

TORIC REDUCTION OF SINGULARITIES FOR NEWTON NONDEGENERATE p -FORMS

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ABSTRACT. We study a class of holomorphic p -forms satisfying nondegeneracy conditions expressed through their Newton polyhedron and called Newton nondegenerate (NND). We give a characterization of NND p -forms by their toric reduction of singularities defined through a regular refinement of their dual fan. We then present an application of this result to the study of singularities of $(n-1)$ -forms on $\mathbf{C}^n$.

1. INTRODUCTION

The study of the reduction of singularities of dimension 2 foliations dates back to the middle of the 19th century, where it was used in particular cases. It was developed further by the likes of Bendixson [Ben01] and Dulac [Dul04], but the general algorithm was only established by Seidenberg in [Sei68]. A simplified proof was published later by Arno van den Essen in [VdE79]. Dumortier resolved this problem in the $\mathcal{C}^\infty$ -case [Dum77]. The modern formulation of this theorem is as follows: let $\mathcal{F}$ be a foliation defined on the germ $(\mathbf{C}^2, 0)$ by a 1-form $\omega = a(x, y)dx + b(x, y)dy$. The origin is called a *singular point* of $\mathcal{F}$ if $a(0, 0) = b(0, 0) = 0$. We associate by duality the vector field $\chi = b(x, y)\frac{\partial}{\partial x} - a(x, y)\frac{\partial}{\partial y}$ to the form ω . Suppose that the origin is a singular point and consider the matrix of the linear part of χ at the origin:

$$\mathcal{J}_0(\chi, x, y) = \begin{pmatrix} \frac{\partial b}{\partial x}(0) & -\frac{\partial a}{\partial x}(0) \\ \frac{\partial b}{\partial y}(0) & -\frac{\partial a}{\partial y}(0) \end{pmatrix}.$$

It is a *nonnilpotent* singularity if the above matrix is nonnilpotent. If in addition, the quotient of its eigenvalues does not belong to $\mathbf{Q}_{>0}$, it is called a *reduced* singularity.

This definition is independent from the coordinates (x, y) and from the generator ω of the foliation $\mathcal{F}$. Seidenberg's theorem guarantees the existence of a finite sequence of blowing ups of points whose composition $\pi : (V, E) \rightarrow (\mathbf{C}^2, 0)$ is such that the strict transform of $\mathcal{F}$ by π has at most reduced singularities in the exceptional divisor E .

For codimension 1 foliations in higher dimensional spaces, nonnilpotent and reduced singularities have been generalized by Cano and Cerveau (see [Can98] for an introduction) respectively as *presimple* and *simple* singularities. For threefolds, the reduction theorem to simple points was first proved by Cano and Cerveau in the nondicritical case [CC92], and later, for arbitrary codimension 1 foliations by Cano in [Can04].

The codimension 2 case for threefolds corresponds to foliations defined by vector fields and was first treated by Panazzolo in the real analytic setting in [Pan06], then by Panazzolo and McQuillan in the complex framework [MP13]. Higher dimensional cases are still open, most of the available literature focuses on codimension 1 foliations. These theorems have applications to dynamical systems, birational geometry and Mori theory ([Bru15] and [CS21]).

The Newton nondegeneracy condition (NND), first introduced in the context of function singularity theory (see [BKK76], [Var76] and [Var77]), ties successfully together toric geometry, Newton polyhedra and the study of singularities: any NND affine hypersurface admits a resolution of

singularities by one toric morphism. Generalizations to higher codimensions of this result were established for complete intersection singularities in [Oka90] and for more general varieties in [AGMS13], [Tev07] and [AGMM24].

In the same spirit, we are concerned with the reduction of singularities of p -forms by the use of only one toric morphism. A version of Newton nondegeneracy has been formulated for plane vector fields by Brunella and Miari in [BM90] to study topological classification problems. In [CC99], without assuming any Newton nondegeneracy condition, the authors characterize germs of foliations on $(\mathbf{C}^2, 0)$ whose singularities become reduced after applying a toric morphism defined via their Newton polyhedron. In [Est05], Esterov defines the Newton polyhedron of a germ of 1-form in $(\mathbf{C}^n, 0)$ and introduces $\mathbf{C}$ -genericity, a variant of the NND condition, together with toric resolution techniques to establish a formula for the index of a 1-form on a complete intersection singularity. In dimension three, topological classification results for vector fields were proven in [AG15], assuming a different NND condition.

More recently, Molina-Samper introduced a definition of NND codimension 1 foliations using Newton polyhedra systems (see [MS19], [MS20] and [MS22]) and gave a characterization of such foliations in terms of the existence of a logarithmic reduction of singularities of combinatorial nature: in particular, it is a sequence of blowing ups with smooth centers of the ambient space. However, this procedure does not provide a construction of a fan whose associated toric variety yields a resolution.

By contrast, the viewpoint adopted in this article — which was first initiated by Brunella and Miari in the case of plane vector fields [BM90] — is the following: starting from the Newton polyhedron of a differential form, we construct directly a toric morphism that reduces the singularities. This geometric viewpoint produces an explicit resolution obtained through a single toric morphism and naturally extends to p -forms with $p > 1$.

This article is motivated by a conjecture of Teissier which states that any affine singular variety can be desingularized by a single toric morphism after reembedding it in a higher dimensional affine space: [Tei03], [Tei14], [Tei23], [Mou23]. While the general case of this conjecture remains open, it has been resolved for plane curve singularities (see [Mou17], [dFGPM23]). We aim to formulate and study similar results, but in the context of foliations instead of algebraic varieties. A step in this direction is the statement 2) of Theorem 4.1 below, where we give a characterization of NND p -forms via toric resolution of singularities.

Theorem. Let η be a nonidentically zero holomorphic p -form defined on $\mathbf{C}^n$. We denote by $\Gamma = \Gamma(\eta)$ its Newton polyhedron and by Σ a regular refinement of the dual fan of η . Let $\pi : X_\Sigma \rightarrow \mathbf{C}^n$ be the associated proper birational toric morphism. For every affine chart $\mathcal{U}_\sigma$ of X_Σ defined by a maximal cone $\sigma \in \Sigma$ with minimal generators $v_1, \dots, v_n$, set $t_k = \min_{I \in \Gamma(\eta)} \langle v_k, I \rangle$, where $k \in \{1, \dots, n\}$ and $T = (t_1, \dots, t_n)$ and denote by $Y = (y_1, \dots, y_n)$ the coordinates which are canonically associated to the vectors $v_1, \dots, v_n$ on $\mathcal{U}_\sigma$. Set $A = \{k \in \{1, \dots, n\}, t_k > 0\}$.

- 1) The pull-back $\pi^*\eta$ can be expressed locally in every affine chart $\mathcal{U}_\sigma$ as

$$\pi^*\eta(Y) = Y^T \underbrace{\left(\sum_{|K|=p} \bar{f}_K(Y) \frac{dY_K}{Y_K} \right)}_{\bar{\eta}(Y)}$$

where each coefficient $\bar{f}_K(Y)$ of $\bar{\eta}(Y)$ is holomorphic and divisible by the monomial $Y_{K \setminus A} = \prod_{k \in K \setminus A} y_k$.

- 2) η is Newton nondegenerate if and only if in every affine chart $\mathcal{U}_\sigma$, the coefficients $\bar{f}_K(Y)$ of $\bar{\eta}(Y)$ have no common zeroes.

For $n = 2$, Seidenberg's theorem gives reduced singularities, which are, in particular, nonnilpotent. This nonnilpotency result is weaker, but we generalize it to any NND $(n - 1)$ -form with arbitrary $n \geq 2$ in the following Theorem 4.3.

Theorem. Let η be a nonidentically zero holomorphic p -form on $\mathbf{C}^n$. If we suppose that η is NND and $p = n - 1$, then the strict transform η' of η by the toric morphism defined via a regular refinement of its dual fan has at worst nonnilpotent singularities.

In the first part of this paper, we generalize Newton polyhedron of 1-forms to p -forms and define associated Newton nondegeneracy conditions. The second part gathers results from toric geometry which are needed to formulate and prove our main result: statement 2) in Theorem 4.1 which characterizes NND p -forms. The proof of the theorem and the application to $(n - 1)$ -forms are detailed in the last part. We don't require any integrability condition.

Acknowledgements. The author wishes to thank Hussein Mourtada, Matteo Ruggiero, Bernard Teissier and Beatriz Molina-Samper for discussions related to this work and the anonymous referee for the careful reading, remarks and suggestions.

2. NEWTON NONDEGENERACY CONDITION

In this paper we adopt the convention that the set $\mathbf{N}$ contains 0: $\mathbf{N} = \{0, 1, 2, 3, \dots\}$. Let n and p be fixed integers such that $n \geq 2$ and $1 \leq p < n$.

2.1. Newton polyhedron of a p -form. We denote by $X = (x_1, \dots, x_n)$ the standard coordinate system on $\mathbf{C}^n$. We identify n -tuples in $\mathbf{C}^n$ with row matrices. The notation $\langle \cdot | \cdot \rangle$ stands for the usual dot product on $\mathbf{C}^n$. Let N be an n -dimensional smooth complex analytic space and $P \in N$, for any local coordinates $Z = (z_1, \dots, z_n)$ centered at P and any $n \times n$ matrix $M = (v_{ij})$ with entries v_{ij} in $\mathbf{N}$, we set $Z^M = (z_1^{v_{11}} \dots z_n^{v_{1n}}, \dots, z_1^{v_{n1}} \dots z_n^{v_{nn}})$. If $I = (i_1, \dots, i_n) \in \mathbf{N}^n$ is a row matrix, we will identify Z^I with the monomial $z_1^{i_1} \dots z_n^{i_n}$. We will also use these notations for points $c = (c_1, \dots, c_n) \in \mathbf{C}^n$, namely: $c^M = (c_1^{v_{11}} \dots c_n^{v_{1n}}, \dots, c_1^{v_{n1}} \dots c_n^{v_{nn}})$ and $c^I = c_1^{i_1} \dots c_n^{i_n}$. For any finite set J , the number of elements of J is denoted by $|J|$. When $J \subset \{1, \dots, n\}$ has q elements $j_1 < \dots < j_q$, we set $Z_J = z_{j_1} \dots z_{j_q}$ and $dZ_J = dz_{j_1} \wedge \dots \wedge dz_{j_q}$. If $q = p$, the logarithmic p -differentials are:

$$\frac{dZ_J}{Z_J} = \frac{dz_{j_1}}{z_{j_1}} \wedge \dots \wedge \frac{dz_{j_p}}{z_{j_p}}.$$

Let η be a holomorphic p -form defined on $\mathbf{C}^n$. Taking $N = \mathbf{C}^n$ and $Z = X$ allows us to express η in a logarithmic way in the standard coordinate system:

$$(1) \quad \eta(X) = \sum_{|J|=p} f_J(X) \frac{dX_J}{X_J}.$$

In the above expression, the subscript $|J| = p$ means that the sum is indexed by all the subsets $J \subset \{1, \dots, n\}$ with $|J| = p$. The coefficients $f_J(X)$ are holomorphic functions on $\mathbf{C}^n$ such that $\frac{f_J(X)}{X_J}$ is holomorphic. Expanding the functions $f_J(X)$ into power series, we obtain

$$(2) \quad \eta(X) = \sum_{|J|=p} \left(\sum_{I \in \mathbf{N}^n} a_{I,J} X^I \right) \frac{dX_J}{X_J},$$

where the coefficients $a_{I,J}$ are complex numbers and X_J divides every nonzero monomial $a_{I,J} X^I$. This means that if a coordinate of I indexed by J is zero, then $a_{I,J} = 0$.

For any subset $A \subset \mathbf{N}^n$, we define η_A by restricting the exponents in the expression of η to A :

$$\eta_A(X) = \sum_{|J|=p} (f_J)_{|A}(X) \frac{dX_J}{X_J},$$

where

$$(f_J)_{|A}(X) = \sum_{I \in A} a_{I,J} X^I.$$

We then define the *support* $\text{Supp } \eta$ of η :

$$\text{Supp } \eta = \{I \in \mathbf{N}^n \mid \exists J \subset \{1, \dots, n\}, |J| = p \text{ and } a_{I,J} \neq 0\}.$$

Definition 2.1 and Definition 2.5 below draw both from [BM90] and [Est05].

Definition 2.1. Let $B \subset \mathbf{R}_{\geq 0}^n$. The *Newton polyhedron* of B , denoted by $\Gamma = \Gamma(B)$ is the convex envelope of

$$\bigcup_{I \in B} (I + \mathbf{R}_{\geq 0}^n).$$

Its boundary is called the *Newton boundary*. The dimension 1 faces of Γ are called *edges* and the codimension 1 one are *facets*. The Newton polyhedron of η is, by definition, the Newton polyhedron of its support and denoted by $\Gamma(\eta)$.

Example 2.2. If $p = 1$ and $n = 2$, we denote by $X = (x, y)$ the standard coordinates on $\mathbf{C}^2$. Consider the 1-form defined by $\eta(x, y) = (2x^2 + xy)dx + (x^2 + y^2 + x)dy$. We express it logarithmically as in Eq. (2):

$$\eta(x, y) = (2x^3 + x^2y) \frac{dx}{x} + (x^2y + y^3 + xy) \frac{dy}{y}.$$

The support is $\text{Supp } \eta = \{(3, 0), (2, 1), (0, 3), (1, 1)\}$. The monomials $xydx$ and x^2dy contribute to the same point $(2, 1)$. In the figure below, the Newton polyhedron of η and its boundary are represented. The latter is the union of four facets F_1, F_2, F_3 and F_4 , of which F_2 and F_3 are the compact ones.

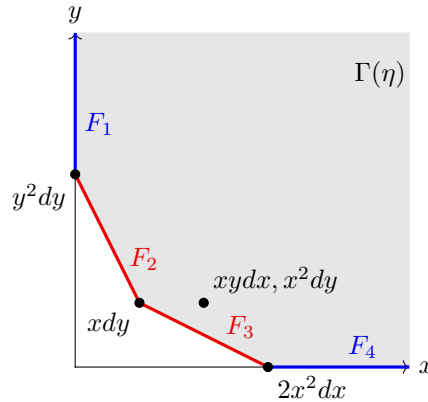


FIGURE 1. Newton polyhedron of η .

Example 2.3. If $p = 2$ and $n = 3$, let $X = (x, y, z)$ be the standard coordinate system on $\mathbf{C}^3$ and ω the 2-form given by $\omega(x, y, z) = (z^6 + xy)dx \wedge dy + xzdx \wedge dz + (x^6 + x^4yz + yz)dy \wedge dz$. We write ω in a logarithmic way:

$$\omega(x, y, z) = (xyz^6 + x^2y^2)\frac{dx}{x} \wedge \frac{dy}{y} + x^2z^2\frac{dx}{x} \wedge \frac{dz}{z} + (x^6yz + x^4y^2z^2 + y^2z^2)\frac{dy}{y} \wedge \frac{dz}{z}.$$

The support of the 2-form ω is $\text{Supp } \omega = \{(1, 1, 6), (2, 2, 0), (2, 0, 2), (6, 1, 1), (4, 2, 2), (0, 2, 2)\}$. The Newton boundary of $\Gamma(\omega)$, represented in the next figure, is the union of a compact facet and six noncompact ones.

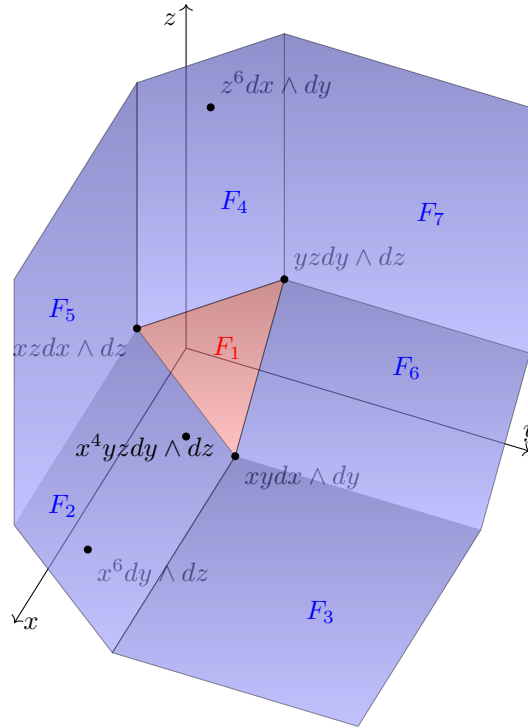


FIGURE 2. Newton boundary of $\Gamma(\omega)$.

2.2. Newton nondegeneracy.

Definition 2.4. Let η be a holomorphic p -form on $\mathbf{C}^n$ written as $\eta(X) = \sum_{|J|=p} g_J(X) dX_J$, where $g_J(X)$ are holomorphic functions. The *singular locus* of η is defined as the common zeroes of the coefficients g_J :

$$\text{Sing } \eta = \{c \in \mathbf{C}^n \mid \forall J \subset \{1, \dots, n\}, |J| = p, g_J(c) = 0\}.$$

Its elements are called *singularities* or *singular points* of η .

Definition 2.5. Let η be a holomorphic p -form on $\mathbf{C}^n$. We say that η is *Newton nondegenerate* (NND) if for any nonempty face $F \subset \Gamma(\eta)$, the so called *initial form* η_F has no singularities in $(\mathbf{C}^*)^n$.

Remark 2.6. Suppose η is expressed logarithmically as in Eq. (1):

$$\eta(X) = \sum_{|J|=p} \frac{f_J(X)}{X_J} dX_J.$$

Let F be a nonempty face of $\Gamma(\eta)$ and $J \subset \{1, \dots, n\}$ with $|J| = p$. As $\left(\frac{f_J}{X_J}\right)|_F = \frac{(f_J)|_F}{X_J}$, for any $c \in (\mathbf{C}^*)^n$, $\left(\frac{f_J}{X_J}\right)|_F(c) = 0$ if and only if $(f_J)|_F(c) = 0$. Thus, Definition 2.5 is equivalent to the following: for any nonempty face $F \subset \Gamma(\eta)$ and $J \subset \{1, \dots, n\}$ with $|J| = p$, the functions $(f_J)|_F$ have no common zeroes in $(\mathbf{C}^*)^n$.

Example 2.7. We examine Example 2.2 and Example 2.3, starting with

$$\eta(x, y) = (2x^3 + x^2y) \frac{dx}{x} + (x^2y + y^3 + xy) \frac{dy}{y}.$$

A practical observation is that the NND condition is automatically verified for vertices, so they don't play a role in the definition. Nevertheless, let's check it this time by computing the initial forms: $\eta_{\{(0,3)\}}(x, y) = y^2 dy$, $\eta_{\{(1,1)\}}(x, y) = x dy$ and $\eta_{\{(3,0)\}}(x, y) = 2x^2 dx$. As expected, they only have singularities in the coordinate hyperplanes. For the facets, the initial forms are $\eta_{F_1}(x, y) = y^2 dy$, $\eta_{F_2}(x, y) = (y^2 + x) dy$, $\eta_{F_3}(x, y) = 2x^2 dx + x dy$ and $\eta_{F_4}(x, y) = 2x^2 dx$. We notice that η_{F_2} has a curve of singularities intersecting $(\mathbf{C}^*)^2$. Let's look at the full face $F = \Gamma(\eta)$, the associated initial form $\eta_{\Gamma(\eta)}$ is η itself. It has two singularities: $(0, 0)$ and $(-\frac{1}{5}, \frac{2}{5}) \in (\mathbf{C}^*)^2$. We conclude that the faces F_2 and $\Gamma(\eta)$ prevent η from being NND.

We can find the initial forms for ω with Fig. 2. Notice that the support point $(4, 2, 2)$ given by the monomial $x^4 y z dy \wedge dz$ does not belong to the Newton boundary, it is in the interior of the Newton polyhedron.

To begin, we consider the edges of $\Gamma(\omega)$. For instance, $\omega_{F_4 \cap F_5}(x, y, z) = x z dx \wedge dz$ is a monomial. This is also true for the initial forms defined by the remaining five noncompact edges. Let's examine the compact ones, for example: $\omega_{F_1 \cap F_2}(x, y, z) = x y dx \wedge dy + x z dx \wedge dz$. Here again the NND condition is fulfilled and this will also be the case for the 2 other segments of F_1 . Now, considering facets: F_3 , F_5 and F_7 define monomials. The point $(4, 2, 2)$ associated to the monomial $x^4 y z dy \wedge dz$ may be in the support of ω , but it is outside of the face F_2 , so $\omega_{F_2}(x, y, z) = x y dx \wedge dy + x z dx \wedge dz + x^6 dy \wedge dz$, whose singular locus is the hyperplane $\{x = 0\}$. We can make the same type of verifications for F_1 , F_4 and F_6 . The initial form associated to the full face $\Gamma(\omega)$ is ω . Its singular locus is defined by a polynomial system whose set of solutions is the y -axis $\{x = z = 0\}$. We conclude that ω is NND.

3. TORIC BACKGROUND

In this section, we collect some results on normal toric varieties. We follow closely the introduction found in [AGMS13]. A general reference on toric geometry is [CLS24]. One can also consult the second chapter in [Oka97] for an introduction to toric morphisms.

3.1. Cones and fans. Let $v_1, \dots, v_q \in \mathbf{R}^n$. The *convex polyhedral cone* generated by $v_1, \dots, v_q$ is the subset

$$\sigma = \{\lambda_1 v_1 + \dots + \lambda_q v_q, \lambda_k \in \mathbf{R}_{\geq 0}, k = 1, \dots, q\} \subset \mathbf{R}^n.$$

We will often shortly refer to these objects as 'cones'. The vectors v_k are called *generators* of the cone. A convex polyhedral cone is said to be *rational* if it has a set of generators in $\mathbf{Z}^n$. A rational polyhedral cone is said to be *strongly convex* if it does not contain any nontrivial linear subspace.

Let $\sigma \subset \mathbf{R}^n$ be a strongly convex rational polyhedral cone. Let ρ be an edge (i.e. 1-dimensional face) of σ . Since σ is strongly convex, ρ is a half-line (also called a *ray*) and since ρ is rational, the semigroup $\rho \cap \mathbf{Z}^n$ is generated by a unique element called the *ray generator* of ρ . A strongly convex rational polyhedral cone is generated by the ray generators of its edges (Lemma 1.2.15. in [CLS24]). We call it the set of *minimal generators* of σ . In this article, whenever we use the expression ‘minimal generators’ for a strongly convex rational polyhedral cone, it should be understood in the previous sense.

A strongly convex rational polyhedral cone $\sigma \subset \mathbf{R}^n$ is *smooth* if its minimal generators form part of a $\mathbf{Z}$ -basis of $\mathbf{Z}^n$.

A fan Σ in $\mathbf{R}^n$ is a finite collection of cones $\sigma \subset \mathbf{R}^n$ such that:

- a) Each $\sigma \in \Sigma$ is a strongly convex rational polyhedral cone;
- b) For all $\sigma \in \Sigma$, each face of σ is also in Σ ;
- c) For all $\sigma_1, \sigma_2 \in \Sigma$, the intersection $\sigma_1 \cap \sigma_2$ is a face of each (hence, also in Σ).

Given a fan Σ in $\mathbf{R}^n$, we will say that $\sigma \in \Sigma$ is a *maximal cone* of Σ if it is maximal for the inclusion among the cones of Σ . The fan Σ is *regular* when each of its cones are smooth (in practice, it is sufficient to check that the maximal cones are smooth). Let Σ and Σ' be fans in $\mathbf{R}^n$, we say that Σ' is a refinement of Σ if every $\sigma \in \Sigma$ is a union of cones in Σ' . We refer to [CLS24] for the proof of the existence of regular refinements.

Theorem 3.1. Any fan in $\mathbf{R}^n$ has a regular refinement.

Definition 3.2. Let η be a nonidentically zero holomorphic p -form on $\mathbf{C}^n$. For any nonempty face $F \subset \Gamma(\eta)$, the normal vectors to the facets of $\Gamma(\eta)$ containing F which are in the orthant $\mathbf{R}_{\geq 0}^n$ generate a cone denoted by C_F . The set of all the such cones

$$\Sigma(\eta) = \{C_F, F \text{ is a nonempty face of } \Gamma(\eta)\}$$

is a fan called the *dual fan* of η (see [AGMS13] and [AGMM24] for an introduction in the algebraic setting, which can be adapted to a p -form since the Newton polyhedron is defined by a set of points).

3.2. Varieties and morphisms. We refer to [CLS24] and [Oka97] for the proofs of the following results. For our purposes, we will only need the expression of a toric morphism in affine charts.

Definition 3.3. A *toric variety* of dimension n is an irreducible complex algebraic variety V containing a torus $T_N \cong (\mathbf{C}^*)^n$ as a Zariski open subset such that the action of T_N on itself extends to an action $T_N \times V \rightarrow V$ given by a morphism of algebraic varieties.

To any fan Σ in $\mathbf{R}^n$, we can associate a normal separated toric variety X_Σ (Theorem 3.1.5 in [CLS24]). Moreover, if the fan Σ is regular, then X_Σ is a smooth variety.

Definition 3.4. Let Σ and Σ' be fans in $\mathbf{R}^n$. A morphism $\pi : X_\Sigma \rightarrow X_{\Sigma'}$ is *toric* if π maps the torus $T \subset X_\Sigma$ into the torus $T' \subset X_{\Sigma'}$ and $\pi|_T$ is a group morphism.

Theorem 3.5. Let Σ and Σ' be fans in $\mathbf{R}^n$. If Σ is a refinement of Σ' , then it induces a proper birational toric morphism $\pi : X_\Sigma \rightarrow X_{\Sigma'}$.

Suppose that in the previous theorem, Σ' is the fan whose only maximal cone is the orthant $\mathbf{R}_{\geq 0}^n$, then $X_{\Sigma'} = \mathbf{C}^n$. Let $\mathcal{U}_\sigma$ be an affine chart of X_Σ defined by a maximal cone σ with minimal generators (i.e. the ray generators of its edges) $v_1, \dots, v_n$. For any $k \in \{1, \dots, n\}$, we can write $v_k = (v_{1k}, \dots, v_{nk}) \in \mathbf{N}^n$. Let M be the matrix whose columns are given by the coordinates of

Each maximal cone of the dual fan has 4 minimal generators and hence, is not smooth (4 vectors cannot be part of a $\mathbf{Z}$ -basis of $\mathbf{Z}^3$). There are several ways to refine the dual fan to get a regular one. In Fig. 4 below, we present two possible refinements, but there are more.

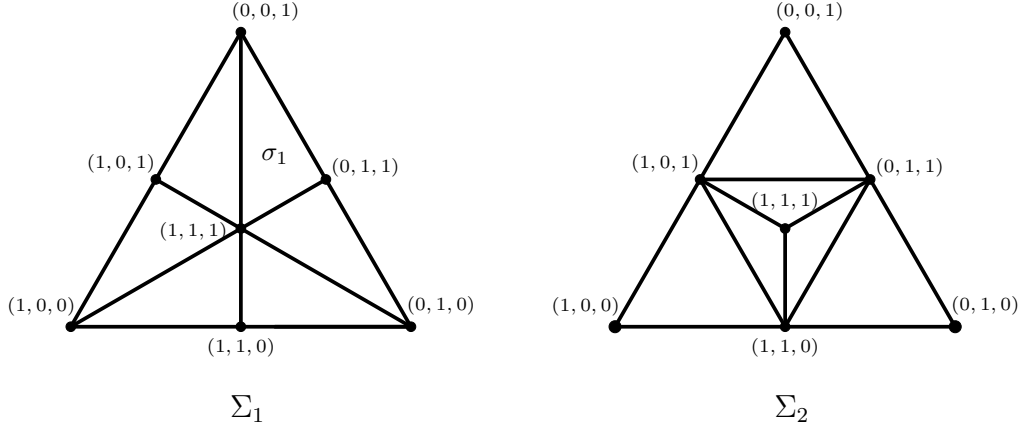


FIGURE 4. Regular refinements of the dual fan of ω .

One can check that any maximal cone σ of Σ_1 or Σ_2 with minimal generators v_1, v_2, v_3 is smooth by verifying that $\det(v_1, v_2, v_3) = \pm 1$ (the order of v_1, v_2, v_3 is not important here).

Consider the refinement Σ_1 and the cone σ_1 with minimal generators $v_1 = (1, 1, 1)$, $v_2 = (0, 1, 1)$ and $v_3 = (0, 0, 1)$.

$$M = \begin{pmatrix} v_1 & v_2 & v_3 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

Denote by $X = (x_1, x_2, x_3)$ the standard coordinates on $\mathbf{C}^3$ and $Y = (y_1, y_2, y_3)$ the coordinates associated to v_1, v_2 and v_3 (in this order) on the affine chart $\mathcal{U}_\sigma$. The morphism $\pi|_{\mathcal{U}_\sigma}$ is described by $X = Y^M$, that is, the following equations:

$$\begin{cases} x_1 = y_1^1 y_2^0 y_3^0 \\ x_2 = y_1^1 y_2^1 y_3^0 \\ x_3 = y_1^1 y_2^1 y_3^1 \end{cases} \quad \text{i.e.} \quad \begin{cases} x_1 = y_1 \\ x_2 = y_1 y_2 \\ x_3 = y_1 y_2 y_3 \end{cases}$$

Consider the following 2-form on $\mathbf{C}^3$:

$$\omega_0(x_1, x_2, x_3) = x_1^2 x_2^2 \frac{dx_1}{x_1} \wedge \frac{dx_2}{x_2} + x_1^2 x_3^2 \frac{dx_1}{x_1} \wedge \frac{dx_3}{x_3} + x_2^2 x_3^2 \frac{dx_2}{x_2} \wedge \frac{dx_3}{x_3}.$$

The form ω_0 has the same Newton polyhedron as ω in Example 2.3, but it has a simpler expression so we prefer to work with ω_0 instead of ω for this example. As ω and ω_0 share the same Newton polyhedron, it follows that they have the same dual fan. One can check that ω_0 is NND. The pull-backs of the logarithmic differentials are given below:

$$\begin{aligned}
\pi^* \left(\frac{dx_1}{x_1} \wedge \frac{dx_2}{x_2} \right) &= \frac{dy_1}{y_1} \wedge \left(\frac{dy_1}{y_1} + \frac{dy_2}{y_2} \right) = \frac{dy_1}{y_1} \wedge \frac{dy_2}{y_2}, \\
\pi^* \left(\frac{dx_1}{x_1} \wedge \frac{dx_3}{x_3} \right) &= \frac{dy_1}{y_1} \wedge \left(\frac{dy_1}{y_1} + \frac{dy_2}{y_2} + \frac{dy_3}{y_3} \right) = \frac{dy_1}{y_1} \wedge \frac{dy_2}{y_2} + \frac{dy_1}{y_1} \wedge \frac{dy_3}{y_3}, \\
\pi^* \left(\frac{dx_2}{x_2} \wedge \frac{dx_3}{x_3} \right) &= \left(\frac{dy_1}{y_1} + \frac{dy_2}{y_2} \right) \wedge \left(\frac{dy_1}{y_1} + \frac{dy_2}{y_2} + \frac{dy_3}{y_3} \right) = \frac{dy_1}{y_1} \wedge \frac{dy_3}{y_3} + \frac{dy_2}{y_2} \wedge \frac{dy_3}{y_3}.
\end{aligned}$$

This leads us to the expression of $\pi^*\omega_0$:

$$\begin{aligned}
\pi^*\omega_0 &= y_1^4 y_2^2 \frac{dy_1}{y_1} \wedge \frac{dy_2}{y_2} + y_1^4 y_2^2 y_3^2 \left(\frac{dy_1}{y_1} \wedge \frac{dy_2}{y_2} + \frac{dy_1}{y_1} \wedge \frac{dy_3}{y_3} \right) + y_1^4 y_2^4 y_3^2 \left(\frac{dy_1}{y_1} \wedge \frac{dy_3}{y_3} + \frac{dy_2}{y_2} \wedge \frac{dy_3}{y_3} \right) \\
\pi^*\omega_0 &= y_1^4 y_2^2 \underbrace{\left[(1 + y_3^2) \frac{dy_1}{y_1} \wedge \frac{dy_2}{y_2} + (y_3^2 + y_2^2 y_3^2) \frac{dy_1}{y_1} \wedge \frac{dy_3}{y_3} + y_2^2 y_3^2 \frac{dy_2}{y_2} \wedge \frac{dy_3}{y_3} \right]}_{\bar{\eta}(Y)}.
\end{aligned}$$

We notice that the coefficients of $\bar{\eta}(Y)$ do not vanish simultaneously, as would be expected from Theorem 4.1. Indeed, if $y_2^2 y_3^2 = 0$ and $y_3^2 + y_2^2 y_3^2 = 0$, then $y_3^2 = 0$, but this contradicts $1 + y_3^2 = 0$.

For $k \in \{1, 2, 3\}$, let $t_k = \min_{I \in \Gamma(\omega_0)} \langle v_k, I \rangle$. We observe on Fig. 2 that for $v_1 = (1, 1, 1)$, the minimum t_1 is obtained on the compact facet F_1 and $t_1 = 4$. For $v_2 = (0, 1, 1)$, we find $t_2 = 2$, which is achieved on F_2 and for $v_3 = (0, 0, 1)$, we get $t_3 = 0$ with F_3 . It follows that $T = (t_1, t_2, t_3) = (4, 2, 0)$ and $Y^T = y_1^4 y_2^2$, which corresponds to the factorization above. Let $A = \{k \in \{1, 2, 3\}, t_k > 0\}$. The strict transform η' of η by π is by definition given by $\eta'(Y) = Y_A \cdot \bar{\eta}(Y)$. As $A = \{1, 2\}$, we get $\eta'(Y) = y_1 y_2 \cdot \bar{\eta}(Y)$ and

$$\eta'(Y) = \underbrace{(1 + y_3^2)}_{a_1} dy_1 \wedge dy_2 + \underbrace{y_2 y_3 (1 + y_2^2)}_{a_2} dy_1 \wedge dy_3 + \underbrace{y_1 y_2^2 y_3}_{a_3} dy_2 \wedge dy_3.$$

Theorem 4.3 guarantees that every $c \in \text{Sing } \eta'$ is a nonnilpotent singularity. This means that the matrix of the linear part of $\chi = a_3 \partial_{y_1} - a_2 \partial_{y_2} + a_1 \partial_{y_3}$ at c is nonnilpotent.

The previous results would still hold with any other regular refinement (as Σ_2 , for instance).

4. REDUCTION OF SINGULARITIES

4.1. Toric reduction for NND p -forms.

Theorem 4.1. Let η be a nonidentically zero holomorphic p -form defined on $\mathbf{C}^n$. We denote by $\Gamma = \Gamma(\eta)$ its Newton polyhedron and by Σ a regular refinement of the dual fan of η . Let $\pi : X_\Sigma \rightarrow \mathbf{C}^n$ be the associated proper birational toric morphism. For every affine chart $\mathcal{U}_\sigma$ of X_Σ defined by a maximal cone $\sigma \in \Sigma$ with minimal generators $v_1, \dots, v_n$, set $t_k = \min_{I \in \Gamma(\eta)} \langle v_k, I \rangle$, where $k \in \{1, \dots, n\}$ and $T = (t_1, \dots, t_n)$ and denote by $Y = (y_1, \dots, y_n)$ the coordinates which are canonically associated to the vectors $v_1, \dots, v_n$ on $\mathcal{U}_\sigma$. Set $A = \{k \in \{1, \dots, n\}, t_k > 0\}$.

1) The pull-back $\pi^*\eta$ can be expressed locally in every affine chart $\mathcal{U}_\sigma$ as

$$\pi^*\eta(Y) = Y^T \underbrace{\left(\sum_{|K|=p} \bar{f}_K(Y) \frac{dY_K}{Y_K} \right)}_{\bar{\eta}(Y)},$$

where each coefficient $\bar{f}_K(Y)$ of $\bar{\eta}(Y)$ is holomorphic and divisible by the monomial $Y_{K \setminus A} = \prod_{k \in K \setminus A} y_k$.

- 2) η is Newton nondegenerate if and only if in every affine chart $\mathcal{U}_\sigma$, the coefficients $\bar{f}_K(Y)$ of $\bar{\eta}(Y)$ have no common zeroes.

Proof. We will keep the same notations as in Section 3.2. We want to compute the pull-back of η by π in the chart $\mathcal{U}_\sigma$. The toric morphism π is expressed in this chart by the change of variables $X = Y^M$:

$$(3) \quad \begin{cases} x_1 = y_1^{v_{11}} \cdots y_n^{v_{1n}} \\ x_2 = y_1^{v_{21}} \cdots y_n^{v_{2n}} \\ \vdots \\ x_n = y_1^{v_{n1}} \cdots y_n^{v_{nn}}. \end{cases}$$

For all $j \in \{1, \dots, n\}$, taking logarithmic differentials in Eq. (3), we obtain a change of base formula:

$$(4) \quad \frac{dx_j}{x_j} = \sum_{k=1}^n v_{jk} \frac{dy_k}{y_k}.$$

Let J be a subset of $\{1, \dots, n\}$ with p elements $j_1 < \dots < j_p$, using Eq. (4), we find

$$\frac{dX_J}{X_J} = \bigwedge_{s=1}^p \left(\sum_{k=1}^n v_{j_s k} \frac{dy_k}{y_k} \right),$$

from which we derive a change of base formula for p -logarithmic differentials:

$$(5) \quad \frac{dX_J}{X_J} = \sum_{|K|=p} |M_{J,K}| \frac{dY_K}{Y_K},$$

where $|M_{J,K}|$ is the determinant of the submatrix of M whose rows and columns are indexed respectively by J and K . Recall that η is expressed in the standard coordinate system X as in Eq. (1):

$$\eta(X) = \sum_{|J|=p} f_J(X) \frac{dX_J}{X_J}.$$

Combining Eq. (5) with the previous one, we start to compute the total transform of η by π :

$$\begin{aligned} \pi^* \eta(Y) &= \sum_{|J|=p} f_J(Y^M) \sum_{|K|=p} |M_{J,K}| \frac{dY_K}{Y_K}, \\ \pi^* \eta(Y) &= \sum_{|K|=p} \left(\sum_{|J|=p} |M_{J,K}| f_J(Y^M) \right) \frac{dY_K}{Y_K}. \end{aligned}$$

Let us expand the coefficients f_J into power series, keeping the same notations as in Eq. (2):

$$\pi^* \eta(Y) = \sum_{|K|=p} \left(\sum_{|J|=p} |M_{J,K}| \left(\sum_{I \in \mathbb{N}^n \cap \Gamma} a_{I,J} Y^{I \cdot M} \right) \right) \frac{dY_K}{Y_K},$$

where the dot $\cdot$ denotes the matrix product.

$$\pi^* \eta(Y) = \sum_{I \in \mathbb{N}^n \cap \Gamma} \left(\sum_{|K|=p} \sum_{|J|=p} a_{I,J} |M_{J,K}| \frac{dY_K}{Y_K} \right) Y^{I \cdot M}.$$

For any $I \in \mathbf{N}^n \cap \Gamma$, recall that $Y^{I \cdot M} = y_1^{\langle v_1, I \rangle} \dots y_n^{\langle v_n, I \rangle}$. For all $k \in \{1, \dots, n\}$, we have $t_k = \min_{I \in \Gamma} \langle v_k, I \rangle$ and we denote by F_k the face of Γ defined by v_k , namely

$$F_k = \{I \in \Gamma, \langle v_k, I \rangle = t_k\}.$$

Let $A = \{k \in \{1, \dots, n\}, t_k > 0\}$. We consider the divisor $(\prod_{k \in A} y_k = 0)$. This leads us to factor the previous expression by Y^T , where $T = (t_1, \dots, t_n)$:

$$\pi^* \eta(Y) = Y^T \left(\underbrace{\sum_{|K|=p} \left(\sum_{I \in \mathbf{N}^n \cap \Gamma} \sum_{|J|=p} a_{I,J} |M_{J,K}| Y^{I \cdot M - T} \right) \frac{dY_K}{Y_K}}_{\bar{f}_K(Y)} \right).$$

For any $K \subset \{1, \dots, n\}$ such that $|K| = p$, we can rewrite $\bar{f}_K(Y)$ with dot products:

$$(6) \quad \bar{f}_K(Y) = \sum_{I \in \mathbf{N}^n \cap \Gamma} \sum_{|J|=p} a_{I,J} |M_{J,K}| y_1^{\langle v_1, I \rangle - t_1} \dots y_n^{\langle v_n, I \rangle - t_n}.$$

• **Proof of statement 1).**

Let $K \subset \{1, \dots, n\}$ such that $|K| = p$ and $k \in K \setminus A$, we will show that y_k divides $\bar{f}_K(Y)$. To do so, we substitute $y_k = 0$ in Eq. (6):

$$\bar{f}_K(Y)_{|y_k=0} = \sum_{I \in F_k} \sum_{|J|=p} a_{I,J} |M_{J,K}| \prod_{s \neq k} y_s^{\langle v_s, I \rangle - t_s}.$$

Let $I \in F_k$ and $J \subset \{1, \dots, n\}$ with $|J| = p$. We will justify that the product $a_{I,J} |M_{J,K}|$ is equal to zero. This will show that the sum above is also zero and that y_k divides $\bar{f}_K(Y)$. Recall that $a_{I,J} = 0$ whenever a coordinate of I indexed by J is zero (this has been stated after Eq. (2)). Suppose now that all coordinates of I indexed by J are nonzero and let us prove that $|M_{J,K}| = 0$.

As $k \notin A$, we have $t_k = \min_{I \in \Gamma} \langle v_k, I \rangle = 0$. This implies that v_k has at least one coordinate which is 0. Let $\Lambda = \{j \in \{1, \dots, n\}, v_{jk} > 0\}$. The face $F_k = \{I \in \Gamma, \langle v_k, I \rangle = 0\}$ is the intersection of Γ with coordinate hyperplanes, namely: $F_k = \{I = (i_1, \dots, i_n) \in \Gamma, \forall s \in \Lambda, i_s = 0\}$. The matrix M can be written as:

$$M = \begin{pmatrix} v_1 & \dots & v_k & \dots & v_n \\ v_{11} & \dots & * & \dots & v_{1n} \\ \vdots & \dots & * & \dots & \vdots \\ \vdots & \dots & \vdots & \dots & \vdots \\ \vdots & \dots & \vdots & \dots & \vdots \\ v_{n1} & \dots & * & \dots & v_{nn} \end{pmatrix}.$$

The coefficient $|M_{J,K}|$ is the determinant of the matrix whose rows are indexed by J and columns are indexed by K . As $k \in K$, the k -th column of M will be part of this extracted matrix. The elements of Λ do not belong to J . Indeed, if there was an element $s \in \Lambda \cap J$, as $F_k = \{I = (i_1, \dots, i_n) \in \Gamma, \forall s \in \Lambda, i_s = 0\}$ and $I \in F_k$, the s -th coordinate of I would be indexed by an element of J and equal to zero. This contradicts our hypothesis. In the end, we get that all nonzero elements of the k -th column are placed in rows indexed by $\{1, \dots, n\} \setminus J$. Thus, the submatrix that we consider has a column of zeroes and its determinant $|M_{J,K}|$ is zero.

For any $k \in K \setminus A$, y_k divides $\bar{f}_K(Y)$, so $Y_{K \setminus A}$ divides $\bar{f}_K(Y)$.

• **Proof of the direct implication in statement 2).**

Suppose that there exists a point $c = (c_1, \dots, c_n) \in \mathbf{C}^n$ which is a common zero for the coefficients $\bar{f}_K(Y)$. Let $S = \{k \in \{1, \dots, n\}, c_k = 0\}$.

Consider the nonempty face F of Γ defined by $F = \cap_{k \in S} F_k$. Since c is a common zero of the coefficients $\bar{f}_K(Y)$, for each subset $K \subset \{1, \dots, n\}$, using Eq. (6), we get:

$$(7) \quad \begin{aligned} & \sum_{I \in F} \sum_{|J|=p} a_{I,J} |M_{J,K}| \prod_{k \notin S} c_k^{\langle v_k, I \rangle - t_k} = 0, \\ & \sum_{|J|=p} \left(\sum_{I \in F} a_{I,J} \prod_{k \notin S} c_k^{\langle v_k, I \rangle} \right) |M_{J,K}| = 0. \end{aligned}$$

We observe that for any $I \in \Gamma \cap \mathbf{N}^n$,

$$\prod_{k \notin S} c_k^{\langle v_k, I \rangle} = \prod_{k \notin S} c_k^{i_1 v_{1k} + \dots + i_n v_{nk}} = \prod_{j=1}^n \prod_{k \notin S} (c_k^{v_{jk}})^{i_j} = C^I,$$

where $C = (\prod_{k \notin S} c_k^{v_{1k}}, \dots, \prod_{k \notin S} c_k^{v_{nk}}) \in (\mathbf{C}^*)^n$. This helps us further simplify Eq. (7):

$$(8) \quad \begin{aligned} & \forall K, \sum_{|J|=p} \left(\sum_{I \in F} a_{I,J} C^I \right) |M_{J,K}| = 0, \\ & \sum_{|J|=p} (f_J)_{|F}(C) |M_{J,K}| = 0. \end{aligned}$$

Let $J_1 \prec \dots \prec J_r$ be an ordering of the subsets of $\{1, \dots, n\}$ with p elements, where $r = \binom{n}{p}$.

We introduce the following matrix:

$$M' = \begin{pmatrix} |M_{J_1, J_1}| & |M_{J_1, J_2}| & \dots & |M_{J_1, J_r}| \\ |M_{J_2, J_1}| & |M_{J_2, J_2}| & \dots & |M_{J_2, J_r}| \\ \vdots & \vdots & \ddots & \vdots \\ |M_{J_r, J_1}| & |M_{J_r, J_2}| & \dots & |M_{J_r, J_r}| \end{pmatrix}.$$

We proceed to reformulate the $r = \binom{n}{p}$ equalities in Eq. (8) as a matrix equation:

$$(9) \quad ((f_{J_1})_{|F}(C), \dots, (f_{J_r})_{|F}(C)) M' = 0.$$

Since σ is a smooth cone, M is invertible. Let us prove that M' is also invertible. Denote by $(e_1, \dots, e_n)$ the canonical basis of $\mathbf{C}^n$ and $u : \mathbf{C}^n \rightarrow \mathbf{C}^n$ the $\mathbf{C}$ -linear map represented by the matrix M in this basis. For any subset $J \subset \{1, \dots, n\}$ such that $J = \{j_1, \dots, j_p\}$ and $j_1 < \dots < j_p$, set $e_J = e_{j_1} \wedge \dots \wedge e_{j_p}$. The p -th exterior power $\wedge^p u$ of the map u is defined by $\wedge^p u(e_J) = u(e_{j_1}) \wedge \dots \wedge u(e_{j_p})$. For any subset $K \subset \{1, \dots, n\}$ with p elements $k_1 < \dots < k_p$,

$$\wedge^p u(e_K) = \bigwedge_{\ell=1}^p u(e_{k_\ell}) = \bigwedge_{\ell=1}^p \left(\sum_{i=1}^n v_{ik_\ell} e_i \right) = \sum_{|J|=p} |M_{J,K}| e_J.$$

The family of vectors $(e_{J_1}, \dots, e_{J_r})$ is a basis of $\wedge^p \mathbf{C}^n$. For any $\ell \in \{1, \dots, r\}$,

$$\wedge^p u(e_{J_\ell}) = \sum_{s=1}^r |M_{J_s, J_\ell}| e_{J_s}.$$

This proves that M' is the matrix of $\wedge^p u$ in the basis $(e_{J_1}, \dots, e_{J_r})$. The exterior power of an invertible linear endomorphism is invertible. Thus, $\wedge^p u$ is invertible and so is the matrix M' .

This implies that the unique solution of Eq. (9) is

$$((f_{J_1})|_F(C), \dots, (f_{J_r})|_F(C)) = (0, \dots, 0)$$

and that $C \in (\mathbf{C}^*)^n$ is a common zero for the initial forms $(f_{J_k})|_F$. This is impossible because η is NND, so the coefficients $\bar{f}_K(Y)$ have no common zeroes.

• **Proof of the inverse implication in statement 2).**

Let's suppose that η is not NND and find an affine chart where the coefficients of the total transform have a common zero. By hypothesis, there exists a nonempty face $F \subset \Gamma(\eta)$ and a point $c = (c_1, \dots, c_n) \in (\mathbf{C}^*)^n$ where all the initial forms $(f_J)|_F$ vanish.

As Σ is a regular refinement of the dual fan of $\Gamma(\eta)$, we can always find an affine chart which "captures" the face F . More precisely: there is a chart $\mathcal{U}_\sigma$ defined by a maximal cone $\sigma \in \Sigma$ with minimal generators $v_1, \dots, v_n$ such that some vectors v_k define F :

$$F = \bigcap_{k \in R} F_k ,$$

where $R \subset \{1, \dots, n\}$. Set $c' = c^{M^{-1}}$. For any $k \in \{1, \dots, n\}$, set $\tilde{c}_k = 0$ if $k \in R$ and $\tilde{c}_k = c'_k$ if $k \notin R$. Let $\tilde{c} = (\tilde{c}_1, \dots, \tilde{c}_n)$.

Using the expression of the coefficients $\bar{f}_K$ in Eq. (6), we evaluate at $\tilde{c}$:

$$\begin{aligned} \forall K, \bar{f}_K(\tilde{c}) &= \sum_{I \in \mathbf{N}^n \cap \Gamma} \sum_{|J|=p} a_{I,J} |M_{J,K}| \tilde{c}_1^{\langle v_1, I \rangle - t_1} \dots \tilde{c}_n^{\langle v_n, I \rangle - t_n}, \\ \bar{f}_K(\tilde{c}) &= \sum_{I \in F} \sum_{|J|=p} a_{I,J} |M_{J,K}| \prod_{k \notin R} \tilde{c}_k^{\langle v_k, I \rangle - t_k}, \\ \bar{f}_K(\tilde{c}) &= \sum_{|J|=p} |M_{J,K}| \sum_{I \in F} a_{I,J} \prod_{k \notin R} c_k'^{\langle v_k, I \rangle - t_k}. \end{aligned}$$

For any $I \in F$ and $k \in R$, $c_k'^{\langle v_k, I \rangle - t_k} = (c'_k)^0 = 1$, hence we have

$$\prod_{k \notin R} c_k'^{\langle v_k, I \rangle - t_k} = \prod_{k=1}^n c_k'^{\langle v_k, I \rangle - t_k}.$$

$$\begin{aligned} \bar{f}_K(\tilde{c}) &= \sum_{|J|=p} |M_{J,K}| \sum_{I \in F} a_{I,J} \prod_{k=1}^n c_k'^{\langle v_k, I \rangle - t_k}, \\ \bar{f}_K(\tilde{c}) &= \sum_{|J|=p} |M_{J,K}| \sum_{I \in F} a_{I,J} (c')^{I \cdot M - T}, \\ \bar{f}_K(\tilde{c}) &= \sum_{|J|=p} |M_{J,K}| \sum_{I \in F} a_{I,J} c^{I \cdot M \cdot M^{-1} - T M^{-1}}, \end{aligned}$$

because $c' = c^{M^{-1}}$. Factoring by $c^{-T M^{-1}}$, we find

$$\begin{aligned}\bar{f}_K(\tilde{c}) &= c^{-TM^{-1}} \sum_{|J|=p} |M_{J,K}| \sum_{I \in F} a_{I,J} c^I, \\ \bar{f}_K(\tilde{c}) &= c^{-TM^{-1}} \sum_{|J|=p} |M_{J,K}| (f_J)_{|F}(c), \\ \bar{f}_K(\tilde{c}) &= 0,\end{aligned}$$

since $(f_J)_{|F}(c) = 0$ by hypothesis. □

4.2. An application to $(n-1)$ -forms.

Definition 4.2. To any holomorphic $(n-1)$ -form η on $\mathbf{C}^n$ expressed in the standard coordinates by

$$\eta(X) = \sum_{|J|=n-1} g_J(X) dX_J,$$

we can associate by duality a vector field χ such that:

$$(10) \quad \chi(X) = \sum_{k=1}^n (-1)^{k-1} g_{J_k}(X) \frac{\partial}{\partial x_k},$$

where $J_k = \{1, \dots, n\} \setminus \{k\}$ for all $k = 1, \dots, n$. A singular point c of η is said to be *nonnilpotent* if the matrix of the linear part of χ at c is nonnilpotent.

In the following theorem, the *strict transform* η' of η by π is, by definition, expressed on an affine chart by $\eta'(Y) = Y_A \cdot \bar{\eta}(Y)$ (with the notations in Theorem 4.1).

Theorem 4.3. Let η be a nonidentically zero holomorphic p -form on $\mathbf{C}^n$. If we suppose that η is NND and $p = n-1$, then the strict transform η' of η by the toric morphism defined via a regular refinement of its dual fan has at worst nonnilpotent singularities.

Proof. We will use the same notations as in Theorem 4.1 and its proof. Recall that the pull-back $\pi^*\eta$ can be expressed locally in an affine chart $\mathcal{U}_\sigma$ as

$$\pi^*\eta(Y) = Y^T \left(\sum_{|K|=p} \bar{f}_K(Y) \frac{dY_K}{Y_K} \right).$$

Let us suppose that η is NND and $p = n-1$. Recall that for all $k = 1, \dots, n$, we have $J_k = \{1, \dots, n\} \setminus \{k\}$. The strict transform η' of η by π can be expressed as:

$$\begin{aligned}\eta'(Y) &= Y_A \cdot \left(\sum_{k=1}^n \bar{f}_{J_k}(Y) \frac{dY_{J_k}}{Y_{J_k}} \right) \\ \eta'(Y) &= \sum_{k \in A} y_k a_k dY_{J_k} + \sum_{k \notin A} a_k dY_{J_k},\end{aligned}$$

where $Y_{J_k \setminus A} \cdot a_k = \bar{f}_{J_k}(Y)$. As η is NND, the coefficients $\bar{f}_{J_k}$ do not have common zeroes and therefore, neither do the coefficients a_k . Now, consider the vector field χ associated by duality to η' as in Eq. (10). Theorem 4.3 follows from the lemma below.

Lemma 4.4. Given $0 \leq \ell \leq n$, let

$$\xi = \sum_{i=1}^{\ell} b_i x_i \frac{\partial}{\partial x_i} + \sum_{i=\ell+1}^n b_i \frac{\partial}{\partial x_i}$$

be an affine vector field in $\mathbf{C}^n$. If the coefficients $b_1, b_2, \dots, b_n$ have no common zeroes, then ξ has at worst nonnilpotent singularities.

Let $c = (c_1, \dots, c_n) \in \mathbf{C}^n$ be a singular point of ξ . By definition, we have the equalities $c_k b_k(c) = 0$ for $k = 1, \dots, \ell$ and $b_k(c) = 0$ for $k = \ell + 1, \dots, n$. By hypothesis, the b_k 's do not all vanish at c , so there exists $s \in \{1, \dots, \ell\}$ such that $b_s(c) \neq 0$, so $c_s = 0$. We remark that $b_s(c)$ is a nonzero eigenvalue of the matrix of the linear part of ξ at c . Indeed, its s -th column is

$$\begin{pmatrix} c_s \partial_{x_1} b_s(c) \\ \vdots \\ c_s \partial_{x_{s-1}} b_s(c) \\ (b_s(c) + c_s \partial_{x_s} b_s(c)) \\ c_s \partial_{x_{s+1}} b_s(c) \\ \vdots \\ c_s \partial_{x_n} b_s(c) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ b_s(c) \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

We conclude that the singular point c is nonnilpotent. □

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