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REPRESENTATIONS OF SOME LATTICES INTO THE GROUP OF ANALYTIC DIFFEOMORPHISMS OF THE SPHERE $\mathbb{S}^2$

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ABSTRACT. In [11] it is proved that any morphism from a subgroup of finite index of $\mathrm{SL}(n,\mathbb{Z})$ to the group of analytic diffeomorphisms of $\mathbb{S}^2$ has a finite image as soon as $n \geq 5$. The case $n = 4$ is also claimed to follow along the same arguments; in fact this is not straightforward and that case indeed needs a modification of the argument. In this paper we recall the strategy for $n \geq 5$ and then focus on the case $n = 4$.

1. Introduction

After the works of Margulis ([15, 20]) on the linear representations of lattices of simple, real Lie groups with $\mathbb{R}$-rank larger than 1, some authors, like Zimmer, suggest to study the actions of lattices on compact manifolds ([22, 23, 24, 25]). One of the main conjectures of this program is the following: let us consider a connected, simple, real Lie group G, and let Γ be a lattice of G of $\mathbb{R}$-rank larger than 1. If there exists a morphism of infinite image from Γ to the group of diffeomorphisms of a compact manifold M, then the $\mathbb{R}$-rank of G is bounded by the dimension of M. There are a lot of contributions in that direction ([3, 4, 5, 8, 9, 10, 11, 12, 17, 18]). In this article we will focus on the embeddings of subgroups of finite index of $\mathrm{SL}(n,\mathbb{Z})$ into the group $\mathrm{Diff}^\omega(\mathbb{S}^2)$ of real analytic diffeomorphisms of $\mathbb{S}^2$ (*see* [11]).

The article is organized as follows. First of all we will recall the strategy of [11]: the study of the nilpotent subgroups of $\mathrm{Diff}^\omega(\mathbb{S}^2)$ implies that such subgroups are metabelian. But subgroups of finite index of $\mathrm{SL}(n,\mathbb{Z})$, for $n \geq 5$, contain nilpotent subgroups of length $n-1$ of finite index which are not metabelian; as a consequence Ghys gets the following statement.

Theorem A ([11])**.** *Let Γ be a subgroup of finite index of $\mathrm{SL}(n,\mathbb{Z})$. As soon as $n \geq 5$ there is no embedding of Γ into $\mathrm{Diff}^\omega(\mathbb{S}^2)$.*

To study nilpotent subgroups of $\mathrm{Diff}^\omega(\mathbb{S}^2)$ one has to study nilpotent subgroups of $\mathrm{Diff}^\omega_+(\mathbb{S}^1)$ (*see* §2), and then nilpotent subgroups of the group of formal diffeomorphisms of $\mathbb{C}^2$ (*see* §3). The last section is devoted to establish the following result.

Theorem B. *Let Γ be a subgroup of finite index of $\mathrm{SL}(n,\mathbb{Z})$. As soon as $n \geq 4$ there is no embedding of Γ into $\mathrm{Diff}^\omega(\mathbb{S}^2)$.*

The proof relies on the characterization, up to isomorphism, of nilpotent subalgebras of length 3 of the algebra of formal vector fields of $\mathbb{C}^2$ that vanish at the origin.

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2. Nilpotent subgroups of the group of analytic diffeomorphisms of $\mathbb{S}^1$

Let G be a group; let us set $\mathrm{G}^{(0)} = \mathrm{G}$ and $\mathrm{G}^{(i)} = [\mathrm{G}, \mathrm{G}^{(i-1)}] \quad \forall\, i \geq 1$. The group G is *nilpotent* if there exists an integer n such that $\mathrm{G}^{(n)} = \{\mathrm{id}\}$; the *length of nilpotence* of G is the smallest integer k such that $\mathrm{G}^{(k)} = \{\mathrm{id}\}$.

Set $\mathrm{G}_{(0)} = \mathrm{G}$ and $\mathrm{G}_{(i)} = [\mathrm{G}_{(i-1)}, \mathrm{G}_{(i-1)}] \quad \forall\, i \geq 1$. The group G is *solvable* if $\mathrm{G}_{(n)} = \{\mathrm{id}\}$ for some integer n; the *length of solvability* of G is the smallest integer k such that $\mathrm{G}_{(k)} = \{\mathrm{id}\}$.

We say that the group G (resp. algebra $\mathfrak{g}$) is *metabelian* if [G, G] (resp. $[\mathfrak{g}, \mathfrak{g}]$) is abelian.

Proposition 2.1 ([11]). *Any nilpotent subgroup of* $\mathrm{Diff}^{\omega}_{+}(\mathbb{S}^1)$ *is abelian.*

Proof. Let G be a nilpotent subgroup of $\mathrm{Diff}^{\omega}_{+}(\mathbb{S}^1)$. Assume that G is not abelian; it thus contains a Heisenberg group

$$\langle f,\, g,\, h \,|\, [f,g] = h,\, [f,h] = [g,h] = \mathrm{id}\rangle.$$

The application "rotation number"

$$\mathrm{Diff}^{\omega}_{+}(\mathbb{S}^1) \to \mathbb{R}/\mathbb{Z}, \qquad\qquad \psi \mapsto \lim_{n\to+\infty} \frac{\psi^n(x) - x}{n}$$

is not a morphism but its restriction to a solvable subgroup is a morphism ([1]). Hence the rotation number of h is zero, and the set $\mathrm{Fix}(h)$ of fixed points of h is non-empty, and finite. Considering some iterates of f and g instead of f and g one can assume that f and g fix any point of $\mathrm{Fix}(h)$. The set of fixed points of a non-trivial element of $\langle f,\, g\rangle$ is finite and invariant by h so the action of $\langle f,\, g\rangle$ is free [1] on each component of $\mathbb{S}^1 \setminus \mathrm{Fix}(h)$. But the action of a free group on $\mathbb{R}$ is abelian: contradiction. □

3. Nilpotent subgroups of the group of formal diffeomorphisms of $\mathbb{C}^2$

Let us denote $\widehat{\mathrm{Diff}}(\mathbb{C}^2, 0)$ the group of formal diffeomorphisms of $\mathbb{C}^2$, *i.e.*, the formal completion of the group of germs of holomorphic diffeomorphisms at 0. Let Diff_i be the quotient of $\widehat{\mathrm{Diff}}(\mathbb{C}^2, 0)$ by the normal subgroups of formal diffeomorphisms tangent to the identity with multiplicity i; it can be viewed as the set of jets of diffeomorphisms at order i with the law of composition with truncation at order i. Note that Diff_i is a complex linear algebraic group. One can see $\widehat{\mathrm{Diff}}(\mathbb{C}^2, 0)$ as the projective limit of the Diff_i's: $\widehat{\mathrm{Diff}}(\mathbb{C}^2, 0) = \varprojlim \mathrm{Diff}_i$. Let us denote by $\widehat{\chi}(\mathbb{C}^2, 0)$ the algebra of formal vector fields in $\mathbb{C}^2$ vanishing at 0. One can define the set χ_i of the i-th jets of vector fields; one has $\varprojlim \chi_i = \widehat{\chi}(\mathbb{C}^2, 0)$.

Let $\widehat{\mathcal{O}}(\mathbb{C}^2)$ be the ring of formal series in two variables, and let $\widehat{K}(\mathbb{C}^2)$ be its fraction field; $\mathcal{O}_i$ is the set of elements of $\widehat{\mathcal{O}}(\mathbb{C}^2)$ truncated at order i.

The family $\big(\exp_i \colon \chi_i \to \mathrm{Diff}_i\big)_i$ is filtered, *i.e.*, compatible with the truncation. We then define the exponential application as follows: $\exp = \varprojlim \exp_i \colon \widehat{\chi}(\mathbb{C}^2, 0) \to \widehat{\mathrm{Diff}}(\mathbb{C}^2, 0)$.

As in the classical case, if X belongs to $\widehat{\chi}(\mathbb{C}^2, 0)$, then $\exp(X)$ can be seen as the "flow at time $t = 1$" of X. Indeed an element X_i of χ_i can be seen as a derivation of $\mathcal{O}_i$; so it can be written $S_i + N_i$ where S_i and N_i are two semi-simple (resp. nilpotent) derivations that commute. Passing to the limit, one gets $X = S + N$ where S is a semi-simple vector field, N a nilpotent one, and $[S, N] = \mathrm{id}$ (*see* [16]). A semi-simple vector field is a formal vector field conjugate to a diagonal linear vector field that is complete. A vector field is nilpotent if and only if its linear

1. The stabilizer of every point is trivial, *i.e.*, the action of a non-trivial element of $\langle f,\, g\rangle$ has no fixed point.

part is; let us remark that the usual flow φ_t of a nilpotent vector field is polynomial in t

$$\varphi_t(x) = \sum_I P_I(t)x^I, \qquad P_I \in (\mathbb{C}[t])^2$$

so $\varphi_1(x)$ is well-defined. As a consequence $\exp(tX) = \exp(tS)\exp(tN)$ is well-defined for $t = 1$. Note that the Jordan decomposition is purely formal: if X is holomorphic, then S and N are not necessary holomorphic.

Proposition 3.1 ([11]). *Any nilpotent subalgebra of $\widehat{\chi}(\mathbb{C}^2, 0)$ is metabelian.*

Proof. Let $\mathfrak{l}$ be a nilpotent subalgebra of $\widehat{\chi}(\mathbb{C}^2, 0)$, and let $Z(\mathfrak{l})$ be its center. Since

$$\widehat{\chi}(\mathbb{C}^2, 0) \otimes \widehat{K}(\mathbb{C}^2)$$

is a vector space of dimension 2 over $\widehat{K}(\mathbb{C}^2)$, one has the following alternatives:

- the dimension of the subspace generated by $Z(\mathfrak{l})$ in $\widehat{\chi}(\mathbb{C}^2, 0) \otimes \widehat{K}(\mathbb{C}^2)$ is 1;
- the dimension of the subspace generated by $Z(\mathfrak{l})$ in $\widehat{\chi}(\mathbb{C}^2, 0) \otimes \widehat{K}(\mathbb{C}^2)$ is 2.

Let us study these different cases.

Under the first assumption there exists an element X of $Z(\mathfrak{l})$ having the following property: any vector field of $Z(\mathfrak{l})$ can be written uX with u in $\widehat{K}(\mathbb{C}^2)$. Let us consider the subalgebra $\mathfrak{g}$ of $\mathfrak{l}$ given by

$$\mathfrak{g} = \big\{ \widetilde{X} \in \mathfrak{l} \,|\, \exists\, u \in \widehat{K}(\mathbb{C}^2),\ \widetilde{X} = uX \big\}.$$

Since X belongs to $Z(\mathfrak{l})$, the algebra $\mathfrak{g}$ is abelian; it is also an ideal of $\mathfrak{l}$. Let us assume that $\mathfrak{l}$ is not abelian: let Y be an element of $\mathfrak{l}$ whose projection on $\mathfrak{l}/\mathfrak{g}$ is non-trivial, and central. Any vector field of $\mathfrak{l}$ can be written as $uX + vY$ with u, v in $\widehat{K}(\mathbb{C}^2)$. As X belongs to $Z(\mathfrak{l})$, and Y is central modulo $\mathfrak{g}$ one has

$$X(u) = X(v) = Y(v) = 0.$$

The vector fields $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ being some linear combinations of X and Y with coefficients in $\widehat{K}(\mathbb{C}^2, 0)$, the partial derivatives of v are zero so v is a constant. Therefore $[\mathfrak{l}, \mathfrak{l}] \subset \mathfrak{g}$; but $\mathfrak{g}$ is abelian thus $\mathfrak{l}$ is metabelian.

In the second case $Z(\mathfrak{l})$ contains two elements X and Y which are linearly independent on $\widehat{K}(\mathbb{C}^2)$. Any vector field of $\mathfrak{l}$ can be written as $uX + vY$ with u and v in $\widehat{K}(\mathbb{C}^2)$. Since X and Y belong to $Z(\mathfrak{l})$ one has

$$X(u) = X(v) = Y(u) = Y(v) = 0.$$

As a consequence u and v are constant, *i.e.*, $\mathfrak{l} \subset \{uX + vY \,|\, u,\, v \in \mathbb{C}\}$; in particular $\mathfrak{l}$ is abelian. $\square$

Proposition 3.2 ([11]). *Any nilpotent subgroup of $\widehat{\mathrm{Diff}}(\mathbb{C}^2, 0)$ is metabelian.*

Proof. Let G be a nilpotent subgroup of $\widehat{\mathrm{Diff}}(\mathbb{C}^2, 0)$ of length k. Let us denote by G_i the projection of G on Diff_i. The Zariski closure $\overline{\mathrm{G}_i}$ of G_i in Diff_i is an algebraic nilpotent subgroup of length k. It is sufficient to prove that $\overline{\mathrm{G}_i}$ is metabelian.

Since $\overline{\mathrm{G}_i}$ is a complex algebraic subgroup it is the direct product of the subgroup $\overline{\mathrm{G}_{i,u}}$ of its unipotent elements and the subgroup $\overline{\mathrm{G}_{i,s}}$ of its semi-simple elements (*see for example* [2]).

An element of Diff_i is unipotent if and only if its linear part, which belongs to $\mathrm{GL}(2, \mathbb{C})$, is; so $\overline{\mathrm{G}_{i,s}}$ projects injectively onto a nilpotent subgroup of $\mathrm{GL}(2, \mathbb{C})$. Therefore $\overline{\mathrm{G}_{i,s}}$ is abelian.

The group $\overline{\mathrm{G}_{i,u}}$ coincides with $\exp \mathfrak{l}_i$ where $\mathfrak{l}_i$ is a nilpotent Lie algebra of χ_i of length k. Passing to the limit one thus obtains the existence of a nilpotent subalgebra $\mathfrak{l}$ of $\widehat{\chi}(\mathbb{C}^2, 0)$ of length k such that $\exp(\mathfrak{l})$ projects onto $\overline{\mathrm{G}_{i,u}}$ for any i. According to Proposition 3.1 the subalgebra $\mathfrak{l}$, and thus $\overline{\mathrm{G}_{i,u}}$ are metabelian. $\square$

4. Nilpotent subgroups of the group of analytic diffeomorphisms of $\mathbb{S}^2$

Proposition 4.1 ([11]). *Any nilpotent subgroup of* $\mathrm{Diff}^\omega(\mathbb{S}^2)$ *has a finite orbit.*

Proof. Let G be a nilpotent subgroup of $\mathrm{Diff}^\omega(\mathbb{S}^2)$; up to finite index one can assume that the elements of G preserve the orientation. Let ϕ be a non-trivial element of G that commutes with G. Let $\mathrm{Fix}(\phi)$ be the set of fixed points of ϕ; it is a non-empty analytic subspace of $\mathbb{S}^2$ invariant by G. If p is an isolated fixed point of ϕ, then the orbit of p under the action of G is finite. So it is sufficient to study the case where $\mathrm{Fix}(\phi)$ only contains curves; there are thus two possibilities:

- $\mathrm{Fix}(\phi)$ is a singular analytic curve whose set of singular points is a finite orbit for G;
- $\mathrm{Fix}(\phi)$ is a smooth analytic curve, not necessary connected. One of the connected component of $\mathbb{S}^2 \setminus \mathrm{Fix}(\phi)$ is a disk denoted by $\mathbb{D}$. Any subgroup Γ of finite index of G which contains ϕ fixes $\mathbb{D}$. Let us consider an element γ of Γ, and a fixed point m of γ that belongs to $\overline{\mathbb{D}}$. By construction ϕ has no fixed point in $\mathbb{D}$ so according to the Brouwer Theorem $(\phi^k(m))_k$ has a limit point on the boundary $\partial\mathbb{D}$ of $\overline{\mathbb{D}}$. Therefore γ has at least one fixed point on $\partial\mathbb{D}$. The group Γ thus acts on $\partial\mathbb{D}$, and any of its elements has a fixed point on $\mathbb{D}$. Then Γ has a fixed point on $\partial\mathbb{D}$ (Proposition 2.1).

□

Theorem 4.2 ([11]). *Any nilpotent subgroup of* $\mathrm{Diff}^\omega(\mathbb{S}^2)$ *is metabelian.*

Proof. Let G be a nilpotent subgroup of $\mathrm{Diff}^\omega(\mathbb{S}^2)$, and let Γ be a subgroup of finite index of G having a fixed point m (such a subgroup exists according to Proposition 4.1). One can embed Γ into $\widehat{\mathrm{Diff}}(\mathbb{R}^2,0)$, and so into $\widehat{\mathrm{Diff}}(\mathbb{C}^2,0)$, by considering the jets of infinite order of elements of Γ in m. According to Proposition 3.2 the group Γ is metabelian.

One can suppose that G is a finitely generated group.

Let us first assume that G has no element of finite order. Then G is a cocompact lattice of the nilpotent, simply-connected Lie group $\mathrm{G}\otimes\mathbb{R}$ (*see* [19]). The group G is metabelian if and only if $\mathrm{G}\otimes\mathbb{R}$ is; but Γ is metabelian so $\mathrm{G}\otimes\mathbb{R}$ also.

Finally let us consider the case where G contains at least one element of finite order. The set of such elements is a normal subgroup of G that thus intersects non-trivially the center $Z(\mathrm{G})$ of G. Let us consider a non-trivial element ϕ of $Z(\mathrm{G})$ which has finite order. Let us recall that a finite group of diffeomorphisms of the sphere is conjugate to a group of isometries. Denote by G^+ the subgroup of elements of G which preserve the orientation. It is thus sufficient to prove that G^+ is metabelian; indeed if ϕ does not preserve the orientation, then ϕ has order 2, and $\mathrm{G} = \mathbb{Z}/2\mathbb{Z}\times\mathrm{G}^+$. So let us assume that ϕ preserves the orientation; ϕ is conjugate to a direct isometry of $\mathbb{S}^2$, and has exactly two fixed points on the sphere. The group G has thus an invariant set of two elements. By considering germs in the neighborhood of these two points, one gets that G can be embedded into $2\cdot\mathrm{Diff}(\mathbb{R}^2,0)$ [2] and thus into $2\cdot\mathrm{Diff}(\mathbb{C}^2,0)$:

$$1 \longrightarrow \mathrm{Diff}(\mathbb{C}^2,0) \longrightarrow 2\cdot\mathrm{Diff}(\mathbb{C}^2,0) \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0.$$

Remark that $2\cdot\mathrm{Diff}(\mathbb{C}^2,0)$ is the projective limit of the algebraic groups $2\cdot\mathrm{Diff}_i$. One can conclude as in the proof of Proposition 3.2 except that the subgroup of the semi-simple elements of $2\cdot\mathrm{Diff}_i$ embeds now in $2\cdot\mathrm{GL}(2,\mathbb{C})$; it is metabelian because it contains an abelian subgroup of index 2. □

Let Γ be a subgroup of finite index of $\mathrm{SL}(n,\mathbb{Z})$ for $n\geq 5$. Since Γ contains nilpotent subgroups of finite index of length $n-1$ (for example the group of upper triangular unipotent matrices) which are not metabelian one gets the following statement.

2. Let G be a group and let q be a positive integer; $q\cdot\mathrm{G}$ denotes the semi-direct product of G^q by $\mathbb{Z}/q\mathbb{Z}$ under the action of the cyclic permutation of the factors.

Corollary 4.3 ([11])**.** *Let Γ be a subgroup of finite index of $\mathrm{SL}(n,\mathbb{Z})$; as soon as $n\geq 5$ there is no embedding of Γ into $\mathrm{Diff}^\omega(\mathbb{S}^2)$.*

5. Nilpotent subgroups of length 3 of the group of analytic diffeomorphisms of $\mathbb{S}^2$

Let us precise Proposition 3.1 for nilpotent subalgebras of length 3 of $\widehat{\chi}(\mathbb{C}^2,0)$. Let $\mathfrak{l}$ be such an algebra. The dimension of the subspace generated by $Z(\mathfrak{l})$ in $\widehat{\chi}(\mathbb{C}^2,0)\otimes\widehat{K}(\mathbb{C}^2)$ has dimension at most 1, for else $\mathfrak{l}$ would be abelian (Proposition 3.1) and this is impossible under our assumptions. So let us assume that the dimension of the subspace generated by $Z(\mathfrak{l})$ in $\widehat{\chi}(\mathbb{C}^2,0)\otimes\widehat{K}(\mathbb{C}^2)$ is 1. There exists an element X in $Z(\mathfrak{l})$ with the following property: any element of $Z(\mathfrak{l})$ can be written uX with u in $\widehat{K}(\mathbb{C}^2)$. Let $\mathfrak{g}$ denote the abelian ideal of $\mathfrak{l}$ defined by

$$\mathfrak{g}=\big\{\widetilde{X}\in\mathfrak{l}\,\big|\,\exists\, u\in\widehat{K}(\mathbb{C}^2),\,\widetilde{X}=uX\big\}.$$

By hypothesis $\mathfrak{l}$ is not abelian. Let Y be in $\mathfrak{l}$; assume that its projection onto $\mathfrak{l}/\mathfrak{g}$ is a non-trivial element of $Z(\mathfrak{l}/\mathfrak{g})$. Any vector field of $\mathfrak{l}$ can be written

$$uX+vY,\qquad u,\,v\in\widehat{K}(\mathbb{C}^2).$$

Since X, resp. Y belongs to $Z(\mathfrak{l})$ (resp. $Z(\mathfrak{l}/\mathfrak{g})$) and since the length of $\mathfrak{l}$ is 3, one has

$$X(u)=Y^3(u)=X(v)=Y(v)=0. \tag{5.1}$$

If X and Y are non-singular, one can choose formal coordinates x and y such that $X=\frac{\partial}{\partial x}$ and $Y=\frac{\partial}{\partial y}$. The previous conditions can be thus translated as follows: v is a constant and u is a polynomial in y of degree 2. We will see that we have a similar property without assumption on X and Y.

Lemma 5.1. *Let X and Y be two vector fields of $\widehat{\chi}(\mathbb{C}^2,0)$ that commute and are not colinear. One can assume that $(X,Y)=\Big(\frac{\partial}{\partial\widetilde{x}},\frac{\partial}{\partial\widetilde{y}}\Big)$ where $\widetilde{x}$ and $\widetilde{y}$ are two independent variables in a Liouvillian extension of $\widehat{K}(\mathbb{C}^2,0)$.*

Proof. Since X and Y are non-colinear, there exist two 1-forms α, β with coefficients in $\widehat{K}(\mathbb{C}^2)$ such that $\alpha(X)=1$, $\alpha(Y)=0$, $\beta(X)=0$, and $\beta(X)=1$. The vector fields X and Y commute if and only if α and β are closed (this statement of linear algebra is true for convergent meromorphic vector fields and is also true in the completion). The 1-form α is closed so according to [7] one has

$$\alpha=\sum_{i=1}^r\lambda_i\frac{d\widehat{\phi}_i}{\widehat{\phi}_i}+d\Big(\frac{\widehat{\psi}_1}{\widehat{\psi}_2}\Big)=d\Big(\sum_{i=1}^r\lambda_i\log\widehat{\phi}_i+\frac{\widehat{\psi}_1}{\widehat{\psi}_2}\Big)$$

where $\widehat{\psi}_1$, $\widehat{\psi}_2$, and the $\widehat{\phi}_i$ denote some formal series and the λ_i some complex numbers. One has a similar expression for β. So there exists a Liouvillian extension κ of $\widehat{K}(\mathbb{C}^2)$ having two elements $\widetilde{x}$ and $\widetilde{y}$ with $\alpha=d\widetilde{x}$ and $\beta=d\widetilde{y}$. One thus has $X(\widetilde{x})=1$, $X(\widetilde{y})=0$, $Y(\widetilde{x})=0$, and $Y(\widetilde{y})=1$. $\square$

From (5.1) one gets: v is a constant, and u is a polynomial in $\widetilde{y}$ of degree 2; so one proves the following statement.

Proposition 5.2. *Let $\mathfrak{l}$ be a nilpotent subalgebra of $\widehat{\chi}(\mathbb{C}^2,0)$ of length 3. Then $\mathfrak{l}$ is isomorphic to a subalgebra of*

$$\mathfrak{n}=\Big\{P(\widetilde{y})\frac{\partial}{\partial\widetilde{x}}+\alpha\frac{\partial}{\partial\widetilde{y}}\;\Big|\;\alpha\in\mathbb{C},\,P\in\mathbb{C}[\widetilde{y}],\,\deg P=2\Big\}.$$

Remark 5.3. We use a real version of this statement whose proof is an adaptation of the previous one: a nilpotent subalgebra $\mathfrak{l}$ of length 3 of $\widehat{\chi}(\mathbb{R}^2,0)$ is isomorphic to a subalgebra of

$$\mathfrak{n}=\Big\{P(\widetilde{y})\frac{\partial}{\partial\tilde{x}}+\alpha\frac{\partial}{\partial\tilde{y}}\,\Big|\,\alpha\in\mathbb{R},\ P\in\mathbb{R}[\widetilde{y}],\ \deg P=2\Big\}.$$

Theorem 5.4. *Let Γ be a subgroup of finite index of $\mathrm{SL}(n,\mathbb{Z})$; as soon as $n\geq 4$ there is no embedding of Γ into $\mathrm{Diff}^\omega(\mathbb{S}^2)$.*

Proof. Let $\mathrm{U}(4,\mathbb{Z})$ (resp. $\mathrm{U}(4,\mathbb{R})$) be the subgroup of unipotent upper triangular matrices of $\mathrm{SL}(4,\mathbb{Z})$ (resp. $\mathrm{SL}(4,\mathbb{R})$); it is a nilpotent subgroup of length 3. Assume that there exists an embedding from a subgroup Γ of finite index of $\mathrm{SL}(4,\mathbb{Z})$ into $\mathrm{Diff}^\omega(\mathbb{S}^2)$. Up to finite index Γ contains $\mathrm{U}(4,\mathbb{Z})$. Let us set $\mathrm{H}=\ \rho(\mathrm{U}(4,\mathbb{Z}))$. Up to finite index H has a fixed point (Proposition 4.1). One can thus see H as a subgroup of $\mathrm{Diff}(\mathbb{R}^2,0)\subset\widehat{\mathrm{Diff}}(\mathbb{R}^2,0)$ up to finite index.

Let us denote by j^1 the morphism from $\widehat{\mathrm{Diff}}(\mathbb{R}^2,0)$ to Diff_i. Up to conjugation, $j^1(\rho(\mathrm{U}(4,\mathbb{Z})))$ is a subgroup of

$$\Big\{\begin{bmatrix}\lambda & t\\ 0 & \lambda\end{bmatrix}\,\Big|\,\lambda\in\mathbb{R}^*,\ t\in\mathbb{R}\Big\}.$$

Up to index 2 one can thus assume that $j^1\circ\rho$ takes values in the connected, simply-connected group T defined by

$$\mathrm{T}=\Big\{\begin{bmatrix}\lambda & t\\ 0 & \lambda\end{bmatrix}\,\Big|\,\lambda,\ t\in\mathbb{R},\ \lambda>0\Big\}.$$

Let us set

$$\mathrm{Diff}_i(\mathrm{T})=\big\{f\in\mathrm{Diff}_i\,|\,j^1(f)\in\mathrm{T}\big\};$$

the group $\mathrm{Diff}_i(\mathrm{T})$ is a connected, simply-connected, nilpotent and algebraic group. The morphism

$$\rho_i\colon\mathrm{U}(4,\mathbb{Z})\to\mathrm{Diff}_i$$

can be extended to a unique continuous morphism $\widetilde{\rho_i}\colon\mathrm{U}(4,\mathbb{R})\to\mathrm{Diff}_i(\mathrm{T})$ (*see* [13, 14]) so to an algebraic morphism [3]. Let us note that $\widetilde{\rho_i}(\mathrm{U}(4,\mathbb{Z}))$ is an algebraic subgroup of $\mathrm{Diff}_i(\mathrm{T})$ that contains $\rho_i(\mathrm{U}(4,\mathbb{Z}))$; in particular $\overline{\mathrm{H}_i}=\overline{\rho_i(\mathrm{U}(4,\mathbb{Z}))}\subset\widetilde{\rho_i}(\mathrm{U}(4,\mathbb{R}))$. By construction the family $(\mathrm{H}_i)_i$ is filtered; since the extension is unique, the family $(\widetilde{\rho_i})_i$ is also filtered. Therefore $\mathrm{K}=\varprojlim\overline{\mathrm{H}_i}$ is well-defined. Since ρ is injective, H is a nilpotent subgroup of length 3; as $\mathrm{H}\subset\mathrm{K}$ and as any $\overline{\mathrm{H}_i}$ is nilpotent of length at most 3 the group K is nilpotent of length at most 3. For i sufficiently large $\widetilde{\rho_i}(\mathrm{U}(4,\mathbb{R}))$ is nilpotent of length 3; this group is connected so its Lie algebra is also nilpotent of length 3. Therefore the image of

$$D\widetilde{\rho}:=\varprojlim D\widetilde{\rho_i}\colon\mathfrak{u}(4,\mathbb{R})\to\widehat{\chi}(\mathbb{R}^2,0)$$

is isomorphic to $\mathfrak{n}$ (Proposition 5.2). So there exists a surjective map ψ from $\mathfrak{u}(4,\mathbb{R})$ onto $\mathfrak{n}$. The kernel of ψ is an ideal of $\mathfrak{u}(4,\mathbb{R})$ of dimension 2; hence $\ker\psi=\langle\delta_{14},a\delta_{13}+b\delta_{24}\rangle$ where the δ_{ij} denote the Kronecker matrices. One concludes by noting that $\dim Z(\mathfrak{u}(4,\mathbb{R})/\ker\psi)=2$ whereas $\dim Z(\mathfrak{n})=1$. □

Corollary 5.5. *The image of a morphism from a subgroup of $\mathrm{SL}(n,\mathbb{Z})$ of finite index to $\mathrm{Diff}^\omega(\mathbb{S}^2)$ is finite as soon as $n\geq 4$.*

3. Let N_1 and N_2 be two connected, simply-connected, nilpotent and algebraic subgroups of $\mathbb{R}$; any continuous morphism from N_1 to N_2 is algebraic.

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