DUALITY OF SINGULARITIES FOR FLAT SURFACES IN EUCLIDEAN SPACE

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Dedicated to Professor Goo Ishikawa on the occasion of his 60th birthday.

Abstract. In this paper, we shall discuss the duality of singularities for a class of flat surfaces in Euclidean space. After introducing the definition of the conjugate of a tangent developable, we show that, if a tangent developable admits a swallowtail, its conjugate has a cuspidal cross cap. Similarly, we prove that the conjugate of a tangent developable having cuspidal $S^1_1$ singularities has cuspidal butterflies, and that cuspidal beaks have self-duality. We also show that cuspidal edges do not possess such a property, by exhibiting an example of a tangent developable with cuspidal edges whose conjugate has 5/2-cuspidal edges. Finally, we prove that conjugates of complete flat fronts with embedded ends cannot be complete flat fronts.

1. Introduction

We denote Euclidean 3-space by $\mathbb{R}^3$. It is well-known that, for a minimal surface $f = (x_1, x_2, x_3): M \to \mathbb{R}^3$, its coordinate functions $x_j$ ($j = 1, 2, 3$) are harmonic functions on $M$. Then, the harmonic conjugates $x_j^\#$ ($j = 1, 2, 3$) define another minimal surface $f^\# = (x_1^\#, x_2^\#, x_3^\#)$, which is called the conjugate minimal surface. Similarly, for maximal surfaces in the Lorentz-Minkowski 3-space $L^3$, we can define the conjugate. Since the only complete maximal surfaces are spacelike planes [2], we need to consider maximal surfaces with singular points. Umehara–Yamada [23] introduced a class of maximal surfaces with admissible singularities called maxfaces, which satisfy the following property so-called the duality of singularities:

Fact 1.1 ([23, 4]). Let $f : M \to L^3$ be a maxface, $f^\# : M \to L^3$ its conjugate, and $p \in M$ a singular point. Then, $f$ at $p$ is $A$-equivalent to the cuspidal edge (resp. swallowtail, cuspidal cross cap) if and only if $f^\#$ at $p$ is $A$-equivalent to the cuspidal edge (resp. cuspidal cross cap, swallowtail).

The property as in Fact 1.1 is called the duality of singularities. Let $S^3_1$ (resp. $S^3$) be the de Sitter 3-space (resp. the 3-sphere) of constant sectional curvature 1. Also, let $H^3_1$ (resp. $H^3$) be the anti-de Sitter 3-space (resp. the hyperbolic 3-space) of constant sectional curvature $-1$, and $Q^3$ be the 3-lightcone. It is known that such a duality of singularities holds for various classes of surfaces as follows:

- timelike minimal surfaces (so-called minfaces) in $L^3$ [21] (cf. [1]),
- spacelike surfaces of non-zero constant mean curvature in $L^3$ [7],
- spacelike surfaces of constant mean curvature 1 in $S^3_1$ [4],
- timelike surfaces of constant mean curvature 1 in $H^3_1$ [24],
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• spacelike surfaces of zero extrinsic curvature in $S^2_1$, $H^3$ and $Q^3$ [13],
• surfaces of zero extrinsic curvature in $S^3$ [12].

We remark that such a duality is known for more degenerate singularities, such as cuspidal beaks, cuspidal butterflies and cuspidal $S^3_1$ singularities ([12, 18]).

In this article, we shall study the duality of singularities in the case of flat surfaces with singularities in $\mathbb{R}^3$. Murata–Umehara [17] investigated the global properties of flat surfaces with singularities called flat fronts (cf. Fact 2.1). In particular, they proved that complete flat fronts with non-empty singular sets must be tangent developables. Ishikawa [10] investigated the singularities of tangent developables from the view point of the (real) projective geometry. In particular, Ishikawa [10] used the Scherbak’s dual curves [20] in the dual projective space to define the dual tangent developables, and proved the duality of singularities. For more details, see [10, 11] (cf. [3]). However, to the best of the author’s knowledge, there was no notion like the conjugate of flat surfaces $\mathbb{R}^3$ in the setting of Euclidean geometry. Thus, we shall find a suitable definition of the conjugate of flat fronts which satisfy the duality of singularities.

This paper is organized as follows. In Section 2, we review some basic facts on flat fronts, singularities of frontals in $\mathbb{R}^3$, and frontals in the 2-sphere $S^2$. Then, in Section 3, after reviewing a-orientable admissible developable frontals introduced by Murata–Umehara [17], we apply the criteria for cuspidal cross caps to such developable frontals. Comparing the condition for swallowtails and that for cuspidal cross caps, we give a definition of the conjugates for tangent developables (Definition 3.6, cf. Corollary 3.9). In Section 4, applying the criteria for other singularities (cuspidal beaks, cuspidal butterfly, cuspidal $S^3_1$, 5/2-cuspidal edge) to such tangent developables (cf. Propositions 4.2, 4.4, 4.6 and 4.9), we obtain the duality of singularities (Theorem 4.10). In the case of the cuspidal edge, we exhibit an example which does not satisfy the desired duality (see Example 4.11). Finally, in Section 5, we glance a global property of such conjugate operation, by proving that the conjugate of a complete flat front with embedded ends cannot be a complete flat front (Proposition 5.1).

2. Preliminaries

We denote by $\mathbb{R}^3$ the Euclidean 3-space. Let $M$ be a connected smooth 2-manifold and
$$f : M \rightarrow \mathbb{R}^3$$
a smooth map. A point $p \in M$ is called a singular point if $f$ is not an immersion at $p$. Otherwise, we say $p$ a regular point. Denote by $S(f) (\subset M)$ the singular set. If $S(f)$ is empty, we call $f$ a (regular) surface. In this case, at least locally, we can take a smooth unit normal vector field $\nu$ along $f$, that is, for every point $p \in M$, there exist an open neighborhood $U$ of $p$ and a smooth map $\nu : U \rightarrow S^2$ such that
$$df_q(v) \cdot \nu(q) = 0 \quad \text{holds for each } q \in U \text{ and } v \in T_qM,$$
where the dot ‘.’ is the canonical inner product on $\mathbb{R}^3$ and $S^2$ is the unit sphere
$$S^2 := \{x \in \mathbb{R}^3; x \cdot x = 1\}.$$

2.1. Flat fronts. A smooth map $f : M \rightarrow \mathbb{R}^3$ is called a frontal if, for each point $p \in M$, there exist a neighborhood $U$ of $p$ and a smooth map $\nu : U \rightarrow S^2$ which satisfies (2.1). Such a $\nu$ is called the unit normal vector field of $f$. If $\nu$ can be defined throughout $M$, $f$ is called co-orientable. If $(L :=) (f, \nu) : U \rightarrow \mathbb{R}^3 \times S^2$ gives an immersion, $f$ is called a wave front (or a front, for short).

A front $f$ with a unit normal $\nu$ is called flat if $\text{rank}(df) \leq 1$ on $M$. Denote by $ds^2 := df \cdot df$ the first fundamental form of $f$. In the case that $f$ is regular, $f$ is flat as a front if and only if
function $S$ (cf. Fact 4.3); [19] for cuspidal
for cuspidal cross cap (cf. Fact 3.4); [14] for cuspidal beaks (cf. Fact 4.1); [13] for cuspidal butterfly
beaks, cuspidal butterfly, cuspidal
respectively, where
(2.3)

Fact 2.1 ([17]). Let $\xi : S^1 \to S^2$ be a regular curve without inflection points, and $\alpha = \alpha(t)dt$ a
1-form on $S^1 = \mathbb{R}/2\pi \mathbb{Z}$ such that $\int_{S^1} \xi \alpha = 0$ holds. Then, $f : S^1 \times \mathbb{R} \to \mathbb{R}^3$ defined by
(2.2)

$f(t,v) := \sigma(t) + v\xi(t)$ \quad $\left(\sigma(t) := \int_0^t \alpha(\tau) \xi(\tau)d\tau\right)$

is a complete flat front with non-empty singular set. Conversely, let $f : M \to \mathbb{R}^3$ be a complete
flat front defined on a connected smooth 2-manifold $M$. If the singular set $S(f)$ of $f$ is not empty, then $f$ is umbilic-free, co-orientable, $M$ is diffeomorphic to $S^1 \times \mathbb{R}$, and $f$ is given by (2.2). Moreover, if the ends of $f$ are embedded, $f$ has at least four singular points other than cuspidal edges.

For the definition of umbilic points, see [17] (cf. [5, 6, 8]). The final statement of Fact 2.1 may
be regarded as a variant of four vertex theorem for plane curves.

2.2. Singularities of frontals. Fix a smooth 2-manifold $M$ and take two points $p_i \in M$
($i = 1,2$). Let $f_i : (M,p_i) \to (\mathbb{R}^3, f(p_i))$ ($i = 1,2$) be two map germs. We say $f_1$ is $A$-equivalent
to $f_2$ if there exist diffeomorphism germs

$\varphi : (M,p_1) \to (M,p_2)$ \quad and \quad $\Phi : (\mathbb{R}^3, f(p_1)) \to (\mathbb{R}^3, f(p_2))$

such that $\Phi \circ f_1 \circ \varphi^{-1} = f_2$. We set $f_{CE}$, $f_{SW}$, $f_{CCR}$, $f_{CBK}$, $f_{CBF}$, $f_{CS^k\pm}$, $f_{rCE}$ to be the germs
from $(\mathbb{R}^2,0)$ to $(\mathbb{R}^3,0)$ given by:

\begin{align}
 f_{CE}(u,v) &:= (u,v^2,v^3), \\
 f_{SW}(u,v) &:= (4u^3 + 2uv, 3u^4 + u^2v, -v), \\
 f_{CCR}(u,v) &:= (u,v^2,uv^3), \\
 f_{CBK}(u,v) &:= (v, -2u^3 + uv^2, -3u^4 + u^2v^2), \\
 f_{CBF}(u,v) &:= (u, 5v^4 + 2uv, 4v^5 + uv^2 - u^2), \\
 f_{CS^k\pm}(u,v) &:= (u,v^2,v^3(u^{k+1} \pm v^2)), \\
 f_{rCE}(u,v) &:= (u,v^2,v^5),
\end{align}

respectively, where $k$ is a positive integer. We call the map germ $f_{CE}$ (resp. $f_{SW}$, $f_{CCR}$,
$f_{CBK}$, $f_{CBF}$, $f_{CS^k\pm}$, $f_{rCE}$) the cuspidal edge (resp. swallowtail, cuspidal cross cap, cuspidal
beaks, cuspidal butterfly, cuspidal $S_k^\pm$ singularity, 5/2-cuspidal edge).

Kokubu–Rossman–Saji–Umehara–Yamada [15] gave a useful criteria for cuspidal edge and
swallowtail. Similar useful criteria for other singularities are given in the following: [4] for
cuspidal cross cap (cf. Fact 3.4); [14] for cuspidal beaks (cf. Fact 4.1); [13] for cuspidal butterfly
(cf. Fact 4.3); [19] for cuspidal $S_k^\pm$ singularity (cf. Fact 4.5); and [9] for 5/2-cuspidal edge (cf.
Fact 4.8). To state such criteria, we shall review some basic notions for frontals.

Let $f : M \to \mathbb{R}^3$ be a frontal with the (locally defined) unit normal $\nu$. Take a point $p \in M$. Let
$(U; u,v)$ be a coordinate neighborhood of $p$. We call $\lambda := \det(f_u, f_v, \nu)$ the signed area density
function. Remark that $p$ is a singular point of $f$ if and only if $\lambda(p) = 0$. If $d\lambda(p) \neq 0$, a singular
point \( p \) is called non-degenerate. We remark that if \( p \) is non-degenerate, then \( \operatorname{rank}(df)_p = 1 \) holds. By the implicit function theorem, there exists a regular curve \( \gamma(t) \) \((|t| < \varepsilon)\) on the \( uv\)-plane such that \( \gamma(0) = p \) and the image of \( \gamma \) coincides with the singular point set \( S(f) \) near \( p \), where \( \varepsilon > 0 \). We call \( \gamma(t) \) the singular curve and \( \gamma' = d\gamma/dt \) the singular direction. Then, there exists a non-zero smooth vector field \( \zeta(t) \) along \( \gamma(t) \) such that \( \zeta(t) \) is a null vector (i.e., \( df(\zeta(t)) = 0 \)) for each \( t \). Such a vector field \( \zeta(t) \) is called a null vector field. On the other hand, a non-vanishing smooth vector field \( \zeta = \zeta(u,v) \) on \( U \) so that \( \zeta|_{S(f)} \) gives a kernel direction of \( f \) is also called a null vector field. We set the functions \( \delta(t) \) and \( \psi_{cerr}(t) \) as

\[
\delta(t) := \det (\gamma'(t), \zeta(t)), \quad \psi_{cerr}(t) := \det ((f \circ \gamma)'(t), (\nu \circ \gamma)(t), d\nu(\zeta(t))),
\]

respectively. Later we use these functions in the criteria for various singularity types (cf. Facts 3.4, 4.1, 4.3, 4.5 and 4.8).

2.3. **Frontals in 2-sphere.** Let \( J \) be an open interval of \( \mathbb{R} \). A smooth map \( \xi : J \rightarrow S^2 \) is called a frontal if there exists a smooth unit vector field \( n \) along \( \xi \) such that \( \xi' \cdot n = 0 \) holds. We call \( n \) the unit normal vector field or the spherical dual. The pair \((\xi, n)\) gives a Legendre curve in the unit tangent bundle

\[
T_1S^2 = \{(p,v) \in S^2 \times S^2; p \cdot v = 0\}
\]
with respect to the canonical contact structure. Since $\xi \cdot n' = 0$, there exist smooth 1-forms $\rho$, $\omega$ such that
\begin{equation}
(2.5) \quad d\xi = \rho \eta, \quad dn = -\omega \eta,
\end{equation}
where we set $\eta := n \times \xi$. Then, the frame $F(t) := \{\xi(t), \eta(t), n(t)\}$ satisfies
\begin{equation}
(2.6) \quad F^{-1}dF = \begin{pmatrix}
0 & -\rho & 0 \\
\rho & 0 & -\omega \\
0 & \omega & 0
\end{pmatrix},
\end{equation}
where we used the identity $d\eta = -\rho \xi + \omega n$. Conversely, the following holds.

**Fact 2.2** ([22, Theorem 2.5]). Let $\rho$, $\omega$ be smooth 1-forms on an interval $J$. Then, there exists a frontal $\xi : J \to S^2$ with the spherical dual $n$ such that (2.5) holds.

Therefore, we may conclude that there exists a one-to-one correspondence between frontals with spherical duals and pairs of smooth 1-forms. We call the pair of 1-forms $(\rho, \omega)$ the data of the frontal $\xi : J \to S^2$.

3. **Conjugates of tangent developables**

In this section, comparing the criteria for swallowtail and cuspidal cross cap, we give a definition of the conjugates of developable frontals.

3.1. **Developable frontals.** Let $J$ be an open interval including $0 \in J$. Take 1-forms $\alpha$, $\beta$ on $J$ and a frontal $\xi : J \to S^2$ with the spherical dual $n$. Then a smooth map $f : J \times R \to R^3$ defined by
\begin{equation}
(3.1) \quad f(t,v) := \sigma(t) + v \xi(t) \quad \sigma(t) := \int_0^t (\alpha \xi + \beta \eta), \quad \eta := n \times \xi
\end{equation}
is a co-orientable frontal in $R^3$ so that $\nu(t,v) := n(t)$ is a unit normal. We shall call $f(t,v)$ an a-orientable admissible developable frontal and $v$ is called the asymptotic parameter. The quadruple of the 1-forms $(\alpha, \beta, \rho, \omega)$ is independent of the choice of the parameter $t$ on $J$ as a 1-dimensional manifold, which we call the data of $f(t,v)$. Here, $(\rho, \omega)$ is the data corresponding to a frontal in $S^2$ with the spherical dual $n$ (cf. (2.5)).

**Remark 3.1.** We remark that Murata–Umehara defined a-orientable admissible developable frontals in [17, Definition 2.3], where ‘a-orientable’ means ‘asymptotically orientable’, see [17, page 289]. They gave a representation formula in [17, Theorem 2.8]. Our definition is based on [17, Theorem 2.8].

If $f$ is a cylinder, then $\xi : J \to S^2$ is a constant map, that is, $r(t) = 0$ holds for all $t \in J$, where $\rho = r(t)dt$. We call a point $p_0 = (t_0, v_0)$ a cylindrical point of $f(t,v)$ if $\xi'(t_0) = 0$ (i.e., $r(t_0) = 0$) holds. We denote by $S_c(f)$ (resp. $S_{nc}(f)$) the set of cylindrical singular points (resp. non-cylindrical singular points).

**Lemma 3.2** (cf. [17, Proposition 2.16]). Let $f(t,v)$ be an a-orientable admissible developable frontal whose data is given by $(\alpha, \beta, \rho, \omega) = (a(t)dt, b(t)dt, r(t)dt, w(t)dt)$. Then, a point $p_0 = (t_0, v_0) \in J \times R$ is a singular point of $f$ if and only if $b(t_0) + v_0 r(t_0) = 0$. Moreover,

- $f$ is a front at a singular point $p_0 = (t_0, v_0)$ if and only if $w(t_0) \neq 0$.  

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1Cylindrical singular points are linear singular points in the sense of [17, Definition 2.15].
• $p_0 = (t_0, v_0)$ is a cylindrical singular point of $f$ if and only if $b(t_0) = r(t_0) = 0$. Such a $p_0 \in S_c(f)$ is non-degenerate if and only if $b'(t_0) + v_0 r'(t_0) \neq 0$. Setting
\begin{equation}
\gamma_c(v) := (t_0, v), \quad \zeta_c(v) := \partial_t - a(t_0) \partial_v,
\end{equation}
we have that $\gamma_c(v)$ is a singular curve passing through $\gamma_c(v_0) = p_0$, and $\zeta_c(v)$ is a null vector field along $\gamma_c(v)$. Moreover, we have (cf. (2.4))
\begin{equation}
\delta_c(v) := \det(\gamma'_c(v), \zeta_c(v)) = -1.
\end{equation}

• $p_0 = (t_0, v_0)$ is a non-cylindrical singular point of $f$ if and only if $r(t_0) \neq 0$ and $v_0 = -b(t_0)/r(t_0)$. Such a $p_0 \in S_{nc}(f)$ is non-degenerate, and setting
\begin{equation}
\gamma_{nc}(t) := \left(t, -\frac{b(t)}{r(t)}\right), \quad \zeta_{nc}(t) := \partial_t - a(t) \partial_v,
\end{equation}
we have that $\gamma_{nc}(t)$ is a singular curve passing through $\gamma_{nc}(t_0) = p_0$, and $\zeta_{nc}(t)$ is a null vector field along $\gamma_{nc}(t)$. Moreover, we have (cf. (2.4))
\begin{equation}
\delta_{nc}(t) := \det(\gamma'_{nc}(t), \zeta_{nc}(t)) = -a(t) + \left(\frac{b(t)}{r(t)}\right)'.
\end{equation}

Proof. By (2.5), we have
\begin{equation}
f_t = a(t) \xi(t) + (b(t) + v r(t)) \eta(t), \quad f_v = \xi(t).
\end{equation}
So, the signed area density function $\lambda$ is given by
\begin{equation}
\lambda = \det(f_t, f_v, \nu) = (b(t) + v r(t)) \det(\eta(t), \xi(t), \nu(t)) = -b(t) - v r(t).
\end{equation}
Thus, we have $S(f) = \{(t, v) \in J \times \mathbb{R} ; b(t) + v r(t) = 0\}$ and
\begin{equation}
-\lambda_v = b'(t) + v r'(t), \quad -\lambda_v = r(t).
\end{equation}
On the singular set $S(f)$, $f_t - a(t) f_v = 0$ holds. Thus, setting $\zeta(t, v) := \partial_t - a(t) \partial_v$, we have $df(\zeta) = 0$ at a singular point $p_0$. Since $f$ is front at $p_0 \in S(f)$ if and only if
\begin{equation}
(dL)_{p_0} = ((df)_{p_0}, (dv)_{p_0})
\end{equation}
is injective, this condition is equivalent to $(dv)_{p_0}(\zeta) \neq 0$. Since $-dv(\zeta) = -\nu^t = w \eta$, $f$ is front at $p_0 \in S(f)$ if and only if $w(t_0) \neq 0$.

If $p_0$ is cylindrical, $r(t_0) = 0$ holds. Thus, $p_0$ is a cylindrical singular point if and only if $r(t_0) = 0$ and $b(t_0) = b(t_0) + v_0 r(t_0) = 0$. By (3.8), $p_0$ is non-degenerate if and only if $b'(t_0) + v_0 r'(t_0) \neq 0$. In this case, $\gamma_c(v)$ given in (3.2) is a singular curve passing through $\gamma_c(v_0) = p_0$. By (3.6), $f_t - a(t) f_v = 0$ holds along $\gamma_c(v)$, and hence we have $\zeta_c(v)$ given in (3.2) is a null vector field along $\gamma_c(v)$.

If $p_0$ is a non-cylindrical singular point, $r(t_0) \neq 0$ holds. By (3.8), $p_0$ must be non-degenerate. Then $\gamma_{nc}(t)$ given in (3.4) is a singular curve passing through $\gamma_{nc}(t_0) = p_0$. By (3.6),
\begin{equation}
f_t - a(t) f_v = 0
\end{equation}
holds along $\gamma_{nc}(t)$, and hence we have $\zeta_{nc}(t)$ given in (3.4) is a null vector field along $\gamma_{nc}(t)$.

As we seen in Lemma 3.2, the cylindrical and non-cylindrical singular sets, $S_c(f)$ and $S_{nc}(f)$, are written as
\begin{equation}
S_c(f) = \{(t, v) \in J \times \mathbb{R} ; b(t) = r(t) = 0\},
\end{equation}
\begin{equation}
S_{nc}(f) = \left\{(t, v) \in J \times \mathbb{R} ; r(t) \neq 0, \ v = \frac{b(t)}{r(t)} \right\},
\end{equation}
respectively.
Murata–Umehara [17] applied the criteria for cuspidal edge and swallowtail given in [15] to developable frontals as follows:

**Fact 3.3** ([17, Proposition 2.16]). Let \( f(t, v) \) be an a-orientable admissible developable frontal whose data is given by \( (\alpha, \beta, \rho, \omega) = (a(t) dt, b(t) dt, r(t) dt, w(t) dt) \). Then, a point

\[
p_0 = (t_0, v_0) \in J \times \mathbb{R}
\]

is a singular point of \( f \) if and only if \( b(t_0) + v_0 r(t_0) = 0 \). Moreover, for a singular point \( p_0 = (t_0, v_0) \) of \( f \), we have that

- \( f \) at \( p_0 \) is \( \mathcal{A} \)-equivalent to the cuspidal edge if and only if

\[
r(t_0) \neq 0, \quad a(t_0) \neq \left( \frac{b(t)}{r(t)} \right)' \bigg|_{t=t_0}, \quad w(t_0) \neq 0,
\]

or

\[
r(t_0) = 0, \quad b'(t_0) + v_0 r'(t_0) \neq 0, \quad w(t_0) \neq 0.
\]

- \( f \) at \( p_0 \) is \( \mathcal{A} \)-equivalent to the swallowtail if and only if

\[
r(t_0) \neq 0, \quad a(t_0) = \left( \frac{b(t)}{r(t)} \right)' \bigg|_{t=t_0}, \quad a'(t_0) \neq \left( \frac{b(t)}{r(t)} \right)'' \bigg|_{t=t_0}, \quad w(t_0) \neq 0.
\]

We can observe that swallowtails never appear on the cylindrical singular set \( S_c(f) \).

### 3.2. Cuspidal cross cap

Here we review the criterion for the cuspidal cross cap given by Fujimori–Saji–Umehara–Yamada [4].

**Fact 3.4** (Criterion for cuspidal cross cap [4]). Let \( f : U \to \mathbb{R}^3 \) be a frontal defined on a domain \( U \) of \( \mathbb{R}^2 \), with the unit normal \( \nu \), and \( p \in U \) a non-degenerate singular point of \( f \). And let \( \gamma(t) \) be a singular curve such that \( \gamma(0) = p \), \( \xi(t) \) a null vector field, \( \delta(t) \) and \( \psi_{ccr}(t) \) be the functions defined by (2.4). Then, the map germ \( f \) at \( p \) is \( \mathcal{A} \)-equivalent to the cuspidal cross cap if and only if \( \delta(0) \neq 0 \), \( \psi_{ccr}(0) = 0 \) and \( \psi'_{ccr}(0) \neq 0 \).

Now, we shall apply Fact 3.4 to a-orientable admissible developable frontals.

**Proposition 3.5.** Let \( f(t, v) \) be an a-orientable admissible developable frontal whose data is given by \( (\alpha, \beta, \rho, \omega) = (a(t) dt, b(t) dt, r(t) dt, w(t) dt) \). For a singular point \( p_0 = (t_0, v_0) \) of \( f \), we have that \( f \) at \( p_0 \) is \( \mathcal{A} \)-equivalent to the cuspidal cross cap if and only if

\[
r(t_0) \neq 0, \quad a(t_0) \neq \left( \frac{b(t)}{r(t)} \right)' \bigg|_{t=t_0}, \quad w(t_0) = 0, \quad w'(t_0) \neq 0.
\]

**Proof.** First, assume that \( p_0 \in S_{nc}(f) \). By Lemma 3.2, \( \gamma_{nc}(t) \) is a singular curve passing through \( \gamma_{nc}(t_0) = p_0 \), and \( \zeta_{nc}(t) \) is a null vector field along \( \gamma_{nc}(t) \), where \( \gamma_{nc}(t) \) and \( \zeta_{nc}(t) \) are given by (3.4). Let \( \delta_{nc}(t) \) be the function given in (3.5). By Lemma 3.2, the function \( \delta \) given in (2.4) coincides with \( \delta_{nc}(t) \). On the other hand, setting \( \gamma_{nc}(t) := f(\gamma_{nc}(t)) \), we have

\[
\gamma'_{nc}(t) = -\delta_{nc}(t) \xi(t).
\]

Hence, the function \( \psi_{ccr} \) given in (2.4) is

\[
\psi_{ccr}(t) = -\delta_{nc}(t) \det (\xi(t), n(t), n'(t)) = -\delta_{nc}(t) w(t).
\]

Therefore, \( f \) at \( p_0 \) is \( \mathcal{A} \)-equivalent to cuspidal cross cap if and only if (3.12) holds.

Next, we shall prove that, if \( p_0 \in S_c(f) \), \( f \) at \( p_0 \) cannot be \( \mathcal{A} \)-equivalent to the cuspidal cross cap. By Lemma 3.2, \( \gamma_c(v) \) is a singular curve passing through \( \gamma_c(v_0) = p_0 \), and \( \zeta_c(v) \) is a null vector field along \( \gamma_c(v) \), where \( \gamma_c(v) \) and \( \zeta_c(v) \) are given by (3.2). By Lemma 3.2, the
function $\delta$ given in (2.4) is identically $-1$. On the other hand, setting $\tilde{\gamma}_c(v) := f(\gamma_c(v))$, we have $\tilde{\gamma}_c'(v) = \xi(t_0)$. Hence, the function $\psi_{ccr}$ given in (2.4) is
\begin{equation}
\psi_{ccr}(v) = \det (\xi(t), n(t_0), w(t_0)\eta(t_0)) = w(t_0).
\end{equation}
Therefore, $\psi_{ccr}(v_0) = 0$ and $\psi'_{ccr}(v_0) \neq 0$ do not occur at the same time. Thus, $f$ at $p_0$ cannot be $A$-equivalent to the cuspidal cross cap.

3.3. Observation and definition. For an $a$-orientable admissible developable frontal $f = f(t, v)$, we would like to find its conjugate $f^\sharp$ which satisfies the so-called duality of singularities as in Fact 1.1 in the introduction.

We shall compare the condition (3.11) for swallowtail and that (3.12) for cuspidal cross cap.

If $\beta = b(t) dt$ is identically zero, (3.11) is equivalent to
\begin{equation}
r(t_0) \neq 0, \quad a(t_0) = 0, \quad a'(t_0) \neq 0, \quad w(t_0) \neq 0,
\end{equation}
and (3.12) is equivalent to
\begin{equation}
r(t_0) \neq 0, \quad a(t_0) \neq 0, \quad w(t_0) = 0, \quad w'(t_0) \neq 0.
\end{equation}
Thus, for an $a$-orientable admissible developable frontal $f = f(t, v)$ with the data $(\alpha, 0, \rho, \omega)$, if we set $f^\sharp$ to be the $a$-orientable admissible developable frontal whose data is given by
\begin{equation}
(\alpha^\sharp, 0, \rho^\sharp, \omega^\sharp) := (\omega, 0, \rho, \alpha),
\end{equation}
we have that $f$ at $p$ is $A$-equivalent to the swallowtail if and only if $f^\sharp$ at $p$ is $A$-equivalent to the cuspidal cross cap. Namely, $f$ and $f^\sharp$ satisfy the duality of singularities.

$A$-orientable admissible developable frontals with $\beta = 0$ are tangent developables. In fact, when $\beta = 0$, $f$ given in (3.1) is written as
\begin{equation}
f(t, v) := \sigma(t) + v \xi(t) \quad \left( \sigma(t) := \int_0^t \alpha \xi, \quad \eta := n \times \xi \right).
\end{equation}
Since $\sigma'(t)$ and $\xi(t)$ are linearly dependent, we may conclude that $f$ is a tangent developable.

Definition 3.6 (A-tangent developable). We call an $a$-orientable admissible developable frontal with $\beta = 0$ an $a$-tangent developable. For an $a$-tangent developable $f$, the triplet of the 1-forms $(\alpha, \rho, \omega)$ is also called the data. Then, the $a$-tangent developable $f^\sharp$ whose data is given by $(\alpha^\sharp, \rho^\sharp, \omega^\sharp) := (\omega, \rho, \alpha)$ is called the conjugate of $f$.

We remark that, by Lemma 3.2 and $\rho = \rho^\sharp$, the singular set of an $a$-tangent developable $f$ coincides with that of the conjugate $f^\sharp$ of $f$, namely $S(f) = S(f^\sharp) = \{(t, v) \in J \times R; \nu r(t) = 0\}$ holds. In the case that the $a$-tangent developable $f = f(t, v)$ is defined on $M := S^1 \times R$, the domain of the conjugate $f^\sharp$ is the universal covering $\tilde{M} = R^2$ of $M$.

Remark 3.7. An $a$-orientable admissible developable frontal without cylindrical points is an a-tangent developable. If $(t, v)$ is non-cylindrical, by changing the parameter $v \mapsto v - b(t)/r(t)$, $f$ can be written as
\begin{equation}
f = \sigma(t) + \left( v - \frac{b(t)}{r(t)} \right) \xi(t) = \tilde{\sigma}(t) + v \xi(t).
\end{equation}
Here we set $\tilde{\sigma}(t) := \sigma(t) - (b(t)/r(t))\xi(t)$, which satisfies that $\tilde{\sigma}'(t)$ and $\xi(t)$ are linearly dependent.

Let $\{e_1, e_2, e_3\}$ be the canonical orthonormal basis of $R^3$, namely, $(e_1, e_2, e_3) = Id$, where $Id$ is the identity matrix $Id := \text{diag}(1, 1, 1)$. The procedure of constructing the $a$-tangent developable from a given data $(\alpha, \rho, \omega)$ is as follows:

- Take $F_0 \in SO(3)$ arbitrarily.
Let \( F = F(t) \) be a solution of (2.6) with the initial value \( F(t_0) = F_0 \).

Setting \( \xi(t) := F(t)e_1 \), then,
\[
  f(t,v) = \sigma(t) + v\xi(t) \quad \left( \sigma(t) := \int_{t_0}^{t} \alpha t \xi \right)
\]
is an a-tangent developable whose data is given by \((\alpha,\rho,\omega)\) such that \( n(t) := F(t)e_3 \) is a unit normal.

Taking account of the data of the conjugate \((\alpha^2,\rho^2,\omega^2) = (\omega,\rho,\alpha)\), we have the following.

**Lemma 3.8.** Let \( f = f(t,v) \) be the a-tangent developable defined on \( J \times R \) whose data is given by \((\alpha,\rho,\omega)\). Fix \( t_0 \in J \). Take a solution \( F^i = F^i(t) \) of the following initial value problem
\[
(F^i)^{-1}dF^i = \begin{pmatrix} 0 & -\rho & 0 \\ \rho & 0 & -\alpha \\ 0 & \alpha & 0 \end{pmatrix}, \quad F^i(t_0) = Id.
\]
Then setting \( \xi^i(t) := F^i(t)e_1 \), the conjugate \( f^i \) is given by
\[
  f^i(t,v) = \sigma^i(t) + v\xi^i(t) \quad \left( \sigma^i(t) := \int_{t_0}^{t} \omega t \xi^i \right)
\]
such that the data of \( f^i \) is given by \((\alpha^2,\rho^2,\omega^2) = (\omega,\rho,\alpha)\), and \( n^i(t) := F^i(t)e_3 \) gives a unit normal of \( f^i \).

By [17, Proposition 2.16] (cf. Fact 3.3) and Proposition 3.5, we have the following:

**Corollary 3.9.** Let \( f(t,v) \) be an a-tangent developable whose data is given by
\[
(\alpha,\rho,\omega) = (a(t) dt, r(t) dt, w(t) dt).
\]
Take a singular point \( p_0 = (t_0, v_0) \in S(f) \). Then,

- \( f \) at \( p_0 \) is \( A \)-equivalent to the cuspidal edge if and only if
  \[
  v_0 = 0, \quad r(t_0) \neq 0, \quad a(t_0) \neq 0, \quad w(t_0) \neq 0,
  \]
  or
  \[
  v_0 \neq 0, \quad r(t_0) = 0, \quad r'(t_0) \neq 0, \quad w(t_0) \neq 0.
  \]
- \( f \) at \( p_0 \) is \( A \)-equivalent to the swallowtail if and only if
  \[
  v_0 = 0, \quad r(t_0) \neq 0, \quad a(t_0) = 0, \quad a'(t_0) \neq 0, \quad w(t_0) \neq 0.
  \]
- \( f \) at \( p_0 \) is \( A \)-equivalent to the cuspidal cross cap if and only if
  \[
  v_0 = 0, \quad r(t_0) \neq 0, \quad a(t_0) \neq 0, \quad w(t_0) = 0, \quad w'(t_0) \neq 0.
  \]

In particular, \( f \) at \( p_0 \) is \( A \)-equivalent to the swallowtail if and only if \( f^i \) at \( p_0 \) is \( A \)-equivalent to the cuspidal cross cap, where \( f^i \) is the conjugate of \( f \).

As an example, we calculate the conjugate of the standard swallowtail.

**Example 3.10** (Conjugate of the standard swallowtail). Let \( f_{SW} \) be the standard swallowtail given in (2.3). By a parameter change \((u,v) \mapsto (t,y - 6t^2)\), we have
\[
  f_{SW}(t,y) = (-8t^3, -3t^4, 6t^2) + y(2t, t^2, -1).
\]
Thus, setting \( v := \sqrt{1 + 4t^2 + t^4} \), \( f_{SW} \) is an a-tangent developable \( f_{SW}(t,v) = \sigma(t) + v\xi(t) \), where
\[
  \sigma(t) := (-8t^3, -3t^4, 6t^2), \quad \xi(t) := \frac{1}{\sqrt{1 + 4t^2 + t^4}}(2t, t^2, -1).
\]
Since $\sigma'(t) = -12t\sqrt{1 + 4t^2 + t^4}\xi(t)$, we have
\begin{equation}
(3.22)
a(t) = -12t\sqrt{1 + 4t^2 + t^4}.
\end{equation}

Then the spherical dual $n(t)$ of $\xi(t)$ and $\eta(t) = n(t) \times \xi(t)$ are given by
\begin{align*}
n(t) &= \frac{1}{\sqrt{1 + t^2 + t^4}}(t, -1, t^2), \\
\eta(t) &= \frac{1}{\sqrt{1 + 4t^2 + t^4} \sqrt{1 + t^2 + t^4}}(1 - t^4, t + 2t^3, 2t + t^3),
\end{align*}
respectively. Hence we have
\begin{equation}
(3.23)
r(t) = \frac{2\sqrt{1 + t^2 + t^4}}{1 + 4t^2 + t^4}, \quad w(t) = -\frac{\sqrt{1 + 4t^2 + t^4}}{1 + t^2 + t^4},
\end{equation}
where $r(t) = \xi'(t) \cdot \eta(t)$, $w(t) = -n'(t) \cdot \eta(t)$.

Then, applying Lemma 3.8 with $(\alpha, \rho, \omega) = (a(t) \ dt, r(t) \ dt, w(t) \ dt)$ and $t_0 = 0$, we obtain the conjugate $f_{SW}^a(t, v)$, where $a(t)$, $r(t)$, $w(t)$ are given by (3.22) and (3.23), respectively (cf. Figure 2).

**Figure 2.** The a-tangent developable $f_{SW}^a$ which is the conjugate of the standard swallowtail $f_{SW}$ given by (2.3) (cf. Figure 1). By Corollary 3.9, we have that $f_{SW}^a$ at $(t, v) = (0, 0)$ is $A$-equivalent to the cuspidal cross cap. This figure is plotted by integrating (3.18) and (3.19) numerically.

### 4. Other singularities

Here, we shall write down the criteria for other singularities (cuspidal beaks, cuspidal butterfly, cuspidal $S_1^+$ singularity, 5/2-cuspidal edge) on a-tangent developables in terms of their data.

#### 4.1. Cuspidal beaks

First, we review the criterion for the cuspidal beaks given by Izumiya–Saji–Takahashi [14].

**Fact 4.1** (Criterion for cuspidal beaks [14]). Let $f: U \to \mathbb{R}^3$ be a front defined on a domain $U$ of $\mathbb{R}^2$ with the unit normal $\nu$. Also let $p \in U$ be a singular point of $f$ and $\zeta$ a null vector field. Then, the map germ $f$ at $p$ is $A$-equivalent to the cuspidal beaks if and only if $\text{rank}(df)_p = 1$, $d\lambda(p) = 0$, $d\zeta(p) < 0$ and $\zeta \lambda(p) \neq 0$ hold.

Applying Fact 4.1 to a-tangent developables, we have the following.
Proposition 4.2. Let \( f(t, v) \) be an \( a \)-tangent developable whose data is given by
\[
(\alpha, \rho, \omega) = (a(t) \, dt, r(t) \, dt, w(t) \, dt).
\]
Then, for a singular point \( p_0 = (t_0, v_0) \) of \( f \), we have that \( f \) at \( p_0 \) is \( \mathcal{A} \)-equivalent to the cuspidal beaks if and only if
\[
(4.1) \quad v_0 = 0, \quad r(t_0) = 0, \quad r'(t_0) \neq 0, \quad a(t_0) \neq 0, \quad w(t_0) \neq 0.
\]

Proof. We remark that for any singular point \( p_0 \) of \( f \), \( \text{rank}(df)_{p_0} = 1 \) holds (cf. (3.6)). Hence, by Fact 4.1, \( f \) at \( p_0 \) is \( \mathcal{A} \)-equivalent to the cuspidal beaks if and only if \( d\lambda(p_0) = 0 \), \( \det \text{Hess}(\lambda(p_0)) < 0 \), \( \zeta \zeta \lambda(p_0) \neq 0 \), and \( f \) is a front at \( p_0 \). By (3.8), \( d\lambda(p_0) = 0 \) if and only if \( r(t_0) = 0 \) (i.e., \( p_0 \) is cylindrical) and \( v_0 \neq 0 \). Since the signed area density function \( \lambda \) is given by
\[
\lambda(t, v) = -v \, r(t)
\]
(cf. (3.7)), we have
\[
\det \text{Hess} \lambda = \det \begin{pmatrix} \lambda_{tt} & \lambda_{tv} \\ \lambda_{tv} & \lambda_{vv} \end{pmatrix} = -\lambda_{tv}^2 = -(r')^2.
\]
Hence, \( \det \text{Hess}(\lambda(p_0)) < 0 \) if and only if \( r'(t_0) \neq 0 \). As we see in the proof of Lemma 3.2, \( \zeta(t, v) := \partial_t - a(t) \partial_v \) gives a null vector field. Since \( \zeta \lambda = v' r(t) - a(t) r(t) \), we have
\[
(4.2) \quad \zeta^2 \lambda = v \, r''(t) - a'(t) r(t) - 2a(t) r'(t).
\]
Therefore, \( f \) at \( p_0 \) is \( \mathcal{A} \)-equivalent to the cuspidal beaks if and only if (4.1) holds. \( \square \)

4.2. Cuspidal butterfly. Next, we review the criterion for the cuspidal butterfly given by Izumiya–Saji [13].

Fact 4.3 (Criterion for cuspidal butterfly [13]). Let \( f : U \to \mathbb{R}^3 \) be a front defined on a domain \( U \) of \( \mathbb{R}^2 \) with the unit normal \( \nu \). Take a non-degenerate singular point \( p \in U \) of \( f \). Let \( \gamma(t) \) be a singular curve such that \( \gamma(0) = p \) and \( \zeta(t) \) a null vector field. Then, the map germ \( f \) at \( p \) is \( \mathcal{A} \)-equivalent to the cuspidal butterfly if and only if \( \delta(0) = \delta'(0) = 0 \) and \( \delta''(0) \neq 0 \) hold.

Applying Fact 4.3 to \( a \)-tangent developables, we have the following.

Proposition 4.4. Let \( f(t, v) \) be an \( a \)-tangent developable whose data is given by
\[
(\alpha, \rho, \omega) = (a(t) \, dt, r(t) \, dt, w(t) \, dt).
\]
Then, for a singular point \( p_0 = (t_0, v_0) \) of \( f \), we have that \( f \) at \( p_0 \) is \( \mathcal{A} \)-equivalent to the cuspidal butterfly if and only if
\[
(4.3) \quad v_0 = 0, \quad r(t_0) \neq 0, \quad a(t_0) = a'(t_0) = a''(t_0) = 0, \quad w(t_0) \neq 0.
\]

Proof. By (4.2), we have
\[
(4.4) \quad \zeta^2 \lambda = v \, r'''(t) - a''(t) r(t) - 3a'(t) r'(t) - 3a(t) r''(t).
\]
Hence, by Fact 4.3, \( f \) at \( p_0 \) is \( \mathcal{A} \)-equivalent to the cuspidal butterfly if and only if
\begin{enumerate}
  \item \( f \) is a front at \( p_0 = (t_0, v_0) \) (i.e., \( w(t_0) \neq 0 \)),
  \item \( p_0 = (t_0, v_0) \) is non-degenerate (i.e., \( r(t_0) \neq 0 \) or \( r(t_0) = 0, v_0 r'(t_0) \neq 0 \)),
  \item \( v_0 r(t_0) = 0 \),
  \item \( v_0 r'(t_0) - a(t_0) r(t_0) = 0 \),
  \item \( v_0 r''(t_0) - a'(t_0) r(t_0) - 2a(t_0) r'(t_0) = 0 \),
  \item \( v_0 r'''(t_0) - a''(t_0) r(t_0) - 3a'(t_0) r'(t_0) - 3a(t_0) r''(t_0) = 0 \).
\end{enumerate}
If we assume that \( p_0 = (t_0, v_0) \) is cylindrical (i.e., \( r(t_0) = 0 \)), the condition (i) implies \( v_0 r'(t_0) \neq 0 \). This contradicts the condition (iv). Thus, we have \( r(t_0) \neq 0 \). Then, we can check that the conditions (i)–(vi) are equivalent to (4.3). \( \square \)
4.3. Cuspidal $S^\pm_1$ singularity. Now, we review the criterion for the cuspidal $S^+_1$ singularity given by Saji [19].

Fact 4.5 (Criterion for cuspidal $S^\pm_1$ singularity [19]). Let $f : U \to \mathbb{R}^3$ be a frontal defined on a domain $U$ of $\mathbb{R}^2$ with the unit normal $\nu$. Take a non-degenerate singular point $p \in U$ of $f$. Let $\gamma(t)$ be a singular curve such that $\gamma(0) = p$ and $\zeta$ a null vector field. Then, the map germ $f$ at $p$ is $A$-equivalent to the cuspidal $S^+_1$ singularity (resp. the cuspidal $S^-_1$ singularity) if and only if the following (i)-(iv) hold:

(i) $\delta(0) \neq 0$,

(ii) $\psi_{ccr}(0) = \psi'_{ccr}(0) = 0$ and

\begin{equation}
(4.5) \quad (d_1 :=) \psi''_{ccr}(0) \neq 0,
\end{equation}

(iii) there exist a regular curve $c : (-\varepsilon, \varepsilon) \to U$ and $\ell \in \mathbb{R}$ such that $c(0) = p$, $c'(0)$ is parallel to $\zeta(0)$, $c''(0) \neq 0$, $c'''(0) = \ell c''(0)$ and

\begin{equation}
(4.6) \quad (d_2 :=) \det \left( df_p(\xi_p), c''(0), 3c^{(5)}(0) - 10\ell c'^{(4)}(0) \right) \neq 0
\end{equation}

hold, where $\dot{c} := f \circ c$ and $\xi_p := \gamma'(0)$,

(iv) the product $d_1d_2$ is positive (resp. negative), where $d_1$, $d_2$ are given by (4.5), (4.6), respectively. Here, we choose $\zeta$ and $c$ so that $c'(0)$ points the same direction as the null vector $\zeta(0)$ and that $\{\gamma'(0), \zeta(0)\}$ is positively oriented.

Applying Fact 4.5 to a-tangent developables, we have the following.

Proposition 4.6. Let $f(t,v)$ be an a-tangent developable whose data is given by

\begin{equation}
(\alpha, \rho, \omega) = (a(t) \, dt, r(t) \, dt, w(t) \, dt).
\end{equation}

Then, for a singular point $p_0 = (t_0, v_0)$ of $f$, we have that $f$ at $p_0$ is $A$-equivalent to the cuspidal $S^+_1$ singularity if and only if

\begin{equation}
(4.7) \quad v_0 = 0, \quad r(t_0) \neq 0, \quad a(t_0) \neq 0, \quad w(t_0) = w'(t_0) = 0, \quad w''(t_0) \neq 0
\end{equation}

Remark 4.7. It is known that, by Ishikawa’s theorem [10], developable surfaces do not admit any cuspidal $S^\pm_k$ singularities for $k > 1$. We also remark that, by Mond [16] and Saji [19, Theorem 4.1], tangent developable surfaces of a regular space curve do not admit cuspidal $S^-_1$ singularity, as in the following proof.

Proof of Proposition 4.6. We first show that $p_0$ is non-cylindrical. If we assume $p_0 = (t_0, v_0)$ is cylindrical, we have $\gamma_\nu(v) = (t_0, v)$ is a singular curve passing through $\gamma_\nu(v_0) = p_0$. Then the function $\psi_{ccr}$ defined as (2.4) is given by $\psi_{ccr}(v) = w(t_0)$ (cf. (3.14)). Thus,

$$\psi_{ccr}(v_0) = \psi'_{ccr}(v_0) = 0$$

and $\psi''_{ccr}(v_0) \neq 0$ do not occur at the same time. Therefore, $p_0$ must be non-cylindrical.

Since $r(t_0) \neq 0$ and $0 = \lambda(t_0, v_0) = -v_0r(t_0)$, we have $v_0 = 0$. Then, $\gamma_{nc}(t) = (t, 0)$ is a singular curve passing through $\gamma_{nc}(t_0) = p_0$, and $\zeta_{nc}(t) = \partial_k - a(t)\partial_r$ is a null vector field along $\gamma_{nc}(t)$. Then we have $\delta(t) = \det(\gamma_{nc}'(t), \zeta_{nc}(t)) = -a(t)$ (cf. (3.5)). Thus, the condition (i) of the criterion in Fact 4.5 implies $a(t_0) \neq 0$.

Now, assume that $a(t_0) < 0$, namely, $\{\gamma_{nc}(t_0), \zeta_{nc}(t_0)\}$ is positively oriented. The function $\psi_{ccr}$ defined as (2.4) is given by $\psi_{ccr}(t) = a(t)w(t)$ (cf. (3.13)). Thus, under the condition (i), the condition (ii) in Fact 4.5 implies $w(t_0) = w'(t_0) = 0$ and $w''(t_0) \neq 0$ hold. The constant $d_1$ in
(4.5) is given by $d_1 = a(t_0)w''(t_0)$. With respect to the condition (iii) in Fact 4.5, by a parallel translation of $\mathbb{R}^3$, we may assume that $\sigma(t_0) = 0$ without loss of generality. Then, setting

\begin{equation}
(4.8) \quad c(\tau) := (\tau, -\varphi(\tau)) \quad \left( \varphi(\tau) := \frac{\sigma(\tau) \cdot \xi(t_0)}{\xi(\tau) \cdot \xi(t_0)} \right),
\end{equation}

we have $c(t_0) = p_0$. Differentiating $\varphi(\tau)$, we have that $c'(t_0) = c'_{nc}(t_0)$. Since

$c''(t_0) = -a(t_0)\rho(t_0)\eta(t_0)$

and $c'''(t_0) = -(2a(t_0)\rho'(t_0) + a'(t_0)\rho(t_0))\eta(t_0)$ under the conditions (i) and (ii) in Fact 4.5, we have

\[ \ell \cdot c''(t_0) = \ell \cdot c''(t_0) = \frac{2a(t_0)\rho'(t_0) + a'(t_0)\rho(t_0)}{(a(t_0)\rho(t_0))}. \]

Moreover, by a direct calculation, we can check that $c^{(4)}(t_0)$ is a constant multiple of $\eta(t_0)$ and

$c^{(5)}(t_0) = k_1\eta(t_0) - 4a(0)\rho(0)\omega''(0)n(t_0)$

holds, where $k_1 \in \mathbb{R}$ is a constant. Thus, the constant $d_2$ in (4.6) is given by

\[ d_2 = \delta(0) = -a(t_0)w''(t_0), \quad -a(0)\rho(0)\omega''(0)n(t_0) \]

Hence, under the conditions (i) and (ii) in Fact 4.5, the condition (iii) is always satisfied.

In the case of $a(t_0) > 0$, we take the null vector field as $\zeta_n(t) := -\partial_t + a(t)\partial_v$ and the curve $c(\tau)$ as $c(\tau) := (-\tau, \varphi(\tau))$, where $\varphi(t)$ is given by (4.8). Then, by a similar calculation as above, the constant $d_1$ in (4.5) is given by $d_1 = -a(t_0)w''(t_0)$, and the constant $d_2$ in (4.6) is $d_2 = -12a(0)^2\rho(0)^2\omega''(t_0)$. Therefore, regardless of the sign of $a(t_0)$, we have

\[ d_1d_2 = 12a(0)^2\rho(0)^2\omega''(t_0)^2 > 0. \]

Thus, Fact 4.5 implies that any $a$-tangent developable does not admit cuspidal $S^1_-$ singularities, and that $f$ at $p_0 = (t_0, v_0)$ is $\mathcal{A}$-equivalent to the cuspidal $S^1_+$ singularity if and only if (4.7) holds.

4.4. 5/2-cuspidal edge. Finally, we review the criterion for the 5/2-cuspidal edge given in [9].

**Fact 4.8 (Criterion for 5/2-cuspidal edge [9]).** Let $f : U \to \mathbb{R}^3$ be a frontal defined on a domain $U$ of $\mathbb{R}^2$ with the unit normal $\nu$. Take a non-degenerate singular point $p \in U$ of $f$. Let $\gamma(t)$ ($|t| < \varepsilon$) be a singular curve such that $\gamma(0) = p$ and $\zeta$ a null vector field. Then, the map germ $f$ at $p$ is $\mathcal{A}$-equivalent to the 5/2-cuspidal edge if and only if the following (i)-(iii) hold:

(i) $\delta(0) \neq 0$,

(ii) $\det (\zeta', \zeta^2 f, \zeta^3 f)_{(w,v) = \gamma(t)} = 0$ holds for each $t \in (-\varepsilon, \varepsilon)$,

(iii) $\det (\zeta'(0), \zeta^2 f(p), 3\zeta^5 f(p) - 10C \zeta^4 f(p)) \neq 0$.

Here $\tilde{\zeta}$ is a special null vector field such that

\begin{equation}
(4.9) \quad \tilde{\gamma}'(0) \cdot \tilde{\zeta}^2 f(p) = \tilde{\gamma}'(0) \cdot \tilde{\zeta}^3 f(p) = 0, \quad \tilde{\zeta}^3 f(p) = C \tilde{\zeta}^2 f(p),
\end{equation}

where $C \in \mathbb{R}$ is a constant.

We remark that if $\zeta'(0) \cdot \zeta^2 f(p) = \zeta'(0) \cdot \zeta^3 f(p) = 0$ holds, then there exists a constant $C \in \mathbb{R}$ which satisfies $\tilde{\zeta}^3 f(p) = C \tilde{\zeta}^2 f(p)$. Applying Fact 4.8 to $a$-tangent developables, we have the following.
Proposition 4.9. Let \( f(t,v) \) be an a-tangent developable whose data is given by
\[
(\alpha, \rho, \omega) = (a(t) \, dt, r(t) \, dt, w(t) \, dt).
\]
Then, for a singular point \( p_0 = (t_0,v_0) \) of \( f \), we have that \( f \) at \( p_0 \) is \( A \)-equivalent to the 5/2-cuspidal edge if and only if
\[
v_0 \neq 0, \quad r'(t_0) \neq 0, \quad r(t_0) = w(t_0) = 0, \quad \left( \frac{w'}{r'} \right)_{t=t_0} \neq -\frac{2a(t_0)w'(t_0)}{v_0r'(t_0)}.
\]

Proof. We first show that \( p_0 \) is cylindrical. If we assume that \( p_0 = (t_0,v_0) \) is non-cylindrical, we have that \( r(t_0) \neq 0 \) and \( v_0 = 0 \). As we have seen in Lemma 3.2, \( \gamma_{nc}(t) = (t,0) \) is a singular curve passing through \( \gamma_{nc}(t_0) = p_0 \) and \( \zeta_{nc}(t) = \partial_t - a(t)\partial_v \) is a null vector field. Since the function \( \delta \) defined as (2.4) is given by \( \delta_{nc}(t) = -a(t) \) (cf. (3.5)), the condition (i) in Fact 4.8 is equivalent to \( a(t_0) \neq 0 \). On the other hand, since \( \zeta_{nc}^2 f(\gamma_{nc}(t)) = -a(t)r(t)\eta(t) \) and
\[
\zeta_{nc}^3 f(\gamma_{nc}(t)) = 2a(t)r(t)^2\xi(t) - (r(t)a'(t) + 2a(t)m'(t))\eta(t) + 2a(t)r(t)w(t)\mathbf{n}(t),
\]
we have that the condition (ii) in Fact 4.8 is equivalent to \( w(t) = 0 \) for all \( t \). Then, setting
\[
\bar{\zeta} := \left( 1 - \frac{r(t_0)^2}{a(t_0)^2} \right) \partial_t - a(t)\partial_v,
\]
we have \( \bar{\zeta}^2 f(p_0) = -a(t_0)r(t_0)\eta(t_0) \) and \( \bar{\zeta}^3 f(p_0) = -(a'(t_0)m(t_0) + 2a(t_0)m'(t_0))\eta(t_0) \). Hence, \( \bar{\zeta} \) is a null vector field satisfying (4.9) with the constant
\[
C := (a'(t_0)m(t_0) + 2a(t_0)m'(t_0))/(a(t_0)m(t_0)).
\]

Then, by a direct calculation, we have \( \bar{\zeta}^4 f(p_0), \bar{\zeta}^5 f(p_0) \in \text{Span}(\xi(t_0), \eta(t_0)) \), which implies
\[
\det \left( df(\gamma_{nc}(0)), \bar{\zeta}^2 f(p_0), 3\bar{\zeta}^5 f(p_0) - 10C \bar{\zeta}^4 f(p_0) \right) = 0.
\]
Hence, \( p_0 \) must be cylindrical.

As we have seen in Lemma 3.2, a non-degenerate cylindrical singular point \( p_0 = (t_0,v_0) \) satisfies \( r(t_0) = 0, r'(t_0) \neq 0, v_0 \neq 0 \). Then, \( \gamma_c(v) = (t_0,v) \) is a singular curve passing through \( \gamma_c(v_0) = p_0 \) and \( \zeta_c = \partial_t - a(t)\partial_v \) is a null vector field. Since the function \( \delta \) defined as (2.4) is given by \( \delta_c(v) = -1 \) (cf. (3.3)), the condition (i) in Fact 4.8 is always satisfied. On the other hand, since \( \zeta_c^2 f(\gamma_c(v)) = \nu m'(t_0) \eta(t_0) \) and
\[
\zeta_c^3 f(\gamma_c(v)) = (-2a(t_0)m'(t_0) + \nu m''(t_0))\eta(t_0) - 2\nu w(t_0)m'(t_0)\mathbf{n}(t_0),
\]
we have that the condition (ii) in Fact 4.8 is equivalent to \( w(t_0) = 0 \). Then, \( \zeta_c = \partial_t - a(t)\partial_v \) is a null vector field satisfying (4.9) with the constant \( C := (\nu m''(t_0) - 2a(t_0)m'(t_0))/(\nu m'(t_0)) \). By a direct calculation, we have
\[
\det \left( df(\gamma_c(0)), \zeta_c^2 f(p_0), 3\zeta_c^5 f(p_0) - 10C \zeta_c^4 f(p_0) \right) = -12\nu_0 m'(t_0) (2a(t_0)m'(t_0)w'(t_0) - \nu_0 m''(t_0)w'(t_0) + \nu_0 m'(t_0)w''(t_0)).
\]
Hence, by Fact 4.8, we have that \( f \) at \( p_0 = (t_0,v_0) \) is \( A \)-equivalent to the 5/2-cuspidal edge if and only if (4.10) holds. \( \square \)
Duality of singularities. Here, we give a summary of the criterion for singularities of a-tangent developables. Let \( f(t, v) \) be an a-tangent developable defined on \( J \times \mathbb{R} \) whose data is given by \( (\alpha, \rho, \omega) = (a(t) dt, r(t) dt, w(t) dt) \). In Corollary 3.9, Propositions 4.2, 4.4, 4.6, and 4.9, we proved that the singularity type of the germ \( f \) at \( p_0 = (t_0, v_0) \in J \times \mathbb{R} \) is determined by the data as in Table 1.

Since the conjugate \( f^\sharp \) of an a-tangent developable \( f \) is given by the data
\[
(\alpha^\sharp, \rho^\sharp, \omega^\sharp) := (\omega, \rho, \alpha),
\]
exchanging the roles \( \alpha \) and \( \omega \) we have the following.

**Theorem 4.10** (Duality of singularities for a-tangent developables). Let \( f : M \to \mathbb{R}^3 \) be an a-tangent developable, \( f^\sharp \) the conjugate of \( f \), and \( p_0 \in M \) a singular point, where \( M := J \times \mathbb{R} \). Then, \( f \) at \( p_0 \) is \( \mathcal{A} \)-equivalent to the swallowtail (resp. cuspidal cross cap, cuspidal beaks, cuspidal butterfly, cuspidal \( S_1^+ \) singularity) if and only if \( f^\sharp \) at \( p_0 \) is \( \mathcal{A} \)-equivalent to the cuspidal cross cap (resp. swallowtail, cuspidal beaks, cuspidal \( S_1^+ \) singularity, cuspidal butterfly).

In the case of the cuspidal edge, there exist examples which do not satisfy the desired duality of singularities.

**Example 4.11.** Let \( f = f(t, v) \) be an a-tangent developable whose data is given by
\[
(\alpha, \rho, \omega) = (2t dt, t dt, dt).
\]
By Corollary 3.9, \( f \) at \( (0, v) \) is cuspidal edge for \( v \neq 0 \) (see Figure 3). The conjugate \( f^\sharp = f^\sharp(t, v) \) of \( f \) is given by the data \((\alpha^\sharp, \rho^\sharp, \omega^\sharp) = (dt, t dt, 2t dt)\). By Proposition 4.9, \( f^\sharp \) at \( (0, v) \) is 5/2-cuspidal edge for \( v \neq 0 \) (see Figure 4).

5. Conjugate of complete flat fronts

Finally, we observe a global behavior of the conjugate operations among a-tangent developables.
Figure 3. The image of the a-tangent developable $f = f(t,v)$ whose data is given by $(\alpha, \rho, \omega) = (2t dt, t dt, dt)$. By Corollary 3.9, we have that $f$ at $(0,v)$ is $\mathcal{A}$-equivalent to the cuspidal edge for $v \neq 0$. This figure is plotted by integrating (2.6) and (3.17) numerically. The black line is the image of the cylindrical singular set $S_c(f) = \{(0,v) ; v \neq 0\}$.

Figure 4. The image of the conjugate $f^\sharp = f^\sharp(t,v)$ of the a-tangent developable with the data $(\alpha, \rho, \omega) = (2t dt, t dt, dt)$. Since the data of $f^\sharp$ is given by $(\alpha^\sharp, \rho^\sharp, \omega^\sharp) = (dt, t dt, 2t dt)$, Proposition 4.9 yields that $f$ at $(0,v)$ is $\mathcal{A}$-equivalent to the $5/2$-cuspidal edge for $v \neq 0$. This figure is plotted by integrating (3.18) and (3.19) numerically. The black line is the image of the cylindrical singular set $S_c(f^\sharp) = \{(0,v) ; v \neq 0\}$.

**Proposition 5.1.** Let $f : M \to \mathbb{R}^3$ be an a-tangent developable such that $f$ is a complete flat front with embedded ends, where $M := S^1 \times \mathbb{R}$. Then, the conjugate $f^\sharp$ of $f$ is not a front. In particular, the conjugate of a complete flat front with embedded ends cannot be a complete flat front.

**Proof.** Let $(\alpha, \rho, \omega)$ be the data of $f$. By Fact 2.1, $f$ has at least four singular points other than cuspidal edges. In fact, if we denote by $\alpha = a(t) dt$, it is proved in [17, pp. 311–312] that $a(t)$ changes signs at least four times on $S^1$. Since the data of $f^\sharp$ is given by $(\alpha^\sharp, \rho^\sharp, \omega^\sharp) := (\omega, \rho, \alpha)$, and $f^\sharp$ is front if and only if $\omega^\sharp$ never vanishes, we have that $f^\sharp$ cannot be a front. □

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