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A COMPLETE CHARACTERIZATION OF $\mathcal{A}_0$-SUFFICIENCY OF PLANE-TO-PLANE JETS OF RANK 1

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Abstract. Sufficient conditions for $\mathcal{A}_0$-sufficiency of plane-to-plane r-jets are known. These conditions are stated in the form of two Łojasiewicz inequalities which have to be satisfied. The first of these inequalities is known to be necessary for $\mathcal{A}_0$-sufficiency, and in this article we prove that the second inequality is also necessary for $\mathcal{A}_0$-sufficiency of all jets of rank 1. We also prove that a simpler Łojasiewicz inequality is equivalent to the second inequality for rank 1 jets.

1. Introduction

Let $\mathcal{E}_{[r]}(n,p)$ be the set of C^r map germs $(\mathbb{R}^n,0) \to (\mathbb{R}^p,0)$. Two map germs f and g in $\mathcal{E}_{[r]}(n,p)$ are $\mathcal{A}_s$-equivalent if there exist germs of C^s diffeomorphisms $h : (\mathbb{R}^n,0) \to (\mathbb{R}^n,0)$ and $k : (\mathbb{R}^p,0) \to (\mathbb{R}^p,0)$ such that $g = k \circ f \circ h^{-1}$. If $f, g \in \mathcal{E}_{[r]}(2,2)$ are $\mathcal{A}_s$-equivalent, then we write $f \sim_{\mathcal{A}_s} g$. If f and g are $\mathcal{A}_0$-equivalent, then we say that they are topologically equivalent, and if f and g are not $\mathcal{A}_0$-equivalent, then they are topologically different. A jet $\omega \in J^r(n,p)$ is $\mathcal{A}_0$-sufficient in $\mathcal{E}_{[r]}(n,p)$ if every $f \in \mathcal{E}_{[r]}(n,p)$ with $j^r f(0) = \omega$ is $\mathcal{A}_0$-equivalent to ω. There exists no general theorem giving necessary and sufficient conditions for $\mathcal{A}_0$-sufficency of r-jets in $\mathcal{E}_{[r]}(n,p)$ for arbitrary n and p. Known results include a characterization of $\mathcal{A}_0$-sufficient jets with 0 as an isolated singular point (see [1]), and a study of $\mathcal{A}_0$-sufficiency in $\mathcal{E}_{[r]}(2,2)$ of jets from $\mathbb{R}^2$ to $\mathbb{R}^2$ (see [2]). The result in [2] gives a complete characterization of $\mathcal{A}_0$-sufficent plane-to-plane jets for a restricted class of jets, and it is the aim of this article to extend the result of [2] to a complete characterization of $\mathcal{A}_0$-sufficient plane-to-plane jets of rank 1.

We identify r-jets in $J^r(2,2)$ with polynomial maps $\mathbb{R}^2 \to \mathbb{R}^2$ of degree $\leq r$ with zero constant term. Let $\omega \in J^r(2,2)$. Let $J\omega(p)$ denote the Jacobian determinant of ω at p and let $\Sigma(\omega) = J\omega^{-1}(0)$ denote the singular set of ω. $\Sigma(\omega)$ is an algebraic set. Let $B(x,\rho)$ denote the open ball in $\mathbb{R}^2$ with center x and radius ρ. If ω is a nonzero singular jet, then there is a real number $\rho_0 > 0$ and a natural number N such that $(\Sigma(\omega) \setminus \{0\}) \cap B(0,\rho)$ has exactly N topological components whenever $0 < \rho < \rho_0$. These components are called *branches* of ω.

Let $C_1, C_2, \ldots, C_N$ denote the branches of ω. Since $\Sigma(\omega)$ is an algebraic set, the Curve Selection Lemma implies that each of these branches has a well defined tangent direction at the origin. We think of these directions as points on S^1. If all these points are distinct, then we say that ω has *different tangent directions* at 0. Note that a line through the origin represents two different tangent directions corresponding to antipodal points on S^1.

Identify $J^1(2,2)$ with $\mathbb{R}^4$ by identifying $(ax+by, cx+dy)$ with (a,b,c,d) and let $\Sigma = \{(a,b,c,d) | ad - bc = 0\} \subset J^1(2,2)$. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be a C^r map with $r \geq 2$. The germ of F at a singular point p is a *fold singularity* if two conditions are satisfied. The first condition is that $j^1 F \pitchfork \Sigma$ at p. If the first condition is satisfied, then $\Sigma(F)$ is a C^{r-1} manifold in a neighbourhood of p. The second condition for fold singularities is that $T_p\Sigma(F) + \ker D(JF)(p) = \mathbb{R}^2$. Whether or not the germ of F at a point p in the source of F is a fold singularity is determined by the non-constant part of the 2-jet extension of F at p, i.e. the 2-jet extension at 0 of the map $q \mapsto F(q+p) - F(p)$ which will be denoted by $J^2F(p)$.

An element of $J^2(2,2)$ is then thought of as a polynomial map as above. We may use the coefficients of these polynomials as coordinates of $J^2(2,2)$, and hence identify $J^2(2,2)$ with $\mathbb{R}^4 \times \mathbb{R}^6$ by identifying the polynomial map given by

$$(x,y) \mapsto (ax + by + ex^2 + 2fxy + gy^2, cx + dy + hx^2 + 2ixy + jy^2)$$

with $(L,H) = ((a,b,c,d),(e,f,g,h,i,j))$. It is shown in [2] that in these coordinates, the set of singular 2-jets which are not folds is given by

$$\Gamma = \left\{ (a,\dots,j) \,|\, ad - bc = 0, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} aj - bi - cg + df \\ -ai + bh + cf - de \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}.$$

For every C^2 map $F : \mathbb{R}^2 \to \mathbb{R}^2$ we may define a map $(L_F, H_F) : \mathbb{R}^2 \to J^2(2,2) = \mathbb{R}^4 \times \mathbb{R}^6$ induced by J^2F via the identifications above.

Let $d(\cdot,\Sigma)$ denote the distance function from a point in $\mathbb{R}^4$ to Σ with respect to the norm on $J^1(2,2)$ induced by the Euclidean norm on $\mathbb{R}^4$. For all $f \in \mathcal{E}_{[r]}(2,2)$, define

$$d_f(p) = d(j^1f(p), \Sigma).$$

For all $\epsilon, \rho > 0$ and $f \in \mathcal{E}_{[r]}(2,2)$, define

$$H_{\epsilon,\rho}(f) = \left\{ p \,|\, d(j^1f(p),\Sigma) \leq \epsilon \|p\|^{r-1}, 0 < \|p\| < \rho \right\}.$$

$H_{\epsilon,\rho}(\omega)$ is a semialgebraic set with $\Sigma(\omega) \cap B(0,\rho) \setminus \{0\} \subset H_{\epsilon,\rho}(\omega)$.

Proposition 1.1 (Proposition 2.1 of [2]). *Let $r \geq 2$ and let $\omega \in J^r(2,2)$ be a singular, nonzero jet such that 0 is not isolated in $\Sigma(\omega)$. Let Γ and $C_1,\dots,C_N$ and $H_{\epsilon,\rho}(\omega)$ be as explained above. Consider the following condition:*

(I) *There is a neighbourhood U of 0 and a constant $C > 0$ such that if $p \in U$ and $(L,H) \in \Gamma$, then*

$$\|L_\omega(p) - L\| + \|H_\omega(p) - H\| \, \|p\| \geq C \|p\|^{r-1}.$$

Assume that condition (I) *is satisfied. Then there exist $\epsilon_0 > 0$ and $\rho_0 > 0$ such that the following is satisfied: For each ρ such that $0 < \rho < \rho_0$, and for each ϵ such that $0 < \epsilon < \epsilon_0$, $H_{\epsilon,\rho}(\omega)$ has exactly N connected components and we can label these components by $H^1_{\epsilon,\rho},\dots,H^N_{\epsilon,\rho}$, such that for $i = 1,\dots,N$, $C_i \subset H^i_{\epsilon,\rho}$.*

Theorem 1.2 (Theorem 2.3 of [2]). *If $\omega \in J^r(2,2)$ has an isolated singularity at the origin, then ω is $\mathcal{A}_0$-sufficient in $\mathcal{E}_{[r]}(2,2)$ if and only if inequality (I) of Proposition 1.1 holds.*

In this article, whenever ω is an r-jet which satisfies (I) and we speak about $H_{\epsilon,\rho}(\omega)$, it is understood that $\epsilon < \epsilon_0$ and $\rho < \rho_0$ where ϵ_0 and ρ_0 have the properties stated in Proposition 1.1.

Theorem 1.3 (Main Theorem of [2]). *Let $r > 2$ and let $\omega \in J^r(2,2)$ be a jet as described in Proposition 1.1 . Let Γ, $C_1,\dots,C_N$ and $H_{\epsilon,\rho}(\omega)$ be as defined above and assume that condition* (I) *from Proposition 1.1 is satisfied. Let ρ_0 and ϵ_0 be as in the conclusion of 1.1. Consider the following condition :*

(II) *There exist $\rho > 0$ with $\rho < \rho_0$ and $\epsilon > 0$ with $\epsilon < \epsilon_0$ and a constant C such that if $H^i_{\epsilon,\rho}(\omega)$*

and $H^j_{\epsilon,\rho}(\omega)$ *are distinct components of* $H_{\epsilon,\rho}(\omega)$ *and* $p \in H^i_{\epsilon,\rho}(\omega) \cup \{0\}$ *and* $q \in H^j_{\epsilon,\rho}(\omega) \cup \{0\}$ *then*

$$\|\omega(p) - \omega(q)\| \geq C(\|p\|^{r-1} + \|q\|^{r-1}) \|p - q\| \,.$$

Assume also that the condition (II) *above is satisfied, then* ω *is* $\mathcal{A}_0$*-sufficient in* $\mathcal{E}_{[r]}(2,2)$.

Moreover, the condition (I) *of Proposition 1.1 is a necessary condition for* $\mathcal{A}_0$*-sufficiency in* $\mathcal{E}_{[r]}(2,2)$ *for all jets in* $J^r(2,2)$ *with* $r > 2$*, and if we consider singular, nonzero jets* ω *where 0 is not isolated in* $\Sigma(\omega)$*, and where* ω *has different tangent directions at 0, then condition* (II) *above is also a necessary condition for* $\mathcal{A}_0$*-sufficiency in* $\mathcal{E}_{[r]}(2,2)$.

Proposition 1.4. *If* $\omega \in J^r(2,2)$ *satisfies (I), then every* C^r *realization of* ω *has only regular points and fold singularities outside the origin. If* ω *does not satisfy (I), then there is a* C^r *realization of* ω *with a sequence of simple cusp points converging to the origin. Furthermore, simple cusps are topologically different from folds and regular points.*

Proof. The first assertion follows from the defining property of Γ and Lemma 4.1 of [2]. The second assertion is the content of Lemma 6.3 of [2]. The last assertion is the content of Lemma 6.6 of [2]. □

Proposition 1.5. *If* $\omega \in J^r(2,2)$ *satisfies (I) and (II), then the restriction of every* C^r *realization of* ω *to its singular set is injective. If* ω *has different tangent directions and satisfies (I) but does not satisfy (II), then there is a* C^r *realization of* ω *having a sequence of singular double points converging to the origin.*

Proof. The first part of the Proposition follows from Lemma 4.12 of [2] and the last part follows from Lemma 6.4 of [2]. □

Definition 1.6. A map germ $z = (z_1, z_2) : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ of rank 1 is in *standard form* if $z_1(x, y) = x$.

Theorem 1.3 can be quite difficult to apply in practice. In the case of rank 1 jets in standard form, the following theorem gives the neat conditions that characterize $\mathcal{A}_0$-sufficient jets.

Theorem 1.7. *Let* $r > 2$ *and let* $\omega(x,y) = (x, f(x,y)) \in J^r(2,2)$ *and let* $C_1, \ldots, C_N$ *be as above. Then* ω *is* $\mathcal{A}_0$*-sufficient in* $\mathcal{E}_{[r]}(2,2)$ *if and only if the conditions (i) and (ii) below are satisfied:*

(i) *There are a neighbourhood* U *of 0 and a constant* $C > 0$ *such that if* $p \in U$*, then*

$$|f_y(p)| + |f_{yy}(p)| \, \|p\| \geq C \, \|p\|^{r-1} \,.$$

(ii) *There are a neighbourhood* U *of 0 and a constant* $C > 0$ *such that if* C_i *and* C_j *are different components of* $\Sigma(\omega) \setminus \{0\}$ *and* $p = (x,y) \in C_i \cup \{0\} \cap U$ *and* $q = (x,v) \in C_j \cup \{0\} \cap U$*, then*

$$|f(p) - f(q)| > C(\|p\|^{r-1} + \|q\|^{r-1})|y - v|.$$

There is also an analogue of Theorem 1.2 for rank 1 jets in standard form.

Theorem 1.8. *If* $\omega \in J^r(2,2)$ *is in standard form and has an isolated singularity at the origin, then* ω *is* $\mathcal{A}_0$*-sufficient in* $\mathcal{E}_{[r]}(2,2)$ *if and only if (i) of Theorem 1.7 holds.*

Proof. This follows immediately from Theorem 1.2 and Lemma 2.2 of Section 2.2 which says that for jets in standard form, (i) and (I) are equivalent. □

From now on we consider only singular jets where 0 is not an isolated singularity. The main step in the proof of Theorem 1.7 is to prove the following proposition:

Proposition 1.9. *For jets of rank 1 in standard form, (II) $\Leftrightarrow$ (ii).*

The virtue of Theorem 1.7 is that both the set Γ and the sets $H_{\epsilon,\rho}$ are left out of the theorem. Also, when verifying (ii) one only needs to consider pairs of points with the same x-components. Finally, the validity of Theorem 1.7 is not restricted to the case of jets with different tangent directions at 0.

Theorem 1.7 holds for rank 1 jets given in a special form. For rank 1 jets in general, the following theorem holds.

Theorem 1.10. *Let $r > 2$ and let $\omega \in J^r(2,2)$ be a jet of rank 1. Then ω is $\mathcal{A}_0$-sufficient in $\mathcal{E}_{[r]}(2,2)$ if and only if (I) of Theorem 1.3 and (II') below hold:*

(II') *There is a neighbourhood U of 0 and a constant $C > 0$ such that if $i \neq j$ and $p \in C_i \cap U$ and $q \in C_j \cap U$, then*

$$\|\omega(p) - \omega(q)\| > C(\|p\|^{r-1} + \|q\|^{r-1})\,\|p - q\|\,.$$

The article is organized as follows: In Section 2 we prove that Theorem 1.7 implies Theorem 1.10. Section 3 contains a thorough study of the hornshaped neighbourhoods $H_{\epsilon,\rho}$. This enables us to prove that inequality (II') implies inequality (II) for rank 1 jets. This is the topic of Section 4. In Section 4 we also give the proof of Proposition 1.9. This proposition is the key to the construction of a certain Whitney field in Section 5. This Whitney field is the main technical tool in the proof of the necessity of (ii) for all rank 1 jets in standard form, and will conclude the demonstration of Theorem 1.7 and Theorem 1.10.

In the rest of the article, $\mathcal{A}_0$-sufficiency of an r-jet is understood to mean $\mathcal{A}_0$-sufficiency in $\mathcal{E}_{[r]}(2,2)$. Sometimes only the term 'sufficiency' will be used.

Notation 1 ($\lesssim$, $\gtrsim$, $\sim$)**.** Let F and G be two nonnegative real-valued functions defined on some subset of some Euclidean space E. We will use the notation $F \gtrsim G$ if there is a constant $a > 0$ such that $F \geq aG$. The notation $F \lesssim G$ means that there is a constant $b > 0$ such that $F \leq bG$. If $F \lesssim G$ and $F \gtrsim G$, then we write $F \sim G$. For two sequences (p_n) and (q_n) in E and positive real valued functions F and G, $F(p_n) \gtrsim G(q_n)$ means that there is a positive constant a and a natural number N such that $F(p_n) \geq aG(q_n)$ when $n > N$. Similarly, $F(p_n) \lesssim G(q_n)$ means that there is a positive constant b and a natural number N such that $F(p_n) \leq bG(q_n)$ when $n > N$. Of course, $F(p_n) \sim G(q_n)$ means that $F(p_n) \gtrsim G(q_n)$ and $F(p_n) \lesssim G(q_n)$.

Notation 2 (O, o)**.** If F and G are real-valued functions defined in a neighbourhood of 0 in some Euclidean space, then $F(x) = o(G(x))$ means that $F(x)/G(x) \to 0$ as $x \to 0$. If (p_n) and (q_n) are sequences converging to 0, then $F(p_n) = o(G(q_n))$ means that $F(p_n)/G(q_n) \to 0$ as $n \to \infty$. For fractional power series β and γ, $O(\beta)$ denotes the order of β and $\beta = o(\gamma)$ means that $O(\beta) > O(\gamma)$.

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2. Coordinate changes

2.1. **Suitable coordinates.** To establish the connection between Theorem 1.7 and Theorem 1.10, we have to investigate how our Łojasiewicz inequalities behave under coordinate changes. Let $\omega \in J^r(2,2)$ and let $\omega' = k \circ \omega \circ h^{-1}$ where h and k are germs of C^r diffeomorphisms $(\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$.

Lemma 2.1. *ω is $\mathcal{A}_0$-sufficient if and only if $j^r\omega'$ is $\mathcal{A}_0$-sufficient.*

Proof. Assume that ω is sufficient and let $\tilde{\omega}$ be a C^r realization of $j^r\omega'$. Then $j^r(k^{-1}\circ\tilde{\omega}\circ h) = j^r(k^{-1}\circ j^r\omega'\circ h) = \omega$. Thus $\tilde{\omega} \sim_{\mathcal{A}_0} j^r\omega'$, and hence, $j^r\omega'$ is sufficient. Conversely, suppose $j^r\omega'$ is sufficient and let $\bar{\omega}$ be a C^r realization of ω. Then clearly $j^r(k\circ\bar{\omega}\circ h^{-1}) = j^r(k\circ\omega\circ h^{-1}) = j^r\omega'$. Thus $\bar{\omega} \sim_{\mathcal{A}_0} k\circ\bar{\omega}\circ h^{-1} \sim_{\mathcal{A}_0} j^r\omega' \sim_{\mathcal{A}_0} \omega' \sim_{\mathcal{A}_0} \omega$, which shows that ω is sufficient. □

Lemma 2.2. *ω satisfies (I) $\Leftrightarrow$ $j^r\omega'$ satisfies (I).*

Proof. Assume that ω satisfies (I) and that $j^r\omega'$ does not satisfy (I). By Proposition 1.4, $j^r\omega'$ has a C^r realization $\tilde{\omega}$ with a sequence of singular points converging to 0, all topologically different from folds. Then $\omega = j^r(k^{-1}\circ\tilde{\omega}\circ h)$ has a realization which has a sequence of singular points converging to 0, all of which are topologically different from folds. This contradicts the assumption that ω satisfies (I).

Let $\omega_2 = j^r\omega'$. Then $\omega = j^r(k^{-1}\circ\omega'\circ h) = j^r(k^{-1}\circ\omega_2\circ h)$, and hence, the other implication follows from the first implication. □

Lemma 2.3. *Let z and z' in $\mathcal{E}_{[r]}(2,2)$ be such that $z' = k\circ z\circ h^{-1}$ for some germs at the origin of origin-preserving C^r diffeomorphisms h and k . For each $\epsilon,\rho > 0$, there are $\epsilon',\rho' > 0$ such that $h(H_{\epsilon',\rho'}(z)) \subset H_{\epsilon,\rho}(z')$.*

Proof. It is enough to show that $\|p\| \sim \|h(p)\|$ and $d_z(p) \sim d_{z'}(h(p))$. An application of Taylor's formula gives $\|p\| \sim \|h(p)\|$. We also have

$$\begin{aligned} d_z(p) &= \inf\{\|Dz(p)v\| \mid \|v\| = 1\} \qquad \text{(by (3.11) in [2])}\\ &\sim \inf\{\|D(k\circ z)(p)v\| \mid \|v\| = 1\}\\ &\sim \inf\{\|Dz'(h(p))v\| \mid \|v\| = 1\}\\ &= d_{z'}(h(p)), \end{aligned}$$

and the lemma follows. □

Lemma 2.4. *Suppose z and z' in $\mathcal{E}_{[r]}(2,2)$ are such that $j^rz(0) = j^rz'(0)$. Let $\epsilon,\rho > 0$. Then there are $\epsilon',\rho' > 0$ such that*

$$H_{\epsilon',\rho'}(z') \subset H_{\epsilon,\rho}(z).$$

Proof. Assume that z and z' satisfy the premises of the lemma. Let $\tilde{z} = z - z'$. Then $j^r\tilde{z}(0) = 0$, and hence, $\|D\tilde{z}(p)\| = o(\|p\|^{r-1})$. Using this, we see that

$$\begin{aligned} d_z(p) &= \inf\{\|Dz(p)v\| \mid \|v\| = 1\} \le \inf\{\|Dz'(p)v\| + \|D\tilde{z}(p)v\| \mid \|v\| = 1\}\\ &\le \inf\{\|Dz'(p)v\| \mid \|v\| = 1\} + \sup\{\|D\tilde{z}(p)v\| \mid \|v\| = 1\} = d_{z'}(p) + o(\|p\|^{r-1}). \end{aligned}$$

The lemma follows. □

Lemma 2.5. *For every sequence (p_n) of points converging to 0 such that $d(j^1\omega(p_n),\Sigma) = o(\|p_n\|^{r-1})$, there is a subsequence $(p_{n(k)})$ of (p_n) and a C^r realization ω_p of ω such that $p_{n(k)} \in \Sigma(\omega_p)$ for every k.*

Proof. Let (p_n) be as in the lemma. Choose $p_{n(k)}$ such that $\|p_{n(k+1)}\| < \frac{1}{2}\|p_{n(k)}\|$. For every k, let M_k be a matrix such that $\|M_k\| = d(j^1\omega(p_{n(k)}),\Sigma)$ and $D\omega(p_{n(k)}) + M_k$ is singular. Let Q be the r-th order Taylor field defined on $K = \{0\}\cup(\cup_k\{p_{n(k)}\})$ with values in $\mathbb{R}^2$ given by $Q^1(p) = M_k$ for $p = p_{n(k)}$ and $Q = 0$ otherwise. It is clear that Q is a Whitney field. Let h be a C^r extension of Q. Then $j^rh(0) = 0$. Let $\omega_p = \omega + h$. It is clear that ω_p satisfies the conditions in the lemma. □

Lemma 2.6. *(I) and (II) hold for ω $\Leftrightarrow$ (I) and (II) hold for $j^r\omega'$.*

Proof. Assume that (I) holds and (II) fails for ω. By Lemma 2.2, (I) holds for $j^r\omega'$ as well. We proceed to show that (II) fails for $j^r\omega'$. Since (II) fails for ω, there are sequences (p_n) and (q_n) of points converging to 0 such that

$$d(j^1\omega(p_n),\Sigma)=o(\|p_n\|^{r-1}) \text{ and } d(j^1\omega(q_n),\Sigma)=o(\|q_n\|^{r-1})$$

and

$$\|\omega(p_n)-\omega(q_n)\|=o(\|p_n\|^{r-1}+\|q_n\|^{r-1})\,\|p_n-q_n\|$$

and p_n and q_n belong to different components of $H_{\tilde\epsilon,\tilde\rho}(\omega)$. Since h and k are germs of diffeomorphisms, an application of Taylor's formula shows that $\|h(p)\|\sim\|k(p)\|\sim\|p\|$ for all p close to 0. Furthermore, since h and k are diffeomorphisms, the definition of differentiability gives $\|h(p)-h(q)\|\sim\|k(p)-k(q)\|\sim\|p-q\|$ for p,q close to 0. Furthermore, $j^r\omega'=\omega'+\tilde\omega$ where $j^r\tilde\omega(0)=0$ and hence, $\|\tilde\omega(p)\|=o(\|p\|^r)$ and $\|D\tilde\omega(p)\|=o(\|p\|^{r-1})$. Using this and the Mean Value Theorem, we get

$$\begin{aligned}
&\|j^r\omega'(h(p_n))-j^r\omega'(h(q_n))\|\\
&\le\|k\circ\omega(p_n)-k\circ\omega(q_n)\|+\|\tilde\omega(h(p_n))-\tilde\omega(h(q_n))\|\\
&\lesssim\|\omega(p_n)-\omega(q_n)\|+\sup_{t\in[0,1]}\|D\tilde\omega(th(p_n)+(1-t)h(q_n))\|\,\|h(p_n)-h(q_n)\|\\
&=o(\|p_n\|^{r-1}+\|q_n\|^{r-1})\,\|p_n-q_n\|+o(\|h(p_n)\|^{r-1}+\|h(q_n)\|^{r-1})\,\|h(p_n)-h(q_n)\|\\
&=o(\|h(p_n)\|^{r-1}+\|h(q_n)\|^{r-1})\,\|h(p_n)-h(q_n)\|\,.
\end{aligned}$$

By Lemma 2.5 there are subsequences $(p_{n(k)})$ and $(q_{n(k)})$ of (p_n) and (q_n) and C^r realizations ω_p and ω_q of ω such that for each k, $p_{n(k)}\in\Sigma(\omega_p)$ and $q_{n(k)}\in\Sigma(\omega_q)$. Hence, for each of the sequences $(h(p_{n(k)}))$ and $(h(q_{n(k)}))$, there are C^r realizations of $j^r(\omega')$ having singular points along the sequence. It follows that, given small positive ϵ and ρ, then eventually the sequences $(h(p_{n(k)}))$ and $(h(q_{n(k)}))$ are in $H_{\epsilon,\rho}(j^r\omega')$.

We need to show that for small ϵ,ρ, eventually the sequences $(h(p_{n(k)}))$ and $(h(q_{n(k)}))$ lie in different components of $H_{\epsilon,\rho}(j^r\omega')$. To this end, use Lemma 2.3 to pick ϵ',ρ' so small that $h^{-1}(H_{\epsilon',\rho'}(\omega'))\subset H_{\tilde\epsilon,\tilde\rho}(\omega)$ where $\tilde\epsilon$ and $\tilde\rho$ are as above, i.e. such that (p_n) and (q_n) lie in different components of $H_{\tilde\epsilon,\tilde\rho}(\omega)$. Then use Lemma 2.4 to pick ϵ,ρ such that $H_{\epsilon,\rho}(j^r\omega')\subset H_{\epsilon',\rho'}(\omega')$. Assume that there are subsequences $(h(p_{n(k(l))}))$ and $(h(q_{n(k(l))}))$ which lie in the same component of $H_{\epsilon,\rho}(j^r\omega')$. Since h^{-1} is a homeomorphism, the component of $H_{\epsilon,\rho}(j^r\omega')$ containing $(h(p_n))$ and $(h(q_n))$ is mapped by h^{-1} into one component of $H_{\tilde\epsilon,\tilde\rho}(\omega)$. This contradicts the assumption that (p_n) and (q_n) lie in different components of $H_{\tilde\epsilon,\tilde\rho}(\omega)$. Hence, (II) fails for $j^r\omega'$.

To finish the proof, observe that $\omega=j^r(k^{-1}\circ j^r\omega'\circ h)$, and hence the other implication follows from the first. □

2.2. Łojasiewicz inequality (I) for rank 1 jets.

When ω is in standard form, we have a particularly convenient version of inequality (I).

Lemma 2.7. *Let $\omega(x,y)=(x,f(x,y))$ be an r-jet in standard form. Then (I) holds for ω if and only if (i) of Theorem 1.7 holds for ω.*

Proof. To prove that (I) implies (i), notice that

$$(L,H)=(1,0,f_x,0,0,0,0,f_{xx},f_{xy},0)(p)\in\Gamma$$

for all p, and hence, if (I) holds, then

$$|f_y(p)|+|f_{yy}(p)|\,\|p\|=\|L_\omega(p)-L\|+\|H_\omega(p)-H\|\,\|p\|\ge C\,\|p\|^{r-1}\,.$$

Conversely, if (I) fails, then there are a sequence (p_n) in $\mathbb{R}^2$ converging to 0 and a sequence $(L_n, H_n) \in \Gamma$ such that

$$\|L_\omega(p_n) - L_n\| + \|H_\omega(p_n) - H_n\| \, \|p_n\| = o(\|p_n\|^{r-1}).$$

Let $(L_n, H_n) = (a_n, \ldots, d_n, e_n, \ldots, j_n) \in \mathbb{R}^{10}$. We get that $a_n = 1 - o(\|p_n\|^{r-1})$ and $b_n = o(\|p_n\|^{r-1})$. Also, since L_n is singular, $d_n = c_n b_n / a_n = o(\|p_n\|^{r-1})$, which implies $|f_y(p_n)| = o(\|p_n\|^{r-1})$. We have $H_\omega(p_n) = (0, 0, 0, f_{xx}, f_{xy}, f_{yy})(p_n)$. Thus we also have $e_n, f_n, g_n = o(\|p_n\|^{r-2})$. Furthermore, from the definition of Γ, we get that

$$a_n(a_n j_n - b_n i_n - c_n g_n + d_n f_n) + b_n(-a_n i_n + b_n h_n + c_n f_n - d_n e_n) = 0.$$

It follows that $j_n = o(\|p_n\|^{r-2})$ and hence, $|f_{yy}(p_n)| = o(\|p_n\|^{r-2})$. This shows that (i) fails. $\square$

Lemma 2.8. *Let a be a real number and let Φ be the diffeomorphism $\Phi(x, y) = (\Phi_1(x, y), \Phi_2(x, y)) = (x, ax + y)$. Let $\omega \in J^r(2, 2)$ be in standard form. Then $\omega_\Phi = \omega \circ \Phi^{-1}$ is an r-jet in standard form and ω satisfies (i) and (ii) if and only if ω_Φ satisfies (i) and (ii).*

Proof. The first assertion is clear from the form of Φ. For the second assertion, assume that ω satisfies (i) but not (ii). Lemma 2.2 and Lemma 2.7 imply that $\omega \circ \Phi^{-1}$ satisfies (i). Since ω does not satisfy (ii), there are distinct components C_i and C_j of $\Sigma(\omega) \setminus \{0\}$ and sequences $p_n = (x_n, y_n) \in C_i$ and $q_n = (x_n, v_n) \in C_j$, both converging to 0 and such that

$$|f(p_n) - f(q_n)| = o(\|p_n\|^{r-1} + \|q_n\|^{r-1})|y_n - v_n|.$$

From the definition of Φ, it is clear that $\omega_\Phi(x, y) = (x, f_\Phi(x, y))$ is in standard form. Furthermore, $\Phi(C_i)$ and $\Phi(C_j)$ are different components of $\Sigma(\omega_\Phi)$ and

$$\begin{aligned} |f_\Phi(\Phi(p_n)) - f_\Phi(\Phi(q_n))| &= |f(p_n) - f(q_n)| = o(\|p_n\|^{r-1} + \|q_n\|^{r-1})|y_n - v_n| \\ &= o(\|\Phi(p_n)\|^{r-1} + \|\Phi(q_n)\|^{r-1})|\Phi_2(p_n) - \Phi_2(q_n)|, \end{aligned}$$

and hence (ii) fails for ω_Φ.

Observe that $\Phi^{-1}(x, y) = (x, -ax + y)$, and hence the other implication follows directly from the argument above. $\square$

Lemma 2.9. *Let ω be an r-jet which satisfies (I), and let ω' be a C^r map germ with $\omega \sim_{\mathcal{A}_r} \omega'$. Then (II') holds for ω if and only if (II') holds for $j^r\omega'$.*

Proposition 2.10. *If ω is an r-jet in standard form satisfying (i), then (ii) and (II') are equivalent for ω.*

The proofs of Lemma 2.9 and Proposition 2.10 will be postponed until Section 4.

Proof that Theorem 1.7 $\Rightarrow$ Theorem 1.10. Assume that Theorem 1.7 is true. Assume now that (I) and (II') hold for an r-jet $\omega \in J^r(2, 2)$ of rank 1. By Lemma 2.2, Lemma 2.7, Lemma 2.9 and Proposition 2.10, we may choose C^r coordinates transforming ω to the standard form $\bar{\omega}(x, y) = (x, f(x, y))$ such that (i) and (ii) hold for $j^r\bar{\omega}$. By Theorem 1.7, $j^r\bar{\omega}$ is $\mathcal{A}_0$-sufficient. Lemma 2.1 implies that ω is $\mathcal{A}_0$-sufficient.

Conversely, if (I) fails for ω, then, by Lemma 2.2 and Lemma 2.7, (i) fails for $j^r\bar{\omega}$ and hence, $j^r\bar{\omega}$ is not sufficient by Theorem 1.7. By Lemma 2.1, ω is not sufficient. If (I) holds and (II') fails for ω, then (II') fails for $j^r\bar{\omega}$ by Lemma 2.9. By Proposition 2.10, (ii) fails for $j^r\bar{\omega}$. Theorem 1.7 shows that $j^r\bar{\omega}$ is not $\mathcal{A}_0$-sufficient, and hence, by Lemma 2.1 again, ω is not $\mathcal{A}_0$-sufficient. $\square$

3. Hornshaped neighbourhoods

3.1. **Consequences of inequality (i).** Let $\omega(x,y) = (x, f(x,y))$ be an r-jet of rank 1 in standard form for which (I), or equivalently (i) holds. By Lemma 2.8, we may choose coordinates such that no branch of $\Sigma(\omega)$ is tangent to the x-axis. Let

$$\tilde{H}_{\epsilon,\rho} = \{p : |f_y(p)| \le \epsilon \|p\|^{r-1}, 0 < \|p\| \le \rho\}.$$

Recall from (3.3) in [2] that

$$d_\omega(p) = d(j^1\omega(p), \Sigma) \sim \frac{|J\omega(p)|}{\|D\omega(p)\|} \sim |f_y(p)|.$$

It follows that for every $\epsilon > 0$ there are $\epsilon_1, \epsilon_2, \epsilon_3 > 0$ such that

$$H_{\epsilon_1,\rho}(\omega) \subseteq \tilde{H}_{\epsilon_2,\rho}(\omega) \subseteq H_{\epsilon,\rho}(\omega) \subseteq \tilde{H}_{\epsilon_3,\rho}(\omega).$$

Lemma 3.1. *Proposition 1.1 holds when we replace $H_{\epsilon,\rho}$ by $\tilde{H}_{\epsilon,\rho}$.*

Proof. Let

$$S = \{(x,y) | \nabla f_y(x,y) \cdot (y, -x) = 0\}.$$

The proof of Proposition 1.1 in [2] applies to $\tilde{H}_{\epsilon,\rho}(\omega)$ once we have shown that

$$\left|(f_y|S)(p)\right| \gtrsim \|p\|^{r-1}. \tag{3.1}$$

This corresponds to Lemma 3.1 in [2]. Let

$$D = \{p \in S : |f_y(p)| \le |f_y(q)| \text{ for all } q \in S \text{ with } \|p\| = \|q\| \ne 0\}.$$

An application of the Tarski-Seidenberg Theorem shows that D is semialgbraic. Assume that (3.1) does not hold. Then $0 \in \overline{D}$ and the Curve Selection Lemma implies that we can find an analytic curve $\gamma = (\gamma_1, \gamma_2) : [0, \delta) \to \mathbb{R}^2$ with $\gamma(0) = 0$, $\gamma(0, \delta) \subset D$ and $|f_y(\gamma(t))| = o(\|\gamma(t)\|^{r-1})$. Assume that $\|\gamma(t)\| \sim t^s$ and $|f_y(\gamma(t))| \sim t^d$. Then $\frac{d}{s} > r - 1$. Also, $\|\gamma'(t)\| \sim t^{s-1}$ and

$$\left|\nabla f_y(\gamma(t)) \cdot \frac{\gamma'(t)}{\|\gamma'(t)\|}\right| \sim t^{d-s}.$$

Let $v(t) = (\gamma_2(t), -\gamma_1(t))/\|\gamma(t)\|$ and $w(t) = \gamma'(t)/\|\gamma'(t)\|$. Then $v(t) \cdot w(t) \to 0$ as $t \to 0^+$. Let $e_2(t) = \frac{\partial}{\partial y} \circ \gamma(t)$. Then $e_2(t) = a(t)v(t) + b(t)w(t)$ where $|a(t)| < 2$ and $|b(t)| < 2$. Using that $\gamma(t) \in S$, it follows that

$$|f_{yy}(\gamma(t))| = |\nabla f_y(\gamma(t)) \cdot e_2(t)| \lesssim t^{d-s} = o(\|\gamma(t)\|^{r-2}),$$

and hence (i) fails along γ, contrary to our assumptions. Therefore (3.1) must hold and the rest of the proof goes as the proof of Proposition 1.1 in [2]. □

In the rest of the article, when we consider jets in standard form, we will only talk about $\tilde{H}_{\epsilon,\rho}$ and by abuse of notation, it will be denoted by $H_{\epsilon,\rho}$. Lemma 3.1 gives very specific geometric information about $H_{\epsilon,\rho}$. The situation for $\epsilon < \epsilon_0$ and $\rho < \rho_0$ is illustrated in Figure 1.

For the proof of Theorem 1.7 we need information about $H_{\epsilon,\rho}$ of more quantitative character. This section and the next contain the results we need.

Lemma 3.2. *There is a $\delta > 0$ and a neighbourhood U of 0 such that*

$$\{(x,y) \in \mathbb{R}^2 \mid |x| \le \delta |y|^{r-1}\} \cap \Sigma(\omega) \cap U \setminus \{0\} = \emptyset.$$

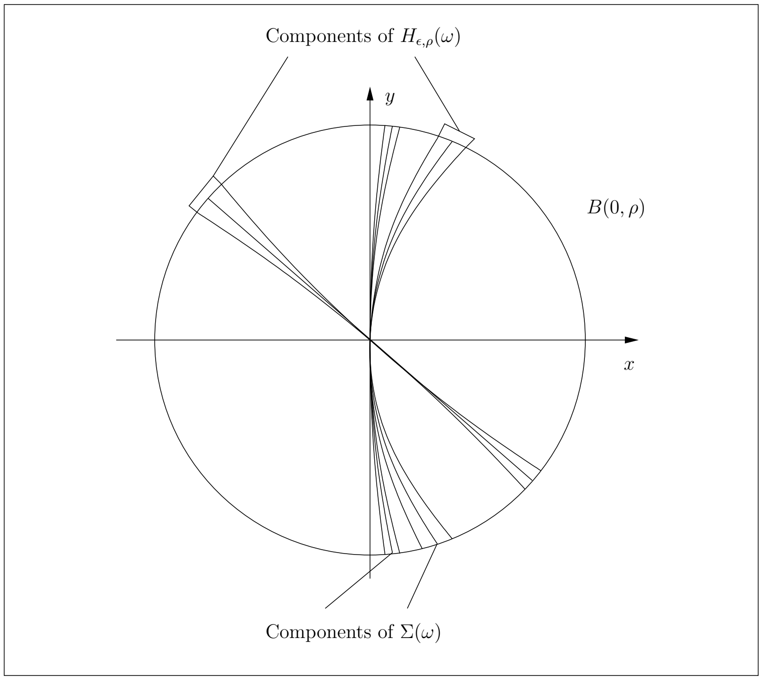

FIGURE 1. The figure shows 6 different components of $H_{\epsilon,\rho}$. The branches of $\Sigma(\omega)$ are contained in different components of $H_{\epsilon,\rho}$.

Proof. Assume that the lemma is false. Then there is a branch of $\Sigma(\omega)$ parametrized by an analytic curve $\alpha(t) = (\alpha_1(t), \alpha_2(t))$ with $\alpha(0) = 0$ and such that $\alpha_1(t) = o(|\alpha_2(t)|^{r-1})$. Let $m = O(\alpha_1(t))$, $n = O(\alpha_2(t))$. Then $m > n(r-1)$. We compute

$$0 = \frac{d}{dt} f_y(\alpha(t)) = \nabla f_y(\alpha(t)) \cdot \alpha'(t) = f_{yx}(\alpha(t))\alpha_1'(t) + f_{yy}(\alpha(t))\alpha_2'(t). \tag{3.2}$$

By (i),

$$|f_{yy}(\alpha(t))\alpha_2'(t)| \gtrsim \|\alpha(t)\|^{r-2} t^{n-1} \sim t^{n(r-2)+n-1} = t^{n(r-1)-1}.$$

By continuity of f_{yx} at 0, we have that $O(f_{yx}(\alpha(t))\alpha_1'(t)) \geq m-1 > n(r-1)-1$. It follows that (3.2) cannot hold, and this contradiction proves the lemma. □

Lemma 3.3. *If ϵ and t are small enough, then $(0,t) \notin H_{\epsilon,\rho}$.*

Proof. It is enough to check that the order in t of $f_y(0,t)$ is not greater than $r-1$. Assume that $O(f_y(0,t)) > r-1$. We have

$$\frac{d}{dt} f_y(0,t) = f_{yy}(0,t),$$

and our assumption implies that $O(f_{yy}(0,t)) > r-2$. This contradicts (i). □

3.2. **Newton-Puiseux roots of $J\omega$.** The real polynomial $J\omega = f_y$ has a Newton-Puiseux factorisation of the form

$$f_y(x,y) = u(x,y) \cdot x^E \cdot \prod_{i=1}^{p} [y - \beta_i(x)]$$

where $u \in \mathbb{C}\{x, y\}$ is a unit, $E \geq 0$ and each β_i is a formal fractional power series in x with complex coefficients. We may assume that $O(\beta_i) > 0$ for all i. Furthermore, all of the fractions occuring as exponents in these formal fractional power series have a common denominator N. This means that for each i, the formal fractional power series obtained by substituting t^N for x is an ordinary formal power series in t. This factorization is a purely algebraic rewriting of the original polynomial, but since the product is a holomorphic function, each of the power series $\beta_i(t^N)$ are in fact convergent power series, and hence, they are holomorphic functions of t for small t. We call the β_i *convergent* fractional power series.

Lemma 3.3 implies that $E = 0$, and we also assume that $O(\beta_i) \leq 1$ for each i. This can always be obtained by composition of ω with a diffeomorphism of the type in Lemma 2.8.

Lemma 3.4. *For each branch C of ω contained in the first quadrant of $\mathbb{R}^2$ there is a uniquely determined index i with $1 \leq i \leq p$ such that $t \mapsto (t^N, \beta_i(t^N)), t > 0$, is a parametrization of C.*

Proof. Let C be a branch of ω contained in the first quadrant of $\mathbb{R}^2$. The Curve Selection Lemma gives an analytic parametrization $\gamma(t)$ of C for $t > 0$. By a change of parameter if necessary, we may assume that $\gamma(t) = (t^{M\cdot N}, \tilde{\gamma}(t^N))$. Now,

$$f_y(\gamma(t)) = u(\gamma(t)) \cdot \prod_{i=1}^{p} [\tilde{\gamma}(t^N) - \beta_i(t^{M\cdot N})] \equiv 0.$$

This is an equality between analytic functions, and hence, for some i, $\tilde{\gamma}(t^N) \equiv \beta_i(t^{M\cdot N})$. It only remains to show that $\beta_i = \beta_j \Rightarrow i = j$. If there are $i \neq j$ such that $\beta_i = \beta_j$, then $f_{yy}(t^N, \beta_i(t^N)) = f_y(t^N, \beta_i(t^N)) = 0$, and this contradicts (i). $\square$

For real $x > 0$, we may think of the β_i as complex valued functions of x. By Lemma 3.4, each branch of ω in the first quadrant is a part of the graph of one of these functions $\beta_i(x)$. Any such fractional power series β_i can have only real coefficients, for we may write $\beta_i(x) = \mathrm{Re}\,\beta_i(x) + \mathfrak{I}\mathrm{Im}\,\beta_i(x)$ where $\mathfrak{I}$ is the imaginary unit and both terms on the right side are convergent fractional power series of x. If $\mathrm{Im}\,\beta_i \neq 0$, then $\mathrm{Im}\,\beta_i(x) \neq 0$ for small x, and this cannot be the case. We may assume that $\beta_1, \beta_2, \ldots, \beta_s$ correspond to the components of $\Sigma(\omega) \setminus \{0\}$ in the first quadrant and that $\beta_1(x) < \beta_2(x) < \ldots < \beta_s(x)$ for small x. The corresponding components will be denoted by $C_1, C_2, \ldots, C_s$.

In our factorisation of f_y, we have in effect solved the equation $f_y(x, y) = 0$ in terms of x. We might equally well have solved the same equation in terms of y and obtained another factorisation

$$f_y(x, y) = u'(x, y) \cdot y^F \cdot \prod_{i=1}^{q} [x - \beta_i^*(y)]$$

where $u' \in \mathbb{C}\{x, y\}$ is a unit, $F \geq 0$ and each β_i^* is a convergent fractional power series in y with $O(\beta_i^*) \geq 0$. As before, we may assume that $y \mapsto (\beta_i^*(y), y), y > 0$ is a parametrization of C_i for $i = 1, \ldots, s$. For $(x, y) \in C_i$, $(x, \beta_i(x)) = (\beta_i^*(y), y)$, and hence, both $\beta_i \circ \beta_i^*$ and $\beta_i^* \circ \beta_i$ are the identity maps. In our case, $F = 0$ and $O(\beta_i^*) \geq 1$ for $i = 1, \ldots s$ because $O(\beta_i) \leq 1$ for $i = 1, \ldots, s$.

We will call the β_i the x-roots of f_y and the β_i^* the y-roots of f_y.

Notice that if $\gamma \neq 0$ is a convergent real fractional power series in x for which the exponents in the powers of x in the terms of γ have a common denominator N and the term of lowest order has positive coefficient, then $\gamma(t^N) = g(t)$ for some real analytic function $g(t) = t^m h(t)$ where h is real analytic and $h(0) > 0$. Then $s = t(h(t))^{\frac{1}{m}}$ is a real analytic change of parameter near $t = 0$, and $t = k(s)$ for some real analytic function k. We have $(t^N, g(t)) = (k(s)^N, s^m)$. Thus, if we set $\gamma^*(y) = k(y^{\frac{1}{m}})^N$, then we get a fractional power series γ^* such that $\gamma^* \circ \gamma = \gamma \circ \gamma^*$ is the identity map.

Lemma 3.5. *Let β be a convergent fractional power series with real coefficients. Let c be the coefficient of the lowest-order term of β. Assume that $c > 0$. Then $O(\beta) \cdot O(\beta^*) = 1$ and the coefficient of the lowest-order term of β^* is $c^{-\frac{1}{O(\beta)}}$.*

Proof. Let d be the coefficient of the lowest-order term of β^*. Since β and β^* are both convergent fractional power series, $y = \beta\circ\beta^*(y) = cd^{O(\beta)}y^{O(\beta)\cdot O(\beta^*)}$+terms of higher order. The conclusion follows immediately from this. □

Lemma 3.6. *Let β_i be one of the x-roots of f_y, and let β_j^* be a y-root of f_y. Let $a \in \mathbb{Q}_+$ and let $t \in \mathbb{R}$ and let $\gamma_s(x) = \beta_i(x) + sx^a + \alpha(x)$ and let $\sigma_t^*(y) = \beta_j^*(y) + ty^a + \alpha(y)$, where α is a convergent fractional power series with $O(\alpha) > a$. Then there are finite sets $S(i, a) \subset \mathbb{R}$ and $T(j, a) \subset \mathbb{R}$, independent of α such that $0 \in S(i, a) \cap T(j, a)$ and $O_x(f_y(x, \gamma_s(x))$ and $O_y(f_y(\sigma_t^*(y), y)$ are constant numbers A and B, respectively, for all $s \notin S(i, a)$ and $t \notin T(j, a)$. If $s \in S(i, a)$, then $O_x(f_y(x, \gamma_s(x)) > A$, and if $t \in T(j, a)$, then $O_y(f_y(\sigma_t^*(y), y)) > B$.*

Proof. We prove only the part of the lemma concerning the x-roots, since the other part is completely analogous. From the factorisation above we get

$$f_y(x, \gamma_s(x)) = u(x, \gamma_s(x)) \cdot (sx^a + \alpha(x)) \cdot \prod_{j\neq i}[\gamma_s(x) - \beta_j(x)].$$

The coefficient of the term of lowest order in this fractional power series is a nonzero polynomial in s. Let $S(i, a)$ be the set of real zeros of this polynomial. It is clear by definition that $s = 0$ has to be a root of this polynomial. □

Definition 3.7. Let β_i be an x-root of f_y and let β_j^* be a y-root of f_y.

We say that a fractional power series γ is an *a-perturbation of* β_i if $\gamma(x) = \beta_i(x) + sx^a + \alpha(x)$ and α is a convergent fractional power series with $O(\alpha) > a$. We say that γ is a *generic* a-perturbation of β_i if $s \notin S(i, a)$ and either $a \neq O(\beta_i)$ or $O(\gamma) = O(\beta_i)$.

We say that a fractional power series σ^* is an *a-perturbation of* β_j^* if $\sigma^*(y) = \beta_j^*(y)+ty^a+\alpha(y)$ and α is a convergent fractional power series with $O(\alpha) > a$. We say that σ^* is a *generic a*-perturbation of β_j^* if $t \notin T(j, a)$ and either $a \neq O(\beta_j^*)$ or $O(\sigma^*) = O(\beta_j^*)$.

Lemma 3.8. *Let $a = O(\beta_j)$ and let γ be a generic a-perturbation of β_j. Then γ^* is a generic $\frac{1}{a}$-perturbation of β_j^*.*

Proof. Assume $\beta_j(x) = cx^a + \beta(x)$ where $O(\beta) > a$. Let $\gamma_s(x) = \beta_j(x) + sx^a + \alpha(x)$. Then $\gamma(x) = \gamma_{\tilde{s}}(x)$ for some $\tilde{s} \notin S(j, a)$. Since γ is a generic a-perturbation of β_j, $\tilde{s} \neq -c$. Therefore $\gamma(x) = (c + \tilde{s})x^a + \beta(x) + \alpha(x)$ is of order a. It follows that

$$\beta_j^*(y) = \frac{1}{c^{1/a}}y^{1/a} + \bar{\beta}(y)$$

and

$$\gamma^*(y) = \frac{1}{(c + \tilde{s})^{1/a}}y^{1/a} + \bar{\alpha}(y).$$

Since $S(j, a)$ is finite and $\tilde{s} \notin S(j, a)$, $\gamma_s(x)$ is generic for s in some small interval I containing $\tilde{s}$ and such that $-c \notin I$. Therefore, $O_x(f_y(x, \gamma_s(x)))$ is constant for $s \in I$, and hence,

$$O_y(f_y(\gamma_s^*(y), y)) = \frac{1}{a}O_x(f_y(x, \gamma_s(x)))$$

is constant for $s \in I$. Since $T(j, \frac{1}{a})$ is finite , this means that $1/(c + \tilde{s})^{1/a} \notin T(j, \frac{1}{a})$. It follows that $\gamma^*(y)$ is a generic $\frac{1}{a}$-perturbation of β_j^*. □

3.3. **Width of $H_{\epsilon,\rho}(\omega)$.** To obtain the necessary estimates of the next section, it is of great importance to know more about how large $H_{\epsilon,\rho}(\omega)$ is and, in some sense, how well separated the components of $H_{\epsilon,\rho}(\omega)$ are.

For every $j = 1, \ldots, s-1$, the map $y \mapsto |f_y(x,y)|$ has a local maximum $\gamma_j(x) \in (\beta_j(x), \beta_{j+1}(x))$. The $\gamma_j(x)$ have to lie in the open intervals because, by (i), $|f_{yy}(p)| > 0$ for all $p \in \Sigma(\omega) \setminus \{0\}$. The functions γ_j have to be Newton-Puiseux roots of f_{yy}, and are therefore convergent fractional power series in x with real coefficients.

For a convergent real fractional power series β, we denote by $G(\beta)$ the set $\{(x, \beta(x))|x > 0\}$ and by $G^*(\beta^*)$ the set $\{(\beta^*(y), y)|y > 0\}$.

Lemma 3.9. *If $i > j$, $a = O(\beta_i - \beta_j)$ and $O(\beta_i) = O(\beta_j)$, then for every generic a-perturbation β of β_i and β_j, there are $\epsilon > 0, \rho > 0$ such that $G(\beta) \cap H_{\epsilon,\rho}(\omega) = \emptyset$.*

Proof. There is a root γ of f_{yy} with $\beta_j(x) < \gamma(x) < \beta_i(x)$. Since $O(\beta_i) = O(\beta_j)$, γ has to be an a-perturbation of β_i and β_j. Łojasiewicz inequality (i) implies that $|f_y(x, \gamma(x))| \gtrsim \|(x, \gamma(x))\|^{r-1}$. Since β is a generic a-perturbation, it follows that $O(f_y(x, \beta(x))) \leq O(f_y(x, \gamma(x)))$. We also have $O(\beta) = O(\beta_i) = O(\beta_j) = O(\gamma)$, and hence, $\|(x, \gamma(x))\| \sim \|(x, \beta(x))\|$. Altogether this shows that $|f_y(x, \beta(x))| \gtrsim \|(x, \beta(x))\|^{r-1}$, and the conclusion follows. □

Lemma 3.10. *Let $b = O(\beta_i^*)$, and let β^* be a generic b-perturbation of β_i^*. Then, for small enough $\epsilon, \rho > 0$, $G^*(\beta^*) \cap H_{\epsilon,\rho}(\omega) = \emptyset$.*

Proof. The fractional power series $\gamma^*(y) = 0$ is a b-perturbation of β_i^*, and from Lemma 3.3 we know that $|f_y(\gamma^*(y), y)| \gtrsim \|(\gamma^*(y), y)\|^{r-1}$. Since β^* is a generic b-perturbation of β_i^*, $O_y(f_y(\beta^*(y), y)) \leq O_y(f_y(\gamma^*(y), y))$, and since we also have $\|(\beta^*(y), y)\| \sim \|(\gamma^*(y), y)\| \sim y$, the lemma follows. □

Lemma 3.11. *Let $a = O(\beta_i)$, and let β be a generic a-perturbation of β_i. Then there are $\epsilon > 0, \rho > 0$ such that $G(\beta) \cap H_{\epsilon,\rho}(\omega) = \emptyset$.*

Proof. Using Lemma 3.8, we see that β^* is a generic $\frac{1}{a}$-perturbation of β_i^*, and by Lemma 3.10, for small $\epsilon > 0, \rho > 0$ we have $G^*(\beta^*) \cap H_{\epsilon,\rho}(\omega) = G(\beta) \cap H_{\epsilon,\rho}(\omega) = \emptyset$. □

Lemma 3.12. *Let ϵ_n, ρ_n be sequences of real numbers such that $\epsilon_n \to 0$ and $\rho_n \to 0$ and let $p_n = (x_n, y_n)$ and $q_n = (u_n, v_n)$ be in $H_{\epsilon_n,\rho_n}(\omega)$. If $u_n < 0 < x_n$, then*

$$\|\omega(p_n) - \omega(q_n)\| \gtrsim \|p_n\|^{r-1} + \|q_n\|^{r-1}.$$

Proof. We claim that $x_n \gtrsim \|p_n\|^{r-1}$ and $|u_n| \gtrsim \|q_n\|^{r-1}$. Any branch of $\Sigma(\omega)$ may be parametrized by some convergent fractional power series $\beta(x)$ which by Lemma 3.2 must satisfy $O(\beta) \geq \frac{1}{r-1}$. By Lemma 3.11 there is a generic $O(\beta_s)$-perturbation $\tilde{\beta}$ of β_s such that $\tilde{\beta}(x) > \beta_s(x)$. By Lemma 3.2, $O(\tilde{\beta}) = O(\beta_s) \geq \frac{1}{r-1}$ and this shows that $y_n < \tilde{\beta}(x_n) < \delta x_n^{\frac{1}{r-1}}$ for some $\delta > 0$. Consider $\omega_\Phi = \omega \circ \Phi$ where $\Phi(x, y) = (x, -y)$. From Lemma 2.8 we know that (i) holds for ω_Φ, and it is clear that $H_{\epsilon,\rho}(\omega) = H_{\epsilon,\rho}(\omega_\Phi)$. It is also obvious that the branches of $\Sigma(\omega_\Phi)$ in the first quadrant correspond to the branches of $\Sigma(\omega)$ in the fourth quadrant. A similar analysis of ω_Φ as the above analysis of ω will show that $-\delta x_n^{\frac{1}{r-1}} < y_n$. This shows that $x_n \gtrsim \|p_n\|^{r-1}$. Let $\Psi(x, y) = (-x, y)$. A similar analysis of $\omega \circ \Psi$ shows that $|u_n| \gtrsim \|q_n\|^{r-1}$. Altogether we get

$$\|\omega(p_n) - \omega(q_n)\| \geq |x_n - u_n| \gtrsim \|p_n\|^{r-1} + \|q_n\|^{r-1}.$$

□

3.4. **Preliminary estimates.** The proof of Theorem 1.7 depends on a number of estimates. The actual proofs of those estimates are a bit lengthy and quite delicate, so we include them here in a separate section.

3.4.1. *The first quadrant.* For $i = 1, \dots, s$, let $H^i_{\epsilon,\rho}(\omega)$ be the component of $H_{\epsilon,\rho}(\omega)$ containing $G(\beta_i) \cap H_{\epsilon,\rho}(\omega)$. Let $\epsilon_n, \tilde{\epsilon}_n, \rho_n$ and $\tilde{\rho}_n$ be sequences of positive real numbers converging to 0. Let $1 \le j < i \le s$ and let $p_n = (x_n, y_n) \in H^i_{\epsilon_n,\rho_n}(\omega)$ and $q_n = (u_n, v_n) \in H^j_{\epsilon_n,\rho_n}(\omega)$ be two sequences. We assume that (II) fails along these sequences, that is,

$$\|\omega(p_n) - \omega(q_n)\| = o(\|p_n\|^{r-1} + \|q_n\|^{r-1}) \, \|p_n - q_n\| . \tag{3.3}$$

Let $\tilde{p}_n = (x_n, \tilde{y}_n) \in H^i_{\tilde{\epsilon}_n,\tilde{\rho}_n}(\omega)$ and $\tilde{q}_n = (u_n, \tilde{v}_n) \in H^j_{\tilde{\epsilon}_n,\tilde{\rho}_n}(\omega)$. We want to see that

$$\|\omega(\tilde{p}_n) - \omega(\tilde{q}_n)\| = o(\|\tilde{p}_n\|^{r-1} + \|\tilde{q}_n\|^{r-1}) \, \|\tilde{p}_n - \tilde{q}_n\| .$$

To this end we need to show that

(1) $\|\tilde{p}_n\| = \|p_n\| + o(\|p_n\|)$
(2) $\|\tilde{q}_n\| = \|q_n\| + o(\|q_n\|)$
(3) $\|p_n - q_n\| = \|\tilde{p}_n - \tilde{q}_n\| + o(\|p_n - q_n\|)$
(4) $\|p_n - \tilde{p}_n\| = o(\|p_n - q_n\|)$
(5) $\|q_n - \tilde{q}_n\| = o(\|p_n - q_n\|)$.

We have assumed that $\beta_i(x) > \beta_j(x)$. Let $\delta > 0$ be a small number. We claim that there are generic $O(\beta_i - \beta_j)$-perturbations $\underline{\beta}_i$ and $\overline{\beta}_i$ of β_i and generic a-perturbations $\underline{\beta}_j$ and $\overline{\beta}_j$ of β_j where $a = O(\beta_i - \beta_j)$ if $O(\beta_i) = O(\beta_j)$ and $a = O(\beta_j)$ if $O(\beta_j) > O(\beta_i)$, such that for small x,

$$\underline{\beta}_j(x) < \beta_j(x) < \overline{\beta}_j(x) < \underline{\beta}_i(x) < \beta_i(x) < \overline{\beta}_i(x), \tag{3.4}$$

$$\overline{\beta}_j(x) - \underline{\beta}_j(x) < \delta(\underline{\beta}_i(x) - \overline{\beta}_j(x)) \tag{3.5}$$

and

$$\overline{\beta}_i(x) - \underline{\beta}_i(x) < \delta(\underline{\beta}_i(x) - \overline{\beta}_j(x)). \tag{3.6}$$

To justify the claim, assume first that $O(\beta_i) = O(\beta_j)$ and let $\gamma_t(x) = t\beta_i(x) + (1-t)\beta_j(x)$. Let

$$\begin{aligned} \overline{\beta}_i(x) &= \gamma_{1+\epsilon}(x) \\ \underline{\beta}_i(x) &= \gamma_{1-\epsilon}(x) \\ \overline{\beta}_j(x) &= \gamma_{\epsilon}(x) \\ \underline{\beta}_j(x) &= \gamma_{-\epsilon}(x). \end{aligned}$$

All these fractional power series are generic $O(\beta_i - \beta_j)$ perturbations of β_i and β_j for all but finitely many choices of ϵ. We compute

$$\overline{\beta}_i - \underline{\beta}_i = \overline{\beta}_j - \underline{\beta}_j = \frac{2\epsilon}{1-2\epsilon}(\underline{\beta}_i - \overline{\beta}_j).$$

The claim follows in this case if we choose $\epsilon < \min\{\frac{1}{4}, \frac{\delta}{4}\}$. If $O(\beta_i) < O(\beta_j)$, then we choose $\overline{\beta}_i$ and $\underline{\beta}_i$ as before, but we choose

$$\begin{aligned} \overline{\beta}_j(x) &= (1+\epsilon)\beta_j(x) \\ \underline{\beta}_j(x) &= (1-\epsilon)\beta_j(x). \end{aligned}$$

Again, for all but finitely many ϵ, these fractional power series are generic $O(\beta_j)$-perturbations of β_j and we compute

$$\begin{aligned}\overline{\beta}_i - \underline{\beta}_i &= 2\epsilon(\beta_i - \beta_j)\\ \overline{\beta}_j - \underline{\beta}_j &= 2\epsilon\beta_j\\ \underline{\beta}_i - \overline{\beta}_j &= (1-\epsilon)\beta_i - \beta_j.\end{aligned}$$

Since $O(\beta_i) < O(\beta_j)$, $\beta_j(x) < \frac{1}{2}\beta_i(x)$ for small x. So for small x,

$$\begin{aligned}\overline{\beta}_i(x) - \underline{\beta}_i(x) &< \frac{2\epsilon}{\frac{1}{2}-\epsilon}(\underline{\beta}_i(x) - \overline{\beta}_j(x))\\ \overline{\beta}_j(x) - \underline{\beta}_j(x) &< \frac{2\epsilon}{\frac{1}{2}-\epsilon}(\underline{\beta}_i(x) - \overline{\beta}_j(x))\end{aligned}$$

and the claim follows from choosing $\epsilon < \min\{\frac{1}{4}, \frac{\delta}{8}\}$.

Lemma 3.13. *There are $\epsilon > 0$ and $\rho > 0$ such that $H^i_{\epsilon,\rho} \cup H^j_{\epsilon,\rho} \subset \{(x,y) \mid \underline{\beta}_i(x) < y < \overline{\beta}_i(x)$ or $\underline{\beta}_j(x) < y < \overline{\beta}_j(x)\}$.*

Proof. It is enough to check that

$$(G(\underline{\beta}_j) \cup G(\overline{\beta}_j) \cup G(\underline{\beta}_i) \cup G(\overline{\beta}_i)) \cap (H^i_{\epsilon,\rho} \cup H^j_{\epsilon,\rho}) = \emptyset.$$

This follows directly from Lemma 3.9 and Lemma 3.11. □

Estimates (1) and (2) above can be shown by the same argument. To show (1), let $\delta > 0$ be arbitrary and notice that by Lemma 3.13 and (3.5), there is an N such that $|\,\|p_n\| - \|\tilde{p}_n\|\,| \leq |\,\|p_n - \tilde{p}_n\|\,| = |y_n - \tilde{y}_n| < |\overline{\beta}_i(x_n) - \underline{\beta}_i(x_n)| \leq \delta(\underline{\beta}_i(x_n) - \overline{\beta}_j(x_n)) < \delta\underline{\beta}_i(x_n) < \delta y_n$ for all $n > N$. Estimate (1) follows since $\|p_n\| \sim y_n$. To justify (3), (4) and (5) we introduce a pair of new sequences which help clarify the geometry of the situation. Let ϵ and ρ be given by Lemma 3.13. Let n be so large that ϵ_n and $\tilde{\epsilon}_n$ are less than ϵ and ρ_n and $\tilde{\rho}_n$ are less than ρ. Let $\bar{p}_n = (u_n, \underline{\beta}_i(u_n))$ and $\bar{q}_n = (x_n, \overline{\beta}_j(x_n))$. One possible configuration of these sequences is illustrated in Figure 2.

We have

$$\begin{aligned}\|\tilde{p}_n - \tilde{q}_n\| &\geq \|p_n - q_n\| - \|p_n - \tilde{p}_n\| - \|q_n - \tilde{q}_n\|\\ &\geq \|p_n - q_n\| - \delta\,\|p_n - \bar{q}_n\| - \delta\,\|q_n - \bar{p}_n\|\,.\end{aligned}$$

We consider the cases $x_n > u_n$ and $x_n \leq u_n$ separately. If $x_n > u_n$, then both $\|p_n - \bar{q}_n\|$ and $\|\bar{p}_n - q_n\|$ are less than or equal to $\|p_n - q_n\|$. In this case, $\|\tilde{p}_n - \tilde{q}_n\| \geq (1-2\delta)\,\|p_n - q_n\|$.

Next is the case $x_n \leq u_n$. If there is a $K > 0$ such that

$$\frac{q_n - \bar{q}_n}{\|q_n - \bar{q}_n\|} \cdot (1,0) > K,$$

then $\|q_n - \bar{q}_n\| < |x_n - u_n|/K = o(\|p_n - q_n\|)$. The last inequality follows from (3.3). If

$$\frac{q_n - \bar{q}_n}{\|q_n - \bar{q}_n\|} \cdot (1,0) \to 0 \text{ as } n \to \infty,$$

then we may assume that either $v_n < \overline{\beta}_j(x_n)$ for all n or that $v_n > \overline{\beta}_j(x_n)$ for all n by passing to a subsequence. If $v_n \leq \overline{\beta}_j(x_n)$, then $\|p_n - \bar{q}_n\| \leq \|p_n - q_n\|$. Now, assume that $v_n > \overline{\beta}_j(x_n)$. In this case, $O(\overline{\beta_j}) < 1$. To see this, let θ_n be the angle between $q_n - \bar{q}_n$ and $(1,0)$. If $O(\overline{\beta_j}) = 1$, then $|\tan\theta_n| \leq 2\overline{\beta_j}'(x_n) < 2M$ for a bound M on $\overline{\beta_j}'$. It follows that $\cos\theta_n$ is bounded away from 0, and that $(q_n - \bar{q}_n)\cdot(1,0)/\,\|q_n - \bar{q}_n\|$ does not converge to 0, contrary to our current assumptions. Therefore $O(\overline{\beta_j}) < 1$. Lemma 3.2 also implies that $\beta^*_k(y) \gtrsim y^{r-1}$ for $k = i,j$. This implies

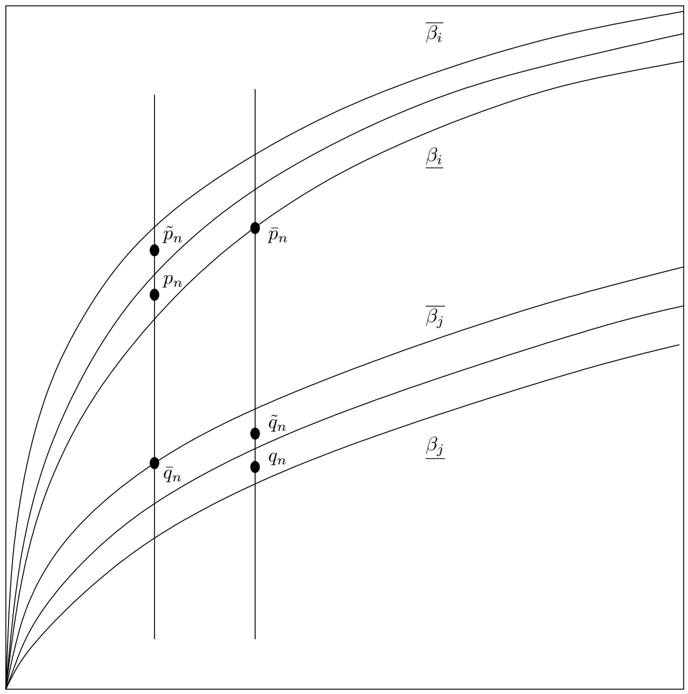

FIGURE 2. Example of a possible configuration of points when $x_n < u_n$.

that $\beta_k(x) \lesssim x^{1/(r-1)}$ and $\beta_k'(x) \lesssim x^{-(r-2)/(r-1)}$ for $k = i, j$. Since $O(\overline{\beta}_j) = O(\beta_j)$, similar inequalities must hold for $\overline{\beta}_j^*$ and $\overline{\beta}_j$ as well. We also claim that $\|q_n\| / \|p_n\|$ is bounded. Assume this is not the case. Then, by passing to a subsequence, we may assume that $\|p_n\| = o(\|q_n\|)$ for large n. Then $y_n = o(v_n)$ for large n, but by Lemma 3.2 again, this implies

$$|x_n - u_n| > |\underline{\beta_i}^*(y_n) - \overline{\beta_j}^*(v_n)| > |\overline{\beta_j}^*(y_n) - \overline{\beta_j}^*(v_n)| \gtrsim v_n^{r-1} \sim \|q_n\|^{r-1}$$

which is false, because, since (II) fails,

$$|x_n - u_n| = o(\|p_n\|^{r-1} + \|q_n\|^{r-1}) \|p_n - q_n\|.$$

This proves the claim. Using these observations, we see that

$$\begin{aligned}
\|q_n - \bar{q}_n\| &\leq |x_n - u_n|(\overline{\beta}_j'(x_n) + 1) \\
&\lesssim |x_n - u_n| \frac{1}{x_n^{\frac{r-2}{r-1}}} && (\text{since } O(\overline{\beta}_j') = O(\beta_j') \geq -\tfrac{r-2}{r-1}) \\
&\lesssim |x_n - u_n| \frac{1}{\|p_n\|^{r-2}} && (\text{since } \|p_n\| \sim y_n \sim \beta_i(x_n) \lesssim x_n^{\frac{1}{r-1}}) \\
&= \frac{o(\|p_n\|^{r-1} + \|q_n\|^{r-1}) \|p_n - q_n\|}{\|p_n\|^{r-2}} && (\text{by } (3.3)) \\
&= o(\|p_n - q_n\|). && (\text{since } \|q_n\| / \|p_n\| \text{ is bounded})
\end{aligned}$$

We conclude that

$$\|p_n - \bar{q}_n\| \leq \|p_n - q_n\| + \|q_n - \bar{q}_n\| = \|p_n - q_n\| + o(\|p_n - q_n\|).$$

Completely analogous arguments show that $\|q_n - \bar{p}_n\| \leq \|p_n - q_n\| + o(\|p_n - q_n\|)$. Altogether we have

$$\|\tilde{p}_n - \tilde{q}_n\| \geq \|p_n - q_n\| - \delta \|p_n - \bar{q}_n\| - \delta \|q_n - \bar{p}_n\| \geq (1 - 3\delta) \|p_n - q_n\|.$$

To finish the justification of (3), let δ_k be a sequence of positive real numbers converging to 0. By the above, for each k there is a natural number $N(k)$ such that $\|\tilde{p}_n - \tilde{q}_n\| \geq (1 - 3\delta_k) \|p_n - q_n\|$ when $n > N(k)$. Since $\delta_k \to 0$, (3) follows. To justify (4), notice that there is a natural number $M(k)$ such that $\|p_n - \tilde{p}_n\| < \delta_k \|p_n - \bar{q}_n\| \leq \delta_k(\|p_n - q_n\| + o(\|p_n - q_n\|))$ when $n > M(k)$. This clearly implies (4), and (5) follows by similar arguments.

3.4.2. *The other quadrants.* Let $\Phi(x, y) = (x, -y)$ and let $\omega_\Phi = \omega \circ \Phi$. By Lemma 2.2 and Lemma 2.7, (i) holds for ω_Φ. Hence, we may parametrize the components of $\Sigma(\omega_\Phi)$ in the first quadrant by Newton-Puiseux roots $\beta_{\Phi,i}, i = 1, \ldots, s_\Phi$ and the analysis of Section 3.4.1 holds for ω_Φ as well.

The fractional power series $-\beta_{\Phi,i}, i = 1, \ldots, s_\Phi$ parametrize the components of $\Sigma(\omega)$ contained in the fourth quadrant, and also, $H_{\epsilon,\rho}(\omega_\Phi) = \Phi(H_{\epsilon,\rho}(\omega))$. Hence, if we instead of β_i and β_j consider $-\beta_{\Phi,i}$ and $-\beta_{\Phi,j}$ in the discussion of Section 3.4.1, we get the same estimates (1)-(5). If we instead of β_i and β_j consider $-\beta_{\Phi,i}$ and β_j, we also obtain (1)-(5) after a minor modification of the justification of (3)-(5). In the latter case the corresponding branches of $\Sigma(\omega)$ have different tangent directions.

To study $H_{\epsilon,\rho}(\omega)$ in the second and third quadrant, let $\Psi(x, y) = (-x, y)$, and study the r-jet $\omega_\Psi = \omega \circ \Psi$. The components of $H_{\epsilon,\rho}(\omega_\Psi)$ contained in the first and fourth quadrant can be studied in the manner explained above, and since $H_{\epsilon,\rho}(\omega_\Psi) = \Psi(H_{\epsilon,\rho}(\omega))$, this gives the estimates (1)-(5) when we consider parametrizations of components of $\Sigma(\omega)$ in the second and/or third quadrant instead of β_i and β_j.

Since, by Lemma 3.12, (II) only fails along pairs of sequences on the same side of the y-axis, this establishes our estimates in all possible cases.

4. Relations between the Łojasiewicz inequalities

Le ω be an r-jet of rank 1 such that (I) holds. Let $\{C_i\}$ be the components of $\Sigma(\omega) \setminus \{0\}$. Recall the second Lojasiewicz inequality of Theorem 1.10:

There is a constant $C > 0$ and a neighbourhood U of 0 such that if $p \in C_i \cap U$ and $q \in C_j \cap U$ for some $i \neq j$, then

$$\|\omega(p) - \omega(q)\| \geq C(\|p\|^{r-1} + \|q\|^{r-1}) \|p - q\| \tag{II'}$$

Proposition 4.1. *If ω is of rank 1 and in standard form, then (II) holds for ω iff (II') holds.*

Proof. (II) $\Rightarrow$ (II') is obvious, (II') being a weakening of (II). We assume $\omega(x, y) = (x, f(x, y))$ and proceed to show that (II') $\Rightarrow$ (II). If (II) fails, then there are $i \neq j$ and sequences ϵ_n and ρ_n of positive real numbers converging to 0 and sequences $p_n = (x_n, y_n) \in H^i_{\epsilon_n,\rho_n}$ and $q_n = (u_n, v_n) \in H^j_{\epsilon_n,\rho_n}$. Then we have $f_y(p_n) = o(\|p_n\|^{r-1})$, $f_y(q_n) = o(\|q_n\|^{r-1})$ and

$$\|\omega(p_n) - \omega(q_n)\| = o(\|p_n\|^{r-1} + \|q_n\|^{r-1}) \|p_n - q_n\|. \tag{4.1}$$

Let $\tilde{p}_n$ and $\tilde{q}_n$ be the points on C_i and C_j having the same x-component as p_n and q_n respectively. These points exist by Lemma 3.2. By the estimates (3)-(5) of the previous section we have

$$\|p_n - q_n\| \sim \|\tilde{p}_n - \tilde{q}_n\| \tag{4.2}$$

and

$$\|p_n - \tilde{p}_n\| = o(\|p_n - q_n\|) \tag{4.3}$$

and

$$\|q_n - \tilde{q}_n\| = o(\|p_n - q_n\|). \tag{4.4}$$

As remarked in Section 3.4.2, (1)-(5) hold regardless of whether (p_n) and (q_n) are in the same quadrant or not. By (1) and (2), $\|p_n\| \sim \|\tilde{p}_n\|$ and $\|q_n\| \sim \|\tilde{q}_n\|$. Let $\epsilon < \epsilon_0$ where ϵ_0 is given by Proposition 1.1. Assume that n is so large that $\epsilon_n < \epsilon$. Then $p_n, \tilde{p}_n \in H^i_\epsilon$ and since H^i_ϵ is semialgebraic and connected, the line segment between p_n and $\tilde{p}_n$ must be contained in H^i_ϵ. If b_n is a sequence such that for every n, b_n lies on the line segment between p_n and $\tilde{p}_n$ or on the line segment between q_n and $\tilde{q}_n$, then $\|b_n\| \sim \|p_n\|$ or $\|b_n\| \sim \|q_n\|$, and since (I), and therefore (i) holds, we must have $|f_{yy}(x_n, y)| > 0$ on the open line segment between p_n and $\tilde{p}_n$. It follows that $|f_y(b_n)| < |f_y(p_n)| = o(\|p_n\|^{r-1}) = o(\|b_n\|^{r-1})$. In a similar fashion we obtain similar inequalities for points on the line segment between q_n and $\tilde{q}_n$. Now, using the Mean Value Theorem, we can find c_n on the line segment between p_n and $\tilde{p}_n$ and d_n on the line segment between q_n and $\tilde{q}_n$ such that

$$\begin{aligned}
\|\omega(\tilde{p}_n) - \omega(\tilde{q}_n)\| &\leq \|\omega(\tilde{p}_n) - \omega(p_n)\| + \|\omega(p_n) - \omega(q_n)\| \\
&\quad + \|\omega(q_n) - \omega(\tilde{q}_n)\| \\
&= |f_y(c_n)| \, \|\tilde{p}_n - p_n\| + o(\|p_n\|^{r-1} + \|q_n\|^{r-1}) \, \|p_n - q_n\| \\
&\quad + |f_y(d_n)| \, \|q_n - \tilde{q}_n\| \\
&= o(\|p_n\|^{r-1}) o(\|p_n - q_n\|) + o(\|p_n\|^{r-1} + \|q_n\|^{r-1}) \, \|p_n - q_n\| \\
&\quad + o(\|q_n\|^{r-1}) o(\|p_n - q_n\|) \\
&= o(\|\tilde{p}_n\|^{r-1} + \|\tilde{q}_n\|^{r-1}) \, \|\tilde{p}_n - \tilde{q}_n\| \, .
\end{aligned}$$

This shows that (II') fails. □

Lemma 4.2 (=Lemma 2.9)**.** *Let ω be an r-jet which satisfies (I), and let ω' be a C^r map germ with $\omega \sim_{\mathcal{A}_r} \omega'$. Then (II') holds for ω if and only if (II') holds for $j^r\omega'$.*

Proof. Let ω be an r-jet, h and k C^r-diffeomorphisms of neighbourhoods of 0 and $\omega' = k \circ \omega \circ h^{-1}$. We may assume that ω is in standard form. Assume that (I) holds for ω and that (II'), and hence (II), fails for ω along sequences in H_i and H_j which are different components of $H_{\epsilon,\rho}(\omega)$. Let C_i and C_j be the branches of $\Sigma(\omega)$ corresponding to H_i and H_j respectively. From the proof of Lemma 2.6 we know that in a small neighbourhood of 0, $h(C_i)$ and $h(C_j)$ are in different components of $H_{\epsilon,\rho}(j^r\omega')$. Let C'_i and C'_j denote the components of $\Sigma(j^r\omega')$ contained in the same components of $H_{\epsilon,\rho}(j^r\omega')$ as $h(C_i)$ and $h(C_j)$ respectively. Since $h^{-1}(C'_i)$ and $h^{-1}(C'_j)$ belong to the singular set of $k^{-1} \circ j^r\omega' \circ h$, which is a C^r realization of ω, $h^{-1}(C'_i)$ and $h^{-1}(C'_j)$ belong to $H^i_{\epsilon,\rho}(\omega)$ and $H^j_{\epsilon,\rho}(\omega)$ for every small ϵ. It now follows from the proof of Proposition 4.1 that (II) fails for ω along sequences in $h^{-1}(C'_i)$ and $h^{-1}(C'_j)$. Then it follows from the proof of Lemma 2.6 again that (II) fails for $j^r\omega'$ along sequences in C'_i and C'_j. This shows that (II') fails for $j^r\omega'$ and finishes the proof of the lemma. □

Proposition 4.3 (=Proposition 2.10)**.** *If ω is in standard form and satisfies (i), then (II') and (ii) are equivalent.*

Proof. (II')$\Rightarrow$ (ii) is obvious, (ii) being a weakening of (II'). Assume that ω is in standard form and satisfies (i), but not (II'). Since (i) is satisfied, Lemma 3.12 implies that (II') fails along sequences on the same side of the y-axis. Assume they are in the 1st or 4th quadrant. Note that Lemma 3.4 also holds for singular branches in the 4th quadrant, and by arguments similar to the arguments in Section 3.4.2, we may parametrize the branches of $\Sigma(\omega)$ in the 4th quadrant by convergent fractional power series. Let now β_i, $i = 1, \ldots, S$ be parametrizations of the S branches of $\Sigma(\omega)$ in these quadrants. Then there are $i \neq j$ and sequences $p_n = (x_n, y_n)$ and $q_n = (u_n, v_n)$ both converging to 0 such that $p_n \in G(\beta_i)$, $q_n \in G(\beta_j)$ and

$$\|\omega(p_n) - \omega(q_n)\| = o(\|p_n\|^{r-1} + \|q_n\|^{r-1}) \|p_n - q_n\| .$$

We may assume that $x_n > u_n > 0$ Let $\tilde{v}_n = \beta_j(x_n)$. Then $\tilde{q}_n = (x_n, \tilde{v}_n) \in G(\beta_j)$. Let $\beta(t) = (\beta^1(t), \beta^2(t))$ be the parametrization of $G(\beta_j)$ by arclength with $\beta(0) = 0$ and $\beta(t) \in G(\beta_j)$ for $t > 0$. Assume that $(u_n, v_n) = \beta(t_{u_n})$ and $(x_n, \tilde{v}_n) = \beta(t_{x_n})$. Then there are parameter values c_n and d_n between t_{u_n} and t_{x_n} such that

$$\begin{aligned}
\|\omega(u_n, v_n) - \omega(x_n, \tilde{v}_n)\| &= \|\omega(\beta(t_{u_n})) - \omega(\beta(t_{x_n}))\| \\
&= \left\| \begin{pmatrix} \beta^1(t_{u_n}) - \beta^1(t_{x_n}) \\ Df(\beta(c_n)) \cdot \beta'(c_n)(t_{u_n} - t_{x_n}) \end{pmatrix} \right\| \\
&= \left\| \begin{pmatrix} \frac{d}{dt}\beta^1(d_n) \\ f_x(\beta(c_n))\frac{d}{dt}\beta^1(c_n) \end{pmatrix} \right\| |t_{u_n} - t_{x_n}| \quad (\text{since } f_y(\beta(t)) \equiv 0) \\
&\lesssim \max\left\{ \left|\frac{d}{dt}\beta^1(c_n)\right|, \left|\frac{d}{dt}\beta^1(d_n)\right| \right\} |t_{u_n} - t_{x_n}|.
\end{aligned}$$

If $O(\beta_j) = 1$, then $t \sim \|\beta(t)\| \sim |\beta^1(t)|$, and in that case,

$$\|\omega(u_n, v_n) - \omega(x_n, \tilde{v}_n)\| \lesssim |x_n - u_n|.$$

If $O(\beta_j) < 1$, then

$$\left|\frac{d}{dt}\beta^1(c_n)\right| \sim \left|\frac{\frac{d}{dt}\beta^1(c_n)}{\frac{d}{dt}\beta^2(c_n)}\right|,$$

since β is parametrised by arclength. Since we have assumed that $x_n > u_n$, we have $t_{x_n} > t_{u_n}$. Then

$$\left|\frac{\frac{d}{dt}\beta^1(c_n)}{\frac{d}{dt}\beta^2(c_n)}\right| < |\beta_j'(x_n)|^{-1}.$$

Now, since $O(\beta_j) < 1$, there is a small $\epsilon > 0$ such that $|\beta_j(x)|$ is a concave function on $[0, \epsilon)$. This implies that for large enough n,

$$|\beta_j'(x_n)| < \left|\frac{v_n - \tilde{v}_n}{x_n - u_n}\right| < \frac{|\beta_j(x_n)|}{|x_n|}.$$

But since β_j is a fractional power series in x, $|\beta_j(x_n)| \sim |x_n||\beta_j'(x_n)|$. Thus

$$|\beta_j'(x_n)| \sim \left|\frac{v_n - \tilde{v}_n}{x_n - u_n}\right|,$$

and hence,

$$\left|\frac{d}{dt}\beta^1(c_n)(t_{u_n} - t_{x_n})\right| \lesssim \left|\frac{x_n - u_n}{v_n - \tilde{v}_n}\right| |v_n - \tilde{v}_n| = |x_n - u_n|.$$

The same holds if we replace c_n with d_n. In any case,

$$\|\omega(u_n, v_n) - \omega(x_n, \tilde{v}_n)\| \lesssim |x_n - u_n| = o(\|p_n\|^{r-1} + \|q_n\|^{r-1}) \|p_n - q_n\| .$$

Using this we get

$$\begin{aligned} \|\omega(p_n) - \omega(\tilde{q}_n)\| &\leq \|\omega(p_n) - \omega(q_n)\| + \|\omega(q_n) - \omega(\tilde{q}_n)\| \\ &= o(\|p_n\|^{r-1} + \|q_n\|^{r-1}) \|p_n - q_n\| \\ &= o(\|p_n\|^{r-1} + \|\tilde{q}_n\|^{r-1}) \|p_n - \tilde{q}_n\| , \end{aligned}$$

which means that (ii) fails to hold. The last equality needs some justification. Notice that $u_n < x_n$ implies that $\|q_n\| < \|\tilde{q}_n\|$. We also have to show that $\|p_n - q_n\| \lesssim \|p_n - \tilde{q}_n\|$. We claim that $u_n = x_n + o(x_n)$. If not, then $|x_n - u_n| = x_n - u_n \sim x_n$. By Lemma 3.2, $x_n \gtrsim \|p_n\|^{r-1}$. This implies that $|x_n - u_n| \gtrsim \|p_n\|^{r-1}$ which contradicts the failure of (II'). Therefore, we may assume that $u_n = x_n + o(x_n)$. This gives $|\beta_j(u_n)| \sim |\beta_j(x_n)|$ and hence, $\|q_n\| \sim \|\tilde{q}_n\|$. Assume that $\|q_n\| = o(\|p_n\|)$. In this case, $\|\tilde{q}_n\| = o(\|p_n\|)$ and it follows that $\|p_n\| \sim \|p_n - q_n\| \sim \|p_n - \tilde{q}_n\|$. Assume now that $\|p_n\| \lesssim \|q_n\|$. We have

$$\|p_n - q_n\| \leq \|p_n - \tilde{q}_n\| + |\beta_j(x_n) - \beta_j(u_n)| + |x_n - u_n|.$$

Using that $|\beta_j(x_n) - \beta_j(u_n)| \leq (|\beta_j'(x_n)| + |\beta_j'(u_n)|)|x_n - u_n|$, we get

$$\|p_n - q_n\| \leq \|p_n - \tilde{q}_n\| + (|\beta_j'(x_n)| + |\beta_j'(u_n)| + 1)|x_n - u_n|.$$

As in the justification of (3) in Section 3.4.1, Lemma 3.2 implies that $|\beta_j'(x_n)| \lesssim x_n^{-\frac{r-2}{r-1}} \lesssim 1/\|\tilde{q}_n\|^{r-2} \sim 1/\|q_n\|^{r-2}$ and similarly, $|\beta_j'(u_n)| \lesssim 1/\|q_n\|^{r-2}$. Now we have

$$\begin{aligned} \|p_n - q_n\| &\leq \|p_n - \tilde{q}_n\| + (1 + \frac{2}{\|q_n\|^{r-2}})|x_n - u_n| \\ &= \|p_n - \tilde{q}_n\| + (1 + \frac{2}{\|q_n\|^{r-2}}) o(\|p_n\|^{r-1} + \|q_n\|^{r-1}) \|p_n - q_n\| \\ &= \|p_n - \tilde{q}_n\| + o(\|p_n - q_n\|). \end{aligned}$$

The last equality follows from the assumption that $\|p_n\| \lesssim \|q_n\|$. This completes the proof of Proposition 4.3. □

Proof of Proposition 1.9. This is a direct consequence of Proposition 4.1 and Proposition 4.3. □

5. Construction of Whitney field and proof of Theorem 1.7

This section deals with the construction of a Whitney field which leads to the proof of the only if part of Theorem 1.7. Let $\omega(x, y) = (x, f(x, y))$ be an r-jet of rank 1 in standard form having no branches of its singular set tangent to the x-axis. Assume that (i) holds and (ii) fails for ω. We only consider the case when (ii) fails along sequences in the first quadrant. Then there are sequences $p_n = (x_n, y_n) \in C_i$ and $q_n = (x_n, v_n) \in C_j$ such that $\|p_n\| \to 0$, $\|q_n\| \to 0$ and

$$|f(p_n) - f(q_n)| = o(\|p_n\|^{r-1} + \|q_n\|^{r-1})|y_n - v_n|.$$

In this case, $\|p_n\| \sim y_n$ and $\|q_n\| \sim v_n$. We assume that $y_n > v_n$ and that $\|p_n - q_n\| = o(\|p_n\| + \|q_n\|)$ and thus, $\|p_n\| \sim \|q_n\|$.

Lemma 5.1. *There are sequences of real positive numbers $\tilde{\epsilon}_n$ and $\tilde{\rho}_n$ converging to 0 and sequences $\tilde{p}_n = (x_n, \tilde{y}_n) \in H^i_{\tilde{\epsilon}_n, \tilde{\rho}_n}$ and $\tilde{q}_n = (x_n, \tilde{v}_n) \in H^j_{\tilde{\epsilon}_n, \tilde{\rho}_n}$ such that*

$$f_y(x_n, \tilde{y}_n) = f_y(x_n, \tilde{v}_n) = \frac{f(x_n, \tilde{y}_n) - f(x_n, \tilde{v}_n)}{\tilde{y}_n - \tilde{v}_n} = o(\|\tilde{p}_n\|^{r-1} + \|\tilde{q}_n\|^{r-1}).$$

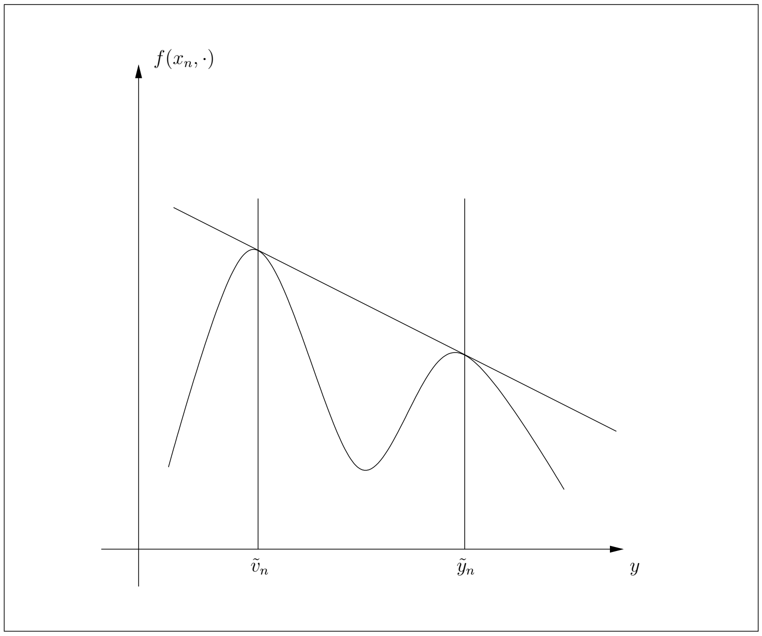

FIGURE 3. Illustration of the geometric idea behind Lemma 5.1.

The points $\tilde{y}_n$ and $\tilde{v}_n$ are chosen such that we obtain the geometric situation illustrated in Figure 3.

Proof of Lemma 5.1. If there are subsequences (p_{n_k}) and (q_{n_k}) of (p_n) and (q_n) respectively such that $f(x_{n_k}, y_{n_k}) = f(x_{n_k}, v_{n_k})$, then, since $f_y(x_n, y_n) = f_y(x_n, v_n) = 0$, we may take $\tilde{y}_n = y_n$ and $\tilde{v}_n = v_n$. If there are no such subsequences, let $p_n(t) = (x_n, y_n + t)$ and $q_n(s) = (x_n, v_n + s)$. Recall that we have assumed that (p_n) and (q_n) are in the first quadrant. We have also assumed that $y_n > v_n$, and hence, $y_n + t > v_n + s$ and $\|p_n(t)\| > \|q_n(s)\|$ for small s and t. In particular, $\|p_n(t)\| > \|q_n(s)\|$ when $p_n(t) \in H^i_{\epsilon,\rho}$ and $q_n(s) \in H^j_{\epsilon,\rho}$. Since (i) holds, there is a constant $C > 0$ such that

$$|f_y(p)| + |f_{yy}(p)| \, \|p\| \geq C \, \|p\|^{r-1}$$

for all p in a neighbourhood $B(0, \rho)$ of 0. Let $\epsilon < C$ and as always, assume that $\epsilon < \epsilon_0$ where ϵ_0 is chosen such that the conclusion of Lemma 3.1 holds. Since $f_{yy}(p) \neq 0$ for all $p \in H_{\epsilon,\rho}$, the restriction of the function $u \mapsto f_y(p + (0, u))$ to any component of the set $\{u | p + (0, u) \in H_{\epsilon,\rho}\}$ is injective. Assume that ρ is large enough to ensure that

$$\sup\{\|p_n(t)\| \mid p_n(t) \in H^i_{\epsilon,\rho}\} < \rho.$$

Since $\|p_n(t)\| > \|q_n(s)\|$, $\{f_y(q_n(s)) \mid q_n(s) \in H^j_{\epsilon,\rho}\} \subset \{f_y(p_n(t)) \mid p_n(t) \in H^i_{\epsilon,\rho}\}$. In fact, both these sets are intervals. Using that $f_{yy}(p) \neq 0$ for all $p \in H_{\epsilon,\rho}$ together with the definition of the $H_{\epsilon,\rho}$ and the assumption on ρ, we see that there are real numbers s_1, s_2, t_1, t_2 such that

$$\{f_y(q_n(s)) \mid q_n(s) \in H^j_{\epsilon,\rho}\} = [-\epsilon \, \|q_n(s_1)\|^{r-1}, \epsilon \, \|q_n(s_2)\|^{r-1}]$$

and

$$\{f_y(p_n(t)) \mid p_n(t) \in H^i_{\epsilon,\rho}\} = [-\epsilon \, \|p_n(t_1)\|^{r-1}, \epsilon \, \|p_n(t_2)\|^{r-1}].$$

It follows that when $q_n(s) \in H^j_{\epsilon,\rho}$, the equation $f_y(q_n(s)) = f_y(p_n(t))$ has a unique solution $t = h(s)$ with $p_n(t) \in H^i_{\epsilon,\rho}$.

Let $F : \mathbb{R}^2 \to \mathbb{R}$ be given by $F(s,t) = f_y(p_n(t)) - f_y(q_n(s))$. We have

$$\frac{\partial F}{\partial t}(s,t) = f_{yy}(p_n(t)) \neq 0$$

when $p_n(t) \in H_{\epsilon,\rho}$. The function $h(s)$ above satisfies $F(s, h(s)) = 0$, and by the Implicit Function Theorem, h is a smooth function.

Define the function G by

$$G(s) = \frac{f(p_n(h(s))) - f(q_n(s))}{p_n(h(s)) - q_n(s)} - f_y(q_n(s)).$$

Clearly G is continuous near $s = 0$. Let ϵ_n be defined by

$$|f(p_n) - f(q_n)| = \epsilon_n(\|p_n\|^{r-1} + \|q_n\|^{r-1})|y_n - v_n|. \tag{5.1}$$

Note that $\epsilon_n > 0$ and that $\epsilon_n \to 0$ as $n \to \infty$. For constants K and indices n such that $|K|\epsilon_n < \epsilon$, let $S(K,n) \in \{s \mid q_n(s) \in H^j_{\epsilon,\rho}\}$ be defined by

$$f_y(q_n(S(K,n))) = K\epsilon_n \|q_n(S(K,n))\|^{r-1}.$$

This definition is unambiguous because $f_{yy} \neq 0$ in $H_{\epsilon,\rho}$. There are eight cases to consider, one for each possible value of

$$\operatorname{Sign}(\omega) = \left(\frac{f(p_n) - f(q_n)}{|f(p_n) - f(q_n)|}, \frac{f_{yy}(p_n)}{|f_{yy}(p_n)|}, \frac{f_{yy}(q_n)}{|f_{yy}(q_n)|}\right) \in \{-1, 1\}^3.$$

The denominators in the definition of $\operatorname{Sign}(\omega)$ cause no problems, because $f_{yy}(p) \neq 0$ when $f_y(p) = 0$ as a consequence of (i). Suppose $\operatorname{Sign}(\omega) = (-1,-1,-1)$. This situation is illustrated in Figure 3. Since $h(0) = 0$, $G(0) < 0$. Let $S = S(K,n)$ for some fixed $K < 0$ and assume that $G(S) < 0$. Notice that necessarily, $S > 0$. There is a sequence ρ_n converging to 0 such that $p_n(h(S)) \in H^i_{|K|\epsilon_n,\rho_n}$ and $q_n(S) \in H^j_{|K|\epsilon_n,\rho_n}$. We get

$$f(p_n(h(S))) - f(q_n(S)) < f_y(q_n(S))(y_n + h(S) - v_n - S) < 0.$$

By our assumption that $\|p_n - q_n\| = o(\|p_n\| + \|q_n\|)$, we have $\|p_n\| = \|q_n\| + o(\|q_n\|)$. By the estimates (1)-(3) of Section 3, $|y_n + h(S) - v_n - S| = |y_n - v_n| + o(|y_n - v_n|)$, $\|p_n(h(S))\| = \|p_n\| + o(\|p_n\|)$ and $\|q_n(S)\| = \|q_n\| + o(\|q_n\|)$. This gives

$$|f(p_n(h(S))) - f(q_n(S))| > \frac{|K|}{4}\epsilon_n(\|p_n\|^{r-1} + \|q_n\|^{r-1})|y_n - v_n|.$$

Let $0 < \delta < \frac{1}{4}$. By estimates (4) and (5) of Section 3, $\|p_n(h(S)) - p_n\| = o(|y_n - v_n|)$ and $\|q_n(S) - q_n\| = o(|y_n - v_n|)$. Furthermore, since $f_{yy} \neq 0$ in $H_{\epsilon,\rho}$, the maximum of $\left|f_y|\overline{p_n p_n(h(S))}\right|$ is $|f_y(p_n(h(S)))|$ and the maximum of $\left|f_y|\overline{q_n q_n(S)}\right|$ is $|f_y(q_n(S))|$. Using this and our assumption that $\|p_n\| = \|q_n\| + o(\|q_n\|)$, we get

$$\begin{aligned}
|f(p_n) - f(q_n)| &\geq |f(p_n(h(S))) - f(q_n(S))| - |f(p_n(h(S))) - f(p_n)| - |f(q_n(S)) - f(q_n)| \\
&\geq |f(p_n(h(S))) - f(q_n(S))| - |K|\epsilon_n \|q_n(S)\|^{r-1} (\|p_n(h(S)) - p_n\| + \|q_n(S) - q_n\|) \\
&\geq |f(p_n(h(S))) - f(q_n(S))| - |K|\epsilon_n\delta(\|q_n\|^{r-1} + \|p_n\|^{r-1})|y_n - v_n| \\
&\geq \frac{|K|}{4}\epsilon_n(1 - 4\delta)(\|p_n\|^{r-1} + \|q_n\|^{r-1})|y_n - v_n|.
\end{aligned}$$

When n is large, we may take $|K| > 4/(1-4\delta)$ and this contradicts (5.1), and hence, $G(S(K,n)) > 0$. By the Intermediate Value Theorem, there is a sequence (s_n) with $0 < s_n < S(K,n)$ such that $G(s_n) \equiv 0$. The proof is finished in this case by choosing $\tilde{\epsilon}_n = |K|\epsilon_n$, $\tilde{y}_n = y_n + h(s_n)$ and $\tilde{v}_n = v_n + s_n$.

All the other seven cases are checked by essentially the same argument. It is just a matter of keeping track of the signs and the directions of the inequalities, so the details are left out. □

Let $\tilde{p}_n = (x_n, \tilde{y}_n)$ and $\tilde{q}_n = (x_n, \tilde{v}_n)$ be the sequences given by Lemma 5.1. We may assume that for all n, $\|\tilde{p}_{n+1}\| < \frac{1}{2}\|\tilde{q}_n\|$. Remember that we have also assumed that $\|\tilde{p}_n\| \geq \|\tilde{q}_n\|$ and $\|\tilde{p}_n - \tilde{q}_n\| = o(\|\tilde{p}_n\|)$. Let $K = \{0\} \cup \bigcup_n \{\tilde{p}_n, \tilde{q}_n\}$. We define an r-th order Taylor field Q on K with values in $\mathbb{R}$ by

$$Q^m(p) = \begin{cases} f(\tilde{q}_n) - f(\tilde{p}_n), & p = \tilde{p}_n,\ m = (0,0); \\ -\frac{f(\tilde{p}_n) - f(\tilde{q}_n)}{\tilde{y}_n - \tilde{v}_n}, & p = \tilde{p}_n, \tilde{q}_n,\ m = (0,1); \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 5.2. *Q is a Whitney field.*

Proof. Let $X = (x, y)$. We have to show that for all $p, q \in K$, $m \in \mathbb{N}^2$,

$$\begin{aligned}(R_q Q)^m(p) &= Q^m(p) - \frac{\partial^{|m|}}{\partial X^m}\left(\sum_\alpha (\frac{1}{\alpha!} Q^\alpha(q)(X-q)^\alpha)\right)\Bigg|_{X=p} \\ &= o(\|p-q\|^{r-|m|}).\end{aligned}$$

There are a number of cases to consider, each of which is straightforward. In any of the cases $(p,q) = (\tilde{p}_n, \tilde{q}_n)$ or $(p,q) = (\tilde{q}_n, \tilde{p}_n)$, the definition of Q gives us that $(R_q Q)^m(p) = 0 = o(\|p-q\|^{r-|m|})$ for $m = (0,0)$ and $m = (0,1)$. In the remaining combinations, $\|p - q\| > \frac{1}{2}\max\{\|p\|, \|q\|\}$ and $(R_q Q)^m(p) = o((\max\{\|p\|, \|q\|\})^{r-|m|})$ for $m = (0,0)$ and $m = (0,1)$. Since $(R_q Q)^m(p) \equiv 0$ when $m = (1,0)$ or $|m| > 1$, it follows that Q is a Whitney field. □

Proof of Theorem 1.7. Assume first that (i) and (ii) hold for ω of rank 1 in standard form $\omega(x,y) = (x, f(x,y))$. By Lemma 2.7 and Proposition 1.9, (I) and (II) holds for ω as well. Then we may use Theorem 1.3 to conclude that ω is $\mathcal{A}_0$-sufficient.

Now, suppose that (i) fails for ω. By Lemma 2.7, (I) also fails for ω, and by Theorem 1.3, ω is not $\mathcal{A}_0$-sufficient.

Finally, suppose that (i) holds and (ii) fails for ω. Then there are distinct components C_i and C_j of $\Sigma(\omega)$ and sequences $p_n = (x_n, y_n) \in C_i$ and $q_n = (x_n, v_n) \in C_j$ such that

$$|f(p_n) - f(q_n)| = o(\|p_n\|^{r-1} + \|q_n\|^{r-1})|y_n - v_n|.$$

By passing to a subsequence, we may also assume that $\|p_n\| \geq \|q_n\|$ and $\|p_{n+1}\| < \frac{1}{2}\|q_n\|$ for all n. If there are subsequences (p_{n_k}) and (q_{n_k}) of (p_n) and (q_n), respectively, with

$$\|p_{n_k} - q_{n_k}\| \sim \max\{\|p_{n_k}\|, \|q_{n_k}\|\},$$

then it is easy to see that the Taylor field

$$Q_1^m(p) = \begin{cases} f(q_{n_k}) - f(p_{n_k}), & p = p_{n_k},\ m = (0,0); \\ 0, & \text{otherwise} \end{cases}$$

is a Whitney field. By Whitney's Extension Theorem ([3]), we may extend Q_1 by a C^r map h_1 defined in a neighbourhood of 0. By construction of Q_1, $j^r h_1(0) = 0$, and hence, $\omega + h_1$ is a C^r realization of ω. However, p_{n_k} and q_{n_k} are singular points of $\omega + h_1$ for every n, and $(\omega + h_1)(p_{n_k}) = (\omega + h_1)(q_{n_k})$. This gives sequences of singular double points of $\omega + h_1$ converging to 0, and it is shown in [2] that a sufficient jet cannot have any such representative. Thus, ω is not $\mathcal{A}_0$-sufficient.

If there are no subsequences as above, then we may assume that $\|p_n - q_n\| = o(\|p_n\| + \|q_n\|)$. By Lemma 2.1, Lemma 2.6, Lemma 2.8 and Proposition 1.9, we may assume that p_n and q_n are in the first quadrant and that $\|p_n\| \sim y_n$ and $\|q_n\| \sim v_n$. In this situation, we can find the sequences $(\tilde{p}_n)$ and $(\tilde{q}_n)$ of Lemma 5.1 and construct the Whitney field Q of Lemma 5.2. By Whitney's Extension Theorem again, we may extend Q by a C^r map h defined in a neighbourhood of 0. By construction of Q, $j^r h(0) = 0$, and hence, $\omega + h$ is a C^r realization of ω. Again, $\tilde{p}_n$ and $\tilde{q}_n$ are singular points of $\omega + h$ for every n, and $(\omega + h)(\tilde{p}_n) = (\omega + h)(\tilde{q}_n)$. This gives sequences of singular double points of $\omega + h$ converging to 0, and hence, ω is not $\mathcal{A}_0$-sufficient. The proof is finished. □

6. Examples

Example 6.1 (Example 1 of [2] revised)**.** Let $r > 3$ and let $\omega(x, y) = (x, f(x, y)) = (x, xy + y^r)$. Since ω is given in standard form, we can apply Theorem 1.7 to prove that ω is $\mathcal{A}_0$-sufficient in $\mathcal{E}_{[r]}(2, 2)$. We have $f_y(x, y) = x + ry^{r-1}$ and $f_{yy}(x, y) = r(r-1)y^{r-2}$.

Assume that (i) does not hold for ω. Then there is a sequence $p_n = (x_n, y_n)$ converging to 0 such that

$$|f_y(p_n)| + |f_{yy}(p_n)|\,\|p_n\| = o(\|p_n\|^{r-1}).$$

Thus, $|f_y(p_n)| = |x_n + ry_n^{r-1}| = o(\|(x_n, y_n)\|^{r-1})$ and this implies that $|x_n| \sim |y_n|^{r-1}$ and $\|p_n\| \sim |y_n|$. But then $f_{yy}(p_n) \geq \|p_n\|^{r-2}$, which contradicts that (i) fails. This proves that ω satisfies (i).

If r is even, then ω has one branch on each side of the y-axis, and (ii) is trivially satisfied. Assume that r is odd. Then $\Sigma(\omega) = \{(x, y) | x = -ry^{r-1}\}$. Let $p = (-ry^{r-1}, y)$ and $q = (-ry^{r-1}, -y)$. Then $\|p - q\| \sim \|p\| = \|q\| \sim |y|$ and we get

$$|f(p) - f(q)| = |2ry^r + 2y^r| \gtrsim (\|p\|^{r-1} + \|q\|^{r-1})|y|.$$

This shows that (ii) holds, and by Theorem 1.7, ω is sufficient as claimed.

Example 6.2. Let $a > b > c > 0$ and let $\omega(x, y) = (x, f(x, y))$ in $J^7(2, 2)$ be such that $f_y(x, y) = (x - ay^2)(x - by^2)(x - cy^2)$. Let

$$F(x, y) = x - y - \frac{1}{3}(a + b + c)(x^3 - y^3) + \frac{1}{5}(ab + ac + bc)(x^5 - y^5) - \frac{1}{7}abc(x^7 - y^7).$$

We claim that ω is $\mathcal{A}_0$-sufficient in $\mathcal{E}_{[7]}(2, 2)$ if $0 \notin \{F(a^{-\frac{1}{2}}, b^{-\frac{1}{2}}), F(a^{-\frac{1}{2}}, c^{-\frac{1}{2}}), F(b^{-\frac{1}{2}}, c^{-\frac{1}{2}})\}$. This means that we need to verify (i) and (ii) of Theorem 1.7 for ω with $r = 7$.

Assume that (i) fails. Then there is a sequence $p_n = (x_n, y_n)$ converging to 0 such that

$$|f_y(p_n)| + |f_{yy}(p_n)|\,\|p_n\| = o(\|p_n\|^6).$$

From the expression for f_y we conclude that (i) can only fail along the sequence if $x_n = dy_n^2 + o(y_n^2)$ for some $d \in \{a, b, c\}$. We also have

$$f_{yy}(x, y) = -2(a + b + c)x^2 y + 4(ab + ac + bc)xy^3 - 6abcy^5.$$

Suppose $x_n = ay_n^2 + o(y_n^2)$. Then $\|p_n\| \sim |y_n|$ and

$$f_{yy}(p_n) = -[2a^2(a + b + c) - 4a(ab + ac + bc) + 6abc]y_n^5 + o(y_n^5).$$

But $f_{yy}(p_n) = o(y_n^5)$ since (i) fails, and hence,

$$2a^2(a + b + c) - 4a(ab + ac + bc) + 6abc = 0.$$

Since $a \neq 0$, this implies the equation

$$(a - b)(a - c) = 0 \tag{6.1}$$

which cannot hold since $a > b > c$. The same argument applies when $x_n = by_n^2 + o(y_n^2)$ and when $x_n = cy_n^2 + o(y_n^2)$ and gives equations

$$(b-c)(b-a) = 0$$

and

$$(c-a)(c-b) = 0.$$

None of these two equations can have a solution with $a > b > c$. Altogether this shows that (i) holds for ω when $r = 7$.

To verify (ii), notice that for $s, t > 0$,

$$\left| f(x, \sqrt{\frac{x}{s}}) - f(x, \sqrt{\frac{x}{t}}) \right| = x^{\frac{7}{2}} |F(s^{-\frac{1}{2}}, t^{-\frac{1}{2}})|.$$

This proves that (ii) holds with $r = 7$, since we have assumed that

$$0 \notin \{F(a^{-\frac{1}{2}}, b^{-\frac{1}{2}}), F(a^{-\frac{1}{2}}, c^{-\frac{1}{2}}), F(b^{-\frac{1}{2}}, c^{-\frac{1}{2}})\}.$$

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