Journal *of* Singularities
Volume 5 (2012), 124-152
DOI: 10.5427/jsing.2012.5j

Proc. of Singularity Theory, Hefei, 2011

MOTIVIC BIVARIANT CHARACTERISTIC CLASSES AND RELATED TOPICS

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ABSTRACT. We have recently constructed a bivariant analogue of the motivic Hirzebruch classes. A key idea is the construction of a suitable universal bivariant theory in the algebraic-geometric (or compact complex analytic) context, together with a corresponding "bivariant blow-up relation" generalizing Bittner's presentation of the Grothendieck group of varieties. Before we already introduced a corresponding universal "oriented" bivariant theory as an intermediate step on the way to a bivariant analogue of Levine–Morel's algebraic cobordism. Switching to the differential topological context of smooth manifolds, we similarly get a new geometric bivariant bordism theory based on the notion of a "fiberwise bordism". In this paper we make a survey on these theories.

1. INTRODUCTION

For the category of finite sets, the number of elements of the set is a basic invariant and the natural numbers $\mathbb{N}$ is the collection of such invariants. The number of elements of a finite set F is called the *cardinality*, denoted by $c(F)$ or $|F|$. The cardinality satisfies the following properties:

(1) if $X \cong X'$ (set-isomorphism), then $c(X) = c(X')$,
(2) $c(X) = c(Y) + c(X \setminus Y)$ for a subset $Y \subset X$ (a *scissor formula*),
(3) $c(X \times Y) = c(X) \times c(Y)$,
(4) $c(pt) = 1$.

The above property (1) is a crucial requirement for counting elements of finite sets. Now, when we consider a similar "cardinality" or invariant on a suitable subcategory of topological spaces, we modify the above requirements (1) and (2) as follows:

(1)' If $X \cong X'$ ($\mathcal{TOP}$-isomorphism), then $c(X) = c(X')$,
(2)' $c(X) = c(Y) + c(X \setminus Y)$ for a closed subset $Y \subset X$,
(3) $c(X \times Y) = c(X) \times c(Y)$,
(4) $c(pt) = 1$.

If such a topological cardinality exists, then it follows that

$$c(\mathbb{R}^1) = c\Big((-\infty, 0) \sqcup \{0\} \sqcup (0, \infty)\Big) = c(\mathbb{R}^1) + 1 + c(\mathbb{R}^1)\,,$$

so that $c(\mathbb{R}^1) = -1$ and $c(\mathbb{R}^n) = (-1)^n$. Thus, for a finite CW-complex X, $c(X)$ is equal to the Euler–Poincaré characteristic $\chi(X)$. The existence of such a topological cardinality is guaranteed by homology theory. To be more precise

$$c(X) := \chi_c(X) = \sum (-1)^i \dim_{\mathbb{R}} H^i_c(X;\mathbb{R}) = \sum (-1)^i \dim_{\mathbb{R}} H^{BM}_i(X;\mathbb{R}) \in \mathbb{Z}\,.$$

Here $H^{BM}_*(X)$ is the Borel–Moore homology group of a locally compact X. Of course to make sense of this, we have to assume that $H^{BM}_*(-)$ is finite dimensional for all spaces considered. Such a very nice

(1) Jörg Schürmann: supported by the SFB 878 "groups, geometry and actions".
(2) Shoji Yokura: partially supported by Grant-in-Aid for Scientific Research (No. 21540088), the Ministry of Education, Culture, Sports, Science and Technology (MEXT), Japan.

context is for example the semi-algebraic (or more generally o-minimal) context (e.g., see [35, Chapter 2]).

Let us consider now a similar "cardinality" or invariant on the category $\mathcal{V}$ of complex algebraic varieties, say "$\mathcal{V}$-cardinality", by modifying (1)' and (2)' as

(1)" If $X \cong X'$ ($\mathcal{V}$-isomorphism), then $c(X) = c(X')$,
(2)" $c(X) = c(Y) + c(X \setminus Y)$ for a closed subvariety $Y \subset X$,
(3) $c(X \times Y) = c(X) \times c(Y)$,
(4) $c(pt) = 1$.

If such an "algebraic-geometric" cardinality exists, then we have

$$c(\mathbb{P}^n) = c(\mathbb{C}^0 \sqcup \mathbb{C}^1 \sqcup \cdots \sqcup \mathbb{C}^n) = 1 + c(\mathbb{C}^1) + \cdots + c(\mathbb{C}^1)^n .$$

Note that we cannot do the same trick as above for $c(\mathbb{R}^1) = -1$. The existence of such an algebraic cardinality is guaranteed by Deligne's theory of mixed Hodge structures. Let u, v be two variables, then the *Deligne–Hodge polynomial* $\chi_{u,v}$ is defined by

$$\chi_{u,v}(X) = \sum (-1)^i (-1)^{p+q} \dim_{\mathbb{C}} Gr^p_{\mathcal{F}} Gr^{\mathcal{W}}_{p+q} (H^i_c(X;\mathbb{C})) u^p v^q \in \mathbb{Z}[u,v] .$$

Here $\mathcal{W}$ is the weight filtration and $\mathcal{F}$ the Hodge filtration of the corresponding mixed Hodge structure. Then $\chi_{u,v}$ is such an algebraic-geometric cardinality with $\chi_{u,v}(\mathbb{C}^1) = uv$. Let us consider the specialization $u = y, v = -1$. Then we have

$$\chi_y(X) := \chi_{y,-1}(X) = \sum (-1)^i (-1)^q \dim_{\mathbb{C}} Gr^p_{\mathcal{F}} (H^i_c(X;\mathbb{C})) y^p ,$$

i.e., only the Hodge (but not the weight) filtration is used. This is called χ_y*-genus* of X.

Let $\mathrm{Iso}(\mathcal{V})$ be the free abelian group generated by the isomorphism classes of complex algebraic varieties. Then the above χ_y can be considered as the homomorphism $\chi_y : \mathrm{Iso}(\mathcal{V}) \to \mathbb{Z}[y]$ defined by $\chi_y([X]) := \chi_y(X)$. Because of the condition (2)" we get

$$\chi_y : K_0(\mathcal{V}) := \frac{\mathrm{Iso}(\mathcal{V})}{\{[X] - [Y] - [X \setminus Y] \mid Y \subset X\}} \to \mathbb{Z}[y] \hookrightarrow \mathbb{Q}[y] ,$$

where Y is a closed algebraic subset of X and $\{[X] - [Y] - [X \setminus Y] \mid Y \subset X\}$ is the abelian subgroup generated by the elements of the form $[X] - [Y] - [X \setminus Y]$. $K_0(\mathcal{V})$ is called the Grothendieck group (or ring) of complex algebraic varieties, with $c(X) = [X]$ the universal motivic "algebraic-geometric" cardinality. $K_0(\mathcal{V}) = K_0(\mathcal{V}/pt)$ can be extended to a covariant (and also contravariant) functor $K_0(\mathcal{V}/-)$ by

$$K_0(\mathcal{V}/X) := \frac{\{[V \to X]\}}{\left\{ [W \xrightarrow{h} X] - [Z \xrightarrow{h|_Z} X] - [W \setminus Z \xrightarrow{h|_{W \setminus Z}} X] \;\middle|\; Z \subset W \right\}} ,$$

where Z is a closed subvariety of W. Here and in the following $\{\cdots\}$ always denotes the corresponding free abelian group (or its subgroup) generated by the listed elements. $K_0(\mathcal{V}/-)$ is covariantly functorial by composition of arrows, whereas for the contravariance one takes the corresponding fiber products. Moreover, these functorialities are compatible with the cross product $\times$ coming from the product of varieties. Note that the same construction works for the category $\mathcal{V} = \mathcal{V}^{(qp)}_k$ of (quasi-projective) algebraic varieties (i.e., reduced separated schemes of finite type) over a base field k, and with a little bit more care (see [7]), also for the "compactifiable" complex analytic context.

Another sort of "algebraic-geometric" cardinality is given by a characteristic number

$$c(M) := \pi_{M*} (c\ell(TM) \cap [M]) \in H_*(pt) \otimes R$$

of the tangent bundle TM of a compact complex algebraic or analytic manifold (or complete smooth algebraic variety) M, with the constant map $\pi_M : M \to pt$ proper and $[M] \in H_*(M)$ the fundamental class of M. Here $H_*(X) = H_{2*}^{BM}(X)$ the even degree Borel-Moore homology or the Chow homology $H_*(X) = CH_*(X)$ (as in [19]). Of course the condition (2)" above does not make sense then. But if

$$c\ell : Vect(-) \to H^*(-) \otimes R$$

is a contravariantly functorial characteristic class from the isomorphism classes of algebraic (or analytic) vector bundles to the appropriate cohomology $H^*(-)$ tensorized with the ring R (i.e., the usual even degree cohomology or the operational Chow cohomology of [19]), then (1)" follows from the projection formula. And if $c\ell$ is also multiplicative (resp., normalized), then this implies (3) (resp., (4)) above. Moreover, as a substitute for (2)", the characteristic number $c(M)$ depends in this case only on the (co)bordism class of M in the algebraic or complex cobordism group of a point (as explained later):

$(2)^{bor}$ If $[M] = [M'] \in \Omega_*^{LM}(pt)$ or $[M] = [M'] \in \Omega_*^{U}(pt)$, then $c(M) = c(M')$.

An important example of such a functorial multiplicative and normalized characteristic class of a complex or algebraic vector bundle E is the Hirzebruch or generalized Todd class of E defined by

$$td_y(E) := \prod_{i=1}^{\operatorname{rank} E} \left(\frac{\alpha_i(1+y)}{1 - e^{-\alpha_i(1+y)}} - \alpha_i y \right) \in H^*(-) \otimes \mathbb{Q}[y]$$

where $\alpha_i \in H^1(-)$ are the Chern roots of E, i.e., the total Chern class of E is given by

$$c(E) = \prod_{i=1}^{\operatorname{rank} E} (1 + \alpha_i) \in H^*(-) \, .$$

The corresponding characteristic number $c(M) =: \chi_y(M)$ is the Hirzebruch χ_y-genus of the manifold M. Note that for a compact complex algebraic manifold M this also agrees with the earlier definition given above in terms of Hodge numbers. And as explained in [7], it is the most general characteristic number having an "additive" extension to singular varieties (over any base field of characteristic zero, and for compactifiable complex analytic varieties), i.e., satisfying the "scissor formula" (2)". Note that the Deligne–Hodge polynomial $\chi_{u,v}(M)$ for a compact complex algebraic manifold M is not a characteristic number in this sense.

Remark 1.1. The Hirzebruch class unifies the following three classes, which are important in geometry and topology:

- $y = -1$: $td_{-1}(E) = \prod_{i=1}^{\operatorname{rank} E} (1 + \alpha_i) = c(E)$, the total Chern class,
- $y = 0$: $td_0(E) = \prod_{i=1}^{\operatorname{rank} E} \frac{\alpha_i}{1 - e^{-\alpha_i}} = td(E)$, the total (original) Todd class,
- $y = 1$: $td_1(E) = \prod_{i=1}^{\operatorname{rank} E} \frac{\alpha_i}{\tanh \alpha_i} = L(E)$, the total Thom–Hirzebruch L-class.

A Grothendieck–Riemann–Roch-type theorem for the χ_y-genus is the following:

Theorem 1.2 ([7] (cf. [36], [46])). *Consider the compact complex analytic or the algebraic context over a base field k of characteristic zero.*

(1) There exists a unique natural transformation (functorial for proper morphisms)

$$T_{y_*} : K_0(\mathcal{V}/-) \to H_*(-) \otimes \mathbb{Q}[y]$$

such that for a smooth variety X

$$T_{y_*}([X \xrightarrow{\mathrm{id}_X} X]) = td_y(TX) \cap [X] \, .$$

Whether X *is singular or not,* $T_{y_*}(X) := T_{y_*}([X \xrightarrow{id_X} X])$ *is called the* motivic Hirzebruch class of X.

(2) *When* $X = pt$ *is a point,* $T_{y_*} : K_0(\mathcal{V}/pt) = K_0(\mathcal{V}) \to \mathbb{Q}[y]$ *equals* χ_y.

The above Hirzebruch class transformation $T_{y_*} : K_0(\mathcal{V}/-) \to H_*(-) \otimes \mathbb{Q}[y]$ "unifies" the following three well-known characteristic classes of singular varieties. Here we work either in the category $\mathcal{V} = \mathcal{V}_k^{(qp)}$ of (quasi-projective) algebraic varieties over a base field k, with $H_*(X) = CH_*(X)$ the Chow homology groups, or in the category $\mathcal{V} = \mathcal{V}_c^{an}$ of compact reduced complex analytic spaces, with $H_*(X) = H_{2*}^{BM}(X)$ the even degree Borel-Moore homology in the complex algebraic or analytic context:

- MacPherson's Chern class transformation [7, 25, 31]:

$$c_* : F(X) \to H_*(X),$$

  defined on the group $F(X)$ of constructible functions in the algebraic context for k of characteritic zero or in the compact complex analytic context. The transformation $c_* : F(-) \to H_*(-)$ is the unique one satisying the smooth condition that for a smooth M, $c_*(\mathbb{1}_M) = c(TM) \cap [M]$ where TM is the tangent bundle of M.
- Baum–Fulton–MacPherson's Todd class or Riemann–Roch transformation [4, 19]:

$$td_* : G_0(X) \to H_*(X) \otimes \mathbb{Q},$$

  defined on the Grothendieck group $G_0(X)$ of coherent sheaves in the algebraic context in any characteristic. In the compact complex analytic context such a transformation can be deduced (compare with [7]) from Levy's K-theoretical Riemann-Roch transformation [30]. The transformation $td_* : G_0(-) \to H_*(-) \otimes \mathbb{Q}$ is the unique one satisying the smooth condition that for a smooth M, $td_*(\mathcal{O}_M) = td(TM) \cap [M]$.
- Goresky– MacPherson's homology L-class [21], which is extended as a natural transformation by Cappell–Shaneson [12] (see also [7, 42, 41]):

$$L_* : \Omega(X) \to H_*(X) \otimes \mathbb{Q}$$

  defined on the cobordism group $\Omega(X)$ of selfdual constructible sheaf complexes (for the Verdier duality).. This transformation is only defined for compact spaces in the complex algebraic or analytic context, with H_* the usual homology, since its definition is based on a corresponding signature invariant together with the Thom–Pontrjagin construction. The transformation $L_* : \Omega(-) \to H_*(-) \otimes \mathbb{Q}$ satisfies the smooth condition that for a smooth M, $L_*(\mathbb{Q}_M[\dim M]) = L(TM) \cap [M]$.

The unification means that there are natural transformations ϵ, mC_0 and sd so that the following diagrams of transformations commute:

$$\begin{tikzcd} & K_0(\mathcal{V}/X) \arrow[dl, "\epsilon"'] \arrow[dr, "T_{-1_*}"] & \\ F(X) \arrow[rr, "c_* \otimes \mathbb{Q}"'] & & H_*(X) \otimes \mathbb{Q}. \end{tikzcd}$$

$$\begin{tikzcd} & K_0(\mathcal{V}/X) \arrow[dl, "mC_0"'] \arrow[dr, "T_{0*}"] & \\ G_0(X) \arrow[rr, "td_*"'] & & H_*(X)\otimes\mathbb{Q}. \end{tikzcd}$$

$$\begin{tikzcd} & K_0(\mathcal{V}/X) \arrow[dl, "sd"'] \arrow[dr, "T_{1*}"] & \\ \Omega(X) \arrow[rr, "L_*"'] & & H_*(X)\otimes\mathbb{Q}. \end{tikzcd}$$

This "unification" could be considered as a positive answer to MacPherson's question posed in [32]. Here the corresponding uniqueness result follows from the surjectivity of ϵ and mC_0, whereas for the L-class transformation this uniqueness only holds on the image of the transformation sd (which is not surjective).

Moreover, in [7] we also constructed in the algebraic context for k of characteristic zero and in the compact complex analytic context, a motivic Chern class transformation (functorial for proper morphisms)

$$mC_y : K_0(\mathcal{V}/X) \to G_0(X)\otimes\mathbb{Z}[y]\ .$$

This satisfies the normalization condition

$$mC_y(M) := mC_y([id_M]) = \sum_{i=0}^{dim(M)} [\Lambda^i T^*M]\cdot y^i =: \lambda_y([T^*M])\cap[\mathcal{O}_M]$$

for M smooth, with λ_y the total λ-class. Then the Hirzebruch class transformation T_{y_*} could also be defined as the composition $td_*\circ mC_y$, renormalized by the multiplication $\times(1+y)^{-i}$ on $H_i(X)\otimes\mathbb{Q}[y]$ to fit with the normalization condition above (see [7]). So mC_y could be considered as a K-theoretical refinement of T_{y_*}.

W. Fulton and R. MacPherson have introduced Bivariant Theory [20] (see also [19]). As reviewed very quickly in §2, a bivariant theory is defined on morphisms, instead of objects, and "unifies" both a covariant functor and a contravariant functor. Important topics in Bivariant Theories are what they call *Grothendieck transformations* between given two bivariant theories. A Grothendieck transformation is a bivariant natural transformation. The main objectives of [20] are bivariant-theoretic Riemann–Roch transformations or bivariant analogues of various theorems of Grothendieck–Riemann–Roch type. A key example of [20, Part II] is the bivariant Riemann–Roch transformation $\tau : \mathbb{K}_{alg} \to \mathbb{H}\otimes\mathbb{Q}$ on the category $\mathcal{V} = \mathcal{V}^{qp}_{\mathbb{C}}$ of complex quasi-projective varieties, with $\mathbb{K}_{alg}(f)$ the bivariant algebraic K-theory of f-perfect complexes and $\mathbb{H}$ the even degree bivariant homology. It unifies the covariant Todd class transformation td_* and the contravariant Chern character ch. An algebraic version on the category $\mathcal{V} = \mathcal{V}^{qp}_k$ of quasi-projective varieties over a base field k of any characteristic was constructed later on in [19, Example 18.3.19], with $\mathbb{H} = CH$ the bivariant operational Chow groups.

As another example, in [20, Part I, §6] Fulton and MacPherson constructed a bivariant Whitney class transformation. And they asked in the complex algebraic context for a corresponding bivariant Chern class transformation $\gamma : \mathbb{F} \to \mathbb{H}$ on their bivariant theory $\mathbb{F}$ of constructible functions satisfying a suitable local Euler condition, which generalizes the covariant MacPherson Chern class transformation $c_* : F \to H_*(-)$. For $\mathbb{H}$ the even degree bivariant homology, this problem was solved by Brasselet [6] in a suitable context (even for compact analytic spaces), whereas Ernström–Yokura [17] solved it for $\mathbb{H} = A^{PI} \supset CH$ another bivariant operational Chow group theory (for the notation A^{PI} see [17]). In [18], by introducing another bivariant theory $\tilde{\mathbb{F}}$ of constructible functions, they also introduced a bivariant

Chern class transformation $\gamma : \tilde{\mathbb{F}} \to CH$. Their approach is based on the usual calculus of constructible functions and the surjectivity of $c_* : F(X) \to CH_*(X)$. Therefore it works in the algebraic context over any base field k of characteristic zero, even though it was stated in [18] only in the complex algebraic context. Here $\tilde{\mathbb{F}}(X \to pt) = F(X)$ follows from the multiplicativity of c_* with respect to cross products $\times$.

In [38] we obtain in the quasi-projective context (over a base field k of any characteristic) two bivariant analogues

$$mC_y = \Lambda_y^{mot} : \mathbb{K}_0(\mathcal{V}_k^{qp}/X \to Y) \to \mathbb{K}_{alg}(X \to Y) \otimes \mathbb{Z}[y]$$

and

$$T_y : \mathbb{K}_0(\mathcal{V}^{qp}/X \to Y) \to \mathbb{H}(X \to Y) \otimes \mathbb{Q}[y]$$

of the motivic Chern and Hirzebruch class transformations mC_y and T_y, with T_y defined as the composition $\tau \circ mC_y$, renormalized by the multiplication $\times(1+y)^i$ on $\mathbb{H}^i(-) \otimes \mathbb{Q}[y]$. Moreover, T_y unifies the bivariant Riemann–Roch transformation $\tau : \mathbb{K}_{alg} \to \mathbb{H} \otimes \mathbb{Q}$ (for $y = 0$) and the bivariant Chern class transformation $\gamma : \tilde{\mathbb{F}} \to CH$ (for $y = -1$). Note that a bivariant L-class transformation (corresponding to $y = 1$) is still missing. In [9, 10] we considered a kind of general construction of a bivariant analogue of a given natural transformation between two covariant functors, but our approach presented in this paper is quite different from it. The former is more "operational", but the latter is more "direct" and very "motivic".

In this paper we make a survey on the above results [38] as well as on a corresponding universal "oriented" bivariant theory [44], which is a first step on the way to a bivariant-theoretic analogue of Levine–Morel's or Levine–Pandharipande's algebraic cobordism [28, 29]. Finally we switch to a differential topological context of smooth manifolds and make a remark on a new geometric bivariant bordism theory based on the notion of a "fiberwise bordism" ([3], [45]).

2. Fulton–MacPherson's bivariant theory

We quickly recall some basic ingredients of Fulton–MacPherson's bivariant theory [20].

Let $\mathcal{V}$ be a category which has a final object pt and on which the fiber product or fiber square is well-defined, e.g. the category $\mathcal{V}_k^{(qp)}$ of (quasi-projective) algebraic varieties (i.e., reduced separated schemes of finite type) over a base field k, or $\mathcal{V}_{(c)}^{an}$ the category of (compact) reduced complex analytic spaces. We also consider a class of maps, called "confined maps" (e.g., proper maps in this algebraic or analytic geometric context), which are closed under composition and base change and contain all the identity maps. Finally, one fixes a class of fiber squares, called "independent squares" (or "confined squares", e.g., "Tor-independent" in algebraic geometry, a fiber square with some extra conditions required on morphisms of the square), which satisfy the following properties:

(i) if the two inside squares in

$$\begin{array}{ccccc} X'' & \xrightarrow{h'} & X' & \xrightarrow{g'} & X \\ \Big\downarrow{\scriptstyle f''} & & \Big\downarrow{\scriptstyle f'} & & \Big\downarrow{\scriptstyle f} \\ Y'' & \xrightarrow[h]{} & Y' & \xrightarrow[g]{} & Y \end{array} \quad \text{or} \quad \begin{array}{ccc} X' & \xrightarrow[h'']{} & X \\ {\scriptstyle f'}\Big\downarrow & & \Big\downarrow{\scriptstyle f} \\ Y' & \xrightarrow[h']{} & Y \\ {\scriptstyle g'}\Big\downarrow & & \Big\downarrow{\scriptstyle g} \\ Z' & \xrightarrow[h]{} & Z \end{array}$$

are independent, then the outside square is also independent.

(ii) for any morphism $f : X \to Y$, the following squares are independent:

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{id}_X} & X \\ {\scriptstyle f}\Big\downarrow & & \Big\downarrow{\scriptstyle f} \\ Y & \xrightarrow[\mathrm{id}_X]{} & Y \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \mathrm{id}_X}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathrm{id}_Y} \\ X & \xrightarrow[f]{} & Y. \end{array}$$

A bivariant theory $\mathbb{B}$ on a category $\mathcal{V}$ with values in the category of (graded) abelian groups is an assignment to each morphism $X \xrightarrow{f} Y$ in the category $\mathcal{V}$ a (graded) abelian group (in most cases we can ignore a possible grading) $\mathbb{B}(X \xrightarrow{f} Y)$, which is equipped with the following three basic operations. The i-th component of $\mathbb{B}(X \xrightarrow{f} Y)$, $i \in \mathbb{Z}$, is denoted by $\mathbb{B}^i(X \xrightarrow{f} Y)$ (with $\mathbb{B}(X \xrightarrow{f} Y) =: \mathbb{B}^0(X \xrightarrow{f} Y)$ in the ungraded context).

Product operations: For morphisms $f : X \to Y$ and $g : Y \to Z$, the ($\mathbb{Z}$-bilinear) product operation

$$\bullet : \mathbb{B}^i(X \xrightarrow{f} Y) \otimes \mathbb{B}^j(Y \xrightarrow{g} Z) \to \mathbb{B}^{i+j}(X \xrightarrow{gf} Z)$$

is defined.

Pushforward operations: For morphisms $f : X \to Y$ and $g : Y \to Z$ with f *confined*, the ($\mathbb{Z}$-linear) pushforward operation

$$f_* : \mathbb{B}^i(X \xrightarrow{gf} Z) \to \mathbb{B}^i(Y \xrightarrow{g} Z)$$

is defined.

Pullback operations: For an *independent* square $\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\Big\downarrow & & \Big\downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y, \end{array}$ the ($\mathbb{Z}$-linear) pullback operation

$$g^* : \mathbb{B}^i(X \xrightarrow{f} Y) \to \mathbb{B}^i(X' \xrightarrow{f'} Y')$$

is defined.

And these three operations are required to satisfy the seven compatibility axioms (see [20, Part I, §2.2] for details):

(B-1) product is associative,
(B-2) pushforward is functorial,
(B-3) pullback is functorial,
(B-4) product and pushforward commute,
(B-5) product and pullback commute,

(B-6) pushforward and pullback commute, and
(B-7) projection formula.

We also assume that $\mathbb{B}$ has *units*, i.e., there is an element $1_X \in \mathbb{B}^0(X \xrightarrow{\mathrm{id}_X} X)$ such that $\alpha \bullet 1_X = \alpha$ for all morphisms $W \to X$ and $\alpha \in \mathbb{B}(W \to X)$; such that $1_X \bullet \beta = \beta$ for all morphisms $X \to Y$ and $\beta \in \mathbb{B}(X \to Y)$; and such that $g^* 1_X = 1_{X'}$ for all $g : X' \to X$.

Let $\mathbb{B}, \mathbb{B}'$ be two bivariant theories on the category $\mathcal{V}$. A *Grothendieck transformation* from $\mathbb{B}$ to $\mathbb{B}'$

$$\gamma : \mathbb{B} \to \mathbb{B}'$$

is a collection of group homomorphisms

$$\mathbb{B}(X \to Y) \to \mathbb{B}'(X \to Y)$$

for all morphisms $X \to Y$ in the category $\mathcal{V}$, which preserves the above three basic operations (as well as the units, but not necessarily possible gradings):

(i) $\gamma(\alpha \bullet_{\mathbb{B}} \beta) = \gamma(\alpha) \bullet_{\mathbb{B}'} \gamma(\beta)$,
(ii) $\gamma(f_* \alpha) = f_* \gamma(\alpha)$, and
(iii) $\gamma(g^* \alpha) = g^* \gamma(\alpha)$.

Most of our bivariant theories in this paper are *commutative* (see [20, §2.2]), i.e., if whenever both

$$\begin{tikzcd} W \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & Z \end{tikzcd} \qquad \begin{tikzcd} W \arrow[r, "f'"] \arrow[d, "g'"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$

are independent squares, then for $\alpha \in \mathbb{B}(X \xrightarrow{f} Z)$ and $\beta \in \mathbb{B}(Y \xrightarrow{g} Z)$

$$g^*(\alpha) \bullet \beta = f^*(\beta) \bullet \alpha.$$

$\mathbb{B}_*(X) := \mathbb{B}(X \to pt)$ becomes a covariant functor for *confined* morphisms and $\mathbb{B}^*(X) := \mathbb{B}(X \xrightarrow{id} X)$ becomes a contravariant ring valued functor for *any* morphisms, with $\mathbb{B}_*(X)$ a left $\mathbb{B}^*(X)$-module under the product $\cap := \bullet : \mathbb{B}^*(X) \otimes \mathbb{B}_*(X) \to \mathbb{B}_*(X)$. As to a possible grading, one sets

$$\mathbb{B}_i(X) := \mathbb{B}^{-i}(X \to pt) \quad \text{and} \quad \mathbb{B}^j(X) := \mathbb{B}^j(X \xrightarrow{id} X)$$

so that $\mathbb{B}^*(X)$ becomes a graded ring with $\cap : \mathbb{B}^j(X) \otimes \mathbb{B}_i(X) \to \mathbb{B}_{i-j}(X)$.

The following notion of a *canonical orientation* makes $\mathbb{B}_*$ a contravariant functor and $\mathbb{B}^*$ a covariant functor with the corresponding Gysin (or transfer) homomorphisms:

Definition 2.1. ([20, Part I, Definition 2.6.2]) Let $\mathcal{S}$ be a class of maps in $\mathcal{V}$, which is closed under compositions and contains all identity maps. Suppose that to each $f : X \to Y$ in $\mathcal{S}$ there is assigned an element $\theta(f) \in \mathbb{B}(X \xrightarrow{f} Y)$ satisfying that

(i) $\theta(g \circ f) = \theta(f) \bullet \theta(g)$ for all $f : X \to Y, g : Y \to Z \in \mathcal{S}$ and
(ii) $\theta(\mathrm{id}_X) = 1_X$ for all X with $1_X \in \mathbb{B}^*(X) := \mathbb{B}(X \xrightarrow{\mathrm{id}_X} X)$ the unit element.

Then $\theta(f)$ is called a *canonical orientation* of f. If we need to refer to which bivariant theory we consider, we denote $\theta_{\mathbb{B}}(f)$ instead of the simple notation $\theta(f)$.

For example the class $\mathcal{S}$ of *smooth* morphisms in the algebraic or analytic geometric context has canonical orientations for all the bivariant theories mentioned in the introduction, with all Cartesian squares independent.

Proposition 2.2. *For the composite $X \xrightarrow{f} Y \xrightarrow{g} Z$, if $f \in \mathcal{S}$ has a canonical orientation $\theta_{\mathbb{B}}(f)$, then we have the Gysin homomorphism (or transfer) defined by $f^!(\alpha) := \theta(f) \bullet \alpha$:*

$$f^! : \mathbb{B}(Y \xrightarrow{g} Z) \to \mathbb{B}(X \xrightarrow{gf} Z),$$

which is functorial. In particular, when $Z = pt$, we have the Gysin homomorphism: $f^! : \mathbb{B}_*(Y) \to \mathbb{B}_*(X)$.

Proposition 2.3. *For an independent square* $\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[g]{} & Y \end{array}$ *, if $g \in \mathcal{C} \cap \mathcal{S}$ and g has a canonical orientation $\theta_{\mathbb{B}}(g)$, then we have the Gysin homomorphism defined by $g_!(\alpha) := g'_*(\alpha \bullet \theta(g))$:*

$$g_! : \mathbb{B}(X' \xrightarrow{f'} Y') \to \mathbb{B}(X \xrightarrow{f} Y),$$

which is functorial. In particular, for an independent square $\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \mathrm{id}_X}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_Y} \\ X & \xrightarrow[f]{} & Y \end{array}$ *with $f \in \mathcal{C} \cap \mathcal{S}$, we have the Gysin homomorphism:$f_! : \mathbb{B}^*(X) \to \mathbb{B}^*(Y)$.*

The symbols $f^!$ and $g_!$ should carry the information of $\mathcal{S}$ and the canonical orientation θ, but we omit them for the sake of simplicity.

A Grothendieck transformation $\gamma : \mathbb{B} \to \mathbb{B}'$ of two bivariant theories $\mathbb{B}$ and $\mathbb{B}'$ induces natural transformations $\gamma_* : \mathbb{B}_* \to \mathbb{B}'_*$ and $\gamma^* : \mathbb{B}^* \to {\mathbb{B}'}^*$, i.e., we have the following commutative diagrams:
For any morphism $f : X \to Y$ we have the commutative diagram

$$\begin{array}{ccc} \mathbb{B}^*(X) & \xrightarrow{\gamma^*} & \mathbb{B}'^*(X) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ \mathbb{B}^*(Y) & \xrightarrow[\gamma^*]{} & \mathbb{B}'^*(Y). \end{array}$$

For a confined morphism $f : X \to Y$ we have the commutative diagram

$$\begin{array}{ccc} \mathbb{B}_*(X) & \xrightarrow{\gamma_*} & \mathbb{B}'_*(X) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*} \\ \mathbb{B}_*(Y) & \xrightarrow[\gamma_*]{} & \mathbb{B}'_*(Y). \end{array}$$

They are related by the *module property*

$$\gamma_*(\beta \cap \alpha) = \gamma^*(\beta) \cap \gamma_*(\alpha) \quad \text{for all} \quad \beta \in \mathbb{B}^*(X), \alpha \in \mathbb{B}_*(X).$$

Suppose that $f : X \to Y$ has a canonical orientation for both bivariant theories. A bivariant element $u_f \in \mathbb{B}'^*(X) = \mathbb{B}'(X \xrightarrow{\mathrm{id}_X} X)$ satisfying

$$\gamma(\theta_{\mathbb{B}}(f)) = u_f \bullet \theta_{\mathbb{B}'}(f)$$

is called a *Riemann–Roch formula* (see [20]). Such a Riemann–Roch formula gives rise to the following (wrong-way) commutative diagrams :

$$\begin{CD} \mathbb{B}^*(X) @>{\gamma^*}>> \mathbb{B}'^*(X) \\ @V{f_!}VV @VV{f_!(\,-\,\bullet u_f)}V \\ \mathbb{B}^*(Y) @>>{\gamma^*}> \mathbb{B}'^*(Y). \end{CD} \qquad \begin{CD} \mathbb{B}_*(Y) @>{\gamma_*}>> \mathbb{B}'_*(Y) \\ @V{f^!}VV @VV{u_f \bullet f^!}V \\ \mathbb{B}_*(X) @>>{\gamma_*}> \mathbb{B}'_*(X). \end{CD}$$

The most important and motivating example of such a Grothendieck transformation is Baum–Fulton–MacPherson's bivariant *Riemann–Roch transformation* ([20, Part II]):

$$\tau : \mathbb{K}_{alg} \to \mathbb{H} \otimes \mathbb{Q}\,,$$

or its algebraic counterpart of [19, Example 18.3.19]. Here $\mathcal{V} = \mathcal{V}_k^{qp}$ is the category of quasi-projective varieties over a base field k of any characteristic, with $\mathbb{H} = CH$ the bivariant operational Chow groups, or $\mathbb{H}$ the even degree bivariant homology in case $k = \mathbb{C}$. The independent squares in this context are the *Tor-independent* fiber squares. $\mathbb{K}_{alg}$ is the bivariant algebraic K-theory of relative perfect complexes, so that ${\mathbb{K}_{alg}}_*(X) = G_0(X)$ is the Grothendieck group of coherent sheaves and ${\mathbb{K}_{alg}}^*(X) = K^0(X)$ is the Grothendieck group of algebraic vector bundles. The associated contravariant transformation is the *Chern character*

$$\tau^* = ch : K^0(X) \to H^*(X) \otimes \mathbb{Q},$$

and the associated covariant transformation is the *Todd class transformation*

$$\tau_* = td_* : G_0(X) \to H_*(X) \otimes \mathbb{Q},$$

which is functorial for proper morphisms $f : X \to Y$. Moreover, they are related by the *module property*

$$td_*(\beta \cap \alpha) = ch(\beta) \cap td_*(\alpha) \quad \text{for all} \quad \beta \in K^0(X), \alpha \in G_0(X).$$

This generalizes the original Grothendieck- and Hirzebruch–Riemann–Roch Theorem. Both bivariant theories $\mathbb{K}_{alg}$ and $H_*(-) \otimes \mathbb{Q}$ are canonically oriented for the class $\mathcal{S}$ of smooth (or more generally of local complete intersection) morphism, with $\theta_{\mathbb{K}}(f) = \mathcal{O}_f := [\mathcal{O}_X] \in \mathbb{K}_{alg}(X \xrightarrow{f} Y)$ the class of the structure sheaf, and $\theta_{\mathbb{H}}(f) = [f] \in \mathbb{H}(X \xrightarrow{f} Y)$ the corresponding "relative fundamental class". They are related by the *Riemann–Roch formula*

$$\tau(\mathcal{O}_f) = td(T_f) \bullet [f]\,,$$

where $u_f := td(T_f) \in H^*(X) \otimes \mathbb{Q}$ and T_f is the (virtual) tangent bundle of f. See [20, (*) on p.124] for $\mathbb{H}$ the bivariant homology in case $k = \mathbb{C}$. For $\mathbb{H} = CH$ the bivariant Chow group and k of any characteristic, the above Riemann–Roch formula follows from [19, Theorem 18.2] as explained in [38]. The Riemann–Roch formula implies the following two results:

SGA 6-Riemann–Roch Theorem: The following diagram commutes for a proper smooth morphism $f : X \to Y$:

$$\begin{CD} K(X) @>{ch}>> H^*(X) \otimes \mathbb{Q} \\ @V{f_!}VV @VV{f_!(td(T_f) \cup \,-\,)}V \\ K(Y) @>>{ch}> H^*(Y) \otimes \mathbb{Q}. \end{CD}$$

Verdier–Riemann–Roch Theorem: The following diagram commutes for a smooth morphism $f : X \to Y$:

$$\begin{array}{ccc} G_0(Y) & \xrightarrow{td_*} & H_*(Y)\otimes\mathbb{Q} \\ {\scriptstyle f^!}\downarrow & & \downarrow{\scriptstyle td(T_f)\cap f^!} \\ G_0(X) & \xrightarrow[td_*]{} & H_*(X)\otimes\mathbb{Q}. \end{array}$$

Both formulae are more generally true for a local complete intersection morphism f, which is special to the Grothendieck transformation τ. In this paper only the case of a smooth morphism will be used, and then similar results are also true for the other considered Grothendieck transformations. It should also be remarked that *one motivation of Fulton–MacPherson's bivariant theory was to unify the above three Riemann–Roch theorems ...* (see [20, Part II, §0.1.4]).

Definition 2.4. (i) Let $\mathcal{S}$ be another class of maps in $\mathcal{V}$, called "specialized maps" (e.g., smooth maps in algebraic geometry), which is closed under composition and under base change and containing all identity maps. Let $\mathbb{B}$ be a bivariant theory. If $\mathcal{S}$ has canonical orientations in $\mathbb{B}$, then we say that $\mathcal{S}$ is *canonical* $\mathbb{B}$*-oriented* and an element of $\mathcal{S}$ is called a *canonical* $\mathbb{B}$*-oriented* morphism.

(ii) Assume furthermore, that the orientation θ on $\mathcal{S}$ satisfies $\theta(f') = g^*\theta(f)$ for any independent square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y \end{array}$$

with $f \in \mathcal{S}$ (which means that the orientation θ is preserved under the pullback operation). Then we call θ a *nice canonical orientation* and say that $\mathcal{S}$ is *nice canonical* $\mathbb{B}$*-oriented*. Similarly an element of $\mathcal{S}$ is called a *nice canonical* $\mathbb{B}$*-oriented* morphism.

Consider for example the class $\mathcal{S}$ of all *smooth* morphisms for $\mathcal{V} = \mathcal{V}_k^{(qp)}$ the category of (quasi-projective) varieties over a base field k of any characteristic, with all fiber squares as the independent squares. Then this class has a nice canonical orientation θ with respect to $\mathbb{K}_{alg}$ or CH in any characteristic (with $\theta(f) = \mathcal{O}_f$ or $[f]$), to $\tilde{\mathbb{F}}$ in characteristic zero (with $\theta(f) = \mathbb{1}_f$) and to $\mathbb{F}$ or bivariant homology $\mathbb{H}$ for $k = \mathbb{C}$ (with $\theta(f) = \mathbb{1}_f$ or $[f]$).

3. A UNIVERSAL BIVARIANT THEORY ON THE CATEGORY OF VARIETIES

Let $\mathcal{V}$ be the category $\mathcal{V} = \mathcal{V}_k^{(qp)}$ of (quasi-projective) varieties over a base field k of any characteristic, or the category $\mathcal{V} = \mathcal{V}_c^{an}$ of compact reduced complex analytic spaces, with all fiber squares as the independent squares. As the "confined" and "specialized" maps we take the class $\mathcal{P}rop$ of *proper* and $\mathcal{S}m$ of *smooth* morphisms, respectively.

Theorem 3.1 ([44], [38])**.** *We define*

$$\mathbb{M}(\mathcal{V}/X \xrightarrow{f} Y)$$

to be the free abelian group generated by the set of isomorphism classes of proper morphisms $h : W \to X$ such that the composite of h and f is a smooth morphism:

$$h \in \mathcal{P}rop \quad and \quad f \circ h : W \to Y \in \mathcal{S}m.$$

Then the association $\mathbb{M}$ *is a* bivariant theory *if the three operations are defined as follows:*
Product operation: For morphisms $f : X \to Y$ and $g : Y \to Z$, the product operation

$$\bullet : \mathbb{M}(\mathcal{V}/X \xrightarrow{f} Y) \otimes \mathbb{M}(\mathcal{V}/Y \xrightarrow{g} Z) \to \mathbb{M}(\mathcal{V}/X \xrightarrow{gf} Z)$$

is defined for $[V \xrightarrow{p} X] \in \mathbb{M}(\mathcal{V}/X \xrightarrow{f} Y)$ *and* $[W \xrightarrow{k} Y] \in \mathbb{M}(\mathcal{V}/Y \xrightarrow{g} Z)$ *by*

$$[V \xrightarrow{p} X] \bullet [W \xrightarrow{k} Y] := [V' \xrightarrow{p \circ k''} X],$$

and bilinearly extended. Here we consider the following fiber squares

$$\begin{CD}
V' @>{p'}>> X' @>{f'}>> W \\
@V{k''}VV @V{k'}VV @V{k}VV \\
V @>>{p}> X @>>{f}> Y @>>{g}> Z.
\end{CD}$$

<u>Pushforward operation</u>*: For morphisms* $f : X \to Y$ *and* $g : Y \to Z$ *with* $f \in \mathcal{P}rop$*, the pushforward operation* $f_* : \mathbb{M}(\mathcal{V}/X \xrightarrow{gf} Z) \to \mathbb{M}(\mathcal{V}/Y \xrightarrow{g} Z)$ *is defined by* $f_*([V \xrightarrow{p} X]) := [V \xrightarrow{f \circ p} Y]$ *and linearly extended.*

<u>Pullback operation</u>*: For an independent square* $\begin{CD} X' @>{g'}>> X \\ @V{f'}VV @VV{f}V \\ Y' @>>{g}> Y \end{CD}$ *the pullback operation*

$g^* : \mathbb{M}(\mathcal{V}/X \xrightarrow{f} Y) \to \mathbb{M}(\mathcal{V}/X' \xrightarrow{f'} Y')$ *is defined by* $g^*([V \xrightarrow{p} X]) := [V' \xrightarrow{p'} X']$ *and linearly extended. Here we consider the following fiber squares:*

$$\begin{CD}
V' @>{g''}>> V \\
@V{p'}VV @VV{p}V \\
X' @>{g'}>> X \\
@V{f'}VV @VV{f}V \\
Y' @>>{g}> Y.
\end{CD}$$

Remark 3.2. (1) The above bivariant theory $\mathbb{M}(\mathcal{V}/-)$ shall be called a *pre-motivic bivariant Grothendieck group* on the category $\mathcal{V}$ of varieties. $\theta(f) := [X \xrightarrow{\mathrm{id}_X} X]$ for the smooth morphism $f : X \to Y$ defines a nice canonical orientation on $\mathbb{M}(\mathcal{V}/-)$.
(2) $\mathbb{M}_*(\mathcal{V}/X) = \mathbb{M}(\mathcal{V}/X \to pt)$ is the free abelian group generated by the isomorphism classes $[V \xrightarrow{h} X]$, where h is proper and V is smooth. $\mathbb{M}_*(\mathcal{V}/-)$ is a covariant functor for proper morphisms and $\mathbb{M}_*(\mathcal{V}/-)$ is a contravariant functor for smooth morphisms.
(3) $\mathbb{M}^*(\mathcal{V}/X) = \mathbb{M}(\mathcal{V}/X \xrightarrow{\mathrm{id}_X} X)$ is the free abelian group generated by the isomorphism classes $[V \xrightarrow{h} X]$, where h is *proper and smooth*. It gets a ring structure $\cup$ by fiber products, with unit $1_X = [X \xrightarrow{\mathrm{id}_X} X]$. Then $\mathbb{M}^*(\mathcal{V}/-)$ is a contravariant functor for any morphisms and $\mathbb{M}^*(\mathcal{V}/-)$ is a covariant functor for morphisms which are smooth and proper.
(4) The bivariant product induces the following "cap product": $\cap : \mathbb{M}^*(\mathcal{V}/X) \times \mathbb{M}_*(\mathcal{V}/X) \to \mathbb{M}_*(\mathcal{V}/X)$. In particular, when X itself is a *smooth* variety, with $[X] := [X \xrightarrow{\mathrm{id}_X} X] \in \mathbb{M}_*(\mathcal{V}/X)$, we have the "Poincaré duality" homomorphism $\cap[X] : \mathbb{M}^*(\mathcal{V}/X) \to \mathbb{M}_*(\mathcal{V}/X)$, which is nothing but

$$[W \xrightarrow{k} X] \cap [X] = [W \xrightarrow{k} X].$$

More generally, the isomorphism class $[V \xrightarrow{h} X] \in \mathbb{M}_*(\mathcal{V}/X)$ of any proper morphism $h: V \to X$ from a *smooth* variety V to X gives rise to the homomorphism

$$\cap[V \xrightarrow{h} X] : \mathbb{M}^*(\mathcal{V}/X) \to \mathbb{M}_*(\mathcal{V}/X)$$

defined by $[W \xrightarrow{k} X] \cap [V \xrightarrow{h} X] = [W \times_X V \to X]$.

The pre-motivic bivariant Grothendieck group $\mathbb{M}(\mathcal{V}/-)$ has the following universal property:

Theorem 3.3 ([44], [38]). *Let $\mathbb{B}$ be a bivariant theory on $\mathcal{V}$ such that a smooth morphism f has a* nice canonical orientation *$\theta(f) \in \mathbb{B}(f)$, and let $c\ell : Vect(-) \to \mathbb{B}^*(-)$ be a contravariantly functorial characteristic class of algebraic (or analytic) vector bundles with values in the associated cohomology theory, which is multiplicative in the sense that $c\ell(V) = c\ell(V')c\ell(V'')$ for any short exact sequence of vector bundles $0 \to V' \to V \to V'' \to 0$. Assume $c\ell$ commutes with the canonical orientation θ, i.e. $\theta(f) \bullet cl(V) = f^*cl(V) \bullet \theta(f)$ for all smooth morphism $f : X \to Y$ and $V \in Vect(Y)$ (e.g. $\mathbb{B}$ is commutative).*

Then there exists a unique Grothendieck transformation $\gamma_{c\ell} : \mathbb{M}(\mathcal{V}/-) \to \mathbb{B}(-)$ satisfying the normalization condition that $\gamma_{c\ell}([X \xrightarrow{\mathrm{id}_X} X]) = c\ell(T_f) \bullet \theta(f)$ for a smooth morphism $f : X \to Y$. Here T_f is the relative tangent bundle of the smooth morphism f.

Remark 3.4. The above Grothendieck transformation $\gamma_{c\ell} : \mathbb{M}(\mathcal{V}/-) \to \mathbb{B}(-)$ satisfies the normalization condition $\gamma_{c\ell}([X \xrightarrow{\mathrm{id}_X} X]) = c\ell(T_f) \bullet \theta(f)$, which is nothing but the *Riemann-Roch formula* with $u_f = c\ell(T_f)$ for a smooth morphism $f : X \to Y$. So by the general theory we get the following Riemann–Roch theorems:

SGA 6 -type Riemann–Roch Theorem: The following diagram commutes for $f : X \to Y$ proper and smooth:

$$\begin{CD}
\mathbb{M}^*(\mathcal{V}/X) @>{\gamma_{c\ell}{}^*}>> \mathbb{B}^*(X) \\
@V{f_!}VV @VV{f_!(c\ell(T_f)\cup -)}V \\
\mathbb{M}^*(\mathcal{V}/Y) @>>{\gamma_{c\ell}{}^*}> \mathbb{B}^*(Y).
\end{CD}$$

Verdier-type Riemann–Roch Theorem: The following diagram commutes for a smooth morphism $f : X \to Y$:

$$\begin{CD}
\mathbb{M}_*(\mathcal{V}/X) @>{\gamma_{c\ell_*}}>> \mathbb{B}_*(X) \\
@V{f^!}VV @VV{c\ell(T_f)\cap f^!}V \\
\mathbb{M}_*(\mathcal{V}/Y) @>>{\gamma_{c\ell_*}}> \mathbb{B}_*(Y).
\end{CD}$$

Remark 3.5. (1) $\gamma_{c\ell} : \mathbb{M}(\mathcal{V}/X \xrightarrow{f} Y) \to \mathbb{B}(X \xrightarrow{f} Y)$ can be called a *bivariant pre-motivic characteristic class transformation*. When Y is a point pt,

$$\gamma_{c\ell *} : \mathbb{M}(\mathcal{V}/X \to pt) \to \mathbb{B}(X \to pt) = \mathbb{B}_*(X)$$

is the *unique natural transformation* satisfying that $\gamma_{c\ell *}([X \xrightarrow{id_X} X]) = c\ell(TX) \cap [X]$ for a smooth variety X. In other words, this gives rise to a *pre-motivic characteristic class transformation for singular varieties*. In a sense, this could be also a very general answer to MacPherson's question about the existence of a unified theory of characteristic classes for singular varieties. We emphasize that for the corresponding universal property of $\mathbb{M}(\mathcal{V}/X)$, we do not have to require the characteristic class $c\ell$ to be

multiplicative or to commute with the canonical orientation θ (since these properties are not used in the proof of Theorem 3.3 (iii), that $\gamma_{c\ell *}$ preserves the pushforward operation).
(2) In particular, we have the following commutative diagrams:

$$\begin{tikzcd} & \mathbb{M}_*(\mathcal{V}/X) \arrow[dl, "\epsilon"'] \arrow[dr, "\gamma_{c*}"] & \\ F(X) \arrow[rr, "c_*"'] & & H_*(X), \end{tikzcd}$$

with $H_*(X) = CH_*(X)$ in the algebraic context over a base field of characteristic zero, or $H_*(X) = H_{2*}^{BM}(X)$ in the complex algebraic or compact complex analytic context. Here $\epsilon([V \xrightarrow{h} X]) := h_* 1\!\!1_V$.

$$\begin{tikzcd} & \mathbb{M}_*(\mathcal{V}/X) \arrow[dl, "mC_0"'] \arrow[dr, "\gamma_{td*}"] & \\ G_0(X) \arrow[rr, "td_*"'] & & H_*(X) \otimes \mathbb{Q}, \end{tikzcd}$$

with $H_*(X) = CH_*(X)$ in the algebraic context over a base field of any characteristic, or $H_*(X) = H_{2*}^{BM}(X)$ in the complex algebraic or compact complex analytic context. Here $mC_0([V \xrightarrow{h} X]) := [h_* \mathcal{O}_V] = h_*[\mathcal{O}_V]$.

$$\begin{tikzcd} & \mathbb{M}_*(\mathcal{V}/X) \arrow[dl, "sd"'] \arrow[dr, "\gamma_{L*}"] & \\ \Omega(X) \arrow[rr, "L_*"'] & & H_{2*}^{BM}(X) \otimes \mathbb{Q}. \end{tikzcd}$$

Here X has to be a compact complex algebraic or analytic variety, with

$$sd([V \xrightarrow{h} X]) := [h_* \mathbb{Q}_V[\dim V]] = h_*[\mathbb{Q}_V[\dim V]] \,.$$

(3) It follows from Hironaka's resolution of singularities ([23]) that there exists a surjection

$$\mathbb{M}_*(\mathcal{V}/X) \to K_0(\mathcal{V}/X)$$

in the algebraic context over a base field of characteristic zero, or in the compact complex analytic context. It turns out that if the natural transformation $\gamma_{c\ell *} : \mathbb{M}_*(\mathcal{V}/X) \to H_*(X) \otimes R$ (with R a $\mathbb{Q}$-algebra) can be pushed down to the relative Grothendieck group $K_0(\mathcal{V}/X)$, then it has to be a specialization of the Hirzebruch class transformation under a ring homomophism $\mathbb{Q}[y] \to R$, i.e., the following diagram commutes (see [7]):

$$\begin{tikzcd} \mathbb{M}_*(\mathcal{V}/X) \arrow[r, "\gamma_{c\ell *}"] \arrow[d, "q"'] & H_*(X) \otimes R \\ K_0(\mathcal{V}/X) \arrow[r, "T_{y*}"'] & H_*(X) \otimes \mathbb{Q}[y] \,. \arrow[u] \end{tikzcd}$$

And one of the main results of our previous paper [7] claims that in this context the above three diagrams also commute with $\mathbb{M}_*(\mathcal{V}/X)$ being replaced by the smaller group $K_0(\mathcal{V}/X)$ (fitting with T_{y*} for $y = -1, 0$ or 1).

Now it is natural to pose the following

Problem 3.6. *Formulate a reasonable bivariant analogue $\mathbb{K}_0(\mathcal{V}/X \xrightarrow{f} Y)$ of the relative Grothendieck group $K_0(\mathcal{V}/X)$ so that the following hold:*
(1) There is a natural group homomorphism $q : \mathbb{K}_0(\mathcal{V}/X \to pt) \to K_0(\mathcal{V}/X)$, which is an isomorphism in the algebraic context over a base field of characteristic zero, or in the compact complex analytic context.
(2) $\mathbb{B}q : \mathbb{M}(\mathcal{V}/X \xrightarrow{f} Y) \to \mathbb{K}_0(\mathcal{V}/X \xrightarrow{f} Y)$ is a certain quotient map, which specializes for Y a point to the quotient map $q : \mathbb{M}_*(\mathcal{V}/X) \to K_0(\mathcal{V}/X)$.
(3) $T_y : \mathbb{K}_0(\mathcal{V}/X \xrightarrow{f} Y) \to \mathbb{H}(X \xrightarrow{f} Y) \otimes \mathbb{Q}[y]$ is a Grothendieck transformation, which specializes for Y a point (in the algebraic context over a base field of characteristic zero, or in the compact complex analytic context) to the motivic Hirzebruch class transformation $T_{y_*} : K_0(\mathcal{V}/X) \to H_*(X) \otimes \mathbb{Q}[y]$.
(4) The following diagram commutes:

$$\begin{tikzcd}
 & \mathbb{M}(\mathcal{V}/X \xrightarrow{f} Y) \arrow[dl, "\mathbb{B}q"'] \arrow[dr, "\gamma_{T^*_{y*}}"] & \\
\mathbb{K}_0(\mathcal{V}/X \xrightarrow{f} Y) \arrow[rr, "T_y"'] & & \mathbb{H}(X \xrightarrow{f} Y) \otimes \mathbb{Q}[y].
\end{tikzcd}$$

Remark 3.7. The associated contravariant functor of such a bivariant theory $\mathbb{K}_0(\mathcal{V}/X \xrightarrow{f} Y)$, namely $K^0(\mathcal{V}/X) := \mathbb{K}_0(\mathcal{V}/X \xrightarrow{\mathrm{id}_X} X)$, can be considered as a contravariant counterpart of the relative Grothen-dieck group $K_0(\mathcal{V}/X)$. The natural transformation $T^*_y : K^0(\mathcal{V}/-) \to H^*(-) \otimes \mathbb{Q}[y]$ can be considered as a contravariant counterpart of the Hirzebruch class transformations T_{y*} satisfying the *module property*.

4. A BIVARIANT GROTHENDIECK GROUP $\mathbb{K}_0(\mathcal{V}/X \xrightarrow{f} Y)$

The following theorem is proved using the "Weak Factorisation Theorem" of [1, 40]:

Theorem 4.1 (Franziska Bittner [5])**.** *Let $K_0(\mathcal{V}/X)$ be the relative Grothendieck group of varieties over $X \in obj(\mathcal{V})$, with $\mathcal{V} = \mathcal{V}_k^{(qp)}$ (resp., $\mathcal{V} = \mathcal{V}_c^{an}$) the category of (quasi-projective) algebraic (resp., compact complex analytic) varieties over a base field k of characteristic zero. Then $K_0(\mathcal{V}/X)$ is isomorphic to $\mathbb{M}_*(X)$ modulo the "blow-up" relation*

$$[\emptyset \to X] = 0 \quad \textit{and} \quad [Bl_Y X' \to X] - [E \to X] = [X' \to X] - [Y \to X]\,,$$

for any cartesian diagram (which shall be called the "blow-up diagram" from here on)

$$\begin{tikzcd}
E \arrow[r, "i'"] \arrow[d, "q'"] & Bl_Y X' \arrow[d, "q"] & \\
Y \arrow[r, "i"] & X' \arrow[r, "f"] & X\,,
\end{tikzcd}$$

with i a closed embedding of smooth spaces and $f : X' \to X$ proper. Here $Bl_Y X' \to X'$ is the blow-up of X' along Y with exceptional divisor E. Note that all these spaces other than X are also smooth (and quasi-projective in case $X', Y \in ob(\mathcal{V}_k^{qp})$).

The kernel of the canonical quotient map $q : \mathbb{M}_*(\mathcal{V}/X) \to K_0(\mathcal{V}/X)$ is the subgroup $BL(\mathcal{V}/X)$ of $\mathbb{M}_*(\mathcal{V}/X)$ generated by $[Bl_Y X' \to X] - [E \to X] - [X' \to X] + [Y \to X]$ for any blow-up diagram as above.

To obtain a bivariant analogue of the subgroup $BL(\mathcal{V}/X)$, we first observe the following result:

Lemma 4.2. *Let $h : X' \to X$ be a* smooth *morphism, with $i : S \to X'$ a closed embedding such that the composite $h \circ i : Z \to X$ is also* smooth *morphism. Consider the cartesian diagram*

$$\begin{array}{ccccc} E & \xrightarrow{i'} & Bl_S X' & & \\ {\scriptstyle q'}\big\downarrow & & \big\downarrow{\scriptstyle q} & & \\ S & \xrightarrow[i]{} & X' & \xrightarrow[h]{} & X\,, \end{array}$$

with $q : Bl_S X' \to X'$ the blow-up of X' along S and $q' : E \to S$ the exceptional divisor map. Then:

(1) $h \circ q : Bl_S X' \to X$ and $h \circ q \circ i' : E \to X$ are also smooth *morphisms, with $Bl_S X', E$ quasi-projective in case $X', Y \in ob(\mathcal{V}_k^{qp})$.*

(2) This blow-up diagram commutes with any base change *in X, i.e. the corresponding fiber-square induced by pullback along a morphism $\tilde{X} \to X$ is isomorphic to the corresponding blow-up diagram of $\tilde{S} \to \tilde{X}'$.*

(3) The closed embeddings i, i' are regular *embeddings, and the projection map q as well as i, i' are of* finite Tor-dimension.

Definition 4.3. For a morphism $f : X \to Y$ in the category $\mathcal{V} = \mathcal{V}_k^{(qp)}$ or $\mathcal{V} = \mathcal{V}_c^{an}$, we consider a blow-up diagram

$$\begin{array}{ccccccc} E & \xrightarrow{i'} & Bl_S X' & & & & \\ \big\downarrow{\scriptstyle q'} & & \big\downarrow{\scriptstyle q} & & & & \\ S & \xrightarrow{i} & X' & \xrightarrow{h} & X & \xrightarrow{f} & Y\,, \end{array}$$

with h proper and i a closed embedding such that $f \circ h$ *as well as* $f \circ h \circ i$ *are smooth.*

Let $\mathbb{BL}(\mathcal{V}/X \xrightarrow{f} Y)$ be the abelian subgroup of $\mathbb{M}(\mathcal{V}/X \xrightarrow{f} Y)$ generated by

$$[Bl_S X' \xrightarrow{hq} X] - [E \xrightarrow{hiq'} X] - [X' \xrightarrow{h} X] + [S \xrightarrow{hi} X]$$

for any such diagram, and we define

$$\mathbb{K}_0(\mathcal{V}/X \xrightarrow{f} Y) := \frac{\mathbb{M}(\mathcal{V}/X \xrightarrow{f} Y)}{\mathbb{BL}(\mathcal{V}/X \xrightarrow{f} Y)}\,.$$

The corresponding equivalence class of $[V \xrightarrow{p} X]$ shall be denoted by $\Big[[V \xrightarrow{p} X]\Big]$.

Theorem 4.4 ([38])**.** *Let $\mathcal{V} = \mathcal{V}_k^{(qp)}$ be the category of (quasi-projective) algebraic varieties (i.e., reduced separated schemes of finite type) over a base field k of any characteristic, or let $\mathcal{V} = \mathcal{V}_c^{an}$ be the category of compact reduced complex analytic spaces. Then $\mathbb{K}_0(\mathcal{V}/X \xrightarrow{f} Y)$ becomes a bivariant theory with the following three operations, so that the canonical projection $\mathbb{B}q : \mathbb{M}(\mathcal{V}/-) \to \mathbb{K}_0(\mathcal{V}/-)$ is a Grothendieck transformation.*

Product operation: For morphisms $f : X \to Y$ and $g : Y \to Z$ the product operation

$$\star : \mathbb{K}_0(\mathcal{V}/X \xrightarrow{f} Y) \otimes \mathbb{K}_0(\mathcal{V}/Y \xrightarrow{g} Z) \to \mathbb{K}_0(\mathcal{V}/X \xrightarrow{gf} Z)$$

is defined by $\Big[[V \xrightarrow{h} X]\Big] \star \Big[[W \xrightarrow{k} Y]\Big] := \Big[[V \xrightarrow{h} X] \bullet [W \xrightarrow{k} Y]\Big]$ *and bilinearly extended.*

Pushforward operation: For morphisms $f : X \to Y$ and $g : Y \to Z$ with $f \in \mathcal{P}rop$ the pushforward operation

$$f_* : \mathbb{K}_0(\mathcal{V}/X \xrightarrow{gf} Z) \to \mathbb{K}_0(\mathcal{V}/Y \xrightarrow{g} Z)$$

is defined by $f_*\Big(\Big[[V \xrightarrow{p} X]\Big]\Big) := \Big[f_*([V \xrightarrow{p} X])\Big]$ *and linearly extended.*

Pullback operation: *For an independent square*
$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y \end{array}$$
the pullback operation

$$g^* : \mathbb{K}_0(\mathcal{V}/X \xrightarrow{f} Y) \to \mathbb{K}_0(\mathcal{V}/X' \xrightarrow{f'} Y')$$

is defined by $g^*\Big(\Big[[V \xrightarrow{p} X]\Big]\Big) := \Big[g^*([V \xrightarrow{p} X])\Big]$ *and linearly extended.*

For the proof of this theorem one only has to show that the three bivariant operations are well defined, i.e. that the subgroup $\mathbb{BL}(\mathcal{V}/X \xrightarrow{f} Y)$ is stable under the bivariant operations. For the pushforward this is clear, but for pullback and product this uses Lemma 4.2(2), as well as the fact that blowing up commutes with smooth (or more generally flat) pullback.

Remark 4.5. In the case when Y is a point, the blow-up diagram defining $\mathbb{BL}(\mathcal{V}/X \xrightarrow{f} pt)$ is nothing but the following:

$$\begin{array}{ccccc} E & \xrightarrow{i'} & Bl_S X' & & \\ \downarrow{\scriptstyle q'} & & \downarrow{\scriptstyle q} & & \\ S & \xrightarrow{i} & X' & \xrightarrow{h} & X\,, \end{array}$$

such that $h : X' \to X$ is proper, X' and S are nonsingular, and $q : Bl_S X' \to X'$ is the blow-up of X' along S with $q' : E \to S$ the exceptional divisor map. Hence $\mathbb{BL}(\mathcal{V}/X \xrightarrow{f} pt)$ is nothing but $BL(\mathcal{V}/X)$, i.e., we have by Bittner's theorem $\mathbb{K}_0(\mathcal{V}/X \to pt) \simeq K_0(\mathcal{V}/X)$ in the compact complex analytic context, as well as in the algebraic context over a base field of characteristic zero. Finally note that we always have a group homomorphism $\mathbb{K}_0(\mathcal{V}/X \to pt) \to K_0(\mathcal{V}/X)$, since $Bl_S X' \backslash E \simeq X' \backslash S$ in the diagram above so that

$$[Bl_S X' \to X] - [E \to X] = [X' \to X] - [S \to X] \in K_0(\mathcal{V}/X)\,.$$

5. Motivic bivariant Chern and Hirzebruch class transformations

Now we are ready to state the following main theorem, which is about the *motivic bivariant Chern and Hirzebruch class transformations*.

Theorem 5.1 ([38])**.** *Let $\mathcal{V} = \mathcal{V}_k^{qp}$ be the category of quasi-projective algebraic varieties over a base field k of any characteristic.*
(1) There exists a unique Grothendieck transformation

$$mC_y = \Lambda_y^{mot} : \mathbb{K}_0(\mathcal{V}_k^{qp}/-) \to \mathbb{K}_{alg}(-) \otimes \mathbb{Z}[y]$$

satisfying the normalization condition that

$$\Lambda_y^{mot}\Big(\Big[[X \xrightarrow{\mathrm{id}_X} X]\Big]\Big) = \Lambda_y(T_f^*) \bullet \mathcal{O}_f$$

for a smooth morphism $f : X \to Y$.
(2) Let $T_y : \mathbb{K}_0(\mathcal{V}_k^{qp}/-) \to \mathbb{H}(-) \otimes \mathbb{Q}[y]$ be defined as the composition $\tau \circ \Lambda_y^{mot}$, renormalized by $\times(1+y)^i$ on $\mathbb{H}^i(-) \otimes \mathbb{Q}[y]$. Here $\mathbb{H}$ is either the operational bivariant Chow group or the even degree

bivariant homology theory for $k = \mathbb{C}$, with τ the corresponding Riemann-Roch transformation. Then T_y is the unique Grothendieck transformation satisfying the normalization condition that

$$T_y\Big(\Big[[X \xrightarrow{\mathrm{id}_X} X]\Big]\Big) = td_y(T_f) \bullet [f]$$

for a smooth morphism $f : X \to Y$.

Remark 5.2. (1) Let $c\ell : Vect(-) \to K^0(-) \otimes \mathbb{Z}[y] = \mathbb{K}^*_{alg}(-) \otimes \mathbb{Z}[y]$ be the characteristic class transformation $c\ell(V) := \lambda_y(V^*)$ given by the total λ-class of the dual vector bundle V^*. Then by Theorem 3.3 there is a unique Grothendieck transformation

$$\gamma_{c\ell} : \mathbb{M}(\mathcal{V}/-) \to \mathbb{K}_{alg}(-) \otimes \mathbb{Z}[y]$$

satisfying the normalization condition in (1) above. So one only has to show that this transformation $\gamma_{c\ell}$ vanishes on all subgroups $\mathbb{BL}(\mathcal{V}/X \xrightarrow{f} Y)$ generated by the "bivariant blow-up relations". The proof of this given in [38] is based on [22, Chapter IV, Theorem 1.2.1 and (1.2.6)].
(2) For the transformation

$$T_y := (1+y)^* \times \tau \circ \Lambda^{mot}_y : \mathbb{K}_0(\mathcal{V}^{qp}_k/-) \to \mathbb{H}(-) \otimes \mathbb{Q}[y, (1+y)^{-1}]$$

one only has to check the normalization condition of (2) above, which (as explained in [38]) follows from the Riemann–Roch formula

$$\tau(\mathcal{O}_f) = td(T_f) \bullet [f] \,,$$

for the bivariant Riemann–Roch transformation τ (with f smooth).

Corollary 5.3 ([38])**.** *Let $\mathcal{V} = \mathcal{V}^{qp}_k$ be the category of quasi-projective algebraic varieties over a base field k of any characteristic. Then we have the following commutative diagrams of Grothendieck transformations:*
(1)

$$\begin{tikzcd} & \mathbb{K}_0(\mathcal{V}^{qp}_k/-) \arrow[dl, "mC_0"'] \arrow[dr, "T_0"] & \\ \mathbb{K}_{alg}(-) \arrow[rr, "\tau"'] & & \mathbb{H}(-) \otimes \mathbb{Q}. \end{tikzcd}$$

(2)

$$\begin{tikzcd} & \mathbb{K}_0(\mathcal{V}^{qp}_k/-) \arrow[dl, "\epsilon"'] \arrow[dr, "T_{-1}"] & \\ \tilde{\mathbb{F}}(-) \arrow[rr, "\gamma"'] & & CH(-) \otimes \mathbb{Q}, \end{tikzcd}$$

if k is of characteristic zero. Here ϵ is the unique Grothendieck transformation satisfying the normalization condition $\epsilon\Big(\Big[[X \xrightarrow{\mathrm{id}_X} X]\Big]\Big) = \mathbb{1}_f$ for a smooth morphism $f : X \to Y$. Similarly for the bivariant Chern class transformation $\gamma : \mathbb{F}(-) \to A^{PI}(-) \otimes \mathbb{Q} \supset CH(-) \otimes \mathbb{Q}$ in case $k = \mathbb{C}$.
(3) Assume k is of characteristic zero. Then the associated covariant transformations in Theorem 5.1 (1) and (2) agree under the identification $\mathbb{K}_0(\mathcal{V}^{qp}_k/X \to pt) \simeq K_0(\mathcal{V}^{qp}_k/X)$ with the motivic Chern and Hirzebruch class transformations mC_y and T_{y*}.

Remark 5.4. We would speculate that Brasselet's bivariant Chern class transformation $\gamma : \mathbb{F}(-) \to \mathbb{H}(-)$ to Fulton–MacPherson's bivariant homology $\mathbb{H}(-)$ (see [6]) satisfies for a smooth morphism $f : X \to Y$ the "strong normalization condition" $\gamma(\mathbb{1}_f) = c(T_f) \bullet [f] \in \mathbb{H}(X \xrightarrow{f} Y)$ with $[f]$ the corresponding relative fundamental class. If this is the case, then Corollary 5.3 (2) would also be true for Brasselet's bivariant Chern class transformation $\gamma : \mathbb{F}(-) \to \mathbb{H}(-)$.

6. Oriented bivariant theories

In [28] Levine and Morel defined the algebraic cobordism (group) $\Omega^*_{LM}(X)$ for a *smooth* variety X, which is a contravariant functor. In fact they first constructed algebraic "bordism" theory $\Omega^{LM}_*(X)$ for *any* variety X as a covariant functor (for projective morphisms), with functorial Gysin maps for local complete intersection morphisms. Then the algebraic cobordism of a smooth (pure-dimensional) variety X is $\Omega^n_{LM}(X) := \Omega^{LM}_{\dim X - n}(X)$.

Our naïve question was

Question 6.1. What is a "real" cobordism theory on varieties or a contravariant version of Levine–Morel's algebraic "bordism" theory $\Omega^{LM}_*(X)$?

Their definition of $\Omega^{LM}_*(X)$ and their main theorem could be put as follows, omitting the things which we do not need in this paper.

Definition-Theorem 6.2. *We consider the category $\mathcal{V} = \mathcal{V}_k$ algebraic varieties over a base field k. For a variety X the "bordism group" $\Omega^{LM}_*(X)$ of algebraic cobordism cycles over X is defined by*

$$\Omega^{LM}_*(X) := \frac{\left\{[V \xrightarrow{h} X; L_1, L_2, \cdots, L_m] \,\middle|\, V \textit{ is smooth}, h \textit{ is projective}, \ L_i \textit{ are line bundles over } V\right\}^+}{\textit{three relations} \begin{cases} \textit{(Dim) Dimension Axiom,} \\ \textit{(Sect) Section Axiom,} \\ \textit{(FGL) Formal Group Law Axiom.} \end{cases}}.$$

Then for k of characteristic zero [1]*, $\Omega^{LM}_*(X)$ is the universal* *oriented* *(graded)* *Borel–Moore functor with products* *of geometric type.*

The line bundles L_i in this definition are related to an "orientation" of this theory via the "first Chern class operator"

$$(1) \quad \widetilde{c_1}(L) : \Omega^{LM}_*(X) \to \Omega^{LM}_{*-1}(X); \ [V \xrightarrow{h} X; L_1, L_2, \cdots, L_m] \mapsto [V \xrightarrow{h} X; L_1, L_2, \cdots, L_m, h^*L]$$

of a line bundle L over X, with $[-]$ denoting an isomorphism class. Here an isomorphism of algebraic cobordism cycles over X of (formal) dimension $dim(V) - m$

$$(V \xrightarrow{h} X; L_1, L_2, \cdots, L_m) \simeq (V' \xrightarrow{h'} X; L'_1, L'_2, \cdots, L'_m)$$

is given by an isomorphism $g : V \simeq V'$ with $h = h' \circ g$, such that $L_i \simeq g^* L'_{\sigma(i)}$ for all i and a permutation σ of $\{1, \ldots, m\}$. Then the set of such isomorphism classes

$$\left\{[V \xrightarrow{h} X; L_1, L_2, \cdots, L_m] \,\middle|\, V \text{ is smooth}, h \text{ is projective}, \ L_i \text{ are line bundles over } V\right\}$$

becomes a monoid with respect to disjoint union $\sqcup$, with unit the empty cobordism cycle. Then $\{\cdots\}^+$ denotes the corresponding group completion (of formal differences), which here is graded by the (formal) dimension of an algebraic cobordism cycle.

The three imposed relations, (D) Dimension Axiom, (S) Section Axiom, and (FGL) Formal Group Law Axiom, are related to the notion "of geometric type", which also involves the first Chern class operator, as shown in Definition 6.5 below. Hence, dropping all things related to first Chern class operators of line bundles, the group completion of the corresponding monoid (with respect to disjoint union $\sqcup$)

$$M_*(X) := \left\{[V \xrightarrow{h} X] \,\middle|\, V \text{ is smooth}, h \text{ is projective}\right\}^+$$

[1] They use resolution of singularities, thus characteristic zero is necessary since the case of resolution of singularities in a positive characteristic is still unresolved.

is the universal (graded) "Borel–Moore functor with products" in the following sense:

Definition 6.3. A covariant functor H_* to the category of (graded) abelian groups is called a *(graded) Borel–Moore functor with products*, if it satisfies the following conditions:

(BM-1) it is covariantly functorial (preserving degrees) for pushforward of projective morphisms (called "projective pushforward").

(BM-2) it is additive, i.e. $H_*(X) \oplus H_*(Y) \simeq H_*(X \sqcup Y)$ via (BM-1) for the closed inclusions $X \to X \sqcup Y$ and $Y \to X \sqcup Y$.

(BM-3) it is contravariantly functorial (shifting degrees by the fiber dimension) for pullback of smooth morphisms (called "smooth pullback").

(BM-4) for any fiber square
$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y \end{array}$$
with f projective and g smooth (hence f' projective and g' smooth), $g^* f_* = f'_*(g')^*$.

(BM-5) there exists a cross or external product $\times : H_*(X) \times H_*(Y) \to H_*(X \times Y)$ (adding degrees), which is commutative and associative together with a unit $1_{pt} \in H_0(pt)$, such that it commutes with projective pushforwards and smooth pullbacks.

Note that a smooth variety M has a "fundamental class" (with $\pi_M : M \to pt$ the constant smooth morphism):
$$[M] := 1_M := \pi_M^* 1_{pt} \in H_*(M) \, .$$
For example, the relative Grothendieck group of algebraic varieties $K_0(\mathcal{V}/-)$ is such a Borel–Moore functor with products. Similarly, $M_*(-)$ is covariantly functorial for projective morphisms by composition of arrows, as well as contravariantly functorial for smooth morphisms by taking fiber products, with the cross product given in the obvious way. It is graded by the dimension of V. Finally, also the Grothendieck group $G_0(-)$ of coherent sheaves, as well as for a base field k of characteristic zero the group of constructible functions $F(-)$, are Borel–Moore functors with products. Similarly for the cobordism group of selfdual constructible sheaf complexes $\Omega(X)$ in the complex algebraic or analytic context. Here we can even consider proper morphisms instead of projective morphisms in the definition above. Of course, $K_0(\mathcal{V}/-), G_0(-)$ and $F(-)$ are ungraded, i.e. $H_*(-) = H_0(-)$, whereas $\Omega(X)$ is $\mathbb{Z}_2$-graded.

Definition 6.4. Let H_* be a (graded) Borel–Moore functor with products. It is called *oriented* if for any line bundle L on X there exists a homomorphism (called "the first Chern class operator")
$$\widetilde{c_1}(L) : H_*(X) \to H_{*-1}(X)$$
such that

(OBM-6) for line bundles L, L' over X the two first Chern class operators commute; i.e.,
$$\widetilde{c_1}(L) \circ \widetilde{c_1}(L') = \widetilde{c_1}(L') \circ \widetilde{c_1}(L) \, .$$
Moreover $\widetilde{c_1}(L) = \widetilde{c_1}(L')$ for isomorphic line bundles L and L' .

(OBM-7) it is compatible with the projective pushforward, i.e., for a projective map $f : X \to Y$ with L a line bundle over Y the following diagram commutes (i.e., $f_* \circ \widetilde{c_1}(f^*L) = \widetilde{c_1}(L) \circ f_*$) :
$$\begin{array}{ccc} H_*(X) & \xrightarrow{f_*} & H_*(Y) \\ {\scriptstyle \widetilde{c_1}(f^*L)}\downarrow & & \downarrow{\scriptstyle \widetilde{c_1}(L)} \\ H_{*-1}(X) & \xrightarrow[f_*]{} & H_{*-1}(Y) \, . \end{array}$$

(OBM-8) it is compatible with the smooth pullback, i.e., for a smooth map $f : X \to Y$ with L a line bundle over Y the following diagram commutes (i.e., $f^* \circ \widetilde{c_1}(L) = \widetilde{c_1}(f^*L) \circ f^*$) :

$$\begin{tikzcd} H_*(X) \arrow[d, "\widetilde{c_1}(f^*L)"'] & H_*(Y) \arrow[l, "f^*"'] \arrow[d, "\widetilde{c_1}(L)"] \\ H_{*-1}(X) & H_{*-1}(Y) \arrow[l, "f^*"] \end{tikzcd} .$$

(OBM-9) the first Chern class operator commutes with the cross product, i.e., for a line bundle L over X, $\pi_1 : X \times Y \to X$ and $\alpha \in H_*(X)$ and $\beta \in H_*(Y)$ we have

$$\widetilde{c_1}(L)(\alpha) \times \beta = \widetilde{c_1}(\pi_1^* L)(\alpha \times \beta) .$$

For example, the universal *oriented* (graded) Borel-Moore functor with products is given by the group completion of the monoid of cobordism cycles (as before):

$$Z_*(X) := \left\{ [V \xrightarrow{h} X; L_1, L_2, \cdots, L_m] \middle| V \text{ is smooth}, h \text{ is projective}, L_i \text{ are line bundles over } V \right\}^+ .$$

Here we are using the obvious notion of a *natural transformation* of (oriented) Borel-Moore functors with product to make sense of the corresponding universal property. A simple example of such a transformation in the complex algebraic context is the "cycle class map" $[-] : CH_* \to H_{2*}^{BM}(-)$.

Definition 6.5. Let H_* be an oriented (graded) Borel–Moore functor with products. It is called *of geometric type* if the following three conditions holds:
(GT-10) (Dimension Axiom): For line bundles $L_1, L_2, \cdots, L_r$ on a smooth scheme M with $r > \dim M$,

$$\widetilde{c_1}(L_1) \circ \widetilde{c_1}(L_2) \circ \cdots \circ \widetilde{c_1}(L_r)([M]) = 0 \in H_*(M) .$$

(GT-11) (Section Axiom): Let L be a line bundle over a smooth scheme M with $s : M \to L$ a section transverse to the zero section of L. Let $Z := s^{-1}(0)$ and $i_Z : Z \to M$ be the inclusion of this submanifold. Then we have

$$\widetilde{c_1}(L)([M]) = (i_Z)_*([Z]) \in H_*(M) .$$

(GT-12) (Formal Group Law Axiom): Let L, L' be line bundles over a smooth scheme M. There is a formal group law $F_{H_*}(u, v) \in H_*(pt)[[u, v]]$ such that

$$F_{H_*}(\widetilde{c_1}(L), \widetilde{c_1}(L')) = \widetilde{c_1}(L \otimes L'),$$

and $F_{H_*}(u, v)$ is the image of the universal formal group law $F_{\mathbb{L}}(u, v) \in \mathbb{L}_*[[u, v]]$ under the homomorphism $\phi_{H_*} : \mathbb{L}_* \to H_*(pt)$ classifying the formal group law F_{H_*} on $H_*(pt)$. Here $\mathbb{L}_*$ is the Lazard ring.

For example the Chow group $CH_*(-)$, or in the complex algebraic context also the even degree Borel-Moore homology $H_{2*}^{BM}(-)$, have a canonical orientation with the *additive* formal group law F_a, whereas the Grothendieck group of coherent sheaves $G_0(-)$ has a canonical orientation with the *multiplicative* formal group law F_m:

$$F_a(u, v) := u + v \quad \text{and} \quad F_m(u, v) := u + v - u \cdot v .$$

Finally, a deep result of Levine–Morel [28] tells us that the formal group law of their algebraic cobordism $\Omega_*^{LM}(-)$ over a base field k of characteristic zero is given by the *universal* formal group of the Lazard ring $\mathbb{L}_*$.

In [44] we introduced (in greater generality) the following notion of an *oriented bivariant theory*.

Definition 6.6. Let $\mathbb{B}$ be a (graded) bivariant theory on the category $\mathcal{V} = \mathcal{V}_k$ of algebraic varieties over a base field k, with all fiber squares as the independent squares, and the projective (or more generally proper) morphisms as the confined maps.

Then $\mathbb{B}$ is called *oriented* if for any morphism $g : X \to Y$ and a line bundle L on X there exists a homomorphism (of degree one in the graded context, called "the first Chern class operator")

$$\widetilde{c_1}(L) : \mathbb{B}(X \xrightarrow{g} Y) \to \mathbb{B}(X \xrightarrow{g} Y)$$

such that

(OB-1) for line bundles L, L' over X the two first Chern class operators commute, i.e.,

$$\widetilde{c_1}(L) \circ \widetilde{c_1}(L') = \widetilde{c_1}(L') \circ \widetilde{c_1}(L) : \mathbb{B}(X \xrightarrow{g} Y) \to \mathbb{B}(X \xrightarrow{g} Y)\,.$$

Moreover $\widetilde{c_1}(L) = \widetilde{c_1}(L')$ for L, L' isomorphic line bundles.

(OB-2) it is compatible with the pushforward, i.e., for a confined map $f : X \to Y$ with L a line bundle over Y one has for all morphisms $g : Y \to Z$:

$$f_* \circ \widetilde{c_1}(f^*L) = \widetilde{c_1}(L) \circ f_* : \mathbb{B}(X \xrightarrow{g\circ f} Z) \to \mathbb{B}(Y \xrightarrow{g} Z)\,.$$

(OB-3) it is compatible with pullback, i.e., for any independent square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\Big\downarrow & & \Big\downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y \end{array}$$

and for L a line bundle over X one has:

$$g^* \circ \widetilde{c_1}(L) = \widetilde{c_1}(g'^*L) \circ g^* : \mathbb{B}(X \xrightarrow{f} Y) \to \mathbb{B}(X' \xrightarrow{f'} Y')\,.$$

(OB-4) the first Chern class operator commutes with the bivariant product, i.e., for all morphisms $f : X \to Y$, resp., $g : Y \to Z$, and a line bundle L over X, resp., L' over Y, one has:

$$\widetilde{c_1}(L)(\alpha \bullet \beta) = \widetilde{c_1}(L)(\alpha) \bullet \beta, \quad \text{resp.,} \quad \widetilde{c_1}(f^*L')(\alpha \bullet \beta) = \alpha \bullet \widetilde{c_1}(L')(\beta)$$

for all $\alpha \in \mathbb{B}(X \xrightarrow{f} Y)$ and $\beta \in \mathbb{B}(Y \xrightarrow{g} Z)$.

Assume, for example, that the oriented (graded) bivariant theory $\mathbb{B}$ on the category $\mathcal{V} = \mathcal{V}_k$ is also commutative, together with a nice canonical orientation for the class of smooth morphisms. Then it is easy to see that the associated covariant functor $\mathbb{B}_*$ is a (graded) oriented Borel-Moore functor with products, except maybe for the "additivity" property (BM-2) (required in Levine–Morel [28]), which in the bivariant context is often not needed. Assume in addition that the bivariant theory $\mathbb{B}$ is also "additive" in the sense that for all morphisms $g : X \sqcup Y \to Z$:

$$i_{X*} \oplus i_{Y*} : \mathbb{B}(X \xrightarrow{g|_X} Z) \oplus \mathbb{B}(Y \xrightarrow{g|_Y} Z) \xrightarrow{\sim} \mathbb{B}(X \sqcup Y \xrightarrow{g} Z)$$

for the closed (and also projective) inclusions $i_X : X \to X \sqcup Y$ and $i_Y : Y \to X \sqcup Y$ into the disjoint union. Then $\mathbb{B}_*$ also satisfies property (BM-2) so that it is a (graded) oriented Borel-Moore functor with products. Finally, the canonical orientation θ is called "additive" if

$$i_{X*}(\theta(g|_X)) + i_{Y*}(\theta(g|_Y)) = \theta(g)$$

for all smooth morphisms $g : X \sqcup Y \to Z$.

As an "oriented" analogue of the pre-motivic bivariant theory $\mathbb{M}(\mathcal{V}/X \xrightarrow{f} Y)$, in [44] we showed the following counterpart of Theorem 3.1, with (more or less) the same definition of the bivariant operations and the first Chern class operator similarly to (1) (given right after Definition-Theorem 6.2):

Theorem 6.7 (Universal oriented bivariant theory)**.** *Let*

$$\mathbb{OB}(X \xrightarrow{f} Y) := \left\{ [V \xrightarrow{h} X; L_1, \cdots L_m] \,\middle|\, h \textit{ is projective}, f \circ h \textit{ is smooth}, \ L_i \textit{ are line bundles over } V \right\}$$

be the free abelian group on isomorphism classes of f-relative cobordism cycles $[V \xrightarrow{h} X; L_1, \cdots L_m]$.

Then we have:

(1) $\mathbb{OB}(X \xrightarrow{f} Y)$ is a universal oriented bivariant theory (graded by m minus the fiber dimension of $f \circ h$).

(2) $\mathbb{OB}_*(X) := \mathbb{OB}(X \to pt)$ is a universal oriented Borel–Moore functor with products (graded by the dimension of V minus m), but without the "additivity" property (BM-2).

If one wants to get the corresponding "additive" counterparts, then one only has to work with the group completion

$$\mathbb{OB}^+(X \xrightarrow{f} Y) := \left\{ [V \xrightarrow{h} X; L_1, \cdots L_m] \,\middle|\, h \text{ is projective}, f \circ h \text{ is smooth}, \ L_i \text{ are line bundles over } V \right\}^+$$

of the monoid of isomorphism classes of f-relative cobordism cycles (with respect to disjoint union in V), so that the associated covariant theory

$$\mathbb{OB}^+_*(X) := \mathbb{OB}^+(X \to pt) = Z_*(X)$$

is nothing other than the corresponding cycle group of Levin–Morel [28].

It remains to see if one can impose suitable "bivariant-theoretic" relations of the geometric type

(b-Dim): "bivariant Dimension Axiom",
(b-Sect): "bivariant Section Axiom",
(b-FGL): "bivariant Formal Group Law Axiom",

on this oriented bivariant group $\mathbb{OB}^+(X \xrightarrow{f} Y)$, so that

$$\mathbb{B}\Omega(X \xrightarrow{f} Y) := \frac{\mathbb{OB}^+(X \xrightarrow{f} Y)}{\{\text{(b-Dim), (b-Sect), (b-FGL)}\}}$$

becomes an oriented bivariant theory, with $\mathbb{B}\Omega(X \to pt) = \Omega^{LM}_*(X)$. Then $\mathbb{B}\Omega(X \xrightarrow{f} Y)$ could be called a **bivariant algebraic cobordism** and $\mathbb{B}\Omega(X \xrightarrow{id} X)$ would be a contravariant analogue of $\Omega^{LM}_*(X)$.

Another possible way to a bivariant algebraic cobordism is the idea to adapt Levine–Pandharipande's new geometric construction of algebraic cobordism to a bivariant context, similarly to the way we extend Bittner's blow-up relation to a bivariant version.

In [29] Levine and Pandharipande gave a new geometric description to the algebraic cobordism (over a base field k of characteristic zero). Let Y be a smooth scheme. A morphism $\pi : Y \to \mathbb{P}^1$ is called a *double point degeneration* over $0 \in \mathbb{P}^1$ if $\pi^{-1}(0)$ can be written as

$$\pi^{-1}(0) = A \cup B$$

where A and B are smooth codimension one closed subschemes of Y, intersecting transversely. The intersection $D = A \cap B$ is the *double point locus* of π over $0 \in \mathbb{P}^1$. Let $N_{A/D}$ and $N_{A/D}$ be the normal bundles of D in A and B respectively. Then the projectivized bundles $\mathbb{P}(\mathcal{O}_D \oplus N_{A/D}) \to D$ and $\mathbb{P}(\mathcal{O}_D \oplus N_{B/D}) \to D$ are isomorphic (see [29]). Either one of these bundles is denoted by $\mathbb{P}(\pi) \to D$.

Definition 6.8. Let Y be smooth, with

$$g : Y \to X \times \mathbb{P}^1$$

a projective morphism such that the composite

$$\pi := \pi_2 \circ g : Y \to X \times \mathbb{P}^1 \to \mathbb{P}^1$$

is a double point degeneration over $0 \in \mathbb{P}^1$. Let $\xi \in \mathbb{P}^1$ be a regular value of π (which exists by "generic smoothness", since we are in characteristic zero). Then the map g is called a *double point cobordism* with degenerate fiber over 0 and smooth fiber over ξ with $Y_\xi := \pi^{-1}(\xi)$. The associated *double point relation* over X is defined by

$$[Y_\xi \to X] - [A \to X] - [B \to X] + [\mathbb{P}(\pi) \to X].$$

Note that one is allowed above to have $B = \emptyset$ (and therefore also $\mathbb{P}(\pi) = \emptyset$) with 0 and ξ both regular values of π. The corresponding relation

$$[Y_\xi \to X] - [A \to X] = 0$$

is called a "naive (co)bordism" between the smooth algebraic manifolds $A = \pi^{-1}(0)$ and $\pi^{-1}(\xi)$. It is just an "algebrization" of Quillen's definition of the complex cobordism relation [34] (and also see the next section). The "naive (co)bordism" relation holds in the algebraic cobordism group $\Omega_*^{LM}(X)$, but it is not enough to divide out the cycle group $M_*(X)$ only by this relation to get the algebraic cobordism group (see [28, Remark 1.2.9]). So this is different from the differential topological context of Quillen [34] in his study of complex cobordism. It is a beautiful and striking result of Levine–Pandharipande [29] that one only has to add the simplest kind of singularities (namely "double points") to get back the algebraic cobordism group $\Omega_*^{LM}(X)$:

Theorem 6.9 (Levine–Pandharipande [29])**.** *Let $\mathcal{R}_*(X) \subset M_*(X)$ be the subgroup generated by all the double point relations over X. One sets*

$$\omega_*^{LP}(X) = \left\{ [V \xrightarrow{h} X] \Big| \, V \textit{is smooth}, h \textit{ is projective} \right\}^+ \Big/ \mathcal{R}_*(X) = M_*(X) / \mathcal{R}_*(X)\,.$$

Then $\omega_*^{LP}(X) \cong \Omega_*^{LM}(X)$.

Motivated by our previous constructions, the group completion (with respect to disjoint union in V)

$$\mathbb{M}(X \xrightarrow{f} Y) := \left\{ [V \xrightarrow{h} X] \,\big|\, h \text{ is projective}, f \circ h \text{ is smooth} \right\}^+$$

is a bivariant analogue of the above group $M_*(X) = \left\{ [V \xrightarrow{h} X] \Big| \, V \text{is smooth}, h \text{ is projective} \right\}^+$. Thus it remains to be seen if one can construct a corresponding bivariant analogue $\mathcal{R}_*(X \xrightarrow{f} Y)$ of the above subgroup $\mathcal{R}_*(X)$ of all the double point relations, such that

(1) $\mathbb{B}\omega^{LP}(X \xrightarrow{f} Y) := \dfrac{\left\{ [V \xrightarrow{h} X] \,\big|\, h \text{ is projective}, f \circ h \text{ is smooth} \right\}^+}{\mathcal{R}_*(X \xrightarrow{f} Y)}$ is a bivariant theory,

(2) $\mathbb{B}\omega_*^{LP}(X) = \mathbb{B}\omega^{LP}(X \to pt) = \omega_*^{LP}(X)$.

If this can be done, it gives us a geometrically defined bivariant algebraic cobordism and thus the contravariant part $\mathbb{B}\omega^{LP}(X \xrightarrow{id_X} X)$ could be considered as a contravariant analogue of Levine–Morel's or Levine–Pandharipande's algebraic bordism.

7. A FINAL REMARK: CLASSICAL (CO)BORDISMS AND FIBERWISE BORDISM GROUPS

Thom's oriented bordism group Ω_n^{SO} is defined by

$$\Omega_n^{SO} := \{n\text{-dimensional closed oriented differentiable manifolds}\} \Big/ \text{bordant relation}.$$

Here two such n-dimensional manifolds M_1^n and M_2^n are called *bordant* if there exists an $(n+1)$-dimensonal compact oriented differentiable manifold W^{n+1} with boundary such that

$$\partial W = M_1 \sqcup (-1)M_2 \,,$$

where the signed one $(-1)M_2$ is the manfiold M_2 with the orientation reversed.

M. Atiyah extended the bordism group to a covariant functor on the category of topological spaces:

Definition 7.1 (Atiyah [2])**.** For a topological space X

$$\Omega_n^{SO}(X) := \left\{ M^n \xrightarrow{h} X \;\middle|\; M^n \in \mathcal{C}^\infty_{or,c},\ h \text{ is continuous} \right\} \Big/ \text{bordant relation.}$$

Here $\mathcal{C}^\infty_{or,c}$ denotes the category of oriented closed $\mathcal{C}^\infty$-manifolds. Two continuous maps $h_1 : M_1^n \to X$ and $h_2 : M_2^n \to X$ are called *bordant* if

(1) there exists a compact $(n+1)$-dimensional oriented differentiable manifold W^{n+1} with boundary such that $\partial W = M_1 \sqcup (-1)M_2$,
(2) there exists a continuous map $H : W^{n+1} \to X$ such that $H|_{M_i} = h_i$ for $i = 1, 2$.

Furthermore, Conner and Floyd [13] extended it as a generalized homology theory so that for a pair (X, A) of CW-complexes, there exists a canonical isomorphism

$$\Phi : \Omega_n^{SO}(X, A) \xrightarrow{\cong} \mathcal{H}_n(X, A; \{MSO(i)\}) : = \lim_{i\to\infty} \left[S^{n+i}, (X/A) \wedge MSO(i)\right]_0$$

Here $\{MSO(i)\}$ is the corresponding Thom spectrum, i.e., the sequence of the Thom complexes $MSO(i)$, which are the Thom spaces of the universal oriented $\mathbb{R}^i$-bundle ξ_i over the classifying space $BSO(i)$. $[A, B]_0$ with the suffix 0 denotes the group of base-point preserving homotopy class of maps.

Definition 7.2. ([2]) For a topological pair (X, A) the *bordism cohomology group* $\Omega^n_{SO}(X, A)$ (called the *cobordism group*) is defined as *the generalized cohomology theory associated to the Thom spectrum* $\{MSO(i)\}$:

$$\Omega^n_{SO}(X, A) := \mathcal{H}^n(X, A; \{MSO(i)\}) : = \lim_{i\to\infty} \left[S^{i-n} \wedge (X/A), MSO(i)\right]_0 \ .$$

A *naïve and fundamental question* on the bordism cohomology group $\Omega^n_{SO}(X) := \Omega^n_{SO}(X, \emptyset)$ is the following:

Question 7.3. Can one give a geometric description of this cohomology group $\Omega^n_{SO}(X)$ like in the definition of the bordism homology group $\Omega_n^{SO}(X)$?

In fact, in the case when M is a closed oriented differentiable manifold, we have a simple solution for the above question:

$$\Omega^i_{SO}(M) = \left\{ N^{\dim M - i} \xrightarrow{h} M \;\middle|\; N^{\dim M-i} \in \mathcal{C}^\infty_{or,c}, h \text{ is continuous} \right\} \Big/ \text{bordant relation .}$$

This is thanks to the following *Atiyah–Thom–Poincaré duality theorem*:

Theorem 7.4. *(*[2]*) For a closed oriented differentiable manifold M there exists canonical isomorphism*

$$\Omega^i_{SO}(M) \xrightarrow{\cong} \Omega^{SO}_{\dim M - i}(M) \,.$$

Remark 7.5. The above definitions together with this duality theorem also hold similarly, if SO is replaced by O or U, i.e., if one considers unoriented or complex (co)bordism (see [14, 34]).

Motivated by our construction of the bivariant Grothendieck group $\mathbb{K}_0(V/X \xrightarrow{f} Y)$ and the oriented bivariant theory $\mathbb{OB}(X \xrightarrow{f} Y)$, we can construct in a similar way a bivariant theory

$$\mathbb{B}\Omega^{SO}(X \xrightarrow{f} Y)$$

on the category $\mathcal{C}^\infty$ of differentiable manifolds (without boundary), such that $\mathbb{B}\Omega^{SO}(X \to pt) \simeq \Omega^{SO}_*(X)$ for X compact. Here the confined morphisms are the proper morphisms, whereas the independent squares are by definition the "transversal squares", i.e., with the differentiable maps f, g transversal so that the corresponding fiber product X' exists in this category $\mathcal{C}^\infty$ of differentiable manifolds. Note that a submersion is transversal to any morphism in the category $\mathcal{C}^\infty$. The corresponding contravariant theory $\mathbb{B}\Omega^{SO}(X \xrightarrow{id_X} X)$ can be seen as a new "cobordism group".

The basic idea for that is the following notion of *f-relative fiberwise bordism*. We set $FM^{-n}(X \xrightarrow{f} Y)$ to be the set of isomorphism classes

$$\left\{[V \xrightarrow{h} X] \,\middle|\, f \circ h \text{ is a proper submersion, whose tangent bundle to the fibers } T_{f\circ h} \text{ is oriented of rank } n\right\}.$$

Definition 7.6. Let $h_1 : V_1 \to X$ and $h_2 : V_2 \to X$ be two morphisms representing elements of the set $FM^{-n}(X \xrightarrow{f} Y)$. They are called *elementary f-relative fiberwise bordant* if there exists a differentiable manifold W with boundary ∂W and a morphism $H : W \to X$ such that

(1) $f \circ H : W \to Y$ is a proper submersion (i.e., also $f \circ H|_{\partial W}$ is a submersion),
(2) the tangent bundle to the fibers $T_{f\circ H}$ is oriented of rank $n+1$ (so that $T_{f\circ H|_{\partial W}}$ gets an induced orientation),
(3) $f \circ H : \partial W \to Y$ is isomorphic to $f \circ h_1 + f \circ h_2 : V_1 \sqcup (-1)V_2 \to Y$, where the signed one $(-1)V_2$ again indicates the reversed orientation of $T_{f\circ h_2}$.

By definition this notion only depends on the corresponding isomorphism classes. Moreover, the corresponding "elementary f-relative fiberwise bordism" relation is symmetric and reflexive. Let "f-relative fiberwise bordism" be the equivalence relation *generated by this*, i.e., $V_1 \xrightarrow{h_1} X$ and $V_2 \xrightarrow{h_2} X$ are "f-relative fiberwise bordant" if they can be *related by a finite string* of "elementary f-relative fiberwise bordant" morphisms.

This equivalence relation is also compatible with the monoid structure on $FM^{-n}(X \xrightarrow{f} Y)$ coming from the disjoint union with respect to V, so that

$$F\Omega^{-n}(X \xrightarrow{f} Y) := FM^{-n}(X \xrightarrow{f} Y)\Big/ f\text{-relative fiberwise bordism}$$

gets an induced monoid structure. The "trivial bordism" $W := V \times [0,1] \to V \xrightarrow{h} X$ shows that it is indeed an abelian group. By definition we have $F\Omega^{-n}(X \to pt) \simeq \Omega^{SO}_n(X)$ for X compact, since any continuous map $V \to X$ between smooth manifolds can be approximated by a differentiable map.

In the case of $F\Omega^{-n}(X \xrightarrow{id_X} X)$, we have

$$FM^{-n}(X \xrightarrow{id_X} X) = \left\{[V \xrightarrow{h} X] \,\middle|\, h \text{ is a proper submersion, with } T_h \text{ is oriented of rank } n\right\}.$$

Then id_X-relative fiberwise bordism is just fiberwise bordism, where two proper oriented submersions $V_1 \xrightarrow{h_1} X$ and $V_2 \xrightarrow{h_2} X$ are *fiberwise bordant*, if there exists a differentiable manifold W with boundary ∂W and a proper oriented submersion $H : W \to X$ of fiber dimension $n+1$ such that

$$H\big|_{\partial W} = h_1 + h_2 : V_1 \sqcup (-1)V_2 \to X\,.$$

Note that in this case "elementary id_X-relative fiberwise bordism" or fiberwise bordism is already an equivalence relation thanks to the "tubular neighborhood theorem". Thus we have

$$F\Omega^{-n}(X) := F\Omega^{-n}(X \xrightarrow{id_X} X) = FM^{-n}(X \xrightarrow{id_X} X)/\text{fiberwise bordant}\,.$$

Note that there is a tautological group homomorphism (commuting with cross products)

$$F\Omega^{-n}(X) \to \Omega^{-n}_{SO}(X)$$

to the usual *oriented cobordism* of the smooth manifold X, once we use the *geometric* definition of the later given by Quillen [34] and Dold [14] in the smooth context (just forget that $H : W \to X$ needs to be a submersion in our case).

Remark 7.7. (1) In the case when $X = S^1$ the 1-dimensional sphere, we have by the "mapping cylinder construction" $F\Omega^{-n}(S^1) \cong \Delta_n$, where Δ_n is the bordism group of orientation-preserving diffeomorphisms of closed oriented C^∞-manifolds of dimension n. It was introduced by Browder [11] and later on studied deeply by Kreck [26, 27].
(2) There is a surjection $F\Omega^{-n}(X \xrightarrow{id_X} X) \twoheadrightarrow \Omega^X_n(pt, \emptyset)$. Here $\Omega^X_n(Y, Y')$ is Weishu Shih's *fiber cobordism group* [39] for a topological pair $Y' \subset Y$.

Imitating the proof of Theorem 3.1, one finally gets the

Theorem 7.8. *Consider the category $\mathcal{C}^\infty$ of differentiable manifolds (without boundary), where the confined morphisms are the proper morphisms, whereas the independent squares are by definition the "transversal squares".*

Then the "f-relative fiberwise bordism group"

$$F\Omega(X \xrightarrow{f} Y) := \bigoplus_{n\geq 0} F\Omega^{-n}(X \xrightarrow{f} Y)$$

becomes a graded bivariant theory with the bivariant operations defined as in Theorem 3.1.

In [3] we will give more details about f-relative fiberwise bordism and the theorem above in the right context, namely in the category of differentiable spaces (e.g., see [33]), where instead of an orientation of $T_{f\circ h}$ we work more generally with a σ-structure in the sense of Dold [14] (e.g. with "structure groups" O or U instead of SO). Then it also becomes in close connection to the recent work of Emerson–Meyer [16] on a topological description of KK-theory.

Acknowledgements: Both of us gave talks at "Singularity Theory Conference", held at the University of Science and Technology of China, Hefei, China, July 25 - 31, 2011:

J. Schümann on "Characteristic classes of Hilbert schemes of points via symmetric products",

S. Yokura on "Fiberwise bordism groups".

We would like to thank the organizers Xiuxiong Chen and Laurentiu Maxim for the invitation, and also the organizers and all the other involved staff for such a wonderful organization of the conference and hospitality given during the conference.

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