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TOPOLOGICAL $\mathcal{K}$ AND $\mathcal{A}$ EQUIVALENCES OF POLYNOMIAL FUNCTIONS

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Abstract. We give a simple proof that the germs of real polynomial functions from $\mathbb{R}^n$ to $\mathbb{R}$ are C^0-$\mathcal{A}$-equivalent if they are PL-$\mathcal{K}$-equivalent (for example, semialgebraically). No restriction on the polynomial functions is needed.

1. Introduction

Given two smooth map germs $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$, if f, g are $\mathcal{A}$-equivalent, then they are $\mathcal{K}$-equivalent and the same is true if we consider any other reasonable category like real or complex analytic, C^k $(k \geq 1)$, C^0, Lipschitz, PL, etc. The converse is known to be true for C^∞-stable map germs (according to the work of J. Mather [8]), but it is also well known that it is false in the general case. Here, we are interested in the function case (i.e., $p = 1$), where it might seem possible to recover the $\mathcal{A}$-class data from the $\mathcal{K}$-class.

In [5] Fukuda proved the finitness theorem for C^0-$\mathcal{A}$- equivalence of polynomial functions from $\mathbb{K}^n$ to $\mathbb{K}$, $\mathbb{K} = \mathbb{R}, \mathbb{C}$. Benedetti and Shiota [2] proved the same result for semialgebraic functions. In [1] the authors gave very simple models for the equivalence classes with respect to C^0-$\mathcal{K}$- equivalence of semialgebraic functions. The motivation of our work is the comparison of C^0-$\mathcal{K}$ and C^0-$\mathcal{A}$ equivalence for function germs.

Notice that for complex analytic functions with isolated singularity, it was shown by Saeki [11] that if $f, g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ are C^0- $\mathcal{V}$-equivalent, then they are C^0-$\mathcal{A}$-equivalent. For real analytic functions with isolated singularity and $n = 2, 3$, it is pointed out by King in [7] that in this case again C^0-$\mathcal{V}$- equivalence implies C^0-$\mathcal{A}$-equivalence. The case $n = 2$ also follows from the works of Prishlyak [10] and Alvarez-Birbrair- Costa-Fernandes [1].

However, it is not true in general that C^0-$\mathcal{K}$-equivalence implies C^0-$\mathcal{A}$-equivalence of functions. For any $n \geq 7$, King [7] gives examples of polynomials $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ with isolated singularity which are C^0-$\mathcal{V}$-equivalent, but not C^0- $\mathcal{A}$-equivalent. This combined with the result of Nishimura [9] that C^0-$\mathcal{V}$-equivalence of smooth functions with isolated singularity implies C^0-$\mathcal{K}$-equivalence provides the desired counterexample. The reason of these counterexamples is that the corresponding zero-sets are homeomorphic but not PL homeomorphic.

In this work we consider the PL classification of polynomial germs $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ with no necessarily isolated singularity. The main result is the following theorem.

Theorem 1.1. *If two polynomials $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ are PL-$\mathcal{K}$-equivalent, then they are PL-$\mathcal{A}$-equivalent.*

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As a consequence, we deduce that for $n = 2, 3$, if two polynomials $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ are C^0-$\mathcal{K}$-equivalent, then they are C^0-$\mathcal{A}$-equivalent. We also obtain another proof of the Finiteness Theorem of Fukuda [5].

2. $\mathcal{A}$-equivalence and triangulation

Let $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be two smooth (or C^∞) map germs. We say that:

- f and g are *$\mathcal{A}$-equivalent* if there exist diffeomorphisms $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ and $k : (\mathbb{R}^p, 0) \to (\mathbb{R}^p, 0)$ such that the following diagram commutes

$$\begin{tikzcd} (\mathbb{R}^n, 0) \arrow[r, "f"] \arrow[d, "h"'] & (\mathbb{R}^p, 0) \arrow[d, "k"'] \\ (\mathbb{R}^n, 0) \arrow[r, "g"] & (\mathbb{R}^p, 0) \end{tikzcd}$$

- f and g are *$\mathcal{K}$-equivalent* if there exist diffeomorphisms $H : (\mathbb{R}^n \times \mathbb{R}^p, 0) \to (\mathbb{R}^n \times \mathbb{R}^p, 0)$ and $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ such that $H(\mathbb{R}^n \times \{0\}) = \mathbb{R}^n \times \{0\}$ and the following diagram is commutative:

$$\begin{tikzcd} (\mathbb{R}^n, 0) \arrow[r, "(\mathrm{id}, f)"] \arrow[d, "h"'] & (\mathbb{R}^n \times \mathbb{R}^p, 0) \arrow[r, "\pi_n"] \arrow[d, "H"'] & (\mathbb{R}^n, 0) \arrow[d, "h"'] \\ (\mathbb{R}^n, 0) \arrow[r, "(\mathrm{id}, g)"] & (\mathbb{R}^n \times \mathbb{R}^p, 0) \arrow[r, "\pi_n"] & (\mathbb{R}^n, 0) \end{tikzcd}$$

  where $\mathrm{id} : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ is the identity mapping of $\mathbb{R}^n$ and $\pi_n : (\mathbb{R}^n \times \mathbb{R}^p, 0) \to (\mathbb{R}^n, 0)$ is the canonical projection germ.
- f and g are *$\mathcal{V}$-equivalent* if there exist a diffeomorphism $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ such that $h(f^{-1}(0)) = g^{-1}(0)$.

In these definitions, if we have homeomorphisms (resp. PL homeomorphisms, semialgebraic homeomorphisms) instead of diffeomorphisms, we say that f and g are C^0-$\mathcal{A}$, C^0-$\mathcal{K}$ or C^0-$\mathcal{V}$-equivalent (resp. PL-$\mathcal{A}$, PL-$\mathcal{K}$ or PL- $\mathcal{V}$-equivalent, semialgebraically $\mathcal{A}$, $\mathcal{K}$ or $\mathcal{V}$-equivalent).

We start with a lemma about PL $\mathcal{A}$-equivalence of PL functions. We consider a PL function $f : X \to \mathbb{R}$, where X is any polyhedron.

Lemma 2.1. *Let $f_1 : X_1 \to \mathbb{R}$ and $f_2 : X_2 \to \mathbb{R}$ be two PL functions. Assume there is a PL homeomorphism $h : X_1 \to X_2$ such that $h(f_1^{-1}(0)) = f_2^{-1}(0)$ and moreover the sign of $f_1(x)f_2(h(x))$ is constant on $X_1 \setminus f_1^{-1}(0)$. Then there are neighbourhoods N_i of $f_i^{-1}(0)$ on X_i and V_i of 0 in $\mathbb{R}$ such that the restrictions $f_i : N_i \to V_i$ are PL-$\mathcal{A}$-equivalent.*

Proof. We assume, for simplicity, that the sign of $f_1(x)$ is equal to the sign of $f_2(h(x))$ on $X_1 \setminus f_1^{-1}(0)$, the other case being analogous.

After subdivision, we can take simplicial complexes K_1, K_2, L_1, L_2 with $|K_i| = X_i$ and $|L_i| = \mathbb{R}$, such that $f_1 : K_1 \to L_1$, $f_2 : K_2 \to L_2$ are simplicial maps and $h : K_1 \to K_2$ is a simplicial isomorphism.

We fix neighbourhoods in the target $V_1 = \mathrm{Star}(0, L_1)$ with vertices $a_1 < 0 < b_1$ and $V_2 = \mathrm{Star}(0, L_2)$ with vertices $a_2 < 0 < b_2$. We also take the corresponding neighbourhoods in the source $N_1 = f_1^{-1}(V_1)$ and $N_2 = f_2^{-1}(V_2)$. We denote by $\beta : V_1 \to V_2$ the simplicial isomorphism given by $\beta(a_1) = a_2$ and $\beta(b_1) = b_2$.

We claim that $h(N_1) = N_2$ and that the following diagram is commutative:

$$\begin{array}{ccc} N_1 & \xrightarrow{f_1} & V_1 \\ {\scriptstyle h}\big\downarrow & & {\scriptstyle \beta}\big\downarrow \\ N_2 & \xrightarrow{f_2} & V_2. \end{array}$$

In fact, let $\sigma \in N_1$ be a simplex such that $f_1(\sigma) = \{0\}$. Then $\sigma \in f_1^{-1}(0)$ and hence $h(\sigma) \in f_2^{-1}(0) \subset N_2$. Moreover, $\beta(0) = 0$ and the above diagram is obviously commutative on σ.

Otherwise, we take a simplex $\sigma \in N_1$ such that $f_1(\sigma) \neq \{0\}$. Since f_1 is simplicial we must have either $f_1(\sigma) = [0, b_1]$ or $f_1(\sigma) = [a_1, 0]$. If $f_1(\sigma) = [0, b_1]$, then $f_2(h(\sigma)) = [0, b_2]$ by the initial assumption and thus, $h(\sigma) \in N_2$. On the other hand, given any vertex v of σ, if $f_1(v) = 0$ then $\beta(f_1(v)) = 0 = f_2(h(v))$ and if if $f_1(v) = b_1$ then again $\beta(f_1(v)) = b_2 = f_2(h(v))$. This shows that the diagram is also commutative on σ in this case. The other case is analogous. $\square$

Proof of theorem 1.1. We first note that since f, g are polynomials, by Shiota Theorem [12], they are triangulable on a small enough neighbourhood of the origin. Hence we can choose triangulations:

$$\begin{array}{ccc} (\mathbb{R}^n, 0) & \xrightarrow{f} & (\mathbb{R}, 0) \\ {\scriptstyle \alpha_1}\big\downarrow & & {\scriptstyle \beta_1}\big\downarrow \\ (X_1, 0) & \xrightarrow{f_1} & (\mathbb{R}, 0), \end{array} \qquad \begin{array}{ccc} (\mathbb{R}^n, 0) & \xrightarrow{g} & (\mathbb{R}, 0) \\ {\scriptstyle \alpha_2}\big\downarrow & & {\scriptstyle \beta_2}\big\downarrow \\ (X_2, 0) & \xrightarrow{f_2} & (\mathbb{R}, 0), \end{array}$$

where X_i are polyhedra, $f_i : X_i \to \mathbb{R}$ are PL-maps and α_i, β_i are homeomorphisms.

Now, the hypothesis that f, g are PL $\mathcal{K}$-equivalent implies that there is a commutative diagram:

$$\begin{array}{ccccc} (X_1, 0) & \xrightarrow{(\mathrm{id}, f_1)} & (X_1 \times \mathbb{R}, 0) & \xrightarrow{\pi_1} & (X_1, 0) \\ {\scriptstyle h}\big\downarrow & & {\scriptstyle H}\big\downarrow & & {\scriptstyle h}\big\downarrow \\ (X_2, 0) & \xrightarrow{(\mathrm{id}, f_2)} & (X_2 \times \mathbb{R}, 0) & \xrightarrow{\pi_1} & (X_2, 0) \end{array}$$

where h, H are PL homeomorphisms, id is the identity mapping and π_1 is the projection onto the first factor.

We write $H(x, y) = (h(x), \theta_x(y))$, then we have that $\theta_x : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ is a family of homeomorphisms depending continuously on x in a neighbourhood of the origin. In particular, we have that either: for any x, θ_x is always increasing, or for any x, θ_x is always decreasing (depending on the local degree of H).

With this notation, the $\mathcal{K}$-equivalence is written as

$$\theta_x(f_1(x)) = f_2(h(x)), \quad \forall x \in X_1.$$

Then we have that $h(f_1^{-1}(0)) = f_2^{-1}(0)$ and that the sign of $f_1(x) f_2(h(x))$ is constant on $X_1 \setminus f_1^{-1}(0)$. The result follows now from lemma 2.1. $\square$

We give now some interesting consequences of this theorem. The first one follows from the fact that in dimensions $n = 2, 3$, any homeomorphism between semialgebraic subsets can be triangulated. Therefore, if two polynomial germs are C^0-$\mathcal{K}$-equivalent, then they are PL-$\mathcal{K}$-equivalent.

Corollary 2.2. *Let $n = 2, 3$, if two polynomials $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ are C^0-$\mathcal{K}$-equivalent, then they are C^0-$\mathcal{A}$-equivalent.*

The second corollary is for the case of isolated singularity. In this case, we have that the C^0-$\mathcal{V}$-equivalence implies the C^0-$\mathcal{K}$-equivalence (see [9]) and the same is true in the PL category.

Corollary 2.3. *If two polynomials $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ with isolated singularity are PL-$\mathcal{V}$-equivalent, then they are PL-$\mathcal{A}$-equivalent.*

Another easy consequence is that the theorem is also true if we consider semialgebraic homeomorphisms instead of PL homeomorphisms. This follows from the fact that any semialgebraic homeomorphism of semialgebraic sets can be triangulated.

Corollary 2.4. *If two polynomials $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ are semialgebraically $\mathcal{K}$-equivalent, then they are semialgebraically $\mathcal{A}$-equivalent.*

Finally, we give another proof of the Finitness Theorem of Fukuda [5] about C^0-$\mathcal{A}$-equivalence of polynomial function germs of a given degree (see also Benedetti-Shiota [2]). It is deduced from Hardt work [6] that there is a finite number of topological types of zero-sets up to semialgebraic homeomorphisms. Moreover, there is a finite number of possible choices for the sign of the function on the complement of the zero-set. By theorem 1.1, we have a finite number of C^0-$\mathcal{A}$-classes.

Corollary 2.5. *There is a finite number of C^0-$\mathcal{A}$-classes in the space of all polynomial map germs $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ of degree $\leq k$.*

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