$$

where we used that $\mu_p(\mathcal{F}, L) = \mu_p(P, y)$. The result follows by noticing that $\mu_p(P, Q_1) \geq 0$. Now, equality holds if and only if $\mu_p(P, Q_1) = 0$. This means that the vector field $P\partial/\partial x + Q_1\partial/\partial y$

is non-singular at p. Since $P(p) = 0$ we must have $Q_1(p) \neq 0$. This gives at least one non-zero eigenvalue for $P\partial/\partial x + Q\partial/\partial y$, which implies (i) or (ii). Reciprocally, if p is a non-degenerate singularity, then $\mu_p(\mathcal{F}) = \mu_p(\mathcal{F}, L) = 1$. In the case of a saddle-node having $y = 0$ as weak separatrix, after an analytic change of coordinates, we may suppose that we have the normal form of the saddle node: $x^{k+1}\partial/\partial x + y(1 + \lambda x^k)\partial/\partial y$, where $\lambda \in \mathbb{C}$ and $k \geq 0$. Its easy to see that $\mu_p(\mathcal{F}) = \mu_p(\mathcal{F}, L) = k + 1$. □

<u>2nd case</u>: L_∞ is $\mathcal{F}$-invariant but not $\widetilde{\mathcal{F}}$-invariant. We have $d_0 = d$ and $\tilde{d}_0 = \tilde{d} - 1$. Equation (8) gives

$$\begin{aligned}\sum_{L_\infty} \mu_p(\mathcal{F}) - m^2 \sum_{L_\infty} \mu_p(\widetilde{\mathcal{F}}) &= (d^2 + d + 1) - m^2((\tilde{d} - 1)^2 + \tilde{d}) \\ &= ((m\tilde{d})^2 + m\tilde{d} + 1) - m^2(\tilde{d}^2 - \tilde{d} + 1) \\ &= m^2(\tilde{d} - 1) + m\tilde{d} + 1.\end{aligned}$$

Let us suppose that the sum of Milnor numbers of $\mathcal{F}$ at L_∞ is minimal, that is $\sum_{L_\infty} \mu_p(\mathcal{F}) = d_0 + 1 = d + 1$. This gives

$$-m^2 \sum_{L_\infty} \mu_p(\widetilde{\mathcal{F}}) \;=\; m^2(\tilde{d} - 1).$$

This implies that $\tilde{d} = 1$ and $\sum_{L_\infty} \mu_p(\widetilde{\mathcal{F}}) = 0$, that is, $\widetilde{\mathcal{F}}$ is the radial foliation. Thus, $\mathcal{F}$ is a homogeneous foliation. As commented on Example 1, for a homogeneous foliation $\mathcal{F}$ of degree d_0, it holds $\sum_{L_\infty} \mu_p(\mathcal{F}) = d_0 + 1$. We can thus state the following result:

Proposition 6. *Let $\mathcal{F}$ be a non-primitive foliation of degree d_0 which leaves L_∞ invariant, having a primitive model $\widetilde{\mathcal{F}}$ for which L_∞ is non-invariant. It holds $\sum_{L_\infty} \mu_p(\mathcal{F}) = d_0 + 1$ if and only if $\mathcal{F}$ is a homogeneous foliation and, in this case, $\widetilde{\mathcal{F}}$ is the radial foliation.*

<u>3rd case</u>: L_∞ is both $\mathcal{F}$-invariant and $\widetilde{\mathcal{F}}$-invariant. We have $d_0 = d$ and $\tilde{d}_0 = \tilde{d}$, thus

$$\begin{aligned}\sum_{L_\infty} \mu_p(\mathcal{F}) - m^2 \sum_{L_\infty} \mu_p(\widetilde{\mathcal{F}}) &= (d^2 + d + 1) - m^2(\tilde{d}^2 + \tilde{d} + 1) \\ &= ((m\tilde{d})^2 + m\tilde{d} + 1) - m^2(\tilde{d}^2 + \tilde{d} + 1) \\ &= -m^2\tilde{d} + m\tilde{d} - m^2 + 1.\end{aligned}$$

Suppose now that $\sum_{L_\infty} \mu_p(\widetilde{\mathcal{F}}) = \tilde{d}_0 + 1 = \tilde{d} + 1$. This is equivalent to

$$\sum_{L_\infty} \mu_p(\mathcal{F}) - m^2(\tilde{d} + 1) \;=\; -m^2\tilde{d} + m\tilde{d} - m^2 + 1,$$

which in its turn is equivalent to

$$\sum_{L_\infty} \mu_p(\mathcal{F}) \;=\; m\tilde{d} + 1 \;=\; d + 1 \;=\; d_0 + 1.$$

We reach the following conclusion:

Proposition 7. *Let $\mathcal{F}$ be a non-primitive foliation of degree d_0 having a primitive model $\widetilde{\mathcal{F}}$ of degree $\tilde{d}_0$. Suppose that both foliations leave L_∞ invariant. Then $\sum_{L_\infty} \mu_p(\widetilde{\mathcal{F}}) = \tilde{d}_0 + 1$ if and only if $\sum_{L_\infty} \mu_p(\mathcal{F}) = d_0 + 1$.*

This results shows an interesting behavior concerning non-primitive foliations and their primitive models. If $\mathcal{F}$ is a non-primitive foliation having $\widetilde{\mathcal{F}}$ as primitive model, both of them having the line at infinity invariant, then the passage from $\widetilde{\mathcal{F}}$ to $\mathcal{F}$ degenerates all singularities in the affine plane $\mathbb{C}^2$, in the sense that $\mu_p(\mathcal{F}) = m^2\mu_p(\widetilde{\mathcal{F}})$ for every $p \in Sing(\mathcal{F})_{|\mathbb{C}^2} = Sing(\widetilde{\mathcal{F}})_{|\mathbb{C}^2}$, where m is the degree of ramification. On the other hand, this process does not degenerate the singularities of $\widetilde{\mathcal{F}}$ lying in L_∞, in the sense that, considering Proposition 5, if all singularities of $\widetilde{\mathcal{F}}$ in L_∞ are either non-degenerate or saddle-nodes with weak separatrix over L_∞, then the same property holds for the singularities of $\mathcal{F}$ in L_∞.

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