Journal *of* Singularities
Volume 20 (2020), 205-231

received: 5 February 2019
in revised form: 7 April 2020
DOI: 10.5427/jsing.2020.20i

BERNOULLI MOMENTS OF SPECTRAL NUMBERS AND HODGE NUMBERS

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Abstract. The distribution of the spectral numbers of an isolated hypersurface singularity is studied in terms of the Bernoulli moments. These are certain rational linear combinations of the higher moments of the spectral numbers. They are related to the generalized Bernoulli polynomials. We conjecture that their signs are alternating and prove this in many cases. One motivation fo the Bernoulli moments comes from the analogy with compact complex manifolds.

1. Conjectures and results

An isolated hypersurface singularity is a holomorphic function germ $f:(\mathbb{C}^{n+1},0)\to(\mathbb{C},0)$ with an isolated singularity at $0\in\mathbb{C}^{n+1}$ (here $n\in\mathbb{N}=\{0,1,2,...\}$). It comes equipped with a Milnor number $\mu\in\mathbb{N}-\{0\}$ and with its spectral numbers, a tuple of μ rational numbers $\alpha_1,...,\alpha_\mu$ which satisfy

$$-1<\alpha_1\leq ...\leq\alpha_\mu<n \quad\text{and}\quad \alpha_i+\alpha_{\mu+1-i}=n-1. \tag{1.1}$$

They come from the Hodge filtration on the middle cohomology of the Milnor fiber and the semisimple part of the monodromy, acting on it [St][AGV].

We are interested in their distribution. We consider the numbers

$$V_{2k}^{sing}(f):=\sum_{i=1}^{\mu}\left(\alpha_i-\frac{n-1}{2}\right)^{2k}\quad\text{for } k\geq 0. \tag{1.2}$$

One should divide them by $\mu=V_0^{sing}(f)$ to get the normalized moments, but we prefer not to do it. So we call $V_2^{sing}(f)$ the *variance* of f and the $V_{2k}^{sing}(f)$ the *higher moments*. In [He1][He2] C. Hertling formulated the following conjecture.

Conjecture 1.1. *Any isolated hypersurface singularity satisfies*

$$V_2^{sing}(f)\leq V_0^{sing}(f)\cdot\frac{\alpha_\mu-\alpha_1}{12}. \tag{1.3}$$

It was proved by M. Saito for irreducible curve singularities [SM2], by T. Brélivet for curve singularities with nondegenerate Newton boundary [Br1], and recently by T. Brélivet for all curve singularities [Br3]. A. Dimca formulated a dual conjecture with $\geq$ instead of $\leq$ for polynomials with isolated singularities [Di]. He considered the global geometry of a polynomial and the spectrum at infinity.

In this paper, the conjecture 1.1 will be extended to a series of inequalities for certain linear combinations of the higher moments, which will be called *Bernoulli moments*. Before explaining this, we consider an analogous situation, where these linear combinations will also be interesting.

2010 *Mathematics Subject Classification.* 32S25, 62E99, 32S35.

Key words and phrases. Spectral numbers, Bernoulli polynomials, higher moments, singularities.

If X is a compact complex manifold of dimension n, one often considers [Hi]

$$h^{p,q} = \dim H^q(X,\Omega^p) \qquad \text{and} \tag{1.4}$$

$$\chi_p = (-1)^p\chi(\Omega^p) = \sum_q (-1)^{p+q}h^{p,q} \tag{1.5}$$

We define

$$V_{2k}^{mfd}(X) := \sum_p \chi_p(p-\frac{n}{2})^{2k} \quad \text{for } k\geq 0. \tag{1.6}$$

At the end of this chapter and in the last chapter, we will consider this situation of a compact complex manifold X and $V_{2k}^{mfd}(X)$. Here $V_0^{mfd}(X)=\int_X c_n$ could be 0; then we cannot normalize the moments $V_{2k}^{mfd}(X)$.

Now let

$$V=\sum_{k=0}^\infty V_{2k}\frac{1}{(2k)!}t^{2k} \tag{1.7}$$

be a formal power series in t^2 with variables $V_0, V_2, V_4, ...$, and let ν be another variable. The Bernoulli moments $\Gamma_{2k}^{Ber}(V,\nu)\in\sum_{l=0}^k \mathbb{Q}[\nu]V_{2l}$ are defined by

$$\Gamma^{Ber}(V,\nu)=\sum_{k=0}^\infty \Gamma_{2k}^{Ber}\frac{1}{(2k)!}t^{2k} = V\cdot\exp\left(\nu\cdot\log\frac{t/2}{\sinh(t/2)}\right). \tag{1.8}$$

The first four of them are

$$\Gamma_0^{Ber}(V,\nu) = V_0, \tag{1.9}$$

$$\Gamma_2^{Ber}(V,\nu) = V_2 - V_0\cdot(\frac{1}{12}\nu), \tag{1.10}$$

$$\Gamma_4^{Ber}(V,\nu) = V_4 - V_2\cdot(\frac{1}{2}\nu)+V_0\cdot(\frac{1}{120}\nu+\frac{1}{48}\nu^2), \tag{1.11}$$

$$\begin{aligned}\Gamma_6^{Ber}(V,\nu) &= V_6 - V_4\cdot(\frac{5}{4}\nu)+V_2\cdot(\frac{1}{8}\nu+\frac{5}{16}\nu^2)\\ &\quad -V_0\cdot(\frac{1}{252}\nu+\frac{1}{96}\nu^2+\frac{5}{576}\nu^3).\end{aligned} \tag{1.12}$$

The Bernoulli moments are closely related to the generalized Bernoulli polynomials. This will be discussed after theorem 1.4. A relation with the Bernoulli numbers B_n is given by

$$\log\frac{t/2}{\sinh(t/2)} = \sum_{k=1}^\infty \frac{-1}{2k}B_{2k}\frac{1}{(2k)!}t^{2k}. \tag{1.13}$$

The Bernoulli numbers B_{2k} for $k\geq 1$ satisfy $B_{2k}\in(-1)^{k-1}\mathbb{Q}_{>0}$. Therefore the coefficient of V_{2j} in Γ_{2k}^{Ber} is a polynomial in ν with all coefficients having the sign $(-1)^{k-j}$. A more precise discussion in chapter 2 shows the following elementary lemma.

Lemma 1.2. *Consider* $V=\sum_{k=0}^\infty V_{2k}\frac{1}{(2k)!}t^{2k}\in\mathbb{R}[[t]]$ *with* $V_0>0$. *Fix* $k_0\in\mathbb{N}\cup\{\infty\}$ *(with* $\mathbb{N}:=\{0,1,2,...\}$ *in this paper).*

a) If $k_0\in\mathbb{N}$, *there exists a number* $\nu\in\mathbb{R}$ *such that*

$$\forall\ k\in\mathbb{N} \text{ with } k\leq k_0 \qquad (-1)^k\Gamma_{2k}^{Ber}(V,\nu)\geq 0. \tag{1.14}$$

b) If a number $\nu\in\mathbb{R}$ *satisfies* (1.14) *for* $k_0\in\mathbb{N}\cup\{\infty\}$ *then also any number* $\nu'\in\mathbb{R}$ *with* $\nu'>\nu$ *satisfies* (1.14).

In view of this lemma, the first of the following two conjectures is weaker than the second. These conjectures are at the center of this paper.

Conjectures 1.3. *Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be an isolated hypersurface singularity.*

(W) (Weak form) Then for all $k \in \mathbb{N}$

$$(-1)^k \Gamma_{2k}^{Ber}(V^{sing}(f), n+1) > 0. \tag{1.15}$$

(S) (Strong form) Then for all $k \in \mathbb{N}$

$$(-1)^k \Gamma_{2k}^{Ber}(V^{sing}(f), \alpha_\mu - \alpha_1) \geq 0. \tag{1.16}$$

The case $k = 1$ of the conjecture (S) is conjecture 1.1. Our evidence for the conjectures is collected in the following theorem.

Theorem 1.4. *a) The conjecture 1.3 (S) is true for all quasihomogeneous singularities.*

b) The conjecture 1.3 (S) is true for all hyperbolic singularities T_{pqr}.

c) The conjecture 1.3 (W) is true for all irreducible curve singularities.

d) [Br3] *The conjecture 1.1 is true for all curve singularities.*

e) If the conjecture (S) [respectively the conjecture (W)] is true for two singularities $f(x)$ and $g(y)$ then it is also true for the sum $f(x) + g(y)$.

f) For any singularity f and any $\nu \in \mathbb{R}_{>0}$ a bound $k_0 \geq 0$ exists such that for all $k \in \mathbb{N}$ with $k \geq k_0$

$$(-1)^k \Gamma_{2k}^{Ber}(V^{sing}(f), \nu) > 0. \tag{1.17}$$

Part a) and b) will be proved in chapter 5. There we will give also precise formulas. The formulas even suggest to consider the numbers $\Gamma_{2k}^{Ber}(V^{sing}(f), \alpha_\mu - \alpha_1)$ as well as the numbers $\Gamma_{2k}^{Ber}(V^{sing}(f), n+1)$ themselves as generalisations of the Bernoulli numbers B_{2k}.

Part c) will be proved in chapter 6. Part e) is an easy consequence of the Thom-Sebastiani formula for spectral numbers; it will be discussed in chapter 2, remark 2.5 b). The proof of part f) will be given after theorem 1.5.

The generating function $V^{sing}(f)$ of the higher moments $V_{2k}^{sing}(f)$ takes a very special form,

$$V^{sing}(f) = \sum_{i=1}^{\mu} \cosh\left(t(\alpha_i - \frac{n-1}{2})\right) = \sum_{i=1}^{\mu} e^{t(\alpha_i - \frac{n-1}{2})}. \tag{1.18}$$

The second equality follows from the symmetry in (1.1). This formula and (1.8) motivate the following definition.

The polynomials $A_k(x, \nu) \in \mathbb{Q}[x, \nu]$ for $k \in \mathbb{N}$ are defined by

$$e^{xt} \cdot \exp\left(\nu \cdot \log \frac{t/2}{\sinh(t/2)}\right) = \sum_{k=0}^{\infty} A_k(x, \nu) \frac{1}{k!} t^k. \tag{1.19}$$

Up to a shift in x they are the generalized Bernoulli polynomials, which were defined by Nörlund [No3][No4][No1][No2]. $A_k(x, \nu)$ is a polynomial of degree k in x and of degree $\left[\frac{k}{2}\right]$ in ν. The classical Bernoulli polynomials are the polynomials $B_k(x) = A_k(x - \frac{1}{2}, 1)$. The Bernoulli numbers are $B_k := B_k(0)$. The polynomials in x for fixed $\nu \in \mathbb{N}$ and especially for $\nu = 1$ have been studied since very long time. We review some properties of the $A_k(x, \nu)$ in chapter 3.

(1.8), (1.18) and (1.19) together show

$$(-1)^k \Gamma_{2k}^{Ber}(V^{sing}(f), \nu) = \sum_{j=1}^{\mu} (-1)^k A_{2k}(\alpha_j - \frac{n-1}{2}, \nu). \tag{1.20}$$

This justifies the name Bernoulli moments. One crucial property of the polynomials $A_k(x, \nu)$ is the following.

Theorem 1.5. [No4] *On any compact interval $I \subset \mathbb{R}$ and for any $\nu \in \mathbb{R} - \mathbb{Z}_{\leq 0}$, the sequence of polynomials*

$$(-1)^k A_{2k}(x,\nu) \cdot \frac{(2\pi)^{2k} \cdot \Gamma(\nu)}{2 \cdot (2k)! \cdot (2k)^{\nu-1}} \tag{1.21}$$

tends uniformly to $\cos(2\pi x)$ as $k \to \infty$ (here Γ is the gamma function).

Therefore for any $\nu \in \mathbb{R} - \mathbb{Z}_{\leq 0}$ the sequence of numbers

$$(-1)^k \Gamma_{2k}^{Ber}(V^{sing}(f),\nu) \cdot \frac{(2\pi)^{2k} \cdot \Gamma(\nu)}{2 \cdot (2k)! \cdot (2k)^{\nu-1}} \tag{1.22}$$

tends with $k \to \infty$ to

$$\begin{aligned} \sum_{j=1}^{\mu} \cos(2\pi(\alpha_j - \frac{n-1}{2})) &= (-1)^{n-1} \cdot \sum_{j=1}^{\mu} e^{2\pi i \alpha_j} \\ &= (-1)^{n-1} \text{trace (monodromy)} \\ &= 1. \end{aligned} \tag{1.23}$$

The last equality is a result of A'Campo [AC1][AC2]. For $\nu > 0$ the factor on the right hand side in (1.22) is positive. This shows part f) of theorem 1.4.

Remarks 1.6. i) If the conjecture (W) is true for a singularity f, one can define a sequence of numbers $\nu_k > 0$ for $k \in \mathbb{N}$ such that ν_k is minimal with the property

$$\forall\ k' \geq k\ \forall\ \nu' > \nu_k\ (-1)^k \Gamma_{2k'}^{Ber}(V^{sing}(f),\nu') > 0. \tag{1.24}$$

In view of part f) of theorem 1.4, this decreasing sequence tends to 0. One could ask how fast.

ii) The conjectures 1.3 and theorem 1.5 together predict the sign of the trace of the monodromy. It is an integer. By A'Campo's result it is the smallest integer with the correct sign. In view of this, it seems that the values $(-1)^k \Gamma_{2k}^{Ber}(V^{sing}(f),\nu)$ are "rather small", up to the factor in (1.22).

The conjecture 1.3 (W) is connected with some work of K. Saito [SK2] on the spectral numbers. He defined the associated distribution

$$\Delta(f)(s) := \sum_{j=1}^{\mu} \delta(s - \alpha_j + \frac{n-1}{2}), \tag{1.25}$$

where $\delta(s)$ is the delta function. Because of (1.18), its Fourier transform is just $V^{sing}(f)(2\pi i t)$. K. Saito proposed to compare $\Delta(f)(s)$ with the continuous distribution

$$\Delta^{(n+1)}(s) := (\Delta^{(1)} * ... * \Delta^{(1)})(s) \tag{1.26}$$

(the convolution n+1 times), where

$$\Delta^{(1)}(s) := \begin{cases} 1 & \text{if} \quad s \in [-\frac{1}{2}, \frac{1}{2}], \\ 0 & \text{if} \quad s \notin [-\frac{1}{2}, \frac{1}{2}] \end{cases} \tag{1.27}$$

He proved that $\Delta^{(n+1)}(s)$ is the limit distribution for quasihomogeneous singularities if the weights tend to zero and for irreducible curve singularities with g Puiseux pairs if the last denominator tends to infinity.

The Fourier transform of $\Delta^{(n+1)}(s)$ is

$$\left(\frac{\sin(\pi t)}{\pi t}\right)^{n+1}. \tag{1.28}$$

Therefore

$$\Gamma^{Ber}(V^{sing}(f),n+1)(2\pi it)=V^{sing}(f)(2\pi it)/\left(\frac{\sin(\pi t)}{\pi t}\right)^{n+1} \tag{1.29}$$

is the quotient of the Fourier transforms of the actual distribution $\Delta(f)(s)$ of spectral numbers and the continuous distribution $\Delta^{(n+1)}(s)$. The conjecture 1.3 (W) simply predicts that all its coefficients are positive. In this sense it confirms a feeling of K. Saito [SK2, p 202, (2.5) ii)] that the limit distribution $\Delta^{(n+1)}(s)$ should not only be a limit, but also a bound for the actual distributions.

But it is difficult to derive from this conjecture on the Fourier transforms concrete information on the distribution $\Delta(f)(s)$. It does not imply a conjecture of K. Saito [SK2, p 203] (and Durfee in the case $n=2$) on the number of spectral numbers in $]-1,0]$. We discuss this in chapter 4.

We presented ample evidence that the Bernoulli moments are natural objects. A characterisation in corollary 2.3 and the explicit formulas in chapter 5 will even strengthen this.

But we found the Bernoulli moments (their shape, not the inequalities in conjecture 1.3) in a different way, by looking at the moments $V_{2k}^{mfd}(X)$ of compact complex manifolds. In chapter 7 the following results will be proved, using the Hirzebruch-Riemann-Roch theorem.

Theorem 1.7. *a) There exist polynomials $q_{kj}(\nu,y_1,...,y_j)\in\mathbb{Q}[\nu,y_1,...,y_j]$ for $k\geq 1$ and $0\leq j\leq 2k-1$ with the following properties. They are quasihomogeneous of degree j with respect to the weights i of y_i. They satisfy $\deg_\nu q_{k0}=k$ and $\deg_\nu q_{kj}\leq k-1-\left[\frac{j}{2}\right]$ for $j\geq 1$. For any compact complex manifold X of any dimension n,*

$$V_{2k}^{mfd}(X)=\sum_{j=0}^{\min(2k-1,n)}\int_X q_{kj}(n,c_1,...,c_j)\cdot c_{n-j} \tag{1.30}$$

if $k\geq 1$ and $V_0^{mfd}(X)=\int_X c_n$.

b) The Bernoulli moments of $V^{mfd}(X)$ with $\nu=n$ are

$$\Gamma_{2k}^{Ber}(V^{mfd}(X),n)=\sum_{j=0}^{\min(2k-1,n)}\int_X q_{kj}(0,c_1,...,c_j)\cdot c_{n-j} \tag{1.31}$$

if $k\geq 1$ and $\Gamma_0^{Ber}(V^{mfd}(X),n)=\int_X c_n$.

The formulas for $k=0,1,2$ are (we omit $\int_X$)

$$V_0^{mfd}(X) = c_n, \tag{1.32}$$

$$V_2^{mfd}(X) = \frac{n}{12}c_n+\frac{1}{6}c_1c_{n-1},, \tag{1.33}$$

$$\begin{aligned} V_4^{mfd}(X) &= \left(\frac{n^2}{48}-\frac{n}{120}\right)\cdot c_n+\left((\frac{n}{12}-\frac{1}{30})c_1\right)\cdot c_{n-1} \\ &+ \left(\frac{c_2}{10}+\frac{c_1^2}{30}\right)\cdot c_{n-2}+\left(\frac{c_1c_2}{10}-\frac{c_3}{10}-\frac{c_1^3}{30}\right)\cdot c_{n-3}. \end{aligned} \tag{1.34}$$

In the case of the projective spaces $\mathbb{P}^n$, the analogues of the conjectures 1.3 are not true for small k, see chapter 7. It would be interesting to understand the significance of the Bernoulli moments for compact complex manifolds.

When some years ago one of us showed Duco van Straten $\Gamma_4^{Ber}(V^{sing},\alpha_\mu-\alpha_1)$ and the observation that it is positive in many examples, he conjectured immediately that there should be a series with signs $(-1)^k$. We thank him for this idea.

2. Deformations of higher moments

Let

$$V = \sum_{k=0}^{\infty} V_{2k} \frac{1}{(2k)!} t^{2k} \tag{2.1}$$

be a formal power series in t^2 with variables $V_0, V_2, V_4, \ldots$, and let ν be another variable. We are interested in formal power series

$$\begin{aligned} \Gamma(V,\nu) &= \sum_{k=0}^{\infty} \Gamma_{2k}(V,\nu) \frac{1}{(2k)!} t^{2k} \quad \text{with} \\ \Gamma_{2k}(V,\nu) &\in \sum_{l=0}^{\infty} \mathbb{C}[\nu] \cdot V_{2l} \end{aligned} \tag{2.2}$$

which satisfy the following property:

$$\Gamma(V,\nu) \cdot \Gamma(V',\nu') = \Gamma(V \cdot V', \nu + \nu'); \tag{2.3}$$

here V' is a second power series in independent variables, and ν' is another variable.

Lemma 2.1. *A power series $\Gamma(V,\nu)$ as in (2.2) satisfies (2.3) if and only if it takes the form*

$$\Gamma(V,\nu) = \left[\sum_{k=0}^{\infty} V_{2k} \frac{1}{(2k)!} (\Psi(t))^k \right] \cdot \exp(\nu \cdot \Theta(t)) \tag{2.4}$$

where

$$\Psi(t) = \sum_{k=1}^{\infty} \Psi_{2k} \frac{1}{(2k)!} t^{2k}, \tag{2.5}$$

$$\Theta(t) = \sum_{k=1}^{\infty} \Theta_{2k} \frac{1}{(2k)!} t^{2k}, \tag{2.6}$$

$\Psi_{2k}, \Theta_{2k} \in \mathbb{C}$, *or if* $\Gamma(V,\nu) = 0$.

Proof: One sees immediately that a power series $\Gamma(V,\nu)$ as in (2.4) satisfies (2.3). The inverse will be carried out in two steps.

(I) We want to prove $\Gamma(V,\nu) = \Gamma(V,0) \cdot \exp(\nu \cdot \Theta(t))$. Define

$$\Phi(t,\nu) := \Gamma(1,\nu)(t) \in \mathbb{C}[\nu][[t]]. \tag{2.7}$$

Then $\Phi(t,0) = 1$,

$$\Phi(t,\nu) \cdot \Phi(t,\nu') = \Gamma(1,\nu) \cdot \Gamma(1,\nu') = \Gamma(1,\nu+\nu') = \Phi(t,\nu+\nu') \tag{2.8}$$

and

$$(\log \Phi)(t,\nu) + (\log \Phi)(t,\nu') = (\log \Phi)(t,\nu+\nu'). \tag{2.9}$$

One sees easily $(\log \Phi)(t,\nu) \in \mathbb{C}[[\nu]][[t]]$. Now $(\log \Phi)(t,0) = \log 1 = 0$ and (2.9) imply

$$(\log \Phi)(t,\nu) = \nu \cdot \Theta(t) \tag{2.10}$$

for some $\Theta(t) \in \mathbb{C}[[t]]$. Setting $V' = 1$ in (2.3) we obtain

$$\Gamma(V,\nu) \cdot \exp(-\nu \cdot \Theta) = \Gamma(V,0). \tag{2.11}$$

(II) We consider the case with $\nu = 0$, that is, without ν.
Claim: $\Gamma_0(V,0) = V_0$ or $\Gamma_0(V,0) = 0$.

Proof: Let $\Gamma_0(V,0) = \lambda_0 V_0 + ... + \lambda_{2l} V_{2l}$ for some $l \geq 0$. First suppose that $l > 0$. Then the special values $V = V' = 1 \cdot \frac{1}{(2l)!} t^{2l}$ in (2.3) yield

$$\begin{aligned} \lambda_{2l}^2 &= \Gamma_0(V,0) \cdot \Gamma_0(V',0) && (2.12) \\ &= \Gamma_0(V \cdot V',0) = \Gamma_0(\frac{1}{((2l)!)^2} t^{4l}, 0) = 0 \end{aligned}$$

because $4l > 2l$. Thus $\lambda_{2l} = 0$. Inductively this yields $\Gamma_0(V,0) = \lambda_0 V_0$. Now the same calculation for $l = 0$ shows $\lambda_0^2 = \lambda_0$, thus $\lambda_0 \in \{0; 1\}$. This finishes the proof of the claim.

Now (2.3) for $V' = 1$ gives

$$\Gamma(V,0) \cdot \Gamma(1,0) = \Gamma(V,0). \tag{2.13}$$

In the case $\Gamma_0(V,0) = 0$ this implies $\Gamma(V,0) = 0$. We restrict ourselves now to the case $\Gamma_0(V,0) = V_0$. Then (2.13) implies $\Gamma(1,0) = 1$. Thus

$$\Gamma_{2k}(V,0) = \sum_{l>0} \lambda_{kl} \cdot V_{2l} \quad \text{for } k > 0 \tag{2.14}$$

is a finite linear combination of terms V_{2l} without the term V_0.

Using (2.14), we can define $\Psi(t) \in \mathbb{C}[[t^2]]$ by

$$\Gamma(V_0 + V_2 \frac{1}{2} t^2, 0) = V_0 + V_2 \frac{1}{2} \Psi(t). \tag{2.15}$$

Now we fix $l \in \mathbb{N}$ and choose a V with values $V_0 = 1$ and $V_{2k} = 0$ for $k > l$. As in [Hi, Lemma 1.2.1] we consider the formal decomposition of the polynomial $V(t)$ of degree $\leq 2l$,

$$V(t) = 1 + \sum_{k=1}^{l} V_{2k} \frac{1}{(2k)!} t^{2k} = \prod_{k=1}^{l} (1 + \beta_{2k} t^2). \tag{2.16}$$

Then

$$\begin{aligned} \Gamma(V(t),0) &= \prod_{k=1}^{l} \Gamma(1 + \beta_{2k} t^2) = \prod_{k=1}^{l} (1 + \beta_{2k} \Psi(t)) \\ &= 1 + \sum_{k=1}^{l} V_{2k} \frac{1}{(2k)!} (\Psi(t))^k. && (2.17) \end{aligned}$$

Because $\Psi(t)$ has no constant term and because the $\Gamma_{2k}(V,0)$ are finite linear combinations of the $V_{2k'}$ and because of (2.14), this shows for general V

$$\Gamma(V(t),0) = \sum_{k=0}^{\infty} V_{2k} \frac{1}{(2k)!} (\Psi(t))^k. \tag{2.18}$$

This completes the proof. □

Remarks 2.2. a) The lemma 2.1 is close to Lemma 1.2.1 and Lemma 1.2.2 in [Hi]. Three differences are the parameter ν here, that we do not necessarily have $V_0 = 1$ and $\Gamma_0(V,\nu) = 1$ here and that here $\Gamma_{2k}(V,\nu)$ is a linear combination of the V_{2l}, not a quasihomogeneous polynomial.

b) (2.3) together with the condition $\Gamma(V,0) = V$ restricts the solutions to the case $\Psi(t) = t^2$. We will only be interested in this case.

Corollary 2.3. *The Bernoulli moments are characterized by the four properties* (2.2), (2.3),

$$\Gamma^{Ber}_{2k}(V,0) = V_{2k}, \tag{2.19}$$

$$\Gamma^{Ber}_{2k}(V^{sing}(A_\mu), \frac{1}{2}) \qquad \text{is a polynomial in } w = \frac{1}{\mu+1} \tag{2.20}$$

for $k \geq 1$.

Proof: The first three conditions show $\Gamma^{Ber}(V,\nu) = V \cdot \exp(\nu \cdot \Theta(t))$ for some $\Theta(t) \in \mathbb{C}[[t]]$. By induction on $k > 0$, the condition (2.20) determines Θ_{2k} uniquely. The formulas (5.4) and (3.9) show $\Theta_{2k} = \Theta^{Ber}_{2k} = \frac{-1}{2k} B_{2k}$. □

The following lemma implies lemma 1.2.

Lemma 2.4. *Consider* $V(t) \in \mathbb{R}[[t^2]]$ *and* $\Theta(t) \in \mathbb{R}[[t^2]]$ *with coefficients* V_{2k} *and* Θ_{2k} *as in* (2.1) *and* (2.6) *and* $V_0 > 0$, $-\Theta_2 > 0$ *and* $(-1)^k \Theta_{2k} \geq 0$ *for all* $k \geq 2$. *Consider*

$$\Gamma(V,\nu)(t) = V \cdot \exp(\nu \cdot \Theta(t)).$$

Fix $k_0 \in \mathbb{N} \cup \{\infty\}$.

a) If $k_0 \in \mathbb{N}$, *there exists a number* $\nu \in \mathbb{R}$ *such that*

$$\forall\ k \in \mathbb{N} \text{ with } k \leq k_0 \qquad (-1)^k \Gamma_{2k}(V,\nu) \geq 0. \tag{2.21}$$

b) If a number $\nu \in \mathbb{R}$ *satisfies* (2.21) *for* $k_0 \in \mathbb{N} \cup \{\infty\}$ *then also any number* $\nu' \in \mathbb{R}$ *with* $\nu' > \nu$ *satisfies* (2.21).

Proof: a) The polynomial $(-1)^k \Gamma_{2k}(V,\nu) \in \mathbb{R}[\nu]$ has degree k. Its term of degree k is

$$(-1)^k V_0 \cdot \Theta_2^k \frac{(2k)!}{k!} \cdot \nu^k. \tag{2.22}$$

It is positive if $\nu > 0$, and for large ν it dominates $(-1)^k \Gamma_{2k}(V,\nu)$.

b) Consider the two power series $\Theta(it) \in \mathbb{R}[[t^2]]$ and $\exp((\nu'-\nu)\cdot\Theta(it)) \in \mathbb{R}[[t^2]]$ for some fixed $\nu' > \nu$. All their coefficients are nonnegative. The numbers $(-1)^k \Gamma_{2k}(V,\nu')$ are the coefficients of

$$\Gamma(V,\nu')(it) = \Gamma(V,\nu)(it) \cdot \exp((\nu'-\nu)\cdot\Theta(it)). \tag{2.23}$$

If the first k_0 coefficients of $\Gamma(V,\nu)(it)$ are nonnegative, then also the first k_0 coefficients of $\Gamma(V,\nu')(it)$ are nonnegative. □

Remarks 2.5. a) In the case of hypersurface singularities, the spectral numbers satisfy a Thom-Sebastiani property [Va][SchS]: Let $f(x_0,...,x_n)$ and $g(y_0,...,y_m)$ be two singularities in different variables with spectral numbers α_i and β_j. Then the spectrum of $f+g$ is the tuple of numbers

$$Sp(f+g) = (\alpha_i + \beta_j + 1 \mid i = 1,...,\mu(f); j = 1,...,\mu(g)). \tag{2.24}$$

This means that the distribution $\Delta(f+g)(s)$ associated to $f+g$ (cf. (1.25)) is the convolution of those associated to f and g,

$$\Delta(f+g) = \Delta(f) * \Delta(g). \tag{2.25}$$

$V^{sing}(f)(2\pi it)$ is the Fourier transform of $\Delta(f)$. Thus

$$V^{sing}(f+g) = V^{sing}(f) \cdot V^{sing}(g). \tag{2.26}$$

With $\nu_1(f) := \alpha_{\mu(f)}(f) - \alpha_1(f)$ and $\nu_2(f) := n+1$, we get for $j = 1,2$

$$\begin{aligned} &\Gamma^{Ber}(V^{sing}(f+g), \nu_j(f+g)) \\ = \ &\Gamma^{Ber}(V^{sing}(f), \nu_j(f)) \cdot \Gamma^{Ber}(V^{sing}(g), \nu_j(g)). \end{aligned} \tag{2.27}$$

b) Conjecture 1.3 (S) [respectively (W)] for a singularity f says that all coefficients of $\Gamma^{Ber}(V^{sing}(f),\alpha_\mu-\alpha_1)(2\pi it)$ [respectively $\Gamma^{Ber}(V^{sing}(f),n+1)(2\pi it)$] are nonnegative [respectively positive]. Formula (2.27) shows part e) of theorem 1.4.

c) Consider a compact complex manifold of dimension n with higher moments $V^{mfd}_{2k}(X)$ as in (1.6) and generating function $V^{mfd}(X)$. By Serre duality (e.g. [Hi, p 123][GH, p 102])

$$h^{pq}=h^{n-p,n-q} \quad \text{and} \quad \chi_p=\chi_{n-p}. \tag{2.28}$$

Therefore

$$V^{mfd}(X)=\sum_{p=0}^{n}\chi_p\cdot\cosh(t(p-\frac{n}{2}))=\sum_{p=0}^{n}\chi_p\cdot e^{t(p-\frac{n}{2})}. \tag{2.29}$$

If Y is a second compact complex manifold, the spaces $H^{pq}(X)=H^q(X,\Omega^p)$ and those of Y and $X\times Y$ satisfy the Künneth formula (e.g. [GH, p 103])

$$H^{*,*}(X\times Y)\cong H^{*,*}(X)\otimes H^{*,*}(Y). \tag{2.30}$$

Therefore

$$\chi_p(X\times Y) = \sum_{a,b:a+b=p}\chi_a(X)\cdot\chi_b(Y) \quad \text{and} \tag{2.31}$$

$$V^{mfd}(X\times Y) = V^{mfd}(X)\cdot V^{mfd}(Y). \tag{2.32}$$

3. Generalized Bernoulli polynomials

Define

$$\Theta^{Ber}(t)=\sum_{k=0}^{\infty}\Theta^{Ber}_{2k}\frac{1}{(2k)!}t^{2k}=\log\frac{t/2}{\sinh(t/2)}. \tag{3.1}$$

The polynomials $A_k(x,\nu)\in\mathbb{Q}[x,\nu]$ for $k\in\mathbb{N}$ are defined by

$$e^{xt}\cdot\exp\big(\nu\cdot\Theta^{Ber}(t)\big)=\sum_{k=0}^{\infty}A_k(x,\nu)\frac{1}{k!}t^k. \tag{3.2}$$

They coincide with the classical generalized Bernoulli polynomials $B^{(\nu)}_k(x)$ up to a shift,

$$A_k(x,\nu)=B^{(\nu)}_k(x+\frac{\nu}{2}). \tag{3.3}$$

The notation $B^{(\nu)}_k(x)$ was established by Nörlund [No1][No2]. He and Milne-Thomson [MT] studied these polynomials systematically for fixed $\nu\in\mathbb{N}$. In [No1, p 177] Nörlund gives references to earlier studies of them for fixed $\nu\in\mathbb{N}$. The Bernoulli polynomials $B_k(x)=B^{(1)}_k(x)$ themselves had first been considered by Jacob Bernoulli, then by Euler. Since the 19th century the literature on them and on the Bernoulli numbers $B_k=B_k(0)$ is huge. Their basic properties are treated for example in [AS][Er][Jo][MT][No1][No2].

In [No1][No2] there are some remarks about the polynomials $B^{(\nu)}_k(0)$ in ν. But a study of the $B^{(\nu)}_k(x)$ as polynomials in x and ν seems to have been started only in the 60ies, in [No3][No4][We]. Weinmann [We] seems to be the only one who shares our point of view that the $A_k(x,\nu)$ have advantages compared with the $B^{(\nu)}_k(x)$: we have $A_k(-x)=(-1)^kA_k(x)$, compared to $B^{(\nu)}_k(\nu-x)=(-1)^kB^{(\nu)}_k(x)$, and $\deg_\nu A_k(x,\nu)=\left[\frac{k}{2}\right]$, compared to $\deg_\nu B^{(\nu)}_k(x)=k$ (both are polynomials of degree k in x).

The following theorem states well-known or elementary properties of the Bernoulli numbers and the $A_k(x,\nu)$.

Theorem 3.1. *a) The Bernoulli numbers satisfy*

$$B_0 = 1,\ B_1 = -\frac{1}{2},\ B_{2k+1} = 0 \quad \text{if } k \geq 1, \tag{3.4}$$

$$B_{2k} = (-1)^{k-1}\frac{2(2k)!}{(2\pi)^{2k}}\zeta(2k) \quad \text{if } k \geq 1, \tag{3.5}$$

$$0 = \sum_{j=0}^{k-1}\binom{k}{j}B_j \quad \text{if } k \geq 2, \tag{3.6}$$

$$(B_{2k} \mid k = 1, ..., 8) = (\frac{1}{6}, -\frac{1}{30}, \frac{1}{42}, -\frac{1}{30}, \frac{5}{66}, -\frac{691}{2730}, \frac{7}{6}, -\frac{3617}{510}). \tag{3.7}$$

(3.5) *shows* $B_{2k} = (-1)^{k-1}|B_{2k}|$ *if* $k \geq 1$ *and gives their asymptotic behaviour because*

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s} \to 1$$

fast if $s \to +\infty$*.* (3.6) *provides an efficient way to calculate them and shows* $B_k \in \mathbb{Q}$*. The usual definition is via the generating function*

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{1}{k!}t^k. \tag{3.8}$$

The Bernoulli numbers turn also up in $\Theta^{Ber}(t)$*,*

$$\Theta^{Ber}(t) = \sum_{k=1}^{\infty}\frac{-1}{2k}B_{2k}\frac{1}{(2k)!}t^{2k}. \tag{3.9}$$

b) The polynomials $A_k(x, \nu)$ *satisfy*

$$A_k(x, 0) = x^k, \tag{3.10}$$

$$A_{2k+1}(0, \nu) = 0, \tag{3.11}$$

$$A_{2k}(0, \nu) \in (-1)^k\mathbb{Q}_{\geq 0}[\nu], \quad \deg_\nu A_{2k}(0, \nu) = k, \tag{3.12}$$

$$A_k(x, \nu) = \sum_{j=0}^{[k/2]}\binom{k}{2j}A_{2j}(0, \nu) \cdot x^{k-2j}, \tag{3.13}$$

$$A_0(x, \nu) = 1,\ A_2(0, \nu) = -\frac{1}{12}\nu,\ A_4(0, \nu) = \frac{1}{120}\nu + \frac{1}{48}\nu^2, \tag{3.14}$$

$$A_6(0, \nu) = -(\frac{1}{252}\nu + \frac{1}{96}\nu^2 + \frac{5}{576}\nu^3), \tag{3.15}$$

$$A_k(x_1 + x_2, \nu_1 + \nu_2) = \sum_{j=0}^{k}\binom{k}{j}A_j(x_1, \nu_1) \cdot A_{k-j}(x_2, \nu_2), \tag{3.16}$$

$$A_k(-x, \nu) = (-1)^k A_k(x, \nu), \tag{3.17}$$

$$\frac{\partial}{\partial x}A_k(x, \nu) = k \cdot A_{k-1}(x, \nu), \tag{3.18}$$

$$\frac{\partial}{\partial \nu}A_k(x, \nu) = \sum_{j=1}^{[k/2]}\binom{k}{2j}\frac{-1}{2j}B_{2j}A_{k-2j}(x, \nu), \tag{3.19}$$

$$A_k(x+\frac{1}{2},\nu+1)-A_k(x-\frac{1}{2},\nu+1)=k\cdot A_{k-1}(x,\nu), \tag{3.20}$$

$$\nu\cdot A_k(x\pm\frac{1}{2},\nu+1)=(\nu-k)A_k(x,\nu)+k(x\pm\frac{\nu}{2})A_{k-1}(x,\nu), \tag{3.21}$$

$$A_k(x,k+1)=\prod_{j=0}^{k-1}(x+\frac{k-1}{2}-j). \tag{3.22}$$

Proof: a) (3.8) follows from $B_k=A_k(-\frac{1}{2},1)$ and (3.2). The calculation

$$t\frac{\partial}{\partial t}\Theta^{Ber}(t)=1-\frac{1}{2}t\frac{\cosh(t/2)}{\sinh(t/2)}=1-\frac{1}{2}t-\frac{t}{e^t-1}, \tag{3.23}$$

(3.8), the fact that $\Theta^{Ber}(t)$ is even and $\Theta^{Ber}(0)=0$ show (3.4) and (3.9).

(3.4) and (3.17) yield $A_k(-\frac{1}{2},1)=A_k(\frac{1}{2},1)$ ($=0$ if k is odd and ≥ 3) for $k\neq 1$. Now (3.10) and (3.16) for $(x_1,x_2,\nu_1,\nu_2)=(-\frac{1}{2},1,1,0)$ imply (3.6). With (3.6) one can calculate (3.7). Finally, (3.5) is well-known and a consequence of (3.24).

b) (3.10), (3.11), (3.16), its special case (3.13) and (3.17) are obvious. (3.12) follows from (3.2), (3.9) and $-B_{2k}\in(-1)^k\mathbb{Q}_{>0}$ for $k\geq 1$. (3.14) and (3.15) can be calculated with (3.19). For (3.18) and (3.19) one differentiates (3.2) by x and ν and uses (3.9). A straightforward calculation yields (3.20). For (3.21) one applies $t\frac{\partial}{\partial t}$ to (3.2) and uses (3.20). By induction one obtains (3.22) from (3.21) for $k=\nu$. □

Especially interesting for us are the behaviour of $A_k(x,\nu)$ for fixed ν and $k\to\infty$ and the relation to Fourier series. Part a) of the following theorem is classical, part b) is a generalization of a) essentially due to Weinmann [We], part c) is essentially due to Nörlund [No4][No3]. Part c) contains theorem 1.5. We do not use part b) later, but it fits well to part c) and to the conjecture 1.3 (W).

Theorem 3.2. *a) Let $f_k:\mathbb{R}\to\mathbb{R}$ be the 1-periodic function with $f_k(x)=A_k(x,1)$ for $x\in]-\frac{1}{2},\frac{1}{2}]$. For $k\geq 1$ its Fourier series $\sigma(f_k)$ is*

$$\sigma(f_{2k}) = (-1)^{k-1}\frac{2(2k)!}{(2\pi)^{2k}}\sum_{n=1}^{\infty}(\frac{(-1)^n}{n^{2k}}\cos(2\pi nx), \tag{3.24}$$

$$\sigma(f_{2k-1}) = (-1)^{k}\frac{2(2k-1)!}{(2\pi)^{2k-1}}\sum_{n=1}^{\infty}(\frac{(-1)^n}{n^{2k-1}}\sin(2\pi nx). \tag{3.25}$$

For $k\geq 2$ the Fourier series $\sigma(f_k)$ converges uniformly to f_k; the Fourier series $\sigma(f_1)$ converges to f_1 on $\mathbb{R}-(\frac{1}{2}+\mathbb{Z})$.

b) For $\nu\in\mathbb{N}_{\geq 1}$ and $k>\nu$ and $x\in[-\frac{\nu}{2},\frac{\nu}{2}]$

$$A_k(x,\nu)=\binom{k-1}{\nu-1}\sum_{j=0}^{\nu-1}(-1)^{\nu-1-j}\binom{\nu-1}{j}\frac{k}{k-j} \tag{3.26}$$
$$\cdot A_j(x,\nu)\cdot f_{k-j}(x+\frac{\nu-1}{2}).$$

Replacing the functions f_{k-j} by their Fourier series, one obtains a series in $\cos(2\pi nx)$ *and* $\sin(2\pi nx)$ *with polynomial coefficients, which converges uniformly to $A_k(x,\nu)$ on $[-\frac{\nu}{2},\frac{\nu}{2}]$*

c) On any compact interval $I \subset \mathbb{R}$ and for any $\nu \in \mathbb{R} - \mathbb{Z}_{\leq 0}$, the sequence of polynomials in (1.21) tends uniformly to $\cos(2\pi x)$ *as $k \to \infty$ and the sequence of polynomials*

$$(-1)^{k-1} A_{2k-1}(x,\nu) \cdot \frac{(2\pi)^{2k-1} \cdot \Gamma(\nu)}{2 \cdot (2k-1)! \cdot (2k-1)^{\nu-1}} \tag{3.27}$$

tends uniformly to $\sin(2\pi x)$ *as $k \to \infty$.*

Proof: a) See for example [Er, p 37] for a proof using a contour integral and [Jo, §82] for a proof in which the Fourier coefficients are calculated inductively.

b) Weinmann [We, p 77] generalized the proof of a) via a contour integral. He obtained the formula which one gets if one replaces the functions f_{k-j} in (3.26) by their Fourier series.

We offer a different proof. Suppose that $\nu \in \mathbb{N}_{\geq 1}$ and $k \geq \nu$. Repeated application of (3.21) for $x - \frac{1}{2}$ yields the formula [No2, p 148 (87)]

$$\begin{aligned} A_k(x,\nu) = \binom{k-1}{\nu-1} \sum_{j=0}^{\nu-1} (-1)^{\nu-1-j} \binom{\nu-1}{j} \frac{k}{k-j} \\ \cdot A_j(x,\nu) \cdot A_{k-j}(x + \frac{\nu-1}{2}, 1). \end{aligned} \tag{3.28}$$

Claim: The formula remains true if one replaces $A_{k-j}(x+\frac{\nu-1}{2},1)$ by $A_{k-j}(x+\frac{\nu-1}{2}-l),1)$ for any $l \in \{0,1,...,\nu-1\}$.
Proof: For $l \in \{1,...,\nu-1\}$ the difference of the formulas for $l-1$ and l is, after dividing by $\binom{k-1}{\nu-1}/k$,

$$\begin{aligned} & \sum_{j=0}^{\nu-1} (-1)^{\nu-1-j} \binom{\nu-1}{j} \frac{1}{k-j} \cdot A_j(x,\nu) \cdot \\ & \qquad \Big(A_{k-j}(x + \frac{\nu-1}{2} - l + 1, 1) - A_{k-j}(x + \frac{\nu-1}{2} - l, 1) \Big) \\ = & \sum_{j=0}^{\nu-1} (-1)^{\nu-1-j} \binom{\nu-1}{j} \cdot A_j(x,\nu) \cdot A_{k-j-1}(x + \frac{\nu}{2} - l, 0) \\ = & \sum_{j=0}^{\nu-1} (-1)^{\nu-1-j} \binom{\nu-1}{j} \cdot A_j(x,\nu) \cdot (x + \frac{\nu}{2} - l)^{k-j-1} \\ = & (x + \frac{\nu}{2} - l)^{k-\nu} \sum_{j=0}^{\nu-1} \binom{\nu-1}{j} \cdot A_j(x,\nu) \cdot (l - x - \frac{\nu}{2})^{\nu-1-j} \\ = & (x + \frac{\nu}{2} - l)^{k-\nu} \sum_{j=0}^{\nu-1} \binom{\nu-1}{j} \cdot A_j(x,\nu) \cdot A_{\nu-1-j}(l - x - \frac{\nu}{2}, 0) \\ = & (x + \frac{\nu}{2} - l)^{k-\nu} \cdot A_{\nu-1}(l - \frac{\nu}{2}, \nu) = 0. \end{aligned} \tag{3.29}$$

Here we used (3.20), (3.10), (3.16) and (3.22). This shows the claim.

Now for any $x \in [-\frac{\nu}{2}, \frac{\nu}{2}]$ there exists an $l \in \{0,1,...,\nu-1\}$ such that $x + \frac{\nu-1}{2} - l \in [-\frac{1}{2}, \frac{1}{2}]$. For $k > \nu$ and $j \leq \nu - 1$, the 1-periodic function f_{k-j} is continuous and equals A_{k-j} on $[-\frac{1}{2}, \frac{1}{2}]$. Therefore we can replace in (3.28) $A_{k-j}(x+\frac{\nu-1}{2},1)$ by $f_{k-j}(x+\frac{\nu-1}{2})$.

c) Let us fix a compact interval $I \subset \mathbb{R}$ and a number $\nu \in \mathbb{R} - \mathbb{Z}_{\leq 0}$. It is sufficient to prove that a bound $b > 0$ exists such that for all $k \in \mathbb{N}$ and all $x \in I$

$$|A_k(x,\nu) \cdot \frac{(2\pi)^k \cdot \Gamma(\nu)}{2 \cdot k! \cdot k^{\nu-1}} - \cos(2\pi x - \frac{\pi}{2}k)| < b \cdot k^{-7/9}. \tag{3.30}$$

Nörlund stated this result [No3][No4], even with k^{-1} instead of $k^{-7/9}$, but for a single x (and with a sign mistake). He gave only a sketch of a proof [No4]. In order to be self-contained, we present an independent and complete proof.

For any $x \in I$ the function

$$t \mapsto e^{xt} \exp(\nu \cdot \Theta^{Ber}(t)) = e^{(x+\frac{1}{2}\nu)t} \left(\frac{t}{e^t - 1}\right)^{\nu} \tag{3.31}$$

is holomorphic on $\mathbb{C} - (2\pi i\mathbb{Z} - \{0\})$. Therefore

$$\frac{1}{k!} A_k(x,\nu) = \frac{1}{2\pi i} \int_{C_0} t^{-1-k} \cdot e^{(x+\frac{1}{2}\nu)t} \left(\frac{t}{e^t - 1}\right)^{\nu} \mathrm{d}t, \tag{3.32}$$

where C_0 is a closed path in $R := \mathbb{C} - \{z \in \mathbb{C} \mid \Re z = 0, \Im z \notin] - 2\pi, 2\pi[\}$ going around 0 once counterclockwise. We replace C_0 by the union $C_1 \cup C_2 \cup C_3 \cup C_4$ of the following paths: C_1 is the circle around $2\pi i$ of radius $2\pi k^{-8/9}$, oriented clockwise, which starts and ends at $2\pi i(1+k^{-8/9})$; C_2 is the half-circle around 0 of radius $2\pi(1+k^{-8/9})$, oriented counterclockwise, which starts at $2\pi i(1+k^{-8/9})$ and ends at $-2\pi i(1+k^{-8/9})$; C_3 and C_4 are obtained from C_1 and C_2 by the map $\mathbb{C} \to \mathbb{C}$, $z \mapsto -z$.

The purpose of $k^{-8/9}$ is that $(1+k^{-8/9})^k \approx \exp(k^{1/9})$ tends to ∞ faster than any power of k if $k \to \infty$, but that $(1+k^{-16/9})^k \approx \exp(k^{-7/9}) \approx 1 + O(k^{-7/9})$ tends to 1. The second property will allow to replace the function $(1+z)^k$ by the function e^{kz} on a disc of radius $k^{-8/9}$ around 0.

We denote by I_j, $j = 1,2,3,4$ the numbers which are obtained if one replaces in the right hand side of (3.32) C_0 by C_j. In the following estimate of $|I_2 + I_4|$ the factor t^{-k} yields the second and the third term and $\left(\frac{1}{e^t-1}\right)^{\nu}$ yields the fourth term;

$$\begin{aligned} |I_2 + I_4| &\leq \mathit{const.} \cdot (2\pi)^{-k} \cdot (1+k^{-8/9})^{-k} \cdot k^{8\nu/9} \\ &\leq \mathit{const.} \cdot (2\pi)^{-k} \cdot \exp(-k^{1/9}) \cdot k^{8\nu/9}. \end{aligned} \tag{3.33}$$

I_3 will give the complex conjugate value of I_1; so we restrict ourselves to I_1. Let C_5 be the circle around 0 of radius $k^{-8/9}$, oriented counterclockwise, which starts and ends at $k^{-8/9}$. With the coordinate change $t = 2\pi i(1+\tau)$ we obtain

$$I_1 = -e^{2\pi i(x+\frac{1}{2}\nu)} \frac{(2\pi i)^{\nu-k}}{2\pi i} \int_{C_5} e^{(x+\frac{1}{2}\nu)2\pi i\tau} \frac{(1+\tau)^{\nu-k-1}}{(e^{2\pi i\tau} - 1)^{\nu}} \mathrm{d}\tau. \tag{3.34}$$

We have for τ in the disc of radius $k^{-8/9}$ around 0

$$e^{(x+\frac{1}{2}\nu)2\pi i\tau}(1+\tau)^{\nu-1} \approx 1 + O(k^{-8/9}), \tag{3.35}$$

$$(1+\tau)^{-k} \approx e^{-\tau k} \cdot (1 + O(k^{-7/9})), \tag{3.36}$$

$$(e^{2\pi i\tau} - 1)^{-\nu} \approx (2\pi i\tau)^{-\nu} \cdot (1 + O(k^{-8/9})). \tag{3.37}$$

Therefore

$$I_1 = -e^{2\pi i(x+\frac{1}{2}\nu)} \frac{(2\pi i)^{-k}}{2\pi i} \int_{C_5} \frac{e^{-\tau k}}{\tau^{\nu}} (1 + k^{-7/9} g_k(x,\tau)) \mathrm{d}\tau, \tag{3.38}$$

where $g_k(x,\tau) : I \times \{z \mid |z| \leq k^{-8/9}\} \to \mathbb{C}$ is real analytic in x and holomorphic in z and bounded independently of k, x, z. Formula (6) in [Er, p 14] says

$$-e^{\pi i\nu}\frac{1}{2\pi i}\int_{C_6}\frac{e^{-\tau k}}{\tau^\nu}\mathrm{d}\tau = \frac{k^{\nu-1}}{\Gamma(\nu)}, \tag{3.39}$$

where C_6 is a path from $+\infty$ to $+\infty$ circulating once counterclockwise around 0. Therefore

$$\begin{aligned} I_1 &= e^{2\pi i x - \frac{\pi}{2}ik}\frac{k^{\nu-1}}{(2\pi)^k\Gamma(\nu)} && (3.40)\\ &- k^{-7/9}e^{2\pi i(x+\frac{1}{2}\nu)}\frac{(2\pi i)^{-k}}{2\pi i}\int_{C_5}\frac{e^{-\tau k}}{\tau^\nu}g_k(x,\tau)\mathrm{d}\tau, && (3.41)\\ &- e^{2\pi i(x+\frac{1}{2}\nu)}\frac{(2\pi i)^{-k}}{2\pi i}\int_{C_6-C_5}\frac{e^{-\tau k}}{\tau^\nu}\mathrm{d}\tau. && (3.42)\end{aligned}$$

The integral (3.42) can be estimated easily. Its vanishing order is dominated by $(2\pi)^{-k}\cdot\exp(-k^{1/9})$. In order to estimate the integral (3.41), we replace C_5 by $(-C_7)\cup C_8\cup C_7$, where C_7 is the straight line from k^{-1} to $k^{-8/9}$ and C_8 is the circle around 0 of radius k^{-1}, oriented counterclockwise, which starts and ends at k^{-1}. With the coordinate change $\widetilde{\tau} = k\tau$, it is easy to see that for $j = 7, 8$ the integral

$$k^{1-\nu}\cdot\int_{C_j}\left|\frac{e^{-\tau k}}{\tau^\nu}\right|\mathrm{d}\tau \tag{3.43}$$

is bounded independently of k. Therefore (3.41) is of order $k^{-7/9}\cdot k^{\nu-1}/(2\pi)^k$.

For I_3 we get the complex conjugate result. Thus

$$\frac{(2\pi)^k\Gamma(\nu)}{2\cdot k^{\nu-1}}(I_1+I_2+I_3+I_4) = \cos(2\pi x - \frac{\pi}{2}k) + O(k^{-7/9}). \tag{3.44}$$

This finishes the proof. □

Remarks 3.3. a) The asymptotic behaviour of the polynomials $A_k(x,\nu)$ has also been studied in [We] and [No4] in the case $k = k_0 + r$ and $\nu = \nu_0 + r$ with $r \to \infty$. Nörlund obtains that a suitable normalization of $B_k^{(\nu)}(x)$ tends to $\frac{1}{\Gamma(1-x)}$, Weinmann finds that a suitable normalization of $A_k(x,\nu)$ tends to a linear combination of $\cos(\pi x)$ and $\sin(\pi x)$ with polynomial coefficients in r. So both find the average interval 1 between neighbouring zeros of $A_k(x,\nu)$. In the case $A_k(x,k+1)$ this is obvious because of formula (3.22). In theorem 3.2 c) we had $\frac{1}{2}$.

b) If k and ν tend to ∞ with a larger [or smaller] fixed quotient ν/k, we expect a larger [or smaller] average interval between neighbouring zeros of $A_k(x,\nu)$.

c) Also, we expect that $A_k(x,\nu)$ has the maximal number k of zeros if $\nu \geq k$. This is clear for $\nu = k+1$ by (3.22). Because of (3.18) it holds for all $\nu \in \mathbb{N}$ with $\nu \geq k+1$: for these ν

$$A_k(x,\nu) = \frac{(\nu-1)!}{k!}\frac{\partial^{\nu-1-k}}{\partial x^{\nu-1-k}}A_{\nu-1}(x,\nu). \tag{3.45}$$

d) The (real) zeros of the Bernoulli polynomials and thus of the polynomials $A_k(x,1)$ are well understood ([De1][De2][In][Le] and references there). Inkeri [In] showed that the number of zeros of the Bernoulli polynomials and of the polynomials $A_k(x,1)$ tends to $\frac{2k}{\pi e}$ as $k \to \infty$. His results are much more precise. Delange [De1][De2] even refined Inkeri's results to such a precision that he can derive without effort that $A_{1000000}(x,1)$ has 234204 zeros. Also the positions of the zeros are well understood.

e) If k is fixed and ν tends to ∞, then the zeros of $A_k(x,\nu)$ tend to $\sqrt{\nu}\cdot c_j$, $j = 1, ..., k$ with $c_1 \leq ... \leq c_k$. This follows from (3.12) and (3.13). We expect that the numbers $c_1, ..., c_k$ are all different. So for large ν the polynomial $A_k(x,\nu)$ is oscillating around 0 only for $|x| \leq c_k \cdot \sqrt{\nu}$. For the conjectures 1.3 the interval $[-\frac{\nu}{2}, \frac{\nu}{2}]$ is relevant.

We conclude with a discussion of $A_2(x,\nu)$ and $A_4(x,\nu)$.

Examples 3.4. a) The polynomial $-A_2(x,\nu) = -x^2 + \frac{1}{12}\nu$ has the zeros $\pm\sqrt{\frac{1}{12}\nu}$. The positive zero is smaller than $\frac{\nu}{2}$ if $\nu > \frac{1}{3}$.

b) The polynomial $A_4(x,\nu) = x^4 - \frac{\nu}{2}x^2 + (\frac{\nu}{120} + \frac{\nu^2}{48})$ has two minima at $\pm x_0 = \pm\sqrt{\frac{\nu}{4}}$ and a local maximum at 0. It has four zeros if $\nu > \frac{1}{5}$. If $\nu > 1$ then $x_0 < \frac{\nu}{2}$. For large ν the zeros are approximately $\pm x_0\sqrt{1+\sqrt{\frac{2}{3}}} = \pm x_0 \cdot 1,3478$ and $\pm x_0\sqrt{1-\sqrt{\frac{2}{3}}} = \pm x_0 \cdot 0,4284$.

4. INTERPRETATION

The conjectures 1.3 are about the higher moments of the spectral numbers of a singularity. Nevertheless it is difficult to derive from them concrete information on the distribution of the spectral numbers. The following remarks point to different aspects of this problem.

Remarks 4.1. a) The meaning of the conjecture 1.1, that is, the case $k = 1$, is clear: the variance is bounded from above. Also for $k \to \infty$ the meaning of the conjectures 1.3 is clear: by the discussion after theorem 1.5 they boil down to the topological statement that the sign of the trace of the monodromy is $(-1)^{n-1}$. But for $k = 2$ and any fixed $k \geq 2$ the meaning of the conjectures 1.3 is not at all clear. If k is small compared to ν, then by remark 3.3 e) the polynomial $(-1)^k A_{2k}(x,\nu)$ is oscillating around 0 only for *const.* $\cdot \sqrt{\nu}$ and has the sign $(-1)^k$ outside, whereas the conjectures 1.3 are concerned with the whole interval $[-\frac{\nu}{2}, \frac{\nu}{2}]$.

b) Because of (1.21) and (1.20), the power series $\Gamma^{Ber}(V^{sing}(f),\nu)(2\pi it)$ has the radius of convergence 1 for any $\nu \in \mathbb{R} - \mathbb{Z}_{\leq 0}$. The conjecture 1.3 (W) [respectively (S)] says that all coefficients are positive [nonnegative] if $\nu = n+1$ [$\nu = \alpha_\mu - \alpha_1$]. What does this say about the function?

c) It would be good to establish an inverse Fourier transform $\mathcal{F}^{(-1)}(f)(s)$ of the function $\Gamma^{Ber}(V^{sing}(f), n+1)(2\pi it)$. Then (1.29) could be rewritten as

$$\Delta(f)(s) = (\mathcal{F}^{(-1)}(f) * \Delta^{(n+1)})(s); \tag{4.1}$$

this could help to give a better answer to K. Saito's hope [SK2, p 202, (2.5) ii)] that the limit distribution $\Delta^{(n+1)}(s)$ should be a bound of the distributions $\Delta(f)(s)$ of the spectral numbers of singularities f.

d) K. Saito formulated some questions connected with this hope [SK2, p 203, (2.8)]: Is

$$|\{j \mid \alpha_j \leq -\frac{1}{2}\}| \quad < \quad \frac{\mu}{(n+1)!2^{n+1}} \ ? \tag{4.2}$$

$$|\{j \mid \alpha_j < 0\}| \quad < \quad \frac{\mu}{(n+1)!} \ ? \tag{4.3}$$

For $n = 2$ a yes to the second question (with $\alpha_j \leq 0$ instead of $\alpha_j < 0$) is equivalent [SM1] to Durfee's conjecture [Du] that the geometric genus of a singularity is $< \mu/6$.

But the conjectures 1.3 do not answer these questions, see the example 4.2. They give only weaker inequalities.

e) The conjectures 1.3 point to relations which should be explored and structures which have yet to be established. On the one hand there is the similarity of $V^{sing}(f)$ and $V^{mfd}(X)$ for compact complex manifolds X. Could one hope to establish for singularities some of the central objects in chapter 7, Chern classes and Hirzebruch-Riemann-Roch theorem?

On the other hand, the conjecture 1.1 was found [He1][He2] by looking at the G-function of Frobenius manifolds [DZ][Gi]. In the singularity case, this is a distinguished holomorphic function on the Frobenius manifold, that is, the base space of a semiuniversal unfolding. Its derivative by the Euler field is just the constant $-\frac{1}{4}\cdot\Gamma_2^{Ber}(V^{sing}(f),\alpha_\mu-\alpha_1)$. In the quantum cohomology case, the G-function is the generating function of the genus 1 Gromov-Witten invariants (the generating function of the genus 0 invariants gives the Frobenius manifold). In that case one has generating functions for the invariants of all genera. Are they related to the higher Bernoulli moments?

These two structures, Chern classes and Frobenius manifolds, might have the potential to provide techniques for proving the conjectures 1.3 in general.

Example 4.2. The conjecture 1.3 (W) does not imply the inequality (4.3) in the case $n=2$. We consider an abstract spectrum with spectral numbers $-\frac{1}{2}$, 0, and $\frac{1}{2}$, with multiplicities r, $\mu-2r$, r, where $0\leq r\leq\frac{\mu}{2}$, $r\in\mathbb{R}$. Then

$$\Gamma_{2k}^{Ber}(V,3)=2rA_{2k}(\frac{1}{2},3)+(\mu-2r)A_{2k}(0,3). \qquad (4.4)$$

For $k=1$ $A_2(\frac{1}{2},3)=0$; so it does not give any restriction on r. For large k $A_{2k}(\frac{1}{2},3)\approx -A_{2k}(0,3)$ by theorem 1.5; this gives in the limit the restriction $2r\leq\mu-2r$, that is, $r\leq\frac{\mu}{4}$, and not $r\leq\frac{\mu}{6}$.

5. QUASIHOMOGENEOUS SINGULARITIES

By [SK1], any quasihomogeneous singularity is right equivalent to a quasihomogeneous singularity $f(x_0,...,x_n)$ which has unique (up to ordering) normalized weights $w_0,...,w_n\in\mathbb{Q}\cap]0,\frac{1}{2}]$ such that f has weighted degree 1. We will restrict to such a singularity, and we will always use these weights.

The starting point of the formulas in this chapter is the following well known generating function of the spectrum $\alpha_1,...,\alpha_\mu$ of a quasihomogeneous singularity:

$$\sum_{j=1}^{\mu}T^{\alpha_j-\frac{n-1}{2}}=\prod_{i=0}^{n}\frac{T^{w_i-\frac{1}{2}}-T^{\frac{1}{2}}}{1-T^{w_i}}. \qquad (5.1)$$

Because of (1.18), $V^{sing}(f)$ is given by the following formula, interpreted as a formal power series in t.

$$V^{sing}(f)=\prod_{i=0}^{n}\frac{e^{(w_i-\frac{1}{2})t}-e^{\frac{1}{2}t}}{1-e^{w_it}}. \qquad (5.2)$$

The proofs of theorem 5.1 to theorem 5.4 will be given after theorem 5.4. The Bernoulli numbers B_{2k} satisfy $B_{2k}\in(-1)^{k-1}\mathbb{Q}_{>0}$ for $k\geq 1$ and $B_0=1$ (theorem 3.1).

Theorem 5.1. *Let $f(x_0,...,x_n)$ be a quasihomogeneous singularity with normalized weights $w_0,...,w_n$. Then*

$$V^{sing}(f)=\prod_{i=0}^{n}\left[\sum_{k=0}^{\infty}\left(w_i^{2k}\frac{2}{2k+1}B_{2k+1}(\frac{1}{2w_i})\right)\frac{1}{(2k)!}t^{2k}\right], \qquad (5.3)$$

$$\Gamma^{Ber}(V^{sing}(f),n+1)=\prod_{i=0}^{n}\left[\sum_{k=0}^{\infty}(-B_{2k})(1-w_i^{2k-1})\frac{1}{(2k)!}t^{2k}\right]. \qquad (5.4)$$

(5.4) shows conjecture 1.3 (W) for f, as $(-B_{2k})(1-w_i^{2k-1})$ has the sign $(-1)^k$ for any $k\geq 0$.

The calculation (5.22) of the formula (5.4) will also be useful for the conjecture 1.3 (W) in the case of curve singularities.

Theorem 5.2. *Conjecture 1.3 (S) is true for the hyperbolic singularities* T_{pqr}*. Then* $\alpha_\mu - \alpha_1 = 1$ *and*

$$\begin{aligned}\Gamma^{Ber}(V^{sing}(T_{pqr}),1) &= \sum_{k=0}^{\infty}\frac{1}{(2k)!}t^{2k}\cdot \\ &\quad \left[B_{2k}\cdot\left(-1+\frac{1}{p^{2k-1}}+\frac{1}{q^{2k-1}}+\frac{1}{r^{2k-1}}\right)\right].\end{aligned} \tag{5.5}$$

Proposition 5.3. *Define* $Q(t,w)\in\mathbb{Q}[w][[t^2]]$ *by*

$$Q(t,w)=\frac{w}{1-w}\left(\frac{e^{(w-\frac{1}{2})t}-e^{\frac{1}{2}t}}{1-e^{wt}}\right)\exp((1-2w)\Theta^{Ber}(t)). \tag{5.6}$$

a) Then

$$\begin{aligned}&Q(t,w)\\ =\;&\exp\left(\Theta^{Ber}(wt)-\Theta^{Ber}((1-w)t)+(1-2w)\Theta^{Ber}(t)\right) &(5.7)\\ =\;&\exp\left(\sum_{k=1}^{\infty}\frac{-1}{2k}B_{2k}p_{2k}(w)\frac{1}{(2k)!}t^{2k}\right), &(5.8)\end{aligned}$$

where

$$p_{2k}(w) \;=\; 1-2w+w^{2k}-(1-w)^{2k}. \tag{5.9}$$

b) The first three of the polynomials p_{2k} *are*

$$\begin{aligned}p_2(w) &= 0, &(5.10)\\ p_4(w) &= 4(\frac{1}{2}-w)w(1-w), &(5.11)\\ p_6(w) &= 6(\frac{1}{2}-w)w(1-w)(\frac{4}{3}-(w(1-w)). &(5.12)\end{aligned}$$

For $k\geq 2$*, the polynomial* p_{2k} *has three simple zeros at* $0,\frac{1}{2},1$ *and no other zeros. It is negative for* $w\in]-\infty,0[\cup]\frac{1}{2},1[$ *and positive for* $w\in]0,\frac{1}{2}[\cup]1,+\infty[$.

c) The polynomials $Q_{2k}(w)$ *in* $Q(t,w)=\sum_k Q_{2k}(w)\frac{1}{(2k)!}t^{2k}$ *satisfy*

$$Q_0=1,\quad Q_2=0,\quad Q_4=\frac{1}{30}\cdot\frac{1}{4}p_4,\quad Q_6=-\frac{1}{42}\cdot\frac{1}{6}p_6, \tag{5.13}$$

and for $k\geq 2$

$$(-1)^kQ_{2k}(w)>0 \qquad \text{if } w\in]0,\frac{1}{2}[\cup]1,+\infty[. \tag{5.14}$$

They have simple zeros at $0,\frac{1}{2},1$.

We expect that they also satisfy

$$(-1)^kQ_{2k}(w)<0 \qquad \text{if } w\in]-\infty,0[\cup]\frac{1}{2},1[, \tag{5.15}$$

but we do not have a proof.

Theorem 5.4. *Let* $f(x_0,...,x_n)$ *be a quasihomogeneous singularity with normalized weights* $w_0,...,w_n$*. Then*

$$\Gamma^{Ber}(V^{sing}(f),\alpha_\mu-\alpha_1)=\mu\prod_{i=0}^{n}Q(t,w_i). \tag{5.16}$$

(5.16) and (5.14) show conjecture 1.3 (S) for f. (5.10) – (5.13) show

$$\begin{aligned}
\Gamma_2^{Ber}(V^{sing}(f),\alpha_\mu-\alpha_1) &= 0, && (5.17)\\
\Gamma_4^{Ber}(V^{sing}(f),\alpha_\mu-\alpha_1) &= \frac{1}{30}\mu\sum_{i=0}^{n}(\frac{1}{2}-w_i)w_i(1-w_i), && (5.18)\\
\Gamma_6^{Ber}(V^{sing}(f),\alpha_\mu-\alpha_1) &= \frac{1}{42}\mu\sum_{i=0}^{n}(\frac{1}{2}-w_i)w_i(1-w_i)\cdot\\
&\quad\cdot(w_i(1-w_i)-\frac{4}{3}). && (5.19)
\end{aligned}$$

(5.17) says that in the case of a quasihomogeneous singularity one has equality in conjecture 1.1. The first proof in [He1][He2] used Frobenius manifolds, the second proof in [Di] was elementary and used the formula (5.1). The third proof here in chapter 5 also uses this formula. But it is more general and yields also the other formulas in the theorems 5.1 to 5.4.

$Q_2 = 0$ is responsable for (5.17) and for the simplicity of the formulas (5.18) and (5.19). For $k \geq 4$ one has also products of the $Q_{2l}(t,w_i)$ in the formulas for the Bernoulli moments $\Gamma_{2k}^{Ber}(V^{sing}(f),\alpha_\mu-\alpha_1)$.

Proof of theorem 5.1: One derives from (3.1) – (3.3) the classical generating function for the Bernoulli polynomials $B_k(x) = B_k^{(1)}(x) = A_k(x-\frac{1}{2},1)$:

$$\frac{te^{xt}}{e^t-1} = \sum_{k=0}^{\infty} B_k(x)\frac{1}{k!}t^k. \tag{5.20}$$

The following calculation shows (5.3).

$$\begin{aligned}
&\sum_{k=0}^{\infty}\left(w^{2k}\frac{2}{2k+1}B_{2k+1}(\frac{1}{2w})\right)\frac{1}{(2k)!}t^{2k} && (5.21)\\
=\ &\frac{2}{wt}\sum_{k=0}^{\infty}B_{2k+1}(\frac{1}{2w})\frac{1}{(2k+1)!}(wt)^{2k+1}\\
=\ &\frac{1}{wt}\left(\frac{wte^{\frac{1}{2w}wt}}{e^{wt}-1}-\frac{-wte^{\frac{1}{2w}(-wt)}}{e^{-wt}-1}\right)\\
=\ &\frac{e^{\frac{1}{2}t}}{e^{wt}-1}+\frac{e^{-\frac{1}{2}t}}{e^{-tw}-1}=\frac{-e^{\frac{1}{2}t}+e^{(w-\frac{1}{2})t}}{1-e^{wt}}.
\end{aligned}$$

The coefficient of $\frac{1}{(2k)!}t^{2k}$ in the first line of (5.21) is not a polynomial in w, but has a pole of order 1 at $w = 0$. The calculation (5.22) shows that the multiplication by $\exp(\frac{1}{2}\Theta^{Ber}(t))$ cancels these poles for $k \geq 1$. The coefficients $\Theta_{2k}^{Ber} = -\frac{1}{2k}B_{2k}$ are inductively determined by this property. This explains the characterisation of the Bernoulli moments in corollary 2.3.

Formula (5.4) is a consequence of (5.2) and the following calculation, which uses at the end (3.8) and $B_1 = -\frac{1}{2}$.

$$\begin{aligned}
&\frac{e^{(w-\frac{1}{2})t}-e^{\frac{1}{2}t}}{1-e^{wt}}\cdot\exp(\Theta^{Ber}(t)) && (5.22)\\
=\ &\frac{e^{(w-\frac{1}{2})t}-e^{\frac{1}{2}t}}{1-e^{wt}}\cdot\frac{t\cdot e^{\frac{1}{2}t}}{e^t-1}=\frac{te^{wt}-te^{t}}{(1-e^{wt})(e^t-1)}
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{t}{e^t-1}+\frac{t}{e^{wt}-1}\\
&= -\left(\frac{t}{e^t-1}+\frac{1}{2}t\right)+\frac{1}{w}\left(\frac{wt}{e^{wt}-1}+\frac{1}{2}wt\right)\\
&= \sum_{k=0}^{\infty}(-B_{2k})(1-w^{2k-1})\frac{1}{(2k)!}t^{2k}.
\end{aligned}
$$

□

Proof of theorem 5.2: The generating function of the spectrum of the hyperbolic surface singularity T_{pqr} is

$$\sum_{j=1}^{\mu}T^{\alpha_j}=T^0+T^1+\frac{T^{1/p}-T}{1-T^{1/p}}+\frac{T^{1/q}-T}{1-T^{1/q}}+\frac{T^{1/r}-T}{1-T^{1/r}}. \tag{5.23}$$

Because of (1.18)

$$V^{sing}(T_{pqr})=e^{-\frac{1}{2}t}\left(1+e^t+\frac{e^{\frac{1}{p}t}-e^t}{1-e^{\frac{1}{p}t}}+\frac{e^{\frac{1}{q}t}-e^t}{1-e^{\frac{1}{q}t}}+\frac{e^{\frac{1}{r}t}-e^t}{1-e^{\frac{1}{r}t}}\right). \tag{5.24}$$

Then, using (5.22) for $w=\frac{1}{p},\frac{1}{q},\frac{1}{r}$, one finds

$$
\begin{aligned}
&\Gamma^{Ber}(V^{sing}(T_{pqr}),1) \qquad (5.25)\\
&= \left(e^{-\frac{1}{2}t}+e^{\frac{1}{2}t}\right)\frac{te^{\frac{1}{2}t}}{e^t-1}+\left(-\frac{t}{e^t-1}+\frac{t}{e^{\frac{1}{p}t}-1}\right)+\ldots\\
&= \left(2\frac{t}{e^t-1}+t\right)+\left(-\frac{t}{e^t-1}+\frac{t}{e^{\frac{1}{p}t}-1}\right)+\ldots\\
&= \sum_{k=0}^{\infty}B_{2k}\left(-1+\frac{1}{p^{2k-1}}+\frac{1}{q^{2k-1}}+\frac{1}{r^{2k-1}}\right)\frac{1}{(2k)!}t^{2k}.
\end{aligned}
$$

□

Proof of proposition 5.3: a) (5.7) follows from

$$\frac{w}{1-w}\left(\frac{e^{(w-\frac{1}{2})t}-e^{\frac{1}{2}t}}{1-e^{wt}}\right)=\frac{\frac{1}{2}wt}{\frac{1}{2}(1-w)t}\frac{\sinh(\frac{1}{2}(1-w)t)}{\sinh(\frac{1}{2}wt)}. \tag{5.26}$$

and the definition (3.1) of $\Theta(t)$. (5.8) follows with (3.9).

b) For $k\geq 2$ one calculates

$$p_{2k}(0)=p_{2k}(\frac{1}{2})=p_{2k}(1)=0, \tag{5.27}$$

$$p'_{2k}(0)=p'_{2k}(1)=2k-2>0, \tag{5.28}$$

$$p'_{2k}(\frac{1}{2})=-2+k\cdot 2^{3-2k}<0, \tag{5.29}$$

$$p'''_{2k}(w)=2k(2k-1)(2k-2)(w^{2k-3}+(1-w)^{2k-3})>0 \quad . \tag{5.30}$$

Because of (5.30) the simple zeros of p_{2k} at $0,\frac{1}{2},1$ are the only zeros of p_{2k} for $k\geq 2$.

c) (5.13) and (5.14) follow from a) and b). The Q_{2k} have simple zeros at $0, \frac{1}{2}, 1$, because a calculation shows for $k \geq 2$

$$Q'_{2k}(0) = Q'_{2k}(1) \quad = \quad -B_{2k}(1-\frac{1}{k}), \tag{5.31}$$

$$Q'_{2k}(\frac{1}{2}) \quad = \quad B_{2k}(\frac{1}{k}-\frac{1}{2^{2k-2}}). \tag{5.32}$$

□

Proof of theorem 5.4: (5.16) follows from (5.6), (5.2) and

$$\alpha_\mu - \alpha_1 = \sum_{i=0}^{n}(1-2w_i), \quad \mu = \prod_{i=0}^{n}\left(\frac{1}{w_i}-1\right). \tag{5.33}$$

The rest is a consequence of proposition 5.3. □

6. Curve singularities

Theorem 6.1. *Conjecture 1.3 (W) is true for any irreducible curve singularity.*

Proof: Suppose that the Puiseux pairs of the irreducible germ of curve f are $(n_1, r_1), \dots, (n_g, r_g)$. Then with $w_1 = r_1$, and for $k \geq 1$, $w_{k+1} = r_{k+1} - r_k n_{k+1} + n_k n_{k+1} w_k$, the Eisenbud and Neumann diagram is given by figure 1 (see [Ne] for a rapid overview). Furthermore let us introduce $n'_k = n_{k+1} \dots n_g$ for $1 \leq k \leq g-1$ and $n'_g = 1$.

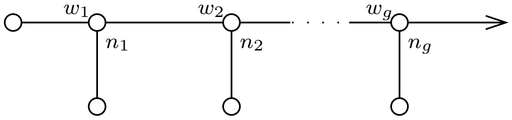

Figure 1. Eisenbud and Neumann diagram of an irreducible germ of a curve

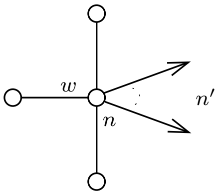

Figure 2. Eisenbud and Neumann diagram of a quasihomogeneous isolated curve singularity

From [Br3], we have a formal decomposition of this diagram in terms of the Newton non-degenerate and commode germs. If we denote by $D(w, n, n')$ the diagram given by figure 2, where w, n are coprime positive integers and n' is a positive integer then the decomposition is

$$D(w_1, n_1, n'_1) + \sum_{k=1}^{g-1}\left(D(w_{k+1}, n_{k+1}, n'_{k+1}) - D(w_k n_k, 1, n'_k)\right). \tag{6.1}$$

This gives

$$Sp(f) = Sp(D(w_1, n_1, n_1')) + \sum_{k=1}^{g-1} \left(Sp(D(w_{k+1}, n_{k+1}, n'_{k+1})) - Sp(D(w_k n_k, 1, n'_k)) \right). \tag{6.2}$$

More precisely, the generating function $\sum_{i=1}^{\mu} T^{\alpha_i + 1}$ is

$$\frac{T^{\frac{1}{n_0'}} - T}{1 - T^{\frac{1}{n_0'}}} \cdot \frac{T^{\frac{1}{w_1 n_1'}} - T}{1 - T^{\frac{1}{w_1 n_1'}}} + \sum_{k=1}^{g-1} \left(\frac{T^{\frac{1}{w_{k+1} n'_{k+1}}} - T}{1 - T^{\frac{1}{w_{k+1} n'_{k+1}}}} - \frac{T^{\frac{1}{w_k n'_{k-1}}} - T}{1 - T^{\frac{1}{w_k n'_{k-1}}}} \right) \frac{T^{\frac{1}{n'_k}} - T}{1 - T^{\frac{1}{n'_k}}}. \tag{6.3}$$

From the quasihomogeneous case (the calculation (5.22) and the formula (5.4)), we know that the first term verifies the conjecture (W). Now to prove the conjecture (W), it is sufficient to prove it for

$$\left(\frac{T^{\frac{1}{w_2}} - T}{1 - T^{\frac{1}{w_2}}} - \frac{T^{\frac{1}{w_1 n_1 n_2}} - T}{1 - T^{\frac{1}{w_1 n_1 n_2}}} \right) \frac{T^{\frac{1}{n_2}} - T}{1 - T^{\frac{1}{n_2}}} \tag{6.4}$$

where n_1, n_2, w_1, w_2 are any positive integers which satisfy $\Delta := w_2 - w_1 n_1 n_2 > 0$. The formula (5.4) of the quasihomogeneous case gives us the term

$$\sum_{i=0}^{k} \binom{2k}{2i} B_{2i} B_{2(k-i)} \left(\frac{1}{(w_1 n_1 n_2)^{2i-1}} - \frac{1}{w_2^{2i-1}} \right) \left(1 - \frac{1}{n_2^{2(k-i)-1}} \right). \tag{6.5}$$

We remark that we can factorise by Δ and $n_2 - 1$ and we get

$$\begin{aligned} &(-B_0) B_{2k} \left(\frac{\sum_{j=0}^{2(k-1)} n_2^j}{n_2^{2k-1}} + \frac{\sum_{j=0}^{2(k-1)} w_2^j (w_1 n_1 n_2)^{2(k-1)-j}}{(w_1 w_2 n_1 n_2)^{2k-1}} \right) \\ &+ \sum_{i=1}^{k-1} \binom{2k}{2i} B_{2i} B_{2(k-i)} \frac{\sum_{j=0}^{2(i-1)} w_2^j (w_1 n_1 n_2)^{2(i-1)-j}}{(w_1 w_2 n_1 n_2)^{2i-1}} \frac{\sum_{j=0}^{2(k-i-1)} n_2^j}{n_2^{2(k-i)-1}}. \end{aligned} \tag{6.6}$$

This permits us to conclude. □

Remark 6.2. For curves we expect in general to get

$$\Gamma^{Ber}(V^{sing}(f), 2) = \Gamma^0 + \sum_{e \in Ed} \Gamma^e \Delta_e, \tag{6.7}$$

where Ed is the set of edges of the Eisenbud and Neumann diagram of f and Δ_e the determinant of the edge e. Because of the local situation, Δ_e is always positive. In [Br3] as well as above and in other cases, we have formulas as (6.7) with Γ^0 of quasihomogeneous type. The difficulty in proving the conjecture 1.3 (W) for other curve singularities lies in the complexity of the coefficients Γ^e.

7. COMPACT COMPLEX MANIFOLDS

The proof of theorem 1.7 will consist of three parts. In the first part (A) we will derive the formula (7.4) for $V^{mfd}(X)$. Motivated by it, we will define and discuss the polynomials $q_{kj}(\nu, y_1, ..., y_j)$

in part (B). In part (C) we will prove theorem 1.7 and the formulas:

$$\Gamma_{2k}^{Ber}(V^{mfd}(X),\nu) = \sum_{j=0}^{\min(2k-1,n)} \int_X q_{kj}(n-\nu,c_1,...,c_j)\cdot c_{n-j} \tag{7.1}$$

if $k\geq 1$,

$$\Gamma_{0}^{Ber}(V^{mfd}(X),\nu) = \int_X c_n \tag{7.2}$$

for any $\nu\in\mathbb{C}$.

After the proof we will make some remarks and finish with three examples.

(A) Let X be a compact complex manifold of dimension n. Let $\alpha_1,...,\alpha_n$ be the Chern roots of the Chern classes $c_1,...,c_n$, that is, $1+c_1+...+c_n=\prod_{j=1}^n(1+\alpha_j)$. The Hirzebruch-Riemann-Roch theorem [Hi][AS] gives

$$\begin{aligned}\chi(\Omega^p) &= \int_X [Td(TM)\cdot ch(\Omega^p)] \\ &= \int_X \left[\left(\prod_{j=1}^n \frac{\alpha_j}{1-e^{-\alpha_j}}\right)\cdot \sum_{j_1<...<j_p} e^{-\alpha_{j_1}-...-\alpha_{j_p}}\right].\end{aligned} \tag{7.3}$$

Here $Td(TM)$ is the Todd class of the tangent bundle TM and $ch(\Omega^p)$ is the exponential Chern character of Ω^p. With (7.3) and (2.29) we can calculate the generating function $V^{mfd}(X)$ of the higher moments $V_{2k}^{mfd}(X)$ which were defined in (1.6).

$$\begin{aligned}&V^{mfd}(X) \\ =\ & \sum_{p=0}^n \chi_p e^{t(p-\frac{n}{2})} = e^{-\frac{n}{2}t}\sum_{p=0}^n \chi(\Omega^p)(-e^t)^p \\ =\ & e^{-\frac{n}{2}t}\int_X \left[\left(\prod_{j=1}^n \frac{\alpha_j}{1-e^{-\alpha_j}}\right)\cdot\prod_{j=1}^n(1-e^te^{-\alpha_j})\right] \\ =\ & \int_X\left[\prod_{j=1}^n\left(\alpha_j\frac{\sinh((\alpha_j-t)/2)}{\sinh(\alpha_j/2)}\right)\right] \\ =\ & \int_X\left[\exp\left(\sum_{j=1}^n\left(\Theta^{Ber}(\alpha_j)-\Theta^{Ber}(\alpha_j-t)\right)\right)\cdot\prod_{j=1}^n(\alpha_j-t)\right].\end{aligned} \tag{7.4}$$

(B) Let $m\in\mathbb{N}_{\geq 1}$ be fixed. We will construct polynomials $a_{k,2k-j}^{(m)}$, $b_{kl}^{(m)}\in\mathbb{Q}[y_1,...,y_m]$ and $c_{kl}^{(m)}$, $d_{kj}^{(m)}\in\mathbb{Q}[\nu,y_1,...,y_m]$. They will all be quasihomogeneous of some degree (the second lower index, $2k-j,l,l,j$) with respect to $y_1,...,y_m$, where $\deg y_j=j$. Those polynomials with weighted degree $\leq m$ will be independent of the choice of m; that means for example in the case of $a_{k,2k-j}^{(m)}$ that $a_{k,2k-j}^{(m)}=a_{k,2k-j}^{(m')}$ for any $m'\geq 2k-j$. At the end we will define $q_{kj}:=d_{2k,j}^{(j)}$.

Let $\sigma_j = \sigma_j(x_1, ..., x_m)$, $j = 1, ..., m$, be the elementary symmetric polynomials in $x_1, ..., x_m$. For $k \geq 1$ and $1 \leq j \leq 2k-1$ a unique polynomial $a^{(m)}_{k,2k-j} \in \mathbb{Q}[y_1, ..., y_m]$ exists such that

$$\sum_{i=1}^{m}(x_i^{2k} - (x_i - t)^{2k} + t^{2k}) = \sum_{j=1}^{2k-1} t^j \cdot a^{(m)}_{k,2k-j}(\sigma_1, ..., \sigma_m). \tag{7.5}$$

It is quasihomogeneous of degree $2k-j$ with respect to $y_1, ..., y_m$. It is independent of m in the sense described above if $2k - j \leq m$.

Because of (3.9), for $k \geq 1$ and $l \geq 1$ unique polynomials $b^{(m)}_{kl} \in \mathbb{Q}[y]$ exist which are quasihomogeneous of weighted degree l with respect to $y_1, ..., y_m$ and which satisfy

$$\begin{aligned} & \exp\left(\sum_{i=1}^{m}\left[\Theta^{Ber}(x_i) - \Theta^{Ber}(x_i - t) + \Theta^{Ber}(t)\right]\right) \\ = \quad & \exp\left(\sum_{k=1}^{\infty}\frac{-1}{2k}B_{2k}\frac{1}{(2k)!}\sum_{j=1}^{2k-1} t^j \cdot a^{(m)}_{k,2k-j}(\sigma_1, ..., \sigma_m)\right) \\ = \quad & 1 + \sum_{k=1}^{\infty} t^k \cdot \sum_{l=1}^{\infty} b^{(m)}_{kl}(\sigma_1, ..., \sigma_m). \end{aligned} \tag{7.6}$$

The polynomials $b^{(m)}_{kl}$ with $l \leq m$ are independent of m.

For $k \in \mathbb{N}$ and $l \in \mathbb{N}$ unique polynomials $c^{(m)}_{kl} \in \mathbb{Q}[\nu, y_1, ..., y_m]$ exist which are quasihomogeneous of degree l with respect to $y_1, ..., y_m$ and which satisfy

$$\begin{aligned} & \exp\left(\sum_{i=1}^{m}\left[\Theta^{Ber}(x_i) - \Theta^{Ber}(x_i - t) + \Theta^{Ber}(t)\right]\right)\exp(-\nu\Theta^{Ber}(t)) \\ = & \sum_{k=0}^{\infty} t^k \cdot \sum_{l=0}^{\infty} c^{(m)}_{kl}(\nu, \sigma_1, ..., \sigma_m). \end{aligned} \tag{7.7}$$

The polynomials $c^{(m)}_{kl}$ with $l \leq m$ are independent of m. Using (3.2) and (7.6) one sees

$$c^{(m)}_{k0}(\nu, y) = \frac{1}{k!}A_k(0, -\nu) \ (= 0 \text{ if } k \text{ is odd}), \tag{7.8}$$

$$c^{(m)}_{0l}(\nu, y) = 0 \quad \text{if } l \geq 1, \tag{7.9}$$

$$c^{(m)}_{kl}(\nu, y) = \sum_{j=0}^{k-1}\frac{1}{j!}A_j(0, -\nu) \cdot b^{(m)}_{k-j,l}(y) \tag{7.10}$$

if $k \geq 1$ and $l \geq 1$. This implies especially

$$\deg_\nu c^{(m)}_{2k,0} = k, \quad \deg_\nu c^{(m)}_{kl} \leq \left[\frac{k-1}{2}\right] \text{ if } l \geq 1. \tag{7.11}$$

We define for $k \geq 1$ and $0 \leq j \leq k-1$

$$d^{(m)}_{kj}(\nu, y) := k!(-1)^j c^{(m)}_{k-j,j}(\nu, y). \tag{7.12}$$

A simple calculation shows that the part of quasihomogeneous degree m with respect to $y_1, ..., y_m$ in

$$\left(\sum_{k=0}^{\infty} t^k \cdot \sum_{l=0}^{\infty} c^{(m)}_{kl}(\nu, y)\right) \cdot \left(\sum_{i=0}^{m} y_{m-i}(-t)^i\right) \tag{7.13}$$

(with $y_0 := 1$) is

$$y_m + \sum_{k=1}^{\infty} \frac{1}{k!} t^k \cdot \sum_{j=0}^{\min(k-1,m)} y_{m-j} d_{kj}^{(m)}(\nu, y). \tag{7.14}$$

Finally, we define for $k \geq 1$ and $1 \leq j \leq 2k-1$ the polynomials $q_{kj}(\nu, y)$ by

$$q_{kj} := d_{2k,j}^{(j)}. \tag{7.15}$$

They are quasihomogeneous of degree j with respect to $y_1, ..., y_m$; the degrees with respect to ν satisfy because of (7.11)

$$\deg_\nu q_{k0} = k, \quad \deg_\nu q_{kj} \leq k - 1 - \left[\frac{j}{2}\right] \text{ if } j \geq 1. \tag{7.16}$$

(C) In (7.4) the last factor is (with $c_0 := 1$)

$$\prod_{j=1}^{n} (\alpha_j - t) = \sum_{i=0}^{n} c_{n-i}(-t)^i. \tag{7.17}$$

$\Gamma^{Ber}(V^{mfd}(X), n-\nu)$ contains only even powers of t. Combining (7.4), (7.7), (7.13) and (7.14), we find

$$\begin{aligned} &\Gamma^{Ber}(V^{mfd}(X), n-\nu) &(7.18)\\ = &\int_X c_n + \sum_{k=1}^{\infty} \frac{1}{(2k)!} t^{2k} \sum_{j=0}^{\min(2k-1,n)} \int_X c_{n-j} q_{kj}(\nu, c_1, ..., c_j). \end{aligned}$$

This shows (7.1), (7.2) and theorem 1.7.

Remarks 7.1. a) The formula (1.33) for $V_2^{mfd}(X)$ was calculated in [LW] and [Bo]. Calculations with some resemblance to those in (A) can be found in [Sal, §3].

b) By (7.2), for any compact complex manifold X $\Gamma_0^{Ber}(V^{mfd}(X), n) = \int_X c_n$. On the other hand, analogously to (1.22) and (1.23), the sequence of numbers

$$(-1)^k \Gamma_{2k}^{Ber}(V^{mfd}(X), n) \cdot \frac{(2\pi)^{2k} \cdot \Gamma(n)}{2 \cdot (2k)! \cdot (2k)^{n-1}} \tag{7.19}$$

tends with $k \to \infty$ to $(-1)^n \sum_p \chi_p = (-1)^n \int_X c_n$. Therefore for odd n and $\int_X c_n \neq 0$ the analogue of the conjectures 1.3 is never satisfied.

c) The example $X = \mathbb{P}^n$ below shows a different rule for the signs if $2k < n$. The example of a K3 surface below shows a behaviour analogous to quasihomogeneous singularities. One could try to classify the compact complex manifolds according to the behaviour of the signs of $(-1)^k \Gamma_{2k}^{Ber}(V^{mfd}(X), n)$.

Examples 7.2. a) $X = \mathbb{P}^n$: We use Nörlunds notation $B_k^{(\nu)}(x)$ of the generalized Bernoulli polynomials, see (3.3), because the generalized Bernoulli numbers $B_k^{(\nu)}(0)$ $(k, \nu \in \mathbb{N})$ will play a role.

$$V^{mfd}(\mathbb{P}^n) = e^{-\frac{n}{2}t} \sum_{p=0}^{n} e^{tp} = e^{-\frac{n}{2}t} \cdot \frac{e^{t(n+1)} - 1}{e^t - 1} \tag{7.20}$$

and

$$\begin{aligned}\Gamma^{Ber}(V^{mfd}(\mathbb{P}^n),n) &= V^{mfd}(\mathbb{P}^n)\cdot\left(\frac{te^{\frac{1}{2}t}}{e^t-1}\right)^n \qquad (7.21)\\ &= \frac{1}{t}\left[\frac{t^{n+1}e^{t(n+1)}}{(e^t-1)^{n+1}}-\frac{t^{n+1}}{(e^t-1)^{n+1}}\right]\\ &= \frac{1}{t}\sum_{k=0}^{\infty}\frac{1}{k!}t^k\left[B_k^{(n+1)}(n+1)-B_k^{(n+1)}(0)\right]\\ &= \sum_{k=0}^{\infty}\frac{1}{(2k)!}t^{2k}\cdot\frac{-2}{2k+1}B_{2k+1}^{(n+1)}(0).\end{aligned}$$

We used the (anti-)symmetry (3.17). Now (3.22) and a special case of (3.16) show (see also [No2, p 148])

$$(x-1)...(x-n)=B_n^{(n+1)}(x)=\sum_{s=0}^{n}\binom{n}{s}x^sB_{n-s}^{(n+1)}(0). \qquad (7.22)$$

Therefore, for $s<n+1$ the sign of the generalized Bernoulli number $B_s^{(n+1)}(0)$ is $(-1)^s$ Thus, $(-1)^k\Gamma_{2k}^{Ber}(V^{mfd}(\mathbb{P}^n),n)=(-1)^k\frac{-2}{2k+1}B_{2k+1}^{(n+1)}(0)$ has for $2k<n$ the sign $(-1)^k$. This behaviour is completely different from that for large k and that in the conjectures 1.3.

b) X a K3 surface:

$$\begin{aligned}V^{mfd}(K3) &= e^{-t}(2+20e^t+2e^{2t}) \qquad (7.23)\\ &= 2\cdot V^{mfd}(\mathbb{P}^2)+18,\end{aligned}$$

$$\begin{aligned}\Gamma^{Ber}(V^{mfd}(K3),2) &= 2\Gamma^{Ber}(V^{mfd}(\mathbb{P}^2),2)+18\cdot\left(\frac{te^{\frac{1}{2}t}}{e^t-1}\right)^2\\ &= \sum_{k=0}^{\infty}\frac{1}{(2k)!}t^{2k}\left[\frac{-4}{2k+1}B_{2k+1}^{(3)}(0)+18\cdot B_{2k}^{(2)}(1)\right]. \qquad (7.24)\end{aligned}$$

One can calculate (7.25) and (7.26) with (3.21), and then (7.27) with (3.20);

$$\begin{aligned}B_k^{(2)}(0) &= (1-k)B_k-kB_{k-1}, \qquad (7.25)\\ B_k^{(3)}(0) &= (2-k)B_k^{(2)}(0)-2kB_{k-1}^{(2)}(0)\\ &= \frac{1}{2}(k-2)(k-1)B_k+\frac{3}{2}(k-2)kB_{k-1} \qquad (7.26)\\ &\quad +(k-1)kB_{k-2},\\ B_k^{(2)}(1) &= B_k^{(2)}(0)+kB_{k-1}=(1-k)B_k. \qquad (7.27)\end{aligned}$$

Therefore

$$\begin{aligned}&(-1)^k\Gamma_{2k}^{Ber}(V^{mfd}(K3),2)\\ =\ &24(2k-1)(-1)^{k-1}B_{2k}+(-1)^{k-1}8kB_{2k-1}\\ =\ &\left\{\begin{array}{rll}0 & \text{if} & k=1,\\ 24(2k-1)(-1)^{k-1}B_{2k}>0 & \text{if} & k\neq 1.\end{array}\right. \qquad (7.28)\end{aligned}$$

This behaviour is similar to that of a quasihomogeneous singularity.

c) X a Riemann surface of genus g:

$$V^{mfd}(X) = (1-g)V^{mfd}(\mathbb{P}^1), \tag{7.29}$$

$$\begin{aligned}\Gamma^{Ber}(V^{mfd}(X),1) &= (1-g)\Gamma^{Ber}(V^{mfd}(\mathbb{P}^1),1) \\ &= (1-g)\sum_{k=0}^{\infty}\frac{1}{(2k)!}t^{2k}\cdot 2B_{2k}.\end{aligned} \tag{7.30}$$

We used (7.25). For $g \geq 2$ $(-1)^k\Gamma^{Ber}_{2k}(V^{mfd}(X),1)$ is positive if $k \geq 1$, but negative if $k = 0$.

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