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THE PUNCTUAL HILBERT SCHEMES FOR THE CURVE SINGULARITIES OF TYPES E_6 AND E_8

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Dedicated to Professor Fumio Sakai on the occasion of his 65th birthday

Abstract. The aim of the present paper is to study the structure of the punctual Hilbert schemes for the curve singularities of types E_6 and E_8. Our analysis uses computational methods to decompose a punctual Hilbert scheme into affine cells. We also use known results about the compactified Jacobians of singular curves.

1. Introduction

Let $\mathcal{O}$ be the complete local ring of a irreducible curve singularity over an algebraically closed field k of characteristic 0. We denote by $\overline{\mathcal{O}}$ and δ the normalization of $\mathcal{O}$ and the δ-invariant of $\mathcal{O}$ respectively. Pfister and Steenbrink [6] defined a special subset $\mathcal{M}$ of the Grassmannian $\mathrm{Gr}\big(\delta, \overline{\mathcal{O}}/I(2\delta)\big)$ where $I(2\delta)$ is the set of all elements in $\mathcal{O}$ whose orders are greater than or equal to 2δ. It is a projective variety defined by Plücker relations and additional equations. We call it the *Pfister-Steenbrink variety* (PS variety) for a given singularity. Using the intersection with Schubert cells, they investigated the structure of $\mathcal{M}$ for certain curve singularities. The punctual Hilbert scheme $\mathcal{M}_r$ of degree r was also constructed as a connected component of $\mathcal{M}$. It is a projective variety which parametrizes the ideals of codimension r in $\mathcal{O}$.

In the present paper, we study the structure of all punctual Hilbert schemes for the curve singularities of types E_6 and E_8 (i.e., the singularities with the local rings $k[[t^3, t^4]]$ and $k[[t^3, t^5]]$ respectively). The PS varieties for these singularities were originally studied in [6]. See also [9] which was the preliminary version of this paper. Our main theorems are stated as follows:

Theorem 1. *Punctual Hilbert schemes $\mathcal{M}_r$ for the curve singularities of type E_6 are given by the following table*:

r	1	2	3	4	5	≥ 6
$\mathcal{M}_r$	$\mathbb{P}^0$	$\mathbb{P}^1$	$\mathbb{P}^2$	$\mathbb{P}^2 \cup X_1$	$\mathbb{P}^2 \cup \mathbb{P}^2$	X_2
$\mathrm{Sing}(\mathcal{M}_r)$		$\emptyset$			$\mathbb{P}^1$	

***Table* 1**

The variety X_1 (resp. X_2) in Table 1 is a rational projective surface (resp. a rational projective threefold).

The defining equations of X_1 and X_2 are listed in Section 4.

Theorem 2. *Punctual Hilbert schemes $\mathcal{M}_r$ for the curve singularity of type E_8 are given by the following table*:

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r	1	2	3	4	5	6	7	≥ 8
$\mathcal{M}_r$	$\mathbb{P}^0$	$\mathbb{P}^1$	$\mathbb{P}^2$	$\mathbb{P}^2 \cup X_3$	$\mathbb{P}^2 \cup \mathbb{P}^2 \cup X_4$	$X_5 \cup X_6$	$X_7 \cup X_8$	X_9
$\mathrm{Sing}(\mathcal{M}_r)$		$\emptyset$		$\mathbb{P}^1$	$\mathbb{P}^1 \cup \mathbb{P}^1$		$\mathbb{P}^2 \cup \mathbb{P}^2$	

***Table* 2**

All the varieties X_i $(i = 3, \cdots, 8)$ in Table 2 are rational and projective. Their dimensions are given by

$$\dim X_i = \begin{cases} 2 & \textit{for } i = 3, \\ 3 & \textit{for } i = 4, \ldots, 8, \\ 4 & \textit{for } i = 9. \end{cases}$$

The PS varieties for curve singularities were studied from another point of view. Rego [8] introduced the compactified Jacobian of singular curves. He also constructed the Jacobi factor for a curve singularity. For a given curve singularity, the Jacobi factor and the PS variety coincide.

The paper is organized as follows. In Section 2, we briefly recall the Pfister-Steenbrink theory for punctual Hilbert schemes for curve singularities. We also fix notations and prove some lemmas needed later. In Section 3, we introduce a computational algorithm to decompose punctual Hilbert schemes into affine cells. It is based on the Gröbner bases theory which was established by Hefetz and Hernandes in [5]. We finally prove Theorem 1 and 2 in Section 4 and 5 respectively. Known results about compactified Jacobians of singular curves are also used in the proof of Theorem 2.

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2. Preliminaries

In the present paper, we restrict ourselves to consider monomial curve singularities defined below. However, the notions in this chapter hold in more general situations (see [6] for details).

Definition 3. *A monomial curve singularity is an irreducible curve singularity whose local ring is isomorphic to $k[[t^{a_1}, \ldots, t^{a_m}]]$ for $a_1, \ldots, a_m \in \mathbb{N}$.*

Remark 4. *Without loss of generality, we may assume that* $\gcd(a_1, \ldots, a_m) = 1$ *in Definition 3.*

Let $\mathcal{O} = k[[t^{a_1}, \ldots, t^{a_m}]]$ be the local ring of a monomial curve singularity. Its normalization $\overline{\mathcal{O}}$ is isomorphic to $k[[t]]$. We call $\Gamma := \{\mathrm{ord}_t(f) \mid f \in \mathcal{O}\}$ the *semigroup* of $\mathcal{O}$. The positive integer $\delta := \dim_k(\overline{\mathcal{O}}/\mathcal{O})$ is called the *δ-invariant* of $\mathcal{O}$. For $n \in \mathbb{N}$, set $\overline{I}(n) := \{\, f \in \overline{\mathcal{O}} | \, \mathrm{ord}_t(f) \geq n\}$ and $I(n) := \overline{I}(n) \cap \mathcal{O}$. Setting $\mathrm{ord}_t(0) = \infty$, we regard $\overline{I}(n)$ (resp. $I(n)$) as an ideal of $\overline{\mathcal{O}}$ (resp. $\mathcal{O}$). For an ideal I in $\mathcal{O}$, we call $\Gamma(I) := \{\mathrm{ord}_t(f) | \, f \in I\}$ the *order set* of I. For $r \in \mathbb{N}$, set

$$\mathcal{I}_r := \{I | \, I \text{ is an ideal of } \mathcal{O} \text{ with } \dim_k \mathcal{O}/I = r\}.$$

Lemma 5. *An ideal I in $\mathcal{O}$ belongs to $\mathcal{I}_r$ if and only if we have $\sharp\{\Gamma \setminus \Gamma(I)\} = r$.*

Proof. It is clear that I belongs to $\mathcal{I}_r$ if and only if we have

$$\mathcal{O}/I = \left\{a_0 + a_1 t^{d_1} + \cdots + a_{r-1} t^{d_{r-1}} + I \,\middle|\, a_i \in k, d_i \in \Gamma \setminus \Gamma(I), d_1 < \cdots < d_{r-1}\right\}$$

Thus the relation $\sharp\{\Gamma \setminus \Gamma(I)\} = r$ holds. $\square$

A subset $\Delta \subset \mathbb{Z}$ is called a Γ-*semi-module*, if $\Delta + \Gamma \subset \Delta$. Note that if Δ is a Γ-semi-module, then $\Delta - r$ is also a Γ-semi-module for any integer r. We write $\Delta = \langle \alpha_1, \cdots, \alpha_p \rangle_\Gamma$ for a Γ-semi-module Δ which is minimally generated by $\alpha_1, \cdots, \alpha_p$ (i.e $\Delta = \sum_{i=1}^{p}(\alpha_i + \Gamma)$ and $\Delta \supsetneq \sum_{i=1, i\neq j}^{p}(\alpha_i + \Gamma)$ for $\forall j \in \{1, \ldots, p\}$). We also denote by $\mathcal{I}(\Delta)$ the set of all ideals of $\mathcal{O}$ whose order sets are Δ. Note that $\mathcal{I}(\Delta) \neq \emptyset$ if and only if $\Delta \subset \Gamma$.

Proposition 6. *There exists a finite number of distinct* Γ*-semi-modules* $\Delta_{r,l}$ *such that*

$$\mathcal{I}_r = \bigcup_{l=1}^{n_r} \mathcal{I}(\Delta_{r,l}). \tag{1}$$

Proof. The finiteness of the number of Γ-semi-modules holds trivially, as there exists only a finite number of semigroups in $\mathbb{N}$ of fixed colength. It is also clear that (1) is a disjoint union. □

Remark 7. *If* Δ *is a* Γ*-semi-module such that* $\mathcal{I}(\Delta) \neq \emptyset$*, then all ideals in* $\mathcal{I}(\Delta)$ *have same codimension by Lemma 5. So the set* $\mathcal{I}(\Delta)$ *is contained in* $\mathcal{I}_r$ *if and only if* $\sharp\{\Gamma \setminus \Delta\} = r$*. It implies that the set of* $\Delta_{r,l}$*'s in (1) is an invariant for the codimension* r*.*

Let $\mathrm{Gr}\left(\delta, \overline{\mathcal{O}}/\mathrm{I}(2\delta)\right)$ be the Grassmannian which consists of δ-dimensional linear subspaces of $\overline{\mathcal{O}}/I(2\delta)$. For $V \in \mathrm{Gr}\left(\delta, \overline{\mathcal{O}}/\mathrm{I}(2\delta)\right)$, we define a multiplication by $\mathcal{O} \times V \ni (f, \overline{v}) \mapsto \overline{fv} \in V$. Set

$$\mathcal{M} := \left\{ V \in \mathrm{Gr}\left(\delta, \overline{\mathcal{O}}/\mathrm{I}(2\delta)\right) \,\middle|\, V \text{ is an } \mathcal{O}\text{-submodule w.r.t. the multiplication} \right\}.$$

Consider the composition map

$$\psi:\ \mathcal{M}\ \to \mathrm{Gr}(\delta, 2\delta) \to \mathrm{M}_{\delta,2\delta}(k)/\sim\ \to \mathbb{P}^N \tag{2}$$

where $\mathrm{Gr}(\delta, 2\delta)$ is the Grassmannian which consists of δ-dimensional linear subspaces of $k^{2\delta}$, $\mathrm{M}_{\delta,2\delta}(k)$ is the set of all $\delta \times 2\delta$ matrices over k and the equivalence relation $\sim$ is the similarity of matrices. For a formal power series $f = \sum_{j=0}^{\infty} a_j t^j$ in $\overline{\mathcal{O}}$, we denote its coset in $\overline{\mathcal{O}}/I(2\delta)$ by $\overline{f} = \sum_{j=0}^{2\delta-1} a_j t^j$. Here we use same notation t for the coset of t. Define $\mathrm{ord}_t(\overline{f})$ by $\mathrm{ord}_t(f)$ (resp. ∞), if $\mathrm{ord}_t(f) \leq 2\delta - 1$ (resp. $\mathrm{ord}_t(f) > 2\delta - 1$). In this paper, we use the notation $[v_1, \cdots, v_\delta]_k$ for a k-vector space generated by $v_1, \ldots, v_\delta$. Let $V = [\overline{f}_1, \cdots, \overline{f}_\delta]_k$ be an element of $\mathcal{M}$ where $\overline{f}_i = \sum_{j=0}^{2\delta-1} a_{ij} t^j$. We identify $\overline{f}_i$ with the point $\boldsymbol{a}_i = (a_{i0}, \cdots, a_{i2\delta-1})$ in $k^{2\delta}$. This identification gives the first map in (2). Let A_V be the $\delta \times 2\delta$ matrix whose ith row is $\boldsymbol{a}_i$. We call it the *representation matrix* of V. We may assume that the coset of A_V in $\mathrm{M}_{\delta,2\delta}(k)/\sim$ is represented by the reduced row echelon form. The second map in (2) sends a k-vector space $[\boldsymbol{a}_1, \cdots, \boldsymbol{a}_\delta]_k$ to the coset of A_V. The third map in (2) is Plücker embedding with $N = \binom{2\delta}{\delta} - 1$.

For $r \in \mathbb{N}$, Pfister and Steenbrink defined a map $\varphi_r : \mathcal{I}_r \to \mathcal{M}$ by $\varphi_r(I) = t^{-r}I/I(2\delta)$.

Proposition 8 ([6], Theorem 3)**.** *The map* φ_r *is injective for any* r*. Furthermore, it is bijective for* $r \geq 2\delta$*. The image* $(\psi \circ \varphi_r)(\mathcal{I}_r)$ *is Zariski closed in* $\psi(\mathcal{M})$*.*

Put $\mathcal{M}_r := \varphi_r(\mathcal{I}_r)$. Since ψ is injective, we identify $\psi(\mathcal{M})$ and $\psi(\mathcal{M}_r)$ with $\mathcal{M}$ and $\mathcal{M}_r$ respectively.

Definition 9. *We call* $\mathcal{M}$ *and* $\mathcal{M}_r$ *the Pfister-Steenbrink variety (PS variety for short) and the punctual Hilbert scheme of degree* r *for a given curve singularity respectively.*

The following fact follows from Proposition 8:

Corollary 10. *Any punctual Hilbert scheme* $\mathcal{M}_r$ *with* $r \geq 2\delta$ *coincides with the PS variety* $\mathcal{M}$*.*

Remark 11. *By virtue of Corollary 10, it is enough to consider codimensions* r *within* $1 \leq r \leq 2\delta$ *for the analysis of* $\mathcal{M}_r$*.*

Set $\mathcal{M}_{r,l} := \varphi_r(\mathcal{I}(\Delta_{r,l}))$ for each component $\mathcal{I}(\Delta_{r,l})$ in (1) and write $\Delta_{r,l} = \langle \alpha_1, \cdots, \alpha_{p_l} \rangle_\Gamma$. Since ψ is injective, we also identify $\psi(\mathcal{M}_{r,l})$ with $\mathcal{M}_{r,l}$. Namely, a component $\mathcal{M}_{r,l}$ is regarded as the subset of the punctual Hilbert scheme $\mathcal{M}_r$ parametrizing ideals in $\mathcal{I}(\Delta_{r,l})$. Set

$$[a, b] := \{x \in \mathbb{Z}_{\geq 0} |\, a \leq x \leq b\}.$$

We have $\Delta_{r,l} - r = \langle \alpha_1 - r, \cdots, \alpha_{p_l} - r \rangle_\Gamma$. Set $A := \{\alpha_1 - r, \cdots, \alpha_{p_l} - r\} \cap [0, 2\delta - 1]$ and $J_\alpha := [\alpha + 1, 2\delta - 1] \setminus \{\Delta_{r,l} - r\}$ for $\alpha \in A$. The following facts are known:

Proposition 12 ([6], Theorem 7)**.** *Let I be an element of $\mathcal{I}(\Delta_{r,l})$. There exist uniquely determined $b_{\alpha j} \in k$ such that the $\mathcal{O}$-submodule $\varphi_r(I)$ is generated by*

$$\overline{h}_\alpha := t^\alpha + \sum_{j \in J_\alpha} b_{\alpha j} t^j \quad (\alpha \in A).$$

Corollary 13 ([6], Corollary of Theorem 11)**.** *The component $\mathcal{M}_{r,l}$ is isomorphic to the affine space k^N where $N = \sum_{\alpha \in A} \sharp J_\alpha$.*

The affine cell decomposition of $\mathcal{M}_r$ follows from Proposition 6 and Corollary 13.

Proposition 14. *The punctual Hilbert scheme $\mathcal{M}_r$ of degree r has an affine cell decomposition*

$$\mathcal{M}_r = \bigcup_{l=1}^{n_r} \mathcal{M}_{r,l}. \tag{3}$$

The following fact also follows from Corollary 13:

Proposition 15. *If $\mathcal{M}_r$ is irreducible, then it is a rational projective variety.*

3. Computational algorithms

The aim of this section is to prove Theorem 24 which decomposes $\mathcal{M}_r$ into affine cells. We freely use the notations introduced in the previous section.

Lemma 16. *Let $\Delta = \langle \alpha_1, \cdots, \alpha_p \rangle_\Gamma$ be a Γ-semi-module. If $\mathcal{I}(\Delta)$ is a component of $\mathcal{I}_r$, then $\mathcal{I}(\Delta \setminus \{\alpha_i\})$ is component of $\mathcal{I}_{r+1}$ for each $i \in \{1, \ldots, p\}$. Conversely, if $\mathcal{I}(\Delta)$ is a component of $\mathcal{I}_{r+1}$, then, for each α_i and $\gamma_1 := \min\{\Gamma \setminus \{0\}\}$, $\mathcal{I}(\Delta \cup \{\alpha_i - \gamma_1\})$ is a component of $\mathcal{I}_r$.*

Proof. Assume that $\mathcal{I}(\Delta)$ is a component of $\mathcal{I}_r$. For any α_i, it is clear that $\Delta \setminus \{\alpha_i\}$ is also a Γ-semi-module. Since $\sharp\{\Gamma \setminus \Delta\} = r$ by Lemma 5, we have $\sharp\{\Gamma \setminus (\Delta \setminus \{\alpha_i\})\} = r + 1$. Hence the set $\mathcal{I}(\Delta \setminus \{\alpha_i\})$ is a component of $\mathcal{I}_{r+1}$. Next assume that the set $\mathcal{I}(\Delta)$ is a component of $\mathcal{I}_{r+1}$. Now we have $\alpha_i - \gamma_1 \notin \Delta$ for any i. Indeed, if $\alpha_i - \gamma_1 \in \Delta$, then there exist α_j and γ in Γ such that $\alpha_i - \gamma_1 = \alpha_j + \gamma$. This fact contradicts the assumption in which $\alpha_1, \cdots, \alpha_p$ are minimal generators for Δ. It is clear that $\Delta \cup \{\alpha_i - \gamma_1\}$ is a Γ-semi-module and $\sharp\{\Gamma \setminus (\Delta \cup \{\alpha_i - \gamma_1\})\} = r$. Hence, $\mathcal{I}(\Delta \cup \{\alpha_i - \gamma_1\})$ is a component of $\mathcal{I}_r$. $\square$

For the decomposition (1) of $\mathcal{I}_r$, set $\mathfrak{D}_r := \{\Delta_{r,1}, \cdots, \Delta_{r,n_r}\}$. The following proposition determines $\mathfrak{D}_r$ from $\mathfrak{D}_{r-1}$:

Proposition 17. *The set $\mathfrak{D}_r$ is constructed from $\mathfrak{D}_{r-1}$ by the following algorithm*:
`INPUT:` $\mathfrak{D}_{r-1} = \{\Delta_{r-1,1}, \cdots, \Delta_{r-1,n_{r-1}}\}$ where $\Delta_{r-1,l} = \langle \alpha_{l1}, \cdots, \alpha_{lp_l} \rangle_\Gamma$ $(l = 1, \ldots, n_{r-1})$
`OUTPUT:` $\mathfrak{D}_r$
`DEFINE:` $\mathfrak{D}_r := \emptyset$
`FOR` each $l \in \{1 \ldots, n_{r-1}\}$ and each $i \in \{1 \ldots, p_l\}$ `DO`
$\Delta := \Delta_{r-1,l} \setminus \{\alpha_{li}\}$
`IF` $\Delta \notin \mathfrak{D}_r$ `THEN` $\mathfrak{D}_r := \mathfrak{D}_r \cup \{\Delta\}$ `ELSE` do nothing

Proof. Our assertion follows from Lemma 16. □

We repeat some facts related to Gröbner Bases, which are needed in our computations. Our main reference is [5]. For a field k, let $\mathcal{O} = k[[x_1(t), \cdots, x_m(t)]]$ be a subring of $k[[t]]$ such that $\dim_k k[[t]]/\mathcal{O} < \infty$ and let $M \subset k[[t]]$ be an $\mathcal{O}$-module. We denote by Γ (resp. $\Gamma(M)$) the set of all orders of elements in $\mathcal{O}$ (resp. M) with respect to t.

Definition 18. *A subset $G = \{g_1, \cdots, g_s\}$ of $\mathcal{O}$ is called a SAGBI basis (Subalgebra Analog to Gröbner Bases for Ideals), if, for all $f \in \mathcal{O}$, there exists a monomial $Q(y_1, \cdots, y_s) \in k[y_1, \cdots, y_s]$ such that $\mathrm{LT}(f) = Q(\mathrm{LT}(g_1), \ldots, \mathrm{LT}(g_s))$.*

Remark 19. *If $\mathcal{O} = k[[t^{a_1}, \ldots, t^{a_m}]]$ where $a_i \in \mathbb{N}$, then $G = \{t^{a_1}, \ldots, t^{a_m}\}$ is a SAGBI basis of $\mathcal{O}$.*

Definition 20. *Let G be a SAGBI basis for $\mathcal{O}$ and let H be a subset of M. The pair (G, H) is called a standard basis of M, if, for any $f \in M$, there exists $h \in H$ and a monomial $Q(y_1, \cdots, y_s) \in k[y_1, \cdots, y_s]$ such that $\mathrm{LT}(f) = Q(\mathrm{LT}(g_1), \ldots, \mathrm{LT}(g_s)) \cdot \mathrm{LT}(h)$.*

The following proposition follows from Definition 20.

Proposition 21. *Let $G = \{g_1, \cdots, g_m\} \subset \mathcal{O}$ be a SAGBI basis for $\mathcal{O}$ and let $H = \{h_1, \ldots, h_n\}$ be a subset of M. The pair (G, H) is a standard basis of M if and only if we have*

$$\Gamma = \langle \mathrm{ord}_t(g_1), \cdots, \mathrm{ord}_t(g_m) \rangle \quad \text{and} \quad \Gamma(M) = \langle \mathrm{ord}_t(h_1), \cdots, \mathrm{ord}_t(h_n) \rangle_\Gamma.$$

For $G \subset \mathcal{O}$ and $H \subset M$, we define the *normal form* of $f \in k[[t]]$ with respect to (G, H) by $\mathrm{NF}(f, G, H) := \sum_{j \notin \Gamma(M)} c_j t^j$ such that $f - \mathrm{NF}(f, G, H) \in M$. We consider the local order $\mathrm{ord}_t(1) \succ \mathrm{ord}_t(t) \succ \mathrm{ord}_t(t^2) \succ \cdots$. For $f \in M$, we denote by $\mathrm{LC}(f)$ (resp. $\mathrm{LT}(f)$) the leading coefficient of f (resp. the leading term of f) with respect to the order. We also define an *S-process* for $f_1, f_2 \in M$ to be

$$S(f_1, f_2) := \mathrm{LC}(f_2) \cdot t^{\gamma_1} \cdot f_1 - \mathrm{LC}(f_1) \cdot t^{\gamma_2} \cdot f_2$$

where $\gamma_1, \gamma_2 \in \Gamma$ satisfy $\gamma_1 + \mathrm{ord}_t(f_1) = \gamma_2 + \mathrm{ord}_t(f_2)$. In particular, we denote by $S_{\min}(f_1, f_2)$ the S-process whose order with respect to t is minimal among all S-processes of f_1 and f_2. We call it the *minimal S-process* of f_1 and f_2. Note that $S_{\min}(f_1, f_2)$ is determined by the pair (γ_1, γ_2) that makes the value $\gamma_1 + \mathrm{ord}_t(f_1) = \gamma_2 + \mathrm{ord}_t(f_2)$ minimal. Such pair is uniquely determined for f_1 and f_2. So is $S_{\min}(f_1, f_2)$. The following fact is known:

Proposition 22 ([5], Theorem 2.3)**.** *Let $G = \{g_1, \cdots, g_m\} \subset \mathcal{O}$ be a SAGBI basis for $\mathcal{O}$ and let $H = \{h_1, \ldots, h_n\}$ be a subset of M. The pair (G, H) is a standard basis of M if and only if the normal form $NF(S_{\min}(h_i, h_j), G, H)$ is zero for any $h_i, h_j \in H$.*

Now we consider the subset $\mathcal{M}_{r,l}$ of $\mathcal{M}_r$. Recall that all elements in $\mathcal{M}_{r,l}$ have an $\mathcal{O}$-submodule structure with $\Gamma(\mathcal{M}_{r,l}) = \{(\Delta_{r,l} - r) \cap [0, 2\delta - 1]\} \cup \{\infty\}$. Note that the Γ-semi-module $\Gamma(\mathcal{M}_{r,l})$ satisfies the following: if $\gamma + \alpha \geqq 2\delta$ for $\gamma \in \Gamma$ and $\alpha \in \Gamma(\mathcal{M}_{r,l})$, then $\gamma + \alpha = \infty$ in $\Gamma(\mathcal{M}_{r,l})$ by the definition of the multiplication of $\mathcal{O}$-submodule. Let $\{\overline{v}_1, \cdots, \overline{v}_\delta\}$ be a basis for an element of $\mathcal{M}_{r,l}$. Writing $(\Delta_{r,l} - r) \cap [0, 2\delta - 1] = \{\beta_1, \ldots, \beta_\delta\}$ ($\beta_i < \beta_{i+1}$), we may assume that each $\overline{v}_i$ has the form

$$\overline{v}_i = t^{\beta_i} + \sum_{j > \beta_i,\, j \notin \Gamma(\mathcal{M}_{r,l})} c_{ij} t^j. \tag{4}$$

Since any element in $\mathcal{M}_{r,l}$ is generated by $\overline{v}_1, \cdots, \overline{v}_\delta$ as k-vector space, we regard $\mathcal{M}_{r,l}$ itself as a k-vector space and just write $\mathcal{M}_{r,l} = [\overline{v}_1, \cdots, \overline{v}_\delta]_k$. Here the coefficients c_{ij}'s in $\overline{v}_1, \cdots, \overline{v}_\delta$ are treated as variables. They may satisfy some conditions to keep the relation

$$\Gamma(\mathcal{M}_{r,l}) = \{\beta_1, \ldots, \beta_\delta\} \cup \{\infty\}.$$

Let T be the set of all such conditions. Setting $K := k(\{c_{ij}\})$ and $\widetilde{\mathcal{O}} := K[[t^{a_1}, \cdots, t^{a_m}]]$, we define the extension $\widetilde{\mathcal{M}}_{r,l}$ of $\mathcal{M}_{r,l}$ to be the $\widetilde{\mathcal{O}}$-submodule generated by $\overline{v}_1, \cdots, \overline{v}_\delta$. We observe that $\Gamma(\mathcal{M}_{r,l}) = \Gamma(\widetilde{\mathcal{M}}_{r,l}) = \{\beta_1, \dots, \beta_\delta\} \cup \{\infty\}$ under the condition set T.

Proposition 23. *The condition set T for $\mathcal{M}_{r,l} = [\overline{v}_1, \cdots, \overline{v}_\delta]_k$ is given by the following algorithm:*
`INPUT:` $G = \{t^{a_1}, \dots, t^{a_m}\}$, $H = \{\overline{v}_1, \cdots, \overline{v}_\delta\}$, $K = k(\{c_{ij}\})$
`OUTPUT:` T
`DEFINE:` $T := \emptyset$
`FOR` each i_1, i_2 in $\{1, \dots, \delta\}$ with $i_1 \neq i_2$ `DO`
$S := S_{\min}(\overline{v}_{i_1}, \overline{v}_{i_2})$
`WHILE` $\mathrm{ord}_t(S) \leqq 2\delta - 1$ `DO`
`IF` $\mathrm{ord}_t(S) \notin \{\mathrm{ord}_t(\overline{v}_i)\}_{i=1,\dots,\delta} \cup \{\infty\}$
`THEN` $S := S - \mathrm{LT}(S)$ `and` $T := T \cup \{\mathrm{LC}(S) = 0\}$
`ELSE` $S := S_{\min}\left(S, \sum_{(t^{a_i}, \overline{v}_j) \in L} t^{a_i} \overline{v}_j\right)$
where $L = \{(t^{a_i}, \overline{v}_j) \in G \times H | a_i + \mathrm{ord}_t(\overline{v}_j) = \mathrm{ord}_t(S)\}$

Proof. For two distinct $\overline{v}_{i_1}$ and $\overline{v}_{i_2}$ in H, we first compute $S_1 := S_{\min}(\overline{v}_{i_1}, \overline{v}_{i_2})$. Note that $\mathrm{LC}(S_1)$ is a polynomial with respect to the coefficients in $\overline{v}_{i_1}$ and $\overline{v}_{i_2}$. If $\mathrm{ord}_t(S_1) \notin \Gamma(\widetilde{\mathcal{M}}_{r,l})$, then we must have $\mathrm{LC}(S_1) = 0$. We add this equation to T and put $S_2 := S_1 - \mathrm{LT}(S_1)$. On the other hand, if $\mathrm{ord}_t(S_1) \in \Gamma(\widetilde{\mathcal{M}}_{r,l})$, then we consider $S_2 := S_{\min}\left(S_1, \sum_{(t^{a_i}, \overline{v}_j) \in L_1} t^{a_i} \overline{v}_j\right)$ for $L_1 := \{(t^{a_i}, \overline{v}_j) \in G \times H |\, a_i + \mathrm{ord}_t(\overline{v}_j) = \mathrm{ord}_t(S_1)\}$. Next we check whether $\mathrm{ord}_t(S_2)$ belongs to $\Gamma(\widetilde{\mathcal{M}}_{r,l})$ or not. Continuing such procedures successively, we get $\mathrm{ord}_t(S_1) < \mathrm{ord}_t(S_2) < \cdots$. So there exists a positive integer q which satisfies $\mathrm{ord}_t(S_q) \leq 2\delta - 1$ and $\mathrm{ord}_t(S_{q+1}) = \infty$. Namely, our procedures terminate in finite steps. Applying these procedures to all distinct pairs in H, we obtain a standard basis (G, H) for $\widetilde{\mathcal{M}}_{r,l}$ by Proposition 22. It also implies $\Gamma(\widetilde{\mathcal{M}}_{r,l}) = \{\mathrm{ord}_t(\overline{v}_i)\}_{i=1,\dots,\delta} \cup \{\infty\}$ by Proposition 21. Hence this algorithm yields the condition set T. □

Let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}$. We finally obtain the following theorem:

Theorem 24 (The computational algorithm for the affine cell decomposition of $\mathcal{M}_r$)**.** *For a given codimenson r, we obtain all affine cells in the decomposition* (3) *of $\mathcal{M}_r$ by the following finite steps.*
Step 1: Set $\mathfrak{D}_1 := \{\Gamma(\mathfrak{m})\}$ and find the set of minimal generators of $\Gamma(\mathfrak{m})$.
Step i ($i = 2, \dots, r$): Compute $\mathfrak{D}_i$ from $\mathfrak{D}_{i-1}$ by Proposition 17 and, for each $\Delta_{i,l}$ in $\mathfrak{D}_i$, find its set of minimal generators.
Step r+1: For each $\mathcal{M}_{r,l}$, find the basis which consists of (4).
Step r+2: For each $\mathcal{M}_{r,l}$, determine the condition set T by Proposition 23.

Proof. Our assertion follows from Proposition 17 and 23. □

4. Proof of Theorem 1

We prove Theorem 1 in this section. Let $\mathcal{O}$ be the local ring $k[[t^3, t^4]]$ of the singularity of type E_6. We have $\Gamma = \{0, 3, 4, 6, 7, 8, 9, \cdots\}$. It follows that $\delta = 3$ and $2\delta = 6$. For each codimension r $(1 \leq r \leq 6)$, we first determine all components in the decomposition (3). Applying Steps $1, \dots, r$ in Theorem 24, we obtain the following datum:

r	Elements of $\mathfrak{D}_r$
1	$\Delta_{1,1} = \langle 3, 4\rangle_\Gamma$
2	$\Delta_{2,1} = \langle 4, 6\rangle_\Gamma$, $\Delta_{2,2} = \langle 3, 8\rangle_\Gamma$
3	$\Delta_{3,1} = \langle 6, 7, 8\rangle_\Gamma$, $\Delta_{3,2} = \langle 4, 9\rangle_\Gamma$, $\Delta_{3,3} = \langle 3\rangle_\Gamma$
4	$\Delta_{4,1} = \langle 7, 8, 9\rangle_\Gamma$, $\Delta_{4,2} = \langle 6, 8\rangle_\Gamma$, $\Delta_{4,3} = \langle 6, 7\rangle_\Gamma$, $\Delta_{4,4} = \langle 4\rangle_\Gamma$
5	$\Delta_{5,1} = \langle 8, 9, 10\rangle_\Gamma$, $\Delta_{5,2} = \langle 7, 9\rangle_\Gamma$, $\Delta_{5,3} = \langle 7, 8\rangle_\Gamma$, $\Delta_{5,4} = \langle 6, 11\rangle_\Gamma$
6	$\Delta_{6,1} = \langle 9, 10, 11\rangle_\Gamma$, $\Delta_{6,2} = \langle 8, 10\rangle_\Gamma$, $\Delta_{6,3} = \langle 8, 9\rangle_\Gamma$, $\Delta_{6,4} = \langle 7, 12\rangle_\Gamma$ $\Delta_{6,5} = \langle 6\rangle_\Gamma$

Table 3

Now we consider $\mathcal{M}_{6,5}$ as an example. As the result of Step 7, we find the k-basis which consists of

$$\overline{v}_1 = 1 + c_{11}t + c_{12}t^2 + c_{15}t^5,\ \overline{v}_2 = t^3 + c_{25}t^5,\ \overline{v}_3 = t^4 + c_{35}t^5.$$

Furthermore, we have $T = \{c_{25} = c_{12} - c_{11}^2, c_{35} = c_{11}\}$ in the consequence of Step 8. It follows that

$$\mathcal{M}_{6,5} = [1 + c_{11}t + c_{12}t^2 + c_{15}t^5, t^3 + (c_{12} - c_{11}^2)t^5, t^4 + c_{11}t^5]_k.$$

In this way, Steps $r+1$ and $r+2$ ($r = 1, \ldots, 6$) yield Table 4 below. For the simplicity, we use notations a, b, c for the coefficients.

r	Components of $\mathcal{M}_r$
1	$\mathcal{M}_{1,1} = [t^2, t^3, t^5]_k$
2	$\mathcal{M}_{2,1} = [t^2, t^4, t^5]_k$, $\mathcal{M}_{2,2} = [t + at^2, t^4, t^5]_k$
3	$\mathcal{M}_{3,1} = [t^3, t^4, t^5]_k$, $\mathcal{M}_{3,2} = [t + at^3, t^4, t^5]_k$ $\mathcal{M}_{3,3} = [1 + at + bt^5, t^3 + at^4, t^4 + at^5]_k$
4	$\mathcal{M}_{4,1} = [t^3, t^4, t^5]_k$, $\mathcal{M}_{4,2} = [t^2 + at^3, t^4, t^5]_k$ $\mathcal{M}_{4,3} = [t^2 + at^4, t^3 + bt^4, t^5]_k$, $\mathcal{M}_{4,4} = [1 + at^2 + bt^5, t^3 + at^5, t^4]_k$
5	$\mathcal{M}_{5,1} = [t^3, t^4, t^5]_k$, $\mathcal{M}_{5,2} = [t^2 + at^3, t^4, t^5]_k$ $\mathcal{M}_{5,3} = [t^2 + at^4, t^3 + bt^4, t^5]_k$, $\mathcal{M}_{5,4} = [t + at^2 + bt^3, t^4, t^5]_k$
6	$\mathcal{M}_{6,1} = [t^3, t^4, t^5]_k$, $\mathcal{M}_{6,2} = [t^2 + at^3, t^4, t^5]_k$ $\mathcal{M}_{6,3} = [t^2 + at^4, t^3 + bt^4, t^5]_k$, $\mathcal{M}_{6,4} = [t + at^2 + bt^3, t^4, t^5]_k$ $\mathcal{M}_{6,5} = [1 + at + bt^2 + ct^5, t^3 + (b - a^2)t^5, t^4 + at^5]_k$

Table 4

Below we analyze the structure of the punctual Hilbert schemes. We only consider the case of $\mathcal{M}_6$ which is the most complicated case. The other cases can be teated in the similar manner. Note that the matrices representing the elements in $\mathcal{M}_{6,l}$ have the same form. So we just express them by A_l.

$$A_1 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},\ A_2 = \begin{pmatrix} 0 & 0 & 1 & a & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} 0 & 0 & 1 & 0 & a & 0 \\ 0 & 0 & 0 & 1 & b & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},\ A_4 = \begin{pmatrix} 0 & 1 & a & b & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$A_5 = \begin{pmatrix} 1 & a & b & 0 & 0 & c \\ 0 & 0 & 0 & 1 & 0 & b - a^2 \\ 0 & 0 & 0 & 0 & 1 & a \end{pmatrix}.$$

The Plücker coordinates π_{ijk} for each $\mathcal{M}_{6,l}$ are given by the determinants which consist of the i, j and kth columns of A_l ($1 \le i < j < k \le 6$). They are calculated as

$$\begin{aligned}
\mathcal{M}_{6,1} :& \pi_{456} = 1,\ \pi_{ijk} = 0 \text{ for } (i,j,k) \neq (4,5,6),\\
\mathcal{M}_{6,2} :& \pi_{356} = 1,\ \pi_{456} = a,\ \pi_{ijk} = 0 \text{ for } (i,j,k) \neq (3,5,6),\ (4,5,6),\\
\mathcal{M}_{6,3} :& \pi_{346} = 1,\ \pi_{356} = b,\ \pi_{456} = -a, \pi_{ijk} = 0 \text{ for } (i,j,k) \neq (3,4,6),\ (3,5,6),\ (4,5,6),\\
\mathcal{M}_{6,4} :& \pi_{256} = 1,\ \pi_{356} = a,\ \pi_{456} = b, \pi_{ijk} = 0 \text{ for } (i,j,k) \neq (2,5,6),\ (3,5,6),\ (4,5,6),\\
\mathcal{M}_{6,5} :& \pi_{145} = 1,\ \pi_{146} = a,\ \pi_{156} = a^2 - b,\ \pi_{245} = a,\ \pi_{246} = a^2,\ \pi_{256} = a^3 - ab,\ \pi_{345} = a,\\
& \quad \pi_{346} = a^2,\ \pi_{345} = a,\ \pi_{346} = a^2,\ \pi_{356} = a^2b - b^2,\ \pi_{456} = c,\ \pi_{ijk} = 0 \text{ for the others.}
\end{aligned}$$

By using these Plücker coordinates, we can check that

$$\overline{\mathcal{M}}_{6,3} = \mathcal{M}_{6,1} \cup \mathcal{M}_{6,2} \cup \mathcal{M}_{6,3} \cong \overline{\mathcal{M}}_{6,4} = \mathcal{M}_{6,1} \cup \mathcal{M}_{6,2} \cup \mathcal{M}_{6,4} \cong \mathbb{P}^2, \tag{5}$$

$$\overline{\mathcal{M}}_{6,3} \cap \overline{\mathcal{M}}_{6,4} = \mathcal{M}_{6,1} \cup \mathcal{M}_{6,2} \cong \mathbb{P}^1. \tag{6}$$

Next we calculate the defining equations of $\mathcal{M}_{6,5}$ to show $\mathcal{M}_6 = \overline{\mathcal{M}}_{6,5}$. Let I be an ideal which is generated by the following polynomials in $k[a, b, c, \pi_{ijk} |\, 1 \le i < j < k \le 6,\ (i,j,k) \neq (1,4,5)]$:

$$\begin{aligned}
&\pi_{146} - a,\ \pi_{156} - a^2 + b,\ \pi_{245} - a,\ \pi_{246} - a^2,\ \pi_{256} - a^3 + ab,\ \pi_{345} - a,\\
&\pi_{346} - a^2,\ \pi_{345} - a,\ \pi_{346} - a^2,\ \pi_{356} - a^2b + b^2,\ \pi_{456} - c
\end{aligned}$$

By the fact which is called Polynomial Implicitization (cf. [3]), the variety $\mathcal{M}_{6,5}$ is defined by the third elimination ideal $I_3 := I \cap k[\pi_{ijk} |\, 1 \le i < j < k \le 6,\ (i,j,k) \neq (1,4,5)]$. We compute a Gröbner basis of I with respect to the lexicographic order $a \succ b \succ c \succ \pi_{123} \succ \cdots \succ \pi_{456}$. The elements which are not involving a, b, c form the Gröbner base of I_3 (see The Elimination Theorem in [3]). This computation was done by the computer algebra system "Singular" (see [4] for the usage of Singular). Furthermore, homogenizing the basis of I_3 at π_{145}, we obtain the defining equations of a projective threefold X_2 as follows:

$$\begin{aligned}
&\pi_{345}^3 + \pi_{145}\pi_{345}\pi_{356} - \pi_{145}\pi_{346}^2 = 0,\ \pi_{256}\pi_{345}^2 - \pi_{145}\pi_{346}\pi_{356} = 0,\\
&\pi_{145}\pi_{256}\pi_{346}^3 - \pi_{345}^3\pi_{356} - \pi_{345}^3\pi_{346}^2 - 2\pi_{145}\pi_{345}^2\pi_{356}^2\\
&\qquad\qquad - \pi_{145}\pi_{345}\pi_{346}^2\pi_{356} + \pi_{145}\pi_{346}^4 - \pi_{145}^2\pi_{356}^3 = 0,\\
&\pi_{145}\pi_{256}\pi_{345}\pi_{356} - \pi_{145}\pi_{256}\pi_{346}^2 + \pi_{345}^3\pi_{346}\\
&\qquad\qquad + 2\pi_{145}\pi_{345}\pi_{346}\pi_{356} - \pi_{145}\pi_{346}^3 = 0,\\
&\pi_{145}\pi_{256}\pi_{345}\pi_{346} - \pi_{345}^4 - 2\pi_{145}\pi_{345}^2\pi_{356} + \pi_{145}\pi_{345}\pi_{346}^2 - \pi_{145}^2\pi_{356}^2 = 0,\\
&\pi_{145}\pi_{246}\pi_{356} - \pi_{145}\pi_{256}\pi_{346} + \pi_{345}^3 + \pi_{145}\pi_{345}\pi_{356} - \pi_{145}\pi_{346}^2 = 0,\\
&\pi_{246}\pi_{346} - \pi_{256}\pi_{345} - \pi_{345}\pi_{346} = 0,\ \pi_{246}\pi_{345} - \pi_{345}^2 - \pi_{145}\pi_{356} = 0,\\
&\pi_{246}^3 - \pi_{246}^2\pi_{345} - \pi_{145}\pi_{256}^2 - \pi_{145}\pi_{256}\pi_{346} = 0, \pi_{245}\pi_{356} - \pi_{256}\pi_{345} = 0,\\
&\pi_{245}\pi_{346} - \pi_{246}\pi_{345} = 0,\ \pi_{245}\pi_{345} - \pi_{145}\pi_{346} = 0,\\
&\pi_{245}\pi_{256} + \pi_{245}\pi_{346} - \pi_{246}^2 = 0,\\
&\pi_{245}\pi_{246} - \pi_{245}\pi_{345} - \pi_{145}\pi_{256} = 0,\ \pi_{245}^2 - \pi_{145}\pi_{246} = 0,\\
&\pi_{145}\pi_{156} - \pi_{245}^2 + \pi_{145}\pi_{345} = 0,\ \pi_{146} - \pi_{245} = 0
\end{aligned}$$

By computer algebra system "Maple", we can check that $\mathcal{M}_6 \setminus \mathcal{M}_{6,5} = \cup_{i=1}^4 \mathcal{M}_{6,i}$ is defined by these equations with $\pi_{145} = 0$. Hence we have $\mathcal{M}_6 = \overline{\mathcal{M}}_{6,5} = X_2$ (i.e X_2 is irreducible). The equalities $\dim \mathcal{M}_6 = \dim \mathcal{M}_{6,5} = 3$ and the rationality of $\mathcal{M}_6$ also follow from Corollary 13 and Proposition 15 respectively. We also conclude that $\mathrm{Sing}(\mathcal{M}_6) \cong \mathbb{P}^1$ by (5) and (6).

We add some comments for $\mathcal{M}_4$ and $\mathcal{M}_5$. We can check that $\mathcal{M}_4$ consists of two components $\overline{\mathcal{M}}_{4,3} = \mathcal{M}_{4,1} \cup \mathcal{M}_{4,2} \cup \mathcal{M}_{4,3} \cong \mathbb{P}^2$ and a rational surface $X_1 = \overline{\mathcal{M}}_{4,4} = \mathcal{M}_{4,1} \cup \mathcal{M}_{4,2} \cup \mathcal{M}_{4,4}$ given by $\pi_{156} + \pi_{345} = 0$, $\pi_{145}\pi_{356} + \pi_{345}^2 = 0$ and $\pi_{ijk} = 0$ for

$$(ijk) \neq (145), (156), (345), (356), (456).$$

It also follows that $\mathrm{Sing}(\mathcal{M}_4) = \overline{\mathcal{M}}_{4,3} \cap \overline{\mathcal{M}}_{4,4} = \mathcal{M}_{4,1} \cup \mathcal{M}_{4,2} \cong \mathbb{P}^1$.

Similarly, we have $\mathcal{M}_5 = \overline{\mathcal{M}}_{5,3} \cup \overline{\mathcal{M}}_{5,4}$ where

$$\overline{\mathcal{M}}_{5,3} = \mathcal{M}_{5,1} \cup \mathcal{M}_{5,2} \cup \mathcal{M}_{5,3} \cong \mathbb{P}^2 \quad \text{and} \quad \overline{\mathcal{M}}_{5,4} = \mathcal{M}_{5,1} \cup \mathcal{M}_{5,2} \cup \mathcal{M}_{5,4} \cong \mathbb{P}^2.$$

It is easy to check $\mathrm{Sing}(\mathcal{M}_5) = \overline{\mathcal{M}}_{5,3} \cap \overline{\mathcal{M}}_{5,4} = \mathcal{M}_{5,1} \cup \mathcal{M}_{5,2} \cong \mathbb{P}^1$. We also see that $\mathcal{M}_4$, $\mathcal{M}_5$ and $\mathcal{M}_6$ possess same $\mathbb{P}^1$ in common, as their singular locus. □

The irreducibility of $\mathcal{M}_6$ also can be proven by known results of the compactified Jacobians. In next section, we use them to show the irreducibility of the PS variety for the singularity of type E_8.

5. Proof of Theorem 2

Consider the curve singularity of type E_8 in this section. In order to prove Theorem 2, we first recall some results about compactified Jacobian $\overline{JC}$ for a singular complete algebraic curve C.

Definition 25 ([8])**.** *The compactified Jacobian $\overline{JC}$ of C consists of all torsion free sheaves $\mathcal{F}$ of rank 1 and degree 0 on C (i.e., $\chi(\mathcal{F}) = 1 - g_a(C)$).*

The following facts about compactified Jacobians are known:

Theorem 26 ([1], [8])**.** *The compactified Jacobian $\overline{JC}$ is irreducible if and only if* $\mathrm{Sing}(C)$ *consits of plane curve singularities.*

Theorem 27 ([2])**.** *For a rational unibranched curve C, its compactified Jacobian is homeomorphic to the direct product of compact spaces, the Jacobi factors $\overline{JC}_p$ where $p \in \mathrm{Sing}(C)$.*

The Jacobi factor for a curve singularity was introduced by Rego in [8] (see also [7]). It coincides with the PS variety for a given singularity.

Proof of Theorem 2. Consider the local ring $\mathcal{O} = k[[t^3, t^5]]$. We have $\Gamma = \{0, 3, 5, 6, 8, 9, \cdots\}$. It also follows that $\delta = 4$ and $2\delta = 8$. Theorem 24 yields the following two tables:

r	Elements of $\mathfrak{D}_i$
1	$\Delta_{1,1} = \langle 3, 5\rangle_\Gamma$
2	$\Delta_{2,1} = \langle 5, 6\rangle_\Gamma$, $\Delta_{2,2} = \langle 3, 10\rangle_\Gamma$
3	$\Delta_{3,1} = \langle 6, 8, 10\rangle_\Gamma$, $\Delta_{3,2} = \langle 5, 9\rangle_\Gamma$, $\Delta_{3,3} = \langle 3\rangle_\Gamma$
4	$\Delta_{4,1} = \langle 8, 9, 10\rangle_\Gamma$, $\Delta_{4,2} = \langle 6, 10\rangle_\Gamma$, $\Delta_{4,3} = \langle 6, 8\rangle_\Gamma$, $\Delta_{4,4} = \langle 5, 12\rangle_\Gamma$
5	$\Delta_{5,1} = \langle 9, 10, 11\rangle_\Gamma$, $\Delta_{5,2} = \langle 8, 10, 12\rangle_\Gamma$, $\Delta_{5,3} = \langle 8, 9\rangle_\Gamma$, $\Delta_{5,4} = \langle 6, 13\rangle_\Gamma$ $\Delta_{5,5} = \langle 5\rangle_\Gamma$
6	$\Delta_{6,1} = \langle 10, 11, 12\rangle_\Gamma$, $\Delta_{6,2} = \langle 9, 11, 13\rangle_\Gamma$, $\Delta_{6,3} = \langle 9, 10\rangle_\Gamma$, $\Delta_{6,4} = \langle 8, 12\rangle_\Gamma$ $\Delta_{6,5} = \langle 8, 10\rangle_\Gamma$, $\Delta_{6,6} = \langle 6\rangle_\Gamma$
7	$\Delta_{7,1} = \langle 11, 12, 13\rangle_\Gamma$, $\Delta_{7,2} = \langle 10, 12, 14\rangle_\Gamma$, $\Delta_{7,3} = \langle 10, 11\rangle_\Gamma$ $\Delta_{7,4} = \langle 9, 13\rangle_\Gamma$, $\Delta_{7,5} = \langle 9, 11\rangle_\Gamma$, $\Delta_{7,6} = \langle 8, 15\rangle_\Gamma$
8	$\Delta_{8,1} = \langle 12, 13, 14\rangle_\Gamma$, $\Delta_{8,2} = \langle 11, 13, 15\rangle_\Gamma$, $\Delta_{8,3} = \langle 11, 12\rangle_\Gamma$ $\Delta_{8,4} = \langle 10, 14\rangle_\Gamma$, $\Delta_{8,5} = \langle 10, 12\rangle_\Gamma$, $\Delta_{8,6} = \langle 9, 16\rangle_\Gamma$, $\Delta_{8,7} = \langle 8\rangle_\Gamma$

Table 5

r	Components of $\mathcal{M}_r$
1	$\mathcal{M}_{1,1} = [t^2, t^4, t^5, t^7]_k$
2	$\mathcal{M}_{2,1} = [t^3, t^4, t^6, t^7]_k$, $\mathcal{M}_{2,2} = [t + at^3, t^4, t^6, t^7]_k$
3	$\mathcal{M}_{3,1} = [t^3, t^5, t^6, t^7]_k$, $\mathcal{M}_{3,2} = [t^2 + at^3, t^5, t^6, t^7]_k$ $\mathcal{M}_{3,3} = [1 + at^2 + bt^7, t^3 - a^2t^7, t^5 + at^7, t^6]_k$
4	$\mathcal{M}_{4,1} = [t^4, t^5, t^6, t^7]_k$, $\mathcal{M}_{4,2} = [t^2 + at^4, t^5, t^6, t^7]_k$ $\mathcal{M}_{4,3} = [t^2 + at^6, t^4 + bt^6, t^5, t^7]_k$, $\mathcal{M}_{4,4} = [t + at^2 + bt^5, t^4 + at^5, t^6, t^7]_k$
5	$\mathcal{M}_{5,1} = [t^4, t^5, t^6, t^7]_k$, $\mathcal{M}_{5,2} = [t^3 + at^4, t^5, t^6, t^7]_k$ $\mathcal{M}_{5,3} = [t^3 + at^5, t^4 + bt^5, t^6, t^7]_k$, $\mathcal{M}_{5,4} = [t + at^3 + bt^5, t^4, t^6, t^7]_k$ $\mathcal{M}_{5,5} = [1 + at + bt^4 + ct^7, t^3 + at^4 + bt^7, t^5 - a^2t^7, t^6 + at^7]_k$
6	$\mathcal{M}_{6,1} = [t^4, t^5, t^6, t^7]_k$, $\mathcal{M}_{6,2} = [t^3 + at^4, t^5, t^6, t^7]_k$ $\mathcal{M}_{6,3} = [t^3 + at^5, t^4 + bt^5, t^6, t^7]_k$, $\mathcal{M}_{6,4} = [t^2 + at^3 + bt^4, t^5, t^6, t^7]_k$ $\mathcal{M}_{6,5} = [t^2 + at^3 + bt^6, t^4 + ct^6, t^5 + at^6, t^7]_k$ $\mathcal{M}_{6,6} = [1 + at^2 + bt^4 + ct^7, t^3 + (b - a^2)t^7, t^5 + at^7, t^6]_k$
7	$\mathcal{M}_{7,1} = [t^4, t^5, t^6, t^7]_k$, $\mathcal{M}_{7,2} = [t^3 + at^4, t^5, t^6, t^7]_k$ $\mathcal{M}_{7,3} = [t^3 + at^5, t^4 + bt^5, t^6, t^7]_k$, $\mathcal{M}_{7,4} = [t^2 + at^3 + bt^4, t^5, t^6, t^7]_k$ $\mathcal{M}_{7,5} = [t^2 + at^3 + bt^6, t^4 + ct^6, t^5 + at^6, t^7]_k$ $\mathcal{M}_{7,6} = [t + at^2 + bt^3 + ct^5, t^4 + at^5, t^6, t^7]_k$
8	$\mathcal{M}_{8,1} = [t^4, t^5, t^6, t^7]_k$, $\mathcal{M}_{8,2} = [t^3 + at^4, t^5, t^6, t^7]_k$ $\mathcal{M}_{8,3} = [t^3 + at^5, t^4 + bt^5, t^6, t^7]_k$, $\mathcal{M}_{8,4} = [t^2 + at^3 + bt^4, t^5, t^6, t^7]_k$ $\mathcal{M}_{8,5} = [t^2 + at^3 + bt^6, t^4 + ct^6, t^5 + at^6, t^7]_k$ $\mathcal{M}_{8,6} = [t + at^2 + bt^3 + ct^5, t^4 + at^5, t^6, t^7]_k$ $\mathcal{M}_{8,7} = [1 + at + bt^2 + ct^4 + dt^7, t^3 + at^4 + (c + a^2b - b^2)t^7,$ $t^5 + (b - a^2)t^7, t^6 + at^7]_k$

Table 6

In Table 6, we have $a, b, c, d \in k$. The analyses for $\mathcal{M}_i$ $(i = 1, \ldots, 8)$ proceed similarly as in the proof of Theorem 1, except the irreducibility of $\mathcal{M}_8$. The defining equations of $\mathcal{M}_{8,7}$ are too many to analyze. So we use Theorem 26 and 27 to show the irreducibility of $\mathcal{M}_8$. Let C be a rational curve with the curve singularity of type E_8 as its unique singularity. Since the compactified Jacobian $\overline{JC}$ is irreducible by Theorem 26, the irreducibility of $\mathcal{M}_8$ also follows from Theorem 27. Finally, we conclude that $\mathcal{M}_8$ is rational by Proposition 15.

For the other cases, we only mention that the punctual Hilbert schemes $\mathcal{M}_6$, $\mathcal{M}_7$ and $\mathcal{M}_8$ possess same $\mathbb{P}^2 \cup \mathbb{P}^2$ with $\mathbb{P}^2 \cap \mathbb{P}^2 = \mathbb{P}^1$, as their singular locus. We omit the defining equations for X_i $(i = 3, \ldots, 9)$. $\square$

Remark 28. *In [2], Beauville proved that the Euler number of the Jacobi factor for the curve singularities of type E_6 (resp. E_8) is 5 (resp. 7). They coincide with the number of affine cells of each $\mathcal{M}_{2\delta}$ (see Table 3 and 5). In general, this fact holds for every irreducible curve singularity with Puiseux exponent (p, q) (see [7]).*

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