VARIETIES OF COMPLEXES AND FOLIATIONS

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Dedicated to Xavier Gómez-Mont on his 60th Birthday.

Abstract. Let $\mathcal{F}(r,d)$ denote the moduli space of algebraic foliations of codimension one and degree $d$ in complex projective space of dimension $r$. We show that $\mathcal{F}(r,d)$ may be represented as a certain linear section of a variety of complexes. From this fact we obtain information on the irreducible components of $\mathcal{F}(r,d)$.

1. Basics on varieties of complexes.

1.1. Let $K$ be a field and let $V_0, \ldots, V_n$ be vector spaces over $K$ of finite dimensions $d_i = \dim_K(V_i)$.

Consider sequences of linear functions

$$V_0 \xrightarrow{f_1} V_1 \xrightarrow{f_2} \ldots \xrightarrow{f_n} V_n,$$

also written

$$f = (f_1, \ldots, f_n) \in V = \prod_{i=1}^n \text{Hom}_K(V_{i-1}, V_i).$$

The variety of differential complexes is defined as

$$\mathcal{C} = \mathcal{C}(V_0, \ldots, V_n) = \{ f = (f_1, \ldots, f_n) \in V/ f_{i+1} \circ f_i = 0, \ i = 1, \ldots, n - 1 \},$$

It is an affine variety in $V$, given as an intersection of quadrics. We intend to study the geometry of this variety (see also e.g., [3], [6]).

1.2. Since the defining equations $f_{i+1} \circ f_i = 0$ are bilinear, we may also consider, when it is convenient, the projective variety of complexes

$$PC \subset \prod_{i=1}^n \mathbb{P}\text{Hom}_K(V_{i-1}, V_i),$$

as a subvariety of a product of projective spaces.

Denoting $V = \bigoplus_{i=0}^n V_i$, each complex $f \in \mathcal{C}$ may be thought as a degree-one homomorphism of graded vector spaces $f : V \to V$ with $f^2 = 0$.

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1.3. For each \( f \in C \) and \( i = 0, \ldots, n \) define
\[
B_i = f_i(V_{i-1}) \subset Z_i = \ker (f_{i+1}) \subset V_i,
\]
and
\[
H_i = Z_i/B_i.
\]
(we understand by convention that \( B_0 = 0 \)).

From the exact sequences
\[
0 \to B_i \to Z_i \to H_i \to 0,
\]
\[
0 \to Z_i \to V_i \to B_{i+1} \to 0,
\]
we obtain for the dimensions
\[
b_i = \dim_K(B_i), \quad z_i = \dim_K(Z_i), \quad h_i = \dim_K(H_i),
\]
the relations
\[
d_i = b_{i+1} + z_i = b_{i+1} + b_i + h_i,
\]
where \( i = 0, \ldots, n \) and \( b_0 = b_{n+1} = 0 \). Therefore,

**Proposition 1.** a) The \( h_i \) and the \( b_i \) determine each other by the formulas:
\[
h_i = d_i - (b_{i+1} + b_i),
\]
\[
b_{j+1} = \chi_j(d) - \chi_j(h),
\]
where for a sequence \( e = (e_0, \ldots, e_n) \) and \( 0 \leq j \leq n \) we denote
\[
\chi_j(e) = (-1)^j \sum_{i=0}^{j} (-1)^i e_i = e_j - e_{j-1} + e_{j-2} + \cdots + (-1)^j e_0,
\]
the \( j \)-th Euler characteristic of \( e \).

b) The inequalities \( b_{i+1} + b_i \leq d_i \) are satisfied for all \( i \).

**Proof.** We write down the \( b_j \) in terms of the \( h_i \): from
\[
\sum_{i=0}^{j} (-1)^i d_i = \sum_{i=0}^{j} (-1)^i (b_{i+1} + b_i + h_i),
\]
we obtain
\[
b_{j+1} = (-1)^j (\sum_{i=0}^{j} (-1)^i d_i - \sum_{i=0}^{j} (-1)^i h_i),
\]
as claimed. \( \square \)

Notice in particular that since \( b_{n+1} = 0 \), we have the usual relation
\[
\sum_{i=0}^{n} (-1)^i d_i = \sum_{i=0}^{n} (-1)^i h_i.
\]
1.4. Now we consider the subvarieties of $C$ obtained by imposing rank conditions on the $f_i$.

**Definition 2.** For each $r = (r_1, \ldots, r_n) \in \mathbb{N}^n$ define

$$C_r = \{ f = (f_1, \ldots, f_n) \in C/ \, \text{rank}(f_i) = r_i, \ i = 1, \ldots, n \}.$$ 

These are locally closed subvarieties of $C$.

**Proposition 3.** a) $C_r \neq \emptyset$ if and only if $r_{i+1} + r_i \leq d_i$ for $0 \leq i \leq n$ (we use the convention $r_0 = r_{n+1} = 0$)

b) In the conditions of a), $C_r$ is smooth and irreducible, of dimension

$$\dim(C_r) = \sum_{i=0}^{n} (d_i - r_i)(r_{i+1} + r_i) = \sum_{i=0}^{n} (d_i - r_i)(d_i - h_i) = \frac{1}{2} \sum_{i=0}^{n} (d_i^2 - h_i^2).$$

**Proof.** a) One implication follows from Proposition 1. Conversely, in the given conditions, we need to define a complex with rank$(f_i) = r_i$ for all $i$. Suppose we constructed

$$V_0 \xrightarrow{f_1} V_1 \xrightarrow{f_2} \ldots \xrightarrow{f_{n-1}} V_{n-1}. $$

We need to define $f_n : V_{n-1} \rightarrow V_n$ such that $f_n \circ f_{n-1} = 0$ and rank$(f_n) = r_n$, that is, a map $V_{n-1}/B_{n-1} \rightarrow V_n$ of rank $r_n$. Such a map exists since $\dim(V_{n-1}/B_{n-1}) = d_{n-1} - r_{n-1} \geq r_n$.

b) Consider the projection (forgetting $f_n$)

$$\pi : C(0, \ldots, V_n) \rightarrow C(0, \ldots, V_{n-1}),$$

where $r = (r_1, \ldots, r_n)$ and $\bar{r} = (r_1, \ldots, r_{n-1})$. Any fiber $\pi^{-1}(f_1, \ldots, f_{n-1})$ is isomorphic to the subvariety in $\text{Hom}(V_{n-1}/B_{n-1}, V_n)$ of maps of rank $r_n$; therefore, it is smooth and irreducible of dimension $r_n(d_{n-1} - r_{n-1} + d_n - r_n)$ (see [1]). The assertion follows by induction on $n$. The various expressions for $\dim(C_r)$ follow by direct calculations.

Another proof of a): Given $r$ such that $r_{i+1} + r_i \leq d_i$, put $h_i = d_i - (r_{i+1} + r_i) \geq 0$ and $z_i = d_i - r_{i+1} = h_i + r_i$. Choose linear subspaces $B_i \subset Z_i \subset V_i$ with $\dim(B_i) = r_i$ and $\dim(Z_i) = z_i$. Since $\dim(V_{i-1}/Z_{i-1}) = \dim(B_i)$, choose an isomorphism $\sigma_i : V_{i-1}/Z_{i-1} \rightarrow B_i$ for each $i$. Composing with the natural projection $V_{i-1} \rightarrow V_{i-1}/Z_{i-1}$ we obtain linear maps $V_{i-1} \rightarrow B_i$ with kernel $Z_{i-1}$ and rank $r_i$, as wanted.

**Remark 4.** In terms of dimension of homology, the condition in Proposition 3 a) translates as follows. Given $h = (h_0, \ldots, h_n) \in \mathbb{N}^{n+1}$, there exists a complex with dimension of homology equal to $h$ if and only if $\chi_i(h) \leq \chi_i(d)$ for $i = 1, \ldots, n-1$ and $\chi_n(h) = \chi_n(d)$.

**Remark 5.** The group $G = \prod_{i=0}^{n} GL(V_i, K)$ acts on $V = \prod_{i=1}^{n} \text{Hom}_K(V_{i-1}, V_i)$ via

$$(g_0, g_1, \ldots, g_n) \cdot (f_1, f_2, \ldots, f_n) = (g_0f_1g_1^{-1}, g_1f_2g_2^{-1}, \ldots, g_{n-1}f_ng_n^{-1}).$$

This action clearly preserves the variety of complexes. It follows from the proof above that the action on each $C_r$ is transitive. Hence, the non-empty $C_r$ are the orbits of $G$ acting on $C(0, \ldots, V_n)$.

**Definition 6.** For $r, s \in \mathbb{N}^n$ we write $s \leq r$ if $s_i \leq r_i$ for $i = 1, \ldots, n$. 
Corollary 7. If $C_r \neq \emptyset$ and $s \leq r$ then $C_s \neq \emptyset$. Also, $\dim(C_s) > 0$ if $s \neq 0$.

Proof. The first assertion follows from Proposition 3 a), and the second from Proposition 3 b).

Proposition 8. With the notation above,

$$\mathcal{C}_r = \bigcup_{s \leq r} C_s = \{ f \in \mathcal{C} / \text{rank}(f_i) \leq r_i, \; i = 1, \ldots, n \}.$$

Proof. Denote $X_r = \bigcup_{s \leq r} C_s$. Since the second equality is clear, $X_r$ is closed. It follows that $\mathcal{C}_r \subset X_r$. To prove the equality, since $\mathcal{C}_r \subset X_r$ is open, it would be enough to show that $X_r$ is irreducible. For this, consider $L = (L_1, \ldots, L_n)$ where $L_i \in \text{Grass}(r_i, V_i)$ and denote

$$X_L = \{ f = (f_1, \ldots, f_n) \in \mathcal{C} / \text{im}(f_i) \subset L_i \subset \ker(f_{i+1}), \; i = 1, \ldots, n \}.$$ 

Consider

$$\tilde{X}_r = \{(L, f) / f \in X_L \} \subset G \times \mathcal{C},$$

where $G = \prod_{i=0}^n \text{Grass}(r_i, V_i)$. The first projection $p_1 : \tilde{X}_r \to G$ has fibers

$$p_1^{-1}(L) = X_L \cong \text{Hom}(V_0, L_1) \times \text{Hom}(V_1/L_1, L_2) \times \cdots \times \text{Hom}(V_{n-1}/L_{n-1}, V_n),$$

which are vector spaces of constant dimension $\sum_{i=0}^n (d_i - r_i)r_{i+1}$. It follows that $\tilde{X}_r$ is irreducible, and hence $X_r = p_2(\tilde{X}_r)$ is also irreducible, as wanted.

Remark 9. In the proof above we find again the formula

$$\dim(X_r) = \dim(X_L) + \dim(G) = \sum_{i=0}^n (d_i - r_i)r_i + \sum_{i=0}^n (d_i - r_i)r_{i+1}.$$ 

Remark 10. The fact that $p_1 : \tilde{X}_r \to G$ is a vector bundle implies that $\tilde{X}_r$ is smooth. On the other hand, since $p_2 : \tilde{X}_r \to X_r$ is birational (an isomorphism over the open set $\mathcal{C}_r$), it is a resolution of singularities.

The following two corollaries are immediate consequences of Proposition 8.

Corollary 11. $C_s \subset \mathcal{C}_r$ if and only if $s \leq r$.

Corollary 12. $\mathcal{C}_r \cap \mathcal{C}_s = \mathcal{C}_t$ where $t_i = \min(r_i, s_i)$ for all $i = 1, \ldots, n$.

Definition 13. For $d = (d_0, \ldots, d_n) \in \mathbb{N}^{n+1}$ let

$$R = R(d) = \{ (r_1, \ldots, r_n) \in \mathbb{N}^n / r_1 \leq d_0, \; r_{i+1} + r_i \leq d_i \; (1 \leq i \leq n-1), \; r_n \leq d_n \}.$$ 

We consider $\mathbb{N}^n$ ordered via $r \leq s$ if $r_i \leq s_i$ for all $i$; the finite set $R$ has the induced order. Notice that $R$ is finite since it is contained in the box $\{ (r_1, \ldots, r_n) \in \mathbb{N}^n / 0 \leq r_i \leq d_i, \; i = 1, \ldots, n \}$.

Proposition 14. With the notation above, the irreducible components of the variety of complexes $\mathcal{C} = \mathcal{C}(V_0, \ldots, V_n)$ are the $\mathcal{C}_r$ with $r \in R(d_0, \ldots, d_n)$ a maximal element.
Proof. From the previous Propositions, we have the equalities
\[ C = \bigcup_{r \in R} C_r = \bigcup_{r \in R} \overline{C}_r = \bigcup_{r \in R^+} \overline{C}_r, \]
where \( R^+ \) denotes the set of maximal elements of \( R \). The result follows because we know that each \( \overline{C}_r \) is irreducible and there are no inclusion relations among the \( \overline{C}_r \) for \( r \in R^+ \) (see Corollary 11). \( \square \)

1.5. Morphisms of complexes. Tangent space of the variety of complexes. Now we would like to compute the dimension of the tangent space of a variety of complexes at each point. With the notation of 1.1 we consider complexes \( f \in \mathcal{C}(V_0, \ldots, V_n) \) and \( f' \in \mathcal{C}(V'_0, \ldots, V'_n) \) (the vector spaces \( V_i \) and \( V'_i \) are not necessarily the same, but the length \( n \) we may assume is the same). We denote
\[ \text{Hom}_\mathcal{C}(f, f'), \]
the set of morphisms of complexes from \( f \) to \( f' \), that is, collections of linear maps \( g_i : V_i \to V'_i \) for \( i = 0, \ldots, n \), such that \( g_i \circ f_i = f'_i \circ g_{i-1} \) for \( i = 1, \ldots, n \). It is a vector subspace of \( \prod_{i=0}^{n} \text{Hom}_R(V_i, V'_i) \), and we would like to calculate its dimension.

For this particular purpose and for its independent interest, we recall the following from [2] (§2 – 5. Complexes scindés):

For \( f \in \mathcal{C}(V_0, \ldots, V_n) \), denote as in 1.1
\[ B_i(f) = f_i(V_{i-1}) \subset Z_i(f) = \ker (f_{i+1}) \subset V_i. \]
Since we are working with vector spaces, we may choose linear subspaces \( \bar{B}_i \) and \( \bar{H}_i \) of \( V_i \) such that
\[ V_i = Z_i(f) \oplus \bar{B}_i \quad \text{and} \quad Z_i(f) = B_i(f) \oplus \bar{H}_i. \]
Then \( V_i = B_i(f) \oplus \bar{H}_i \oplus \bar{B}_i \) and clearly \( f_{i+1} \) takes \( \bar{B}_i \) isomorphically onto \( B_{i+1}(f) \). Notice also that
\[ \dim(\bar{B}_i) = \dim(B_{i+1}(f)) = \text{rank}(f_{i+1}) = r_{i+1}(f), \]
and
\[ \dim(\bar{H}_i) = \dim(Z_i(f)/B_i(f)) = h_i(f). \]

Next, define the following complexes:
\( \bar{H}(i) \) the complex of length zero consisting of the vector space \( \bar{H}_i \) in degree \( i \), the vector space zero in degrees \( \neq i \), and all differentials equal to zero.
\( \bar{B}(i) \) the complex of length one consisting of the vector space \( \bar{B}_{i-1} \) in degree \( i - 1 \), the vector space \( B_i(f) \) in degree \( i \), with the map \( f_i : \bar{B}_{i-1} \to B_i(f) \), and zeroes everywhere else.

**Proposition 15.** With the notation just introduced, \( \bar{H}(i) \) and \( \bar{B}(i) \) are subcomplexes of \( f \) and we have a direct sum decomposition of complexes:
\[ f = \bigoplus_{0 \leq i \leq n} \bar{H}(i) \oplus \bigoplus_{0 \leq i \leq n} \bar{B}(i). \]

**Proof.** Clear from the discussion above; see also [2], loc. cit. \( \square \)

Now we are ready for the calculation of \( \dim_K \text{Hom}_\mathcal{C}(f, f') \).
Proposition 16. With the previous notation, we have:

\[ \dim_K \text{Hom}_C(f, f') = \sum_i h_i h'_i + h_i r'_i + r_i h'_{i-1} + r_i r'_{i-1} \]

\[ = \sum_i h_i (h'_i + r'_i) + r_i d'_{i-1} \]

Proof. We may decompose \( f \) and \( f' \) as in Proposition 15:

\[ \text{Hom}_C(f, f') = \text{Hom}_C(\oplus_i \tilde{H}(i) \oplus \oplus_i \tilde{B}(i), \oplus_i \tilde{H}(i)' \oplus \oplus_i \tilde{B}(i)') \]

\[ = \oplus_{i,j} \text{Hom}_C(\tilde{H}(i), \tilde{H}(j)') \oplus \oplus_{i,j} \text{Hom}_C(\tilde{H}(i), \tilde{B}(j)') \]

\[ \oplus_{i,j} \text{Hom}_C(\tilde{B}(i), \tilde{H}(j)') \oplus \oplus_{i,j} \text{Hom}_C(\tilde{B}(i), \tilde{B}(j)') \]

It is easy to check the following:

\[ \text{Hom}_C(\tilde{H}(i), \tilde{H}(j)') = 0 \text{ for } i \neq j \]

\[ \text{Hom}_C(\tilde{H}(i), \tilde{H}(i)') = \text{Hom}_K(\tilde{H}_i, \tilde{H}_i') \]

\[ \text{Hom}_C(\tilde{H}(i), \tilde{B}(j)') = 0 \text{ for } i \neq j \]

\[ \text{Hom}_C(\tilde{H}(i), \tilde{B}(i)') = \text{Hom}_K(\tilde{H}_i, \tilde{B}_i') \]

(the case \( j = i + 1 \) requires special attention)

\[ \text{Hom}_C(\tilde{B}(i), \tilde{H}(j)') = 0 \text{ for } i - 1 \neq j \]

\[ \text{Hom}_C(\tilde{B}(i), \tilde{H}(i - 1)') = \text{Hom}_K(\tilde{B}_{i-1}, \tilde{H}'_{i-1}) \cong \text{Hom}_K(\tilde{B}_i(f), \tilde{H}'_{i-1}) \]

(the case \( j = i \) requires special attention)

\[ \text{Hom}_C(\tilde{B}(i), \tilde{B}(i)') \cong \text{Hom}_K(\tilde{B}_i(f), \tilde{B}'_i(f)) \]

\[ \text{Hom}_C(\tilde{B}(i), \tilde{B}(i - 1)') = \text{Hom}_K(\tilde{B}_{i-1}, \tilde{B}'_{i-1}) \cong \text{Hom}_K(\tilde{B}_i(f), \tilde{B}'_{i-1}) \]

\[ \text{Hom}_C(\tilde{B}(i), \tilde{B}(j)') = 0 \text{ otherwise} \]

Taking dimensions we obtain the stated formula. \( \square \)

Now we deduce the dimension of the tangent space to a variety of complexes at any point.

Proposition 17. For \( f \in C = C(V_0, \ldots, V_n) \) we have a canonical isomorphism

\[ TC(f) = \text{Hom}_C(f, f(1)), \]

where \( TC(f) \) is the Zariski tangent space to \( C \) at the point \( f \), and \( f(1) \) denotes de shifted complex \( f(1)_i = (-1)^i f_{i+1}, \quad i = -1, 0, \ldots, n. \)

Proof. Since \( C \) is an algebraic subvariety of the vector space \( V = \prod_{i=1}^n \text{Hom}_K(V_{i-1}, V_i) \), an element of \( TC(f) \) is a \( g = (g_1, \ldots, g_n) \in V \) such that \( f + \epsilon g \) satisfies the equations defining \( C \) (i.e., a \( K[\epsilon] \)-valued point of \( C \)), that is,

\[ (f + \epsilon g)_{i+1} \circ (f + \epsilon g)_i = 0, \quad i = 1, \ldots, n - 1 \quad (\text{modulo } \epsilon^2), \]
which is equivalent to
\[ f_{i+1} \circ g_i + g_{i+1} \circ f_i = 0, \quad i = 1, \ldots, n - 1, \]
and this means precisely that \( g \in \text{Hom}_C(f, f(1)). \) 

Corollary 18. For \( f \in C = C(V_0, \ldots, V_n), \)
\[
\dim_K TC(f) = \sum_i h_i (h_{i+1} + r_{i+1}) + r_i d_i
\]
\[
= \sum_i (d_i - r_i - r_{i+1})(d_{i+1} - r_{i+2}) + r_i d_i
\]

Proof. From Proposition 17 we know that \( \dim_K TC(f) = \dim_K \text{Hom}_C(f, f(1)). \) Next we apply Proposition 16 with \( f' = f(1), \) that is, replacing \( d'_i = d_{i+1}, \ r'_i = r_{i+1}, \ h'_i = h_{i+1}, \) to obtain the result. \qed

1.6. Varieties of exact complexes. Now we apply the previous results to the case of exact complexes.

Let us fix \( (d_0, \ldots, d_n) \in \mathbb{N}^n \) so that
\[
\chi_j(d) = (-1)^j \sum_{i=0}^j (-1)^i d_i \geq 0, \quad j = 1, \ldots, n - 1,
\]
\[
\chi_n(d) = (-1)^n \sum_{i=0}^n (-1)^i d_i = 0.
\]

Denoting \( \chi = \chi(d) = (\chi_1(d), \ldots, \chi_n(d)) \in \mathbb{N}^n, \) let us consider the variety \( C_\chi \) of complexes of rank \( \chi \) as in Definition 2. Since \( \chi_i(d) + \chi_{i-1}(d) = d_i \) for all \( i, \) it follows from Proposition 3 that \( C_\chi \) is non-empty of dimension
\[
\frac{1}{2} \sum_{i=0}^n d_i^2.
\]
It follows from Proposition 1 that any complex \( f \in C_\chi \) is exact. Also, since \( \chi \in R \) is clearly maximal, \( C_\chi \) is an irreducible component of \( C \) (see Proposition 14). Let us denote
\[
E = E(d_0, \ldots, d_n) = C_\chi = \{ f \in C/ \ \text{rank}(f_i) \leq \chi_i, \quad i = 1, \ldots, n \},
\]
the closure of the variety \( C_\chi \) of exact complexes. Denote also, for \( i = 1, \ldots, n \)
\[
\chi^i = \chi - e_i = (\chi_1, \ldots, \chi_i-1, \chi_i-1, \chi_{i+1}, \ldots, \chi_n),
\]
and
\[
\Delta_i = C_{\chi^i} = \{ f \in C/ \ \text{rank}(f) \leq \chi - e_i \},
\]
the variety of complexes where the \( i \)-th matrix drops rank by one.

Proposition 19. The codimension of \( \Delta_i \) in \( E \) is equal to one, and
\[
E = C_\chi \cup \Delta_1 \cup \cdots \cup \Delta_n.
\]

Proof. This follows from Proposition 8 and the fact that \( s \in \mathbb{N}^n \) satisfies \( s < \chi \) if and only if \( s \leq \chi - e_i \) for some \( i = 1, \ldots, n. \) \qed
2. Moduli space of foliations.

2.1. Let $X$ denote a (smooth, complete) algebraic variety over the complex numbers, let $L$ be a line bundle on $X$ and let $\omega$ denote a global section of $\Omega^1_X \otimes L$ (a twisted differential 1-form). A simple local calculation shows that $\omega \wedge d\omega$ is a section of $\Omega^3_X \otimes L^\otimes 2$. We say that $\omega$ is integrable if it satisfies the Frobenius condition $\omega \wedge d\omega = 0$. We denote

$$\mathcal{F}(X, L) \subset \mathbb{P}H^0(X, \Omega^1_X \otimes L),$$

the projective classes of integrable 1-forms. The map

$$\varphi : H^0(X, \Omega^1_X \otimes L) \to H^0(X, \Omega^3_X \otimes L^\otimes 2),$$

such that $\varphi(\omega) = \omega \wedge d\omega$ is a homogeneous quadratic map between vector spaces and hence $\varphi^{-1}(0) = \mathcal{F}(X, L)$ is an algebraic variety defined by homogeneous quadratic equations.

Our purpose is to understand the geometry of $\mathcal{F}(X, L)$. In particular, we are interested in the problem of describing its irreducible components. For a survey on this problem see for example [7].

2.2. Let $r$ and $d$ be natural numbers. Consider a differential 1-form in $\mathbb{C}^{r+1}$

$$\omega = \sum_{i=0}^{r} a_i dx_i,$$

where the $a_i$ are homogeneous polynomials of degree $d - 1$ in variables $x_0, \ldots, x_r$, with complex coefficients. We say that $\omega$ has degree $d$ (in particular the 1-forms $dx_i$ have degree one). Denoting $R$ the radial vector field, let us assume that

$$<\omega, R> = \sum_{i=0}^{r} a_i x_i = 0,$$

so that $\omega$ descends to the complex projective space $\mathbb{P}^r$ as a global section of the twisted sheaf of 1-forms $\Omega^1_{\mathbb{P}^r}(d)$. We denote

$$\mathcal{F}(r, d) = \mathcal{F}(\mathbb{P}^r, \mathcal{O}(d)),$$

parametrizing 1-forms of degree $d$ on $\mathbb{P}^r$ that satisfy the Frobenius integrability condition.
3. Complexes associated to an integrable form.

Let us denote
\[ H^0(\mathbb{P}^r, \Omega^k_{\mathbb{P}^r}(d)) = \Omega^k_r(d), \]
and
\[ \Omega_r = \bigoplus_{d \in \mathbb{N}} \bigoplus_{0 \leq k \leq r} \Omega^k_r(d), \]
with structure of bi-graded supercommutative associative algebra given by exterior product \( \wedge \) of differential forms.

**Definition 20.** Gelfand, Kapranov and Zelevinsky defined in [5] another product in \( \Omega_r \), the second multiplication \( * \), as follows:
\[
\omega_1 * \omega_2 = \frac{d_1}{d_1 + d_2} \omega_1 \wedge d\omega_2 + (-1)^{(k_1+1)(k_2+1)} \frac{d_2}{d_1 + d_2} \omega_2 \wedge d\omega_1, \\
= \frac{d_1}{d_1 + d_2} \omega_1 \wedge d\omega_2 + (-1)^{(k_1+1)} \frac{d_2}{d_1 + d_2} d\omega_1 \wedge \omega_2,
\]
where \( \omega_i \in \Omega^k_r(d_i) \) for \( i = 1, 2 \) and \( d_1 + d_2 \neq 0 \). In case \( (d_1, d_2) = (0, 0) \) one defines \( \omega_1 * \omega_2 = 0 \).

It follows that \( \omega_1 * \omega_2 = 0 \) if \( d_1 = 0 \) or \( d_2 = 0 \).

**Remark 21.** For \( \omega_i \in \Omega^k_r(d_i) \) for \( i = 1, 2 \) as above,

a) \( \omega_1 * \omega_2 \) belongs to \( \Omega^{(k_1+k_2+1)}_r(d_1 + d_2) \).

b) \( \omega_1 * \omega_2 = (-1)^{(k_1+1)(k_2+1)} \omega_2 * \omega_1 \).

c) It follows from an easy direct calculation that \( * \) is associative (see [5]).

d) For any \( \omega \in \Omega^1_r(d) \) we have \( \omega * \omega = \omega \wedge d\omega \). In particular, \( \omega \) is integrable if and only if \( \omega * \omega = 0 \).

**Definition 22.** For \( \omega \in \Omega^k_r(d) \) we consider the operator \( \delta_\omega \)
\[
\delta_\omega : \Omega_r \rightarrow \Omega_r,
\]
such that \( \delta_\omega(\eta) = \omega * \eta \) for \( \eta \in \Omega_r \).

**Remark 23.** From Remark 21 a), if \( \omega \in \Omega^k_r(d_1) \) then
\[
\delta_\omega(\Omega^{k_2}_r(d_2)) \subset \Omega^{(k_1+k_2+1)}_r(d_1 + d_2).
\]
In particular, if \( \omega \in \Omega^1_r(d_1) \),
\[
\delta_\omega(\Omega^{k_2}_r(d_2)) \subset \Omega^{(k_2+2)}_r(d_1 + d_2).
\]

**Definition 24.** For \( \omega \in \Omega^1_r(d) \) and \( e \in \mathbb{Z} \) we define two differential graded vector spaces
\[
C^+_\omega(e) : \Omega^0_r(e) \rightarrow \Omega^2_r(e + d) \rightarrow \Omega^1_r(e + 2d) \rightarrow \cdots \rightarrow \Omega^{2k}_r(e + kd) \rightarrow \ldots,
\]
\[
C^-\omega(e) : \Omega^1_r(e) \rightarrow \Omega^3_r(e + d) \rightarrow \Omega^2_r(e + 2d) \rightarrow \cdots \rightarrow \Omega^{2k+1}_r(e + kd) \rightarrow \ldots,
\]
where all maps are \( \delta_\omega \) as in Remark 23.
Proposition 25. Let \( \omega \in \Omega^1_r(d), e \in \mathbb{Z} \) and \( k \in \mathbb{N} \) such that \( k + 2 \leq r \). Then \( \omega \ast \eta = 0 \) for all \( \eta \in \Omega^k_r(e) \) if and only if \( \omega = 0 \). In other words, the linear map
\[
\delta : \Omega^1_r(d) \to \text{Hom}_K(\Omega^k_r(e), \Omega^{k+2}_r(e + d)),
\]
sending \( \omega \to \delta_\omega \), is injective.

Proof. First remark that \( \omega \wedge \eta = 0 \) for all \( \eta \in \Omega^k_r(e) \) (with \( k + 1 \leq r \)) easily implies \( \omega = 0 \). Now suppose \( \omega \ast \eta = 0 \), that is, \( d \omega \wedge d \eta + \eta \wedge d \omega = 0 \), for all \( \eta \in \Omega^k_r(e) \). Take
\[
\eta = x_i^{e-k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}
\]
(here \( x_i \) denote affine coordinates and \( 1 < i_1 < \ldots < i_k < n \)). Since \( d \eta = 0 \), we have
\[
dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge d \omega = 0.
\]
Hence \( d \omega = 0 \) by the first remark. Using the hypothesis again, we know \( \omega \wedge d \eta = 0 \) for all \( \eta \in \Omega^k_r(e) \). Now take \( \eta = x_i^{e-k} dx_{i_1} \wedge \cdots \wedge dx_{i_k} \) (where \( 1 < i_1 < \cdots < i_k < n \)). It follows that \( dx_{i_1} \wedge \cdots \wedge dx_{i_k+1} \wedge \omega = 0 \) and hence \( \omega = 0 \).

\( \square \)

Proposition 26. \( \omega \in \Omega^1_r(d) \) is integrable if and only if \( \delta^2_\omega = 0 \)

Proof. The associativity stated in Remark 21 c) implies that \( \delta_{\omega_1} \circ \delta_{\omega_2} = \delta_{\omega_1 \ast \omega_2} \). In particular, \( \delta^2_\omega = \delta_{\omega \ast \omega} \) and hence the claim follows from Remark 21 d) and Proposition 25.

\( \square \)

Remark 27. It follows from Proposition 26 that \( C^+(r,e) \) and \( C^-(r,e) \) (Definition 24) are differential complexes (for any \( e \in \mathbb{Z} \)) if and only if \( \omega \) is integrable.

Remark 28. To fix ideas we shall mostly discuss \( C^+(r,e) \), but similar considerations apply to \( C^-(r,e) \). If no confusion seems to arise we shall denote \( C^-(r,e) = C^-(r,e) \).

Theorem 29. Fix \( e \in \mathbb{Z} \). Let us consider the graded vector space
\[
\Omega_r(e) = \bigoplus_{0 \leq k \leq \lfloor \frac{r-2}{2} \rfloor} \Omega^m_r(e + kd),
\]
(direct sum of the spaces appearing in \( C^-(r,e) \) above). Define the linear map
\[
\delta(e) = \delta : \Omega^1_r(d) \to \prod_{k=1}^{\lfloor \frac{r-1}{2} \rfloor} \text{Hom}_K(\Omega^{2k-1}_r(e + (k - 1)d), \Omega^{2k+1}_r(e + kd)),
\]
such that \( \delta(\omega) = \delta_\omega \) for each \( \omega \in \Omega^1_r(d) \), and its projectivization
\[
\mathbb{P}\delta : \mathbb{P}\Omega^1_r(d) \to \prod_{k=1}^{\lfloor \frac{r-1}{2} \rfloor} \mathbb{P}\text{Hom}_K(\Omega^{2k-1}_r(e + (k - 1)d), \Omega^{2k+1}_r(e + kd)).
\]
Denote \( C = \mathcal{C}(\Omega^1_r(e), \Omega^3_r(e + d), \Omega^5_r(e + 2d), \ldots, \Omega^{2\lfloor \frac{r-1}{2} \rfloor+1}_r(e + \lfloor \frac{r-1}{2} \rfloor d)) \) the variety of complexes as in 1.1 and \( \mathcal{F}(r,d) \) the variety of foliations as in 2.2. Then
\[
\mathcal{F}(r,d) = (\mathbb{P}\delta)^{-1}(C).
\]
In other terms, \( \mathbb{P}\delta(\mathcal{F}(r,d)) = \mathcal{L} \cap \mathcal{C} \), that is, the variety of foliations \( \mathcal{F}(r,d) \) corresponds via the linear injective map \( \mathbb{P}\delta \) to the intersection of the variety of complexes with the linear space \( \mathcal{L} = \text{im}(\mathbb{P}\delta) \).
Proof. The statement is a rephrasing of Remark 27. □

Proposition 30. Let us denote

\[ d^k_r(e) = \dim \Omega^k_r(e) = \binom{r - k + e}{r - k} \binom{d - 1}{k}, \]

(see [8]) and in particular

\[ d_k = d^{2k+1}_r(e + kd) = \dim \Omega^{2k+1}_r(e + kd), \quad 0 \leq k \leq \left\lfloor \frac{r - 1}{2} \right\rfloor. \]

For this \( d = (d_0, d_1, \ldots, d_{\left\lfloor \frac{r - 1}{2} \right\rfloor}) \) we consider the finite ordered set \( R = R(d) \) as in Proposition 14. Then each irreducible component of the variety of foliations \( \mathcal{F}(r, d) \) is an irreducible component of the linear section \( (\mathbb{P}\delta)^{-1}(\mathcal{T}_r) \) for a unique \( r \in R^+ \).

Proof. From Proposition 14, we have the decomposition into irreducible components

\[ \mathcal{C} = \bigcup_{r \in R^+} \mathcal{T}_r. \]

From Theorem 29 we obtain:

\[ \mathcal{F}(r, d) = (\mathbb{P}\delta)^{-1}(\mathcal{C}) = \bigcup_{r \in R^+} (\mathbb{P}\delta)^{-1}(\mathcal{T}_r). \]

This implies that each irreducible component \( X \) of \( \mathcal{F}(r, d) \) is an irreducible component of \( (\mathbb{P}\delta)^{-1}(\mathcal{T}_r) \) for some \( r \in R^+ \). This element \( r \) is the sequence of ranks of \( \delta_\omega \) for a general \( \omega \in X \), hence it is unique. □
References


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