

COBORDISM OF ALGEBRAIC KNOTS DEFINED BY BRIESKORN POLYNOMIALS, II

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ABSTRACT. In our previous paper, we obtained several results concerning cobordisms of algebraic knots associated with Brieskorn polynomials: for example, under certain conditions, we showed that the exponents are cobordism invariants. In this paper, we further obtain new results concerning the Fox–Milnor relations, decomposition of the algebraic cobordism class of an algebraic knot associated with a Brieskorn polynomial that has a null-cobordant factor over the field of rational numbers, and cyclic suspensions of knots. We also show that a certain infinite family of spherical algebraic knots associated with Brieskorn polynomials are linearly independent in the knot cobordism group.

1. INTRODUCTION

Let $f : (\mathbf{C}^{n+1}, \mathbf{0}) \rightarrow (\mathbf{C}, 0)$, $n \geq 1$, be a holomorphic function germ with an isolated critical point at the origin. For a sufficiently small positive real number $\varepsilon > 0$, set $K_f = S_\varepsilon^{2n+1} \cap V_f$, where $V_f = f^{-1}(0)$ is the complex hypersurface in $\mathbf{C}^{n+1}$ with an isolated singularity at the origin and S_ε^{2n+1} is the sphere of radius ε centered at the origin in $\mathbf{C}^{n+1}$ (see Fig. 1). It is known that K_f is an $(n-2)$ -connected, oriented $(2n-1)$ -dimensional submanifold of $S_\varepsilon^{2n+1} = S^{2n+1}$, that its complement fibers over the circle S^1 , and that the isotopy class of K_f in S^{2n+1} is independent of the choice of ε as long as it is sufficiently small (see [19]). Note also that the embedded topology of $V_f \subset \mathbf{C}^{n+1}$ around the origin determines and is determined by the (oriented) isotopy class of $K_f \subset S^{2n+1}$ (see [24]). We call K_f the *algebraic knot* associated with f . In this paper, a *knot* (or a $(2n-1)$ -*knot*) refers to (the isotopy class of) an $(n-2)$ -connected, oriented $(2n-1)$ -dimensional submanifold of S^{2n+1} . (Here, when $n = 1$, a submanifold is (-1) -connected if it is nonempty.) A $(2n-1)$ -knot K is *spherical* if it is homeomorphic to the sphere S^{2n-1} .

In this paper, we consider Brieskorn polynomials

$$(1.1) \quad f(z_1, z_2, \dots, z_{n+1}) = z_1^{a_1} + z_2^{a_2} + \dots + z_{n+1}^{a_{n+1}}$$

and their associated algebraic knots K_f [4], where we basically assume that the exponents satisfy $a_j \geq 2$, $1 \leq j \leq n+1$, since otherwise V_f is non-singular at the origin and the associated algebraic knot is trivial. We especially focus on the study of their properties concerning cobordisms. Two knots K_0 and K_1 in S^{2n+1} are said to be *cobordant* if there exists a properly embedded oriented submanifold X , abstractly diffeomorphic to $K_0 \times [0, 1]$, of $S^{2n+1} \times [0, 1]$ such that $X \cap (S^{2n+1} \times \{0\}) = K_0$, and $X \cap (S^{2n+1} \times \{1\}) = -K_1^!$, where $-K_1^!$ is the mirror image of K_1 with reversed orientation (see [1, 2]). We say that a $(2n-1)$ -knot K in S^{2n+1} is *null-cobordant* if it bounds a smoothly embedded $2n$ -dimensional disk in D^{2n+2} (note that $\partial D^{2n+2} = S^{2n+1}$).

In our previous paper [3], we obtained several results concerning cobordisms of algebraic knots associated with Brieskorn polynomials: for example, under certain conditions, we showed that the exponents are cobordism invariants. In this paper, we further obtain new results concerning the Fox–Milnor relations for Alexander polynomials, decomposition of the algebraic cobordism class

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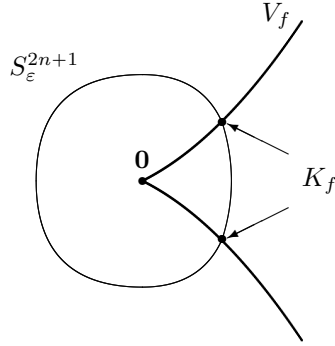


FIGURE 1. The algebraic knot K_f associated with the singularity at 0 of a germ f

of a spherical algebraic knot associated with a Brieskorn polynomial that has a null-cobordant factor over the field of rational numbers, and cyclic suspensions of knots.

The present paper is organized as follows. In §2, we recall several basic definitions and properties concerning invariants and cobordisms of algebraic knots such as Alexander polynomials and Seifert forms.

In §3, we focus on the Fox–Milnor relations for Alexander polynomials [9, 10] and give a complete characterization of Brieskorn polynomials that give algebraic knots whose Alexander polynomials satisfy the Fox–Milnor relation in terms of their exponents. As a consequence, we show that a (non-trivial) algebraic knot associated with a Brieskorn polynomial is never null-cobordant: moreover, it turns out that a spherical (non-trivial) algebraic knot associated with a Brieskorn polynomial always has infinite order in the knot cobordism group. In fact, Michel [18] has shown that such a result holds for (non-trivial) algebraic knots in general: our proof shows that an argument based on the Fox–Milnor relation serves well for deducing such results at least for Brieskorn polynomials.

In §4, we consider the linear independence of a family of spherical algebraic knots associated with certain Brieskorn polynomials in the knot cobordism group. In fact, Litherland [16] has shown that the spherical algebraic knots in S^3 associated with Brieskorn polynomials of two variables (in fact, such knots are torus knots) are linearly independent in the 1-dimensional knot cobordism group by using a certain signature invariant. We will use the same idea to prove a similar linear independence result for higher dimensions.

In §5, we consider the group of algebraic cobordism classes of spherical knots which has been introduced and studied by Levine [14, 15] (see also [6]). We give an explicit example of a spherical algebraic knot associated with a Brieskorn polynomial such that its algebraic cobordism class has a decomposition into those corresponding to the irreducible factors of its Alexander polynomial over the field of rational numbers and that one of them is algebraically null-cobordant. This shows that cobordant spherical algebraic knots associated with Brieskorn polynomials may not share the same irreducible factors of their Alexander polynomials, and therefore the study of cobordism classes of algebraic knots associated with Brieskorn polynomials might be more complicated than is expected.

Finally in §6, we consider cyclic suspensions of knots [12, 22] and study its relationship to the cobordisms. Note that the algebraic knot associated with a polynomial of the form $f(z_1, z_2, \dots, z_{n+1}) + z_{n+2}^d$ is the d -fold cyclic suspension of the algebraic knot associated with f . We will see that the cyclic suspension of knots often behaves very badly with respect to cobordisms. For example we show that certain cyclic suspensions of the algebraic knots constructed by Du Bois–Michel in [7], which are cobordant to each other, are not diffeomorphic and are not cobordant.

Throughout the paper, all manifolds and maps between them are smooth of class C^∞ . The symbol “ $\cong$ ” means a diffeomorphism between manifolds or an appropriate isomorphism between algebraic objects.

2. PRELIMINARIES

This paper is a sequel of our previous paper [3]. The reader is expected to be familiar with the notions and basic results explained in that paper, although we will repeat them when necessary in the present paper.

Let K be a $(2n-1)$ -knot in S^{2n+1} . Suppose that there exists a locally trivial fibration $\varphi : S^{2n+1} \setminus K \rightarrow S^1$. We also assume that there is a trivialization $\tau : N(K) \rightarrow K \times D^2$ of the normal disk bundle neighborhood $N(K)$ of K in S^{2n+1} such that the composition

$$N(K) \setminus K \xrightarrow{\tau|_{N(K) \setminus K}} K \times (D^2 \setminus \{0\}) \xrightarrow{pr_2} D^2 \setminus \{0\} \xrightarrow{r} S^1$$

coincides with $\varphi|_{N(K) \setminus K}$, where pr_2 is the projection to the second factor and r is the radial projection. Then, we say that K is a *fibred knot*. We call the closure F of a fiber of φ a *fiber*. Note that it is a $2n$ -dimensional compact oriented submanifold of S^{2n+1} whose boundary coincides with K . A $(2n-1)$ -dimensional fibred knot K is *simple* if it is $(n-2)$ -connected and F is $(n-1)$ -connected. (Here, for $n=1$, a manifold is (-1) -connected if it is nonempty.) In this case, F is homotopy equivalent to a bouquet of n -dimensional spheres (for example, see [19, Theorem 6.5]). Note that an algebraic knot associated with a holomorphic function germ $f : (\mathbf{C}^{n+1}, \mathbf{0}) \rightarrow (\mathbf{C}, 0)$ with an isolated critical point at the origin is a simple fibred knot [19]. In this case, a fiber of such an algebraic knot is called a *Milnor fiber* for f .

Let $\psi : F \rightarrow F$ be a *geometric monodromy* of the fibration φ ; i.e., it is a diffeomorphism which is constructed by integrating an appropriate horizontal vector field on $S^{2n+1} \setminus K$ with respect to φ and which is the identity on the boundary. In other words, $S^{2n+1} \setminus \text{Int } N(K)$ is diffeomorphic to the manifold

$$F \times [0, 1] / (x, 1) \sim (\psi(x), 0), \quad x \in F,$$

obtained by identifying $F \times \{1\}$ and $F \times \{0\}$ by ψ . It is known that the geometric monodromy is well-defined up to isotopy. Either of the isomorphisms

$$\psi_* : H_n(F; \mathbf{Z}) \rightarrow H_n(F; \mathbf{Z}) \quad \text{or} \quad \psi^* : H^n(F; \mathbf{Z}) \rightarrow H^n(F; \mathbf{Z})$$

is called the *algebraic monodromy*. Its characteristic polynomial $\Delta_K(t) \in \mathbf{Z}[t]$, which is well-defined for both of ψ_* and ψ^* , is often called the *Alexander polynomial* of K . When K is an algebraic knot associated with a holomorphic function germ f , we often denote $\Delta_{K_f}(t)$ by $\Delta_f(t)$.

Let us consider the multiplicative group $\mathbf{C}^*$ and its group ring $\mathbf{Z}\mathbf{C}^*$ over the integers. For a monic polynomial $\Delta(t)$ with nonzero constant term, we denote by $\text{divisor } \Delta$ the element

$$\sum m_\xi \langle \xi \rangle \in \mathbf{Z}\mathbf{C}^*,$$

where $\xi \in \mathbf{C}^*$ runs over all roots of $\Delta(t)$ and $m_\xi \in \mathbf{Z}$ is its multiplicity. If $\Delta(t) = 1$, then we set $\text{divisor } \Delta = 0$.

We also set

$$\Lambda_a = \text{divisor}(t^a - 1)$$

for a positive integer a . Note that the family of such elements $\{\Lambda_a\}_a$, where a runs over all positive integers, is linearly independent over $\mathbf{Z}$.

Now, let us consider a Brieskorn polynomial as in (1.1). Then, by Brieskorn [4], it is known that

$$(2.1) \quad \text{divisor } \Delta_f = (\Lambda_{a_1} - 1)(\Lambda_{a_2} - 1) \cdots (\Lambda_{a_{n+1}} - 1),$$

where $1 \in \mathbf{Z}\mathbf{C}^*$ means $\Lambda_1 = 1\langle 1 \rangle$. This implies that the roots of $\Delta_f(t)$ are all roots of unity and that $\Delta_f(t)$ is a product of cyclotomic polynomials. In particular, each irreducible factor

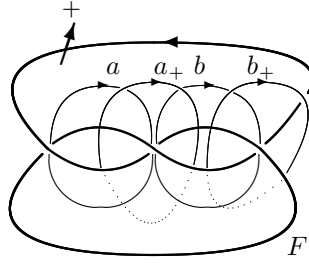


FIGURE 2. Computing a Seifert matrix for the trefoil knot

$\gamma(t) \in \mathbf{Z}[t]$ of $\Delta_f(t)$ is *symmetric*, i.e., we have $\gamma(t) = \pm t^{\deg \gamma} \gamma(t^{-1})$, where $\deg \gamma \in \mathbf{Z}$ is the degree of the polynomial γ .

Let K be a $(2n-1)$ -knot. We say that K is *spherical* if K is homeomorphic to the $(2n-1)$ -dimensional sphere. This means that K may be an exotic sphere, which is homeomorphic but not diffeomorphic to the standard sphere. When K is a simple fibered $(2n-1)$ -knot with $n \neq 2$, it is known that K is spherical if and only if $\Delta_K(1) = \pm 1$ (for example, see [19, Theorem 8.5]). For algebraic knots associated with a Brieskorn polynomial, there is a characterization of spherical knots due to Brieskorn [4] in terms of the exponents (for details, see Theorem 3.18 and Remark 3.19 of the present paper).

Let K be a simple fibered $(2n-1)$ -knot with fiber F . We define the bilinear form

$$\theta_K : H_n(F; \mathbf{Z}) \times H_n(F; \mathbf{Z}) \rightarrow \mathbf{Z}$$

by $\theta_K(\alpha, \beta) = \text{lk}(a_+, b)$, where a and b are n -cycles representing α and β , respectively, a_+ is the n -cycle in S^{2n+1} obtained by pushing a into the positive normal direction of F , and lk denotes the linking number of n -cycles in S^{2n+1} (see Fig. 2). The bilinear form θ_K is called the *Seifert form* of K and its representative matrix is called a *Seifert matrix*. It is known that a Seifert form is unimodular, i.e., the determinant of the Seifert matrix L_K is equal to ± 1 , due to Alexander duality (see [8]). Furthermore, it is also known that the Alexander polynomial $\Delta_K(t)$ coincides with $\pm \det(tL_K + (-1)^n L_K^T)$, where L_K^T denotes the transpose of L_K (for example, see [25]).

It is known that for $n \geq 3$, there is a one-to-one correspondence, through Seifert forms, between the set of isomorphism classes of unimodular bilinear forms over the integers and the set of isotopy classes of simple fibered $(2n-1)$ -knots [8, 11].

Recall that the set of cobordism classes of spherical $(2n-1)$ -knots forms an additive group under the connected sum operation. This is called the $(2n-1)$ -dimensional knot cobordism group and is denoted by C_{2n-1} (for example, see [6, 14, 15]). The class of the trivial knot is the neutral element, and the inverse of the cobordism class of a knot K is the class of $-K$. Note that for $n > 1$, it is known that $C_{2n-1} \cong \mathbf{Z}^\infty \oplus \mathbf{Z}_2^\infty \oplus \mathbf{Z}_4^\infty$ (see [14, 15]).

Let us now recall the algebraic cobordism group G_ε , where $\varepsilon = (-1)^n$ (for details, see [14]). We consider square integer matrices A such that $A + \varepsilon A^T$ is unimodular: such a matrix is called an ε -matrix. An ε -matrix N is *null-cobordant* if N is congruent to a matrix of the form

$$\begin{pmatrix} 0 & N_1 \\ N_2 & N_3 \end{pmatrix},$$

where N_1, N_2 and N_3 are square matrices of the same size. Two ε -matrices A_1 and A_2 are *cobordant* if $A_1 \oplus (-A_2)$ is null-cobordant. Then, it is known that this defines an equivalence relation for ε -matrices, and the set G_ε of cobordism classes of ε -matrices forms an abelian group under block sum “ $\oplus$ ”. Then, by Levine [14], it has been proved that for $n > 2$, the knot cobordism group C_{2n-1} is isomorphic to G_ε with $\varepsilon = (-1)^n$.

Now, let us consider square matrices B with entries in the field $\mathbf{Q}$ of rational numbers. We say that B is *admissible* if

$$(B - B^T)(B + B^T)$$

is nonsingular. Then, the cobordism relation is also defined for admissible matrices, and the set $G^{\mathbf{Q}}$ of cobordism classes of admissible matrices again forms an abelian group under block sum. As $\det(A - \varepsilon A^T) \equiv \det(A + \varepsilon A^T) \pmod{2}$ holds for square integer matrices A , we see that if $A + \varepsilon A^T$ is unimodular, then $A - \varepsilon A^T$ has odd determinant, which implies that ε -matrices are admissible. Moreover, it is known that the natural inclusion $G_{\varepsilon} \rightarrow G^{\mathbf{Q}}$ is a monomorphism. Furthermore, a complete set of invariants for $G^{\mathbf{Q}}$ has been given by Levine [15].

It is also known that $G^{\mathbf{Q}}$ is isomorphic to the group $G_{\mathbf{Q}}$ of cobordism classes of *isometric structures* over $\mathbf{Q}$ (see [6]).

Let us now consider the case of algebraic knots. Let $f : (\mathbf{C}^{n+1}, \mathbf{0}) \rightarrow (\mathbf{C}, 0)$ be a holomorphic function germ with an isolated critical point at the origin. Let us define several notions concerning the Seifert form θ_{K_f} of the algebraic knot K_f associated with f .

DEFINITION 2.1. Two bilinear forms $\theta_i : H_i \times H_i \rightarrow \mathbf{Z}$, $i = 0, 1$, defined on free abelian groups H_i of finite ranks are said to be *Witt equivalent* (or *cobordant*) if there exists a direct summand M of $H_0 \oplus H_1$ such that $(\theta_0 \oplus (-\theta_1))(x, y) = 0$ for all $x, y \in M$ and twice the rank of M is equal to the rank of $H_0 \oplus H_1$. In this case, M is called a *metabolizer*.

Furthermore, we say that θ_0 and θ_1 are *Witt equivalent over the real numbers* if there exists a vector subspace $M_{\mathbf{R}}$ of $(H_0 \otimes \mathbf{R}) \oplus (H_1 \otimes \mathbf{R})$ such that $(\theta_0^{\mathbf{R}} \oplus (-\theta_1^{\mathbf{R}}))(x, y) = 0$ for all $x, y \in M_{\mathbf{R}}$ and $2 \dim_{\mathbf{R}} M_{\mathbf{R}} = \dim_{\mathbf{R}}(H_0 \otimes \mathbf{R}) + \dim_{\mathbf{R}}(H_1 \otimes \mathbf{R})$, where $\theta_i^{\mathbf{R}} : (H_i \otimes \mathbf{R}) \times (H_i \otimes \mathbf{R}) \rightarrow \mathbf{R}$ is the real bilinear form associated with θ_i , $i = 0, 1$.

Now, if n is odd, let us consider

$$\tilde{f}(z_1, z_2, \dots, z_{n+1}, z_{n+2}) = f(z_1, z_2, \dots, z_{n+1}) + z_{n+2}^2.$$

It is known that the Seifert form θ_{K_f} for K_f coincides with $\theta_{K_{\tilde{f}}}$ for $K_{\tilde{f}}$ (for example, see [26]). In the following, we may assume that n is even. We have the decomposition

$$H^n(F_f; \mathbf{C}) = \oplus_{\lambda} H^n(F_f; \mathbf{C})_{\lambda},$$

where F_f is the Milnor fiber for f , λ runs over all the complex roots of the Alexander polynomial $\Delta_f(t)$, and $H^n(F_f; \mathbf{C})_{\lambda}$ is the eigenspace of the algebraic monodromy $H^n(F_f; \mathbf{C}) \rightarrow H^n(F_f; \mathbf{C})$ with respect to the complex coefficients corresponding to the eigenvalue λ . It is known that the intersection form S_f of F_f on $H^n(F_f; \mathbf{C})$ is given by $S_f = L_f + L_f^T$ (refer to (5.1) of §5). Furthermore, this decomposes as the orthogonal direct sum of $(S_f)|_{H^n(F_f; \mathbf{C})_{\lambda}}$. Let $\mu(f)_{\lambda}^+$ (resp. $\mu(f)_{\lambda}^-$) denote the number of positive (resp. negative) eigenvalues of $(S_f)|_{H^n(F_f; \mathbf{C})_{\lambda}}$. Then, the integer

$$\sigma_{\lambda}(f) = \mu(f)_{\lambda}^+ - \mu(f)_{\lambda}^-$$

is called the *equivariant signature* of K_f with respect to λ (for details, see [23, 27]). Note that by the same construction, the equivariant signatures are defined for fibered $(2n-1)$ -knots in general. For spherical $(2n-1)$ -knots, by appropriately defining an *isometric structure* by using a Seifert form, one can also define their equivariant signatures (see [15]).

Then, it is known that the equivariant signatures are integer-valued cobordism invariants. Furthermore, they are additive under connected sum and for a spherical knot, all the equivariant signatures vanish if and only if it is of finite order in the knot cobordism group (see [15, §22]). In fact, it has been known that if two knots are cobordant, then their Seifert forms are Witt equivalent over the real numbers, and that two Seifert forms are Witt equivalent over the real numbers if and only if all the equivariant signatures coincide. For details, the reader is referred to [3].

3. FOX–MILNOR RELATION

Let $\Delta_f(t)$ and $\Delta_g(t)$ denote the Alexander polynomials for the algebraic knots K_f and K_g associated with f and g , respectively, where f and $g : (\mathbf{C}^{n+1}, \mathbf{0}) \rightarrow (\mathbf{C}, 0)$, $n \geq 1$, are holomorphic function germs with an isolated critical point at the origin. We say that the Alexander polynomials satisfy the *Fox–Milnor relation* if there exists a polynomial $\gamma(t)$ with integer coefficients such that $\Delta_f(t)\Delta_g(t) = \pm t^{\deg \gamma} \gamma(t)\gamma(t^{-1})$ ([9, 10]). It is known that if K_f and K_g are cobordant, then their Alexander polynomials satisfy the Fox–Milnor relation (for details, see [9, 10, 1]. See also [2], for example).

We have the following characterization of Alexander polynomials which satisfy the Fox–Milnor relation. In the following, for two elements d_1 and $d_2 \in \mathbf{Z}\mathbf{C}^*$, we write $d_1 \equiv d_2 \pmod{2}$ if there exists an element $d_3 \in \mathbf{Z}\mathbf{C}^*$ such that $d_1 - d_2 = 2d_3$ holds in $\mathbf{Z}\mathbf{C}^*$.

LEMMA 3.1. *For algebraic knots K_f and K_g as above, the following three are equivalent to each other.*

- (1) *The Alexander polynomials $\Delta_f(t)$ and $\Delta_g(t)$ satisfy the Fox–Milnor relation.*
- (2) *We have $\Delta_f(t)\Delta_g(t) = \gamma(t)^2$ for some $\gamma(t) \in \mathbf{Z}[t]$.*
- (3) *We have $\text{divisor } \Delta_f(t) \equiv \text{divisor } \Delta_g(t) \pmod{2}$.*

Proof. By [5], the Alexander polynomials $\Delta_f(t)$ and $\Delta_g(t)$ are products of cyclotomic polynomials; in particular, each of their irreducible factors is symmetric. Therefore, their Alexander polynomials satisfy the Fox–Milnor relation if and only if $\Delta_f(t)\Delta_g(t) = \gamma(t)^2$ for some $\gamma(t) \in \mathbf{Z}[t]$.

On the other hand, if $\Delta_f(t)\Delta_g(t) = \gamma(t)^2$ for some $\gamma(t) \in \mathbf{Z}[t]$, then obviously the congruence in (3) holds. Conversely, if the congruence in (3) holds, then as $\Delta_f(t)\Delta_g(t)$ is a product of cyclotomic polynomials, each irreducible factor appears an even number of times, so that (2) holds. This completes the proof. $\square$

Let

$$f(z) = z_1^{a_1} + z_2^{a_2} + \cdots + z_{n+1}^{a_{n+1}}$$

be a Brieskorn polynomial, $a_j \geq 1$ for all j . Set $E_f = \{a_1, a_2, \dots, a_{n+1}\}$, which may contain the same integer multiple times and is considered to be a *multi-set*.

DEFINITION 3.2. From E_f , we construct the (non-multi-)subset $\overline{E}_f \subset E_f$ by the successive procedure as follows.

- (1) Take off all those even integers which appear an even number of times.
- (2) Take off the multiple elements except for one in such a way that we get a non-multi-set.
- (3) Take off a_j if it is an integer multiple of an odd a_k with $k \neq j$.

We call the set $\overline{E}_f$ thus obtained the *essential exponent set* of f . Note that $\overline{E}_f$ can be empty and that $\overline{E}_f = \{1\}$ if some exponent is equal to 1.

The first main result of this paper is the following.

THEOREM 3.3. *Let*

$$f(z) = z_1^{a_1} + z_2^{a_2} + \cdots + z_{n+1}^{a_{n+1}} \text{ and } g(z) = z_1^{b_1} + z_2^{b_2} + \cdots + z_{n+1}^{b_{n+1}}$$

be Brieskorn polynomials with $a_j \geq 2$ and $b_j \geq 2$ for all j . Then, the Alexander polynomials $\Delta_f(t)$ and $\Delta_g(t)$ satisfy the Fox–Milnor relation if and only if their essential exponent sets coincide, i.e. $\overline{E}_f = \overline{E}_g$.

EXAMPLE 3.4. For example, consider

$$f(z) = z_1^3 + z_2^4 + z_3^4 + z_4^6 + z_5^9 \text{ and } g(z) = z_1^2 + z_2^2 + z_3^3 + z_4^3 + z_5^{12}.$$

Then, we have

$$E_f = \{3, 4, 4, 6, 9\} \text{ and } E_g = \{2, 2, 3, 3, 12\}.$$

In the process of Definition 3.2, after (1), we get the multi-sets $\{3, 6, 9\}$ and $\{3, 3, 12\}$ for f and g , respectively. After (2), we get the sets $\{3, 6, 9\}$ and $\{3, 12\}$. Finally, after (3), we get the sets $\{3\}$ and $\{3\}$. Hence, we get $\overline{E}_f = \overline{E}_g = \{3\}$ and $\Delta_f(t)$ and $\Delta_g(t)$ satisfy the Fox–Milnor relation. In fact, by the formula (2.1) with the help of Lemma 3.5 below, we have

$$\begin{aligned} \text{divisor } \Delta_f(t) &= 4\Lambda_{12} - \Lambda_3 - 1, \\ \text{divisor } \Delta_g(t) &= 24\Lambda_{36} + 6\Lambda_{18} - 6\Lambda_{12} - 2\Lambda_9 - 2\Lambda_6 - 2\Lambda_4 + \Lambda_3 - 1, \end{aligned}$$

so we can verify that $\Delta_f(t)$ and $\Delta_g(t)$ satisfy the Fox–Milnor relation by virtue of Lemma 3.1.

Note that by the signature formula due to Brieskorn [4], we see that the signatures of the 8-dimensional Milnor fibers for f and g are equal to 274 and 30, respectively. Thus, K_f and K_g are not cobordant, since the signature of a fiber of a fibered knot is a cobordism invariant, which can be proved by using the fact that cobordant fibered knots have algebraically cobordant Seifert forms [1]. Nevertheless, their Alexander polynomials satisfy the Fox–Milnor relation.

On the other hand, for

$$h(z) = z_1^3 + z_2^4 + z_3^4 + z_4^6 + z_5^8,$$

we have $\overline{E}_h = \{3, 8\}$, so $\Delta_f(t)$ (or $\Delta_g(t)$) and $\Delta_h(t)$ do not satisfy the Fox–Milnor relation. In fact, we have

$$\text{divisor } \Delta_h(t) = 27\Lambda_{24} - 6\Lambda_{12} + 9\Lambda_8 - 2\Lambda_6 - 2\Lambda_4 + \Lambda_3 - 1,$$

which verifies the above assertion by virtue of Lemma 3.1.

In order to prove Theorem 3.3, let us prepare some preliminary lemmas. Recall that we have

$$\text{divisor } \Delta_f(t) = \prod_{j=1}^{n+1} (\Lambda_{a_j} - 1) \quad \text{and} \quad \text{divisor } \Delta_g(t) = \prod_{j=1}^{n+1} (\Lambda_{b_j} - 1)$$

by [4] and that $\Delta_f(t)$ and $\Delta_g(t)$ satisfy the Fox–Milnor relation if and only if

$$\text{divisor } \Delta_f(t) \equiv \text{divisor } \Delta_g(t) \pmod{2}$$

by Lemma 3.1.

In the following, we will use the basic formula as follows (for example, see [20]). For completeness, we give a proof below.

LEMMA 3.5. *For positive integers a and b , we have*

$$(3.1) \quad \Lambda_a \Lambda_b = (a, b) \Lambda_{[a, b]},$$

where (a, b) denotes the greatest common divisor of a and b , and $[a, b]$ denotes the least common multiple of a and b .

Proof. For $a = 1$ or $b = 1$, the conclusion is obvious. So, we may assume $a > 1$ and $b > 1$. Set $d = (a, b)$, $a' = a/d$ and $b' = b/d$, which are positive integers. Note that then $(a', b') = 1$ and $[a, b] = da'b'$. Let ξ be a primitive $(da'b')$ -th root of unity. Then, we have

$$(3.2) \quad \Lambda_a \Lambda_b = \left(\sum_{k=0}^{a-1} \langle \xi^{b'k} \rangle \right) \left(\sum_{\ell=0}^{b-1} \langle \xi^{a'\ell} \rangle \right) = \sum_{k=0}^{a-1} \sum_{\ell=0}^{b-1} \langle \xi^{b'k+a'\ell} \rangle.$$

Now, we see easily that $\langle \xi^{b'k+a'\ell} \rangle = \langle 1 \rangle$ for

$$(k, \ell) = (0, 0), (a', b - b'), (2a', b - 2b'), \dots, ((d-1)a', b - (d-1)b').$$

Therefore $\langle 1 \rangle$ appears at least d times in the rightmost summation in (3.2). Similarly, given a pair of integers (α, β) with $0 \leq \alpha \leq a-1$ and $0 \leq \beta \leq b-1$, we have $\langle \xi^{b'k+a'\ell} \rangle = \langle \xi^{b'\alpha+a'\beta} \rangle$ for

$$(3.3) \quad \begin{aligned} (k, \ell) &\equiv (\alpha, \beta), (\alpha + a', \beta - b'), (\alpha + 2a', \beta - 2b'), \\ &\dots, (\alpha + (d-1)a', \beta - (d-1)b') \pmod{(a, b)} \end{aligned}$$

with $0 \leq k \leq a-1$ and $0 \leq \ell \leq b-1$, where for integers r, r', s and s' , we write $(r, s) \equiv (r', s') \pmod{(a, b)}$ when $r \equiv r' \pmod{a}$ and $s \equiv s' \pmod{b}$. Note that the integer pairs in (3.3) are all distinct modulo (a, b) . Therefore, $\langle \xi^{b'\alpha+a'\beta} \rangle$ also appears at least d times in the rightmost summation in (3.2).

When (α, β) runs over all integer pairs with $0 \leq \alpha \leq a-1$ and $0 \leq \beta \leq b-1$, we see that $\langle \xi^{b'\alpha+a'\beta} \rangle$ runs over all $\langle \xi^\gamma \rangle$ with $0 \leq \gamma \leq da'b' - 1$, since a' and b' are relatively prime. Hence, we see that each $\langle \xi^\gamma \rangle$ appears exactly d times. Therefore, we have the desired conclusion. This completes the proof. $\square$

Using the basic formula in Lemma 3.5, we can show the following.

LEMMA 3.6. *For positive integers a, b and m , we have the following.*

(1) *If a is even, then we have*

$$(\Lambda_a - 1)^m \equiv \begin{cases} 1 \pmod{2}, & m: \text{ even}, \\ \Lambda_a - 1 \pmod{2}, & m: \text{ odd}. \end{cases}$$

(2) *If a is odd, then we have*

$$(\Lambda_a - 1)^m \equiv \Lambda_a - 1 \pmod{2}$$

for all m .

(3) *If a is odd, then we have*

$$(\Lambda_a - 1)(\Lambda_{ab} - 1) \equiv \Lambda_a - 1 \pmod{2}.$$

(4) *If $a_j, j = 1, 2, \dots, m$, are positive even integers, then we have*

$$\prod_{j=1}^m (\Lambda_{a_j} - 1) \equiv \sum_{j=1}^m \Lambda_{a_j} - 1 \pmod{2}.$$

(5) *If a is even and b is odd, then we have*

$$\Lambda_a(\Lambda_b - 1) \equiv \Lambda_{[a,b]} - \Lambda_a \pmod{2}.$$

Proof. (1) Let us prove the assertion by induction on m . For $m = 1$, it is obvious. For $m = 2$, by Lemma 3.5 we have

$$(\Lambda_a - 1)^2 = a\Lambda_a - 2\Lambda_a + 1 \equiv 1 \pmod{2},$$

since a is even. So, the result holds. Then, suppose $m \geq 3$ and that the assertion holds for $m-1$. If m is even, then $m-1$ is odd and by induction hypothesis we have

$$(\Lambda_a - 1)^m = (\Lambda_a - 1)^{m-1}(\Lambda_a - 1) \equiv (\Lambda_a - 1)^2 \equiv 1 \pmod{2},$$

which shows the assertion for m . If m is odd, then $m-1$ is even and we have

$$(\Lambda_a - 1)^m = (\Lambda_a - 1)^{m-1}(\Lambda_a - 1) \equiv \Lambda_a - 1 \pmod{2},$$

which proves the assertion for m .

(2) For $m = 1$, the assertion is obvious. Suppose $m \geq 2$ and that the assertion holds for $m-1$. Then, we have

$$(\Lambda_a - 1)^m = (\Lambda_a - 1)^{m-1}(\Lambda_a - 1) \equiv (\Lambda_a - 1)^2 = a\Lambda_a - 2\Lambda_a + 1 \equiv \Lambda_a - 1 \pmod{2},$$

since a is odd. This proves the assertion for m .

(3) We have, by Lemma 3.5,

$$(\Lambda_a - 1)(\Lambda_{ab} - 1) = a\Lambda_{ab} - \Lambda_a - \Lambda_{ab} + 1 = (a-1)\Lambda_{ab} - \Lambda_a + 1 \equiv \Lambda_a - 1 \pmod{2},$$

since a is odd, which shows the assertion.

(4) When $m = 1$, the assertion is obvious. Suppose $m \geq 2$ and that the assertion holds for $m - 1$. Then, we have, by Lemma 3.5,

$$\begin{aligned}
 \prod_{j=1}^m (\Lambda_{a_j} - 1) &\equiv \left(\sum_{j=1}^{m-1} \Lambda_{a_j} - 1 \right) (\Lambda_{a_m} - 1) \pmod{2} \\
 &= \sum_{j=1}^{m-1} \Lambda_{a_j} \Lambda_{a_m} - \Lambda_{a_m} - \sum_{j=1}^{m-1} \Lambda_{a_j} + 1 \\
 &= \sum_{j=1}^{m-1} (a_j, a_m) \Lambda_{[a_j, a_m]} - \sum_{j=1}^m \Lambda_{a_j} + 1 \\
 &\equiv \sum_{j=1}^m \Lambda_{a_j} - 1 \pmod{2},
 \end{aligned}$$

since a_j are all even, which proves the assertion.

(5) We have, by Lemma 3.5,

$$\Lambda_a (\Lambda_b - 1) = (a, b) \Lambda_{[a, b]} - \Lambda_a \equiv \Lambda_{[a, b]} - \Lambda_a \pmod{2},$$

since (a, b) is odd, which proves the assertion.

This completes the proof. $\square$

Then, we have the following.

LEMMA 3.7. *We have*

$$\prod_{a \in E_f} (\Lambda_a - 1) \equiv \prod_{a \in \bar{E}_f} (\Lambda_a - 1) \pmod{2}.$$

REMARK 3.8. When $\bar{E}_f = \emptyset$,

$$\prod_{a \in \bar{E}_f} (\Lambda_a - 1)$$

is understood to be equal to 1 in the group ring $\mathbf{Z}\mathbf{C}^*$. This is because we have $\bar{E}_f = \emptyset$ if and only if all the exponents are even and each such number appears an even number of times, and in such a case, by Lemma 3.6 (1), the result is 1 modulo 2.

Proof of Lemma 3.7. By Lemma 3.6 (1) for m even, even if we perform the procedure as in Definition 3.2 (1), the modulo 2 class of the product of $\Lambda_a - 1$ over all elements a of the relevant set does not change. Then, by Lemma 3.6 (1) for m odd and (2), the same holds with the procedure of Definition 3.2 (2). Finally, by Lemma 3.6 (3), after the procedure as in Definition 3.2 (3), we see that

$$\prod_{a \in E_f} (\Lambda_a - 1) \equiv \prod_{a \in \bar{E}_f} (\Lambda_a - 1) \pmod{2}.$$

This completes the proof. $\square$

Proof of Theorem 3.3. We will prove the equivalence based on Brieskorn's formula (2.1).

Suppose that $\bar{E}_f = \bar{E}_g$ holds. Then, by Lemma 3.7 together with Lemma 3.1, we see that $\Delta_f(t)$ and $\Delta_g(t)$ satisfy the Fox–Milnor relation.

Conversely, suppose that $\Delta_f(t)$ and $\Delta_g(t)$ satisfy the Fox–Milnor relation. Then, again by Lemmas 3.7 and 3.1, we have

$$\prod_{a \in \bar{E}_f} (\Lambda_a - 1) \equiv \prod_{b \in \bar{E}_g} (\Lambda_b - 1) \pmod{2}.$$

Let $\overline{E}_f^0$ (resp. $\overline{E}_f^1$) be the subset of $\overline{E}_f$ consisting of even (resp. odd) integers. We also define $\overline{E}_g^0$ and $\overline{E}_g^1$ similarly. Then, we have

$$\begin{aligned} & \left(\prod_{a \in \overline{E}_f^0} (\Lambda_a - 1) \right) \left(\prod_{a \in \overline{E}_f^1} (\Lambda_a - 1) \right) \\ \equiv & \left(\prod_{b \in \overline{E}_g^0} (\Lambda_b - 1) \right) \left(\prod_{b \in \overline{E}_g^1} (\Lambda_b - 1) \right) \pmod{2}. \end{aligned}$$

By Lemma 3.6 (4), we have

$$\begin{aligned} (3.4) \quad & \left(\sum_{a \in \overline{E}_f^0} \Lambda_a - 1 \right) \left(\prod_{a \in \overline{E}_f^1} (\Lambda_a - 1) \right) \\ \equiv & \left(\sum_{b \in \overline{E}_g^0} \Lambda_b - 1 \right) \left(\prod_{b \in \overline{E}_g^1} (\Lambda_b - 1) \right) \pmod{2}. \end{aligned}$$

Now, by (3.1), every term of the form $\Lambda_k \Lambda_\ell$ with k even will result in a term of the form $c \Lambda_m$ with m even for some c . Therefore, the odd subscript terms in the left hand side that appear after expanding are exactly those that come from $1 \cdot \left(\prod_{a \in \overline{E}_f^1} (\Lambda_a - 1) \right)$, and similarly for the right hand side. Then, by comparing the terms of the forms Λ_k with k odd on both sides of the above congruence, with the help of the linear independence of $\{\Lambda_a\}_a$ modulo 2 as described in [3, Lemma 3.3], we have

$$\prod_{a \in \overline{E}_f^1} (\Lambda_a - 1) \equiv \prod_{b \in \overline{E}_g^1} (\Lambda_b - 1) \pmod{2}.$$

As no integer in $\overline{E}_f^1$ (or $\overline{E}_g^1$) is a multiple of another one, by the same argument as in the proof of [3, Theorem 2.7], we see that

$$(3.5) \quad \overline{E}_f^1 = \overline{E}_g^1.$$

By (3.4), we have

$$\begin{aligned} (3.6) \quad & \left(\sum_{a \in \overline{E}_f^0} \Lambda_a \right) \left(\prod_{a \in \overline{E}_f^1} (\Lambda_a - 1) \right) \\ \equiv & \left(\sum_{b \in \overline{E}_g^0} \Lambda_b \right) \left(\prod_{b \in \overline{E}_g^1} (\Lambda_b - 1) \right) \pmod{2}. \end{aligned}$$

Now, let us consider the terms of the forms Λ_k with k minimal on both sides. Since $a \geq 2$ and $b \geq 2$ by our assumption, we see that such a term has to be the product of the Λ_k with the smallest k in each left factor and the term 1 after the expansion of the right factor. Therefore, the minimum term on the left hand side can be clearly identified as the Λ_a with a even and minimal, and similarly for the right hand side. Hence, we see that

$$\min \overline{E}_f^0 = \min \overline{E}_g^0,$$

which we set as m_0 . Consequently, by subtracting

$$\Lambda_{m_0} \left(\prod_{a \in \overline{E}_f^1} (\Lambda_a - 1) \right) \equiv \Lambda_{m_0} \left(\prod_{b \in \overline{E}_g^1} (\Lambda_b - 1) \right) \pmod{2}$$

from both sides of (3.6), we get

$$\begin{aligned} & \left(\sum_{a \in \overline{E}_f^0 \setminus \{m_0\}} \Lambda_a \right) \left(\prod_{a \in \overline{E}_f^1} (\Lambda_a - 1) \right) \\ & \equiv \left(\sum_{b \in \overline{E}_g^0 \setminus \{m_0\}} \Lambda_b \right) \left(\prod_{b \in \overline{E}_g^1} (\Lambda_b - 1) \right) \pmod{2}. \end{aligned}$$

Repeating this procedure, we finally get $\overline{E}_f^0 = \overline{E}_g^0$. This together with (3.5) implies $\overline{E}_f = \overline{E}_g$. This completes the proof of Theorem 3.3. $\square$

REMARK 3.9. In fact, in Theorem 3.3, the hypothesis that $a_j \geq 2$ and $b_j \geq 2$ for all j is not necessary. Even without this hypothesis, the same proof works for the fact that $\overline{E}_f = \overline{E}_g$ implies that $\Delta_f(t)$ and $\Delta_g(t)$ satisfy the Fox–Milnor relation. Conversely, suppose that $\Delta_f(t)$ and $\Delta_g(t)$ satisfy the Fox–Milnor relation and that $a_j = 1$ for some j . If $b_{j'} = 1$ for some j' , then we have $\overline{E}_f = \{1\} = \overline{E}_g$. On the other hand, if $b_{j'} \geq 2$ for all j' , then by Corollary 3.12 below (see also the proof of Corollary 3.13), this leads to a contradiction.

REMARK 3.10. By [3, Proposition 2.6], if the Seifert forms of K_f and K_g are Witt equivalent over the real numbers (i.e., if they have the same equivariant signatures), then their Alexander polynomials satisfy the Fox–Milnor relation. So, by Theorem 3.3, we have $\overline{E}_f = \overline{E}_g$.

COROLLARY 3.11. *Suppose that the exponents of a Brieskorn polynomial f are all distinct and that no exponent is a multiple of another odd exponent. Let g be an arbitrary Brieskorn polynomial with the same number of variables as f . Then K_f and K_g are cobordant if and only if they have the same set of exponents; i.e., if and only if f and g are identical possibly after a permutation of variables. In particular, if the exponents of f are all even and all distinct, the same conclusion holds.*

Proof. First, if f and g have the same set of exponents, then their algebraic knots K_f and K_g are isotopic, and hence they are cobordant. Conversely, suppose that K_f and K_g are cobordant. Then, their Alexander polynomials satisfy the Fox–Milnor relation, and hence, by Theorem 3.3 (see also Remark 3.9), we have $\overline{E}_f = \overline{E}_g$. Under the assumption for f , we see that $E_f = \overline{E}_f$ holds by the definition of the essential exponent set. As $E_f = \overline{E}_f$ has $n + 1$ distinct elements, so does $\overline{E}_g$. As this is a subset of E_g , we must have $\overline{E}_g = E_g$. Hence we have $E_f = E_g$. This completes the proof. $\square$

We also have the following.

COROLLARY 3.12. *Let f be a Brieskorn polynomial such that the exponents satisfy $a_j \geq 2$ for all j . Then, the Alexander polynomial $\Delta_f(t)$ of the algebraic knot K_f associated with f is never a square.*

Proof. First, by the same argument as in the proof of Lemma 3.1, we see that $\Delta_f(t)$ is a square if and only if divisor $\Delta_f(t) \equiv 0 \pmod{2}$.

If all the exponents are even and each of them appears an even number of times, then we have $\overline{E}_f = \emptyset$, and $\Delta_f(t)$ is not a square by Lemma 3.7 and Remark 3.8.

On the other hand, if E_f contains an odd integer greater than or equal to 3, then one of its odd divisors greater than or equal to 3 persists in $\overline{E}_f$. Furthermore, if there are no odd exponents and an even integer appears exactly an odd number of times, then it persists in $\overline{E}_f$. Hence, in these cases, divisor $\Delta_f(t) \not\equiv 0 \pmod{2}$ by Lemma 3.7 and [3, Lemma 3.3], since there are no repetitions in the set $\overline{E}_f \neq \emptyset$ and the term 1 persists modulo 2. Therefore, $\Delta_f(t)$ is not a square.

This completes the proof. $\square$

COROLLARY 3.13. *The algebraic knot K_f associated with a Brieskorn polynomial f such that the exponents satisfy $a_j \geq 2$ for all j , is never null-cobordant.*

Proof. Suppose that K_f is null-cobordant. Then its Alexander polynomial $\Delta_f(t)$ and that of the trivial knot satisfy the Fox–Milnor relation. As the Alexander polynomial of the trivial knot is equal to 1, by Lemma 3.1, we see that $\Delta_f(t)$ must be a square. This contradicts Corollary 3.12. This completes the proof. $\square$

In fact, we have a stronger result as follows.

PROPOSITION 3.14. *Let K_f be the algebraic knot associated with a Brieskorn polynomial f such that the exponents satisfy $a_j \geq 2$ for all j . If it is spherical, then it always has infinite order in the knot cobordism group.*

Proof. Suppose K_f is of finite order. Then, its equivariant signatures all vanish, since they are integer-valued additive invariants of the cobordism classes. Therefore, by Remark 3.10, its Alexander polynomial and that of the trivial knot satisfies the Fox–Milnor relation. Hence, by Lemma 3.1, we see that $\Delta_f(t)$ must be a square. This contradicts Corollary 3.12. This completes the proof. $\square$

REMARK 3.15. In fact, Michel [18] proves the following results for a general holomorphic function germ $f : (\mathbb{C}^{n+1}, \mathbf{0}) \rightarrow (\mathbb{C}, 0)$ possibly with an isolated critical point at $\mathbf{0}$.

(1) If the algebraic knot K_f is spherical, then it is null-cobordant if and only if f does not have a singularity at $\mathbf{0}$.

(2) If f has a singularity at the origin and K_f is spherical, then K_f has infinite order in the knot cobordism group C_{2n-1} .

Our results show that although the Fox–Milnor relation seems to be weak, it leads to the above important results in the case of Brieskorn polynomials.

PROPOSITION 3.16. *Let K_f and K_g be the algebraic knots associated with Brieskorn polynomials f and g , respectively. We assume that they are spherical.*

(1) *If $K_f \sharp (-K_g^1)$ is of finite order in the knot cobordism group, then the order must be equal to 1 or 2.*

(2) *If K_f and K_g have the same equivariant signatures, then $K_f \sharp K_g$ is cobordant to $K_g \sharp K_g$.*

Proof. (1) It is known that $K_f \sharp (-K_g^1)$ is of finite order if and only if its equivariant signatures all vanish. (For example, see [6, 15].) Therefore, by our assumption, the equivariant signatures of K_f and K_g coincide, and by Remark 3.10, the Alexander polynomials of K_f and K_g satisfy the Fox–Milnor relation. Hence, $\Delta_f(t)\Delta_g(t)$ is a square. Then, by [6, Theorem 3.4.8], $K_f \sharp (-K_g^1)$ cannot have order 4 in the knot cobordism group, since its Alexander polynomial coincides with $\Delta_f(t)\Delta_g(t)$, which is a square. Hence, the order must be equal to 1 or 2.

(2) Since $K_f \sharp (-K_g^1)$ has order 1 or 2, we see that

$$2(K_f \sharp (-K_g^1)) = (K_f \sharp K_f) \sharp (-(K_g \sharp K_g)^1)$$

is null-cobordant, and the result follows. $\square$

We also have the following.

PROPOSITION 3.17. *Let f and g be Brieskorn polynomials such that all the exponents are greater than or equal to 2. If the algebraic knots K_f and K_g are cobordant, then their Alexander polynomials $\Delta_f(t)$ and $\Delta_g(t)$ share at least one irreducible cyclotomic polynomial factor.*

Before proving the above proposition, let us recall the following result due to Brieskorn [4]. For a Brieskorn polynomial f with the exponent set E_f , we construct a finite graph Γ_f as follows: the vertices correspond to the elements of E_f , and for $a, b \in E_f$, we connect them by an edge if their greatest common divisor satisfies $(a, b) > 1$. A connected component of Γ_f is called an *odd 2-component* if its vertex set consists of an odd number of even integers such that each pair of vertices are connected by an edge and their greatest common divisor is always equal to 2. Then we have the following.

THEOREM 3.18 (Brieskorn [4]). *Let f be a Brieskorn polynomial of $n + 1$ variables such that the exponents satisfy $a_j \geq 2$ for all j . For $n \neq 2$, the algebraic knot K_f is spherical if and only if Γ_f satisfies one of the following.*

- (1) *The graph Γ_f contains at least two isolated vertices.*
- (2) *The graph Γ_f contains one isolated vertex and an odd 2-component.*

REMARK 3.19. In the above theorem, for $n = 2$, we have $H_*(K_f; \mathbf{Z}) \cong H_*(S^3; \mathbf{Z})$ if and only if the above conditions (1) and (2) hold, which is implicit in the proof due to Brieskorn [4]. On the other hand, by a result of Mumford [21], K_f is homeomorphic to S^3 if and only if $f^{-1}(0)$ does not have a singularity at the origin.

Proof of Proposition 3.17. Let us first show that an equivariant signature for K_f does not vanish.

By Theorem 3.18, we see that by adding appropriate powers of extra two variables to f , we get a Brieskorn polynomial $\tilde{f}$ of $n + 3$ variables such that $K_{\tilde{f}}$ is spherical. Now, by contradiction, suppose that the equivariant signatures for K_f all vanish. Then, its Seifert form is Witt equivalent to 0 over the real numbers (see [25, §4]). Since the Seifert form for $K_{\tilde{f}}$ is the tensor product of that for K_f and a certain matrix (see [26]), we see that it is also Witt equivalent to 0 over the real numbers. Hence, its equivariant signatures all vanish. Since $K_{\tilde{f}}$ is spherical, this implies that it has finite order in the knot cobordism group (see [15, §22]), which contradicts Proposition 3.14. Hence, an equivariant signature of K_f with respect to a root λ of $\Delta_f(t)$ does not vanish.

As an equivariant signature is a cobordism invariant, the equivariant signature of K_g with respect to λ does not vanish, either. This implies that λ is a root of $\Delta_g(t)$. As the Alexander polynomials $\Delta_f(t)$ and $\Delta_g(t)$ are products of cyclotomic polynomials, the result follows. $\square$

4. LINEAR INDEPENDENCE IN THE KNOT COBORDISM GROUP

Litherland [16] has shown that the algebraic knots associated with the Brieskorn polynomials $z_1^p + z_2^q$ with $2 \leq p < q$ and $(p, q) = 1$ are linearly independent in the knot cobordism group of dimension 1. Note that these algebraic knots in S^3 are the so-called *torus knots*.

In order to prove a similar result in higher dimensions, let us prepare the following. For a fixed integer $n \geq 1$, let $\mathcal{B}$ be a set of exponent sets of $n + 1$ elements such that for each exponent set belonging to $\mathcal{B}$, the exponents are greater than or equal to 2 and are pairwise relatively prime, and that no two of the exponent sets of $\mathcal{B}$ have equal product. In other words, for $\{p_j\}_{j=1}^{n+1} \neq \{q_j\}_{j=1}^{n+1} \in \mathcal{B}$, we have

$$p_1 p_2 \cdots p_{n+1} \neq q_1 q_2 \cdots q_{n+1}.$$

We call such a set $\mathcal{B}$ a *good family of exponent sets*. For example, the set $\mathcal{P}$ of all exponent sets such that the exponents are distinct prime numbers is a good family of exponent sets.

THEOREM 4.1. *Let $\mathcal{B}$ be a good family of exponent sets of $n + 1$ elements, and consider the family of Brieskorn polynomials whose exponent sets correspond bijectively to the elements of $\mathcal{B}$. Then,*

for $n \neq 2$, the associated algebraic knots are spherical and are linearly independent in the knot cobordism group of dimension $2n - 1$.

Note that the corresponding algebraic knots are easily seen to be spherical by Theorem 3.18.

For the proof of Theorem 4.1, let us prepare some materials. Let K be a spherical $(2n - 1)$ -knot and L its Seifert matrix. For a complex number ζ of modulus 1, let us consider the signature of the Hermitian matrix

$$(1 - \zeta)L + (1 - \bar{\zeta})L^T.$$

This is independent of the choice of Seifert matrix L . This gives rise to a function $S^1 \rightarrow \mathbf{Z}$, where S^1 is the unit circle in $\mathbf{C}$, and it is known to be continuous (and therefore constant) everywhere except at $(-1)^{n+1}$ times the unit roots of the Alexander polynomial $\Delta_K(t)$ (for example, see [6, Chapter 9]). This function is not a cobordism invariant in general: however, the jumps at $(-1)^{n+1}$ times the unit roots of the Alexander polynomial are cobordism invariants. This is called the *signature jump function*. It is known that it can be written in terms of the equivariant signatures (see [14, 17] or [6, Theorem 3.4.7]).

Now let $\{p_j\} = \{p_1, p_2, \dots, p_{n+1}\}$ be an exponent set in $\mathcal{B}$ and set

$$P = p_1 p_2 \cdots p_{n+1}.$$

Note that the integers $p_1, p_2, \dots, p_{n+1}$ are relatively prime to each other. For a positive integer r , set

$$\begin{aligned} L_+\left(\frac{r}{P}\right) &= \left\{ (k_1, k_2, \dots, k_{n+1}) \in \mathbf{Z}^{n+1} \left| \sum_{j=1}^{n+1} \frac{k_j}{p_j} \equiv \frac{r}{P} \pmod{2}, \right. \right. \\ &\quad \left. \left. 0 < k_j < p_j, j = 1, 2, \dots, n+1 \right\}, \\ L_-\left(\frac{r}{P}\right) &= \left\{ (k_1, k_2, \dots, k_{n+1}) \in \mathbf{Z}^{n+1} \left| \sum_{j=1}^{n+1} \frac{k_j}{p_j} \equiv \frac{r}{P} + 1 \pmod{2}, \right. \right. \\ &\quad \left. \left. 0 < k_j < p_j, j = 1, 2, \dots, n+1 \right\}, \end{aligned}$$

where for rational numbers s and t , and a positive integer u , we write $s \equiv t \pmod{u}$ if the difference $s - t$ is an integer multiple of u .

Then, we have the following.

LEMMA 4.2. *The set $L_+(r/P) \cup L_-(r/P)$ contains at most one element, and we have that $L_+(r/P) = L_-(r/P) = \emptyset$ if and only if r is a multiple of some p_j .*

Proof. Since $p_1, p_2, \dots, p_{n+1}$ are pairwise relatively prime, the following three are equivalent to each other:

- $$\begin{aligned} (1) \quad & \sum_{j=1}^{n+1} \frac{k_j}{p_j} \equiv \frac{r}{P} \pmod{1}, \\ (2) \quad & \sum_{j=1}^{n+1} k_j p_1 p_2 \cdots \check{p}_j \cdots p_{n+1} \equiv r \pmod{P}, \\ (3) \quad & k_j p_1 p_2 \cdots \check{p}_j \cdots p_{n+1} \equiv r \pmod{p_j}, \quad 1 \leq j \leq n+1, \end{aligned}$$

where “ $\check{p}_j$ ” means that it is deleted from the product. (For example, the implication “(3) $\Rightarrow$ (2)” follows from the Chinese Remainder Theorem.) Since for each j , p_j and $p_1 p_2 \cdots \check{p}_j \cdots p_{n+1}$ are relatively prime, if $r \not\equiv 0 \pmod{p_j}$, then there exists a unique $k_j \in \{1, 2, \dots, p_j - 1\}$ satisfying the above property (3) for that j . On the other hand, if $r \equiv 0 \pmod{p_j}$ for some j , there does

not exist $k_j \in \{1, 2, \dots, p_j - 1\}$ with the above property (3) for that j . These observations lead to the desired conclusion. This completes the proof. $\square$

Furthermore, the following follows from [6, §9.3].

LEMMA 4.3. *The signature jump at $\exp(2\pi\sqrt{-1}r/P)$ is equal to 1 if $L_+(r/P) \neq \emptyset$, is equal to -1 if $L_-(r/P) \neq \emptyset$, and is equal to 0 if r is a multiple of some p_j .*

Proof of Theorem 4.1. Let us show that the signature jump functions $J_{\{p_j\}}$ for the algebraic knots corresponding to the exponent sets $\{p_j\}$ are linearly independent over $\mathbf{Z}$ for $\{p_j\} \in \mathcal{B}$. Suppose there is a nontrivial dependence relation among $J_{\{p_j\}}$ over $\mathbf{Z}$. Let M be the maximum of $P = p_1 p_2 \cdots p_{n+1}$ appearing in a nontrivial dependence relation. Note that by the definition of a good family of exponent sets, such maximum is attained only by a unique element $\{q_j\}$ in $\mathcal{B}$. Since $J_{\{p_j\}}(\exp(2\pi\sqrt{-1}/M)) = 0$ for $\{p_j\}$ with $p_1 p_2 \cdots p_{n+1} < M$, we see that the dependence relation implies that $J_{\{q_j\}}(\exp(2\pi\sqrt{-1}/M)) = 0$. This contradicts Lemma 4.3. Therefore, the jump functions corresponding to the elements of $\mathcal{B}$ are linearly independent over $\mathbf{Z}$. Since the jump functions are additive cobordism invariants, the result follows. $\square$

REMARK 4.4. The above proof is based on the idea used in [16] for $n = 1$. In Theorem 4.1, we imposed the condition that no two of the exponent sets of $\mathcal{B}$ have equal product. We do not know if this condition is redundant or not.

5. DECOMPOSITION OF SEIFERT FORM

As has been noted in §2, for $n > 1$, we have that the knot cobordism group C_{2n-1} is isomorphic to $\mathbf{Z}^\infty \oplus \mathbf{Z}_2^\infty \oplus \mathbf{Z}_4^\infty$. Clearly such a decomposition is not unique: however, according to [14, 15], there is a family of explicitly defined cobordism invariants that induce such a decomposition canonically (for details, see [6, §3.2 and §3.4], for example). For instance, for a representative of a given element of C_{2n-1} , for each irreducible symmetric polynomial $\delta(t) \in \mathbf{Q}[t]$ that is a factor of the Alexander polynomial, we have an associated element in the Witt group $G_{\mathbf{Q}}^\delta$ of isometric structures over $\mathbf{Q}$ corresponding to a power of δ . It is also known that $G_{\mathbf{Q}} \cong \bigoplus_\delta G_{\mathbf{Q}}^\delta$, where the sum is over all irreducible symmetric polynomials δ .

In this section, we show that for some elements in C_{2n-1} with $n = 3$ associated with Brieskorn polynomials

$$g(z_1, z_2, z_3, z_4) = z_1^3 + z_2^4 + z_3^4 + z_4^p$$

with $p \geq 5$ prime to 2 and 3, the associated factor in $G_{\mathbf{Q}}^\delta$ can be trivial for some nontrivial factor $\delta(t)$ of the Alexander polynomial, although the associated algebraic knots are of infinite order in the knot cobordism group. Our strategy for the proof is to first decompose the Seifert matrix for the algebraic knot associated with $z_1^3 + z_2^4 + z_3^4$ over $\mathbf{Q}$ and then consider its tensor product with the Seifert matrix for z_4^p , using a result due to [26].

For $a \geq 2$, let M_a be the $(a-1) \times (a-1)$ unimodular matrix

$$M_a = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & \ddots & \vdots \\ 0 & 0 & 1 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & -1 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Note that the Seifert form of the algebraic knot associated with the Brieskorn polynomial

$$f = z_1^{a_1} + z_2^{a_2} + \cdots + z_{n+1}^{a_{n+1}}$$

is given by the tensor product $L = M_{a_1} \otimes M_{a_2} \otimes \cdots \otimes M_{a_{n+1}}$, see [26], in which the matrix appears to be different from the above one: this is due to the difference in the definition of the Seifert form.

Note that we have

$$(5.1) \quad \begin{aligned} S &= L + (-1)^n \overline{L^T}, & H &= (-1)^{n+1} \overline{L^{-1}} L^T, \\ T &= (-1)^{n+1} L (\overline{L^{-1}})^T, & S &= L(I - \overline{H}), \end{aligned}$$

where L is the Seifert matrix, S is the sesquilinearized intersection form of the Milnor fiber, H is the homological monodromy matrix, T is the cohomological monodromy matrix, and I is the identity matrix (for example, see [25]). Here, we have given the above formula in such a way that they work also over the complex numbers.

Let us consider an explicit example:

$$f(z_1, z_2, z_3) = z_1^3 + z_2^4 + z_3^4.$$

The divisor of its characteristic polynomial of the monodromy (or Alexander polynomial) $\Delta_f(t)$ is given by

$$\text{divisor } \Delta_f = (\Lambda_3 - 1)(\Lambda_4 - 1)(\Lambda_4 - 1) = 2\Lambda_{12} + \Lambda_3 - 2\Lambda_4 - 1,$$

and hence we have

$$\begin{aligned} \Delta_f(t) &= \frac{(t^{12} - 1)^2 (t^3 - 1)}{(t^4 - 1)^2 (t - 1)} \\ &= \frac{\phi_{12}^2 \phi_6^2 \phi_4^2 \phi_3^2 \phi_2^2 \phi_1^2 \phi_3 \phi_1}{\phi_4^2 \phi_2^2 \phi_1^2 \phi_1} \\ &= \phi_{12}^2 \phi_6^2 \phi_3^3, \end{aligned}$$

where for a positive integer m , $\phi_m(t)$ denotes the m -th cyclotomic polynomial. Note that the degrees of $\phi_{12}, \phi_6, \phi_3$ are equal to 4, 2, 2, respectively. According to Steenbrink's formula [29], the equivariant signatures corresponding to the roots of $\phi_{12}, \phi_6, \phi_3$ are equal to 8, 0, 6, respectively. (This means, for example, that the sum of the equivariant signatures corresponding to the roots of ϕ_{12} is equal to 8.) Therefore, the ϕ_{12} - and ϕ_3 -primary components of the Seifert form of f are not zero in the Witt groups $G_{\mathbf{Q}}^{\phi_{12}}$ and $G_{\mathbf{Q}}^{\phi_3}$, respectively, in the sense of [15, §10] or [6]. In the following, we will show that the ϕ_6 -primary component is algebraically null-cobordant over $\mathbf{Q}$.

Let us analyze the ϕ_6 -primary component of the Seifert form L of f in the Witt group $G_{\mathbf{Q}}^{\phi_6}$. The Seifert form for f is given by the unimodular (18×18) -matrix

$$L = M_3 \otimes M_4 \otimes M_4.$$

The form M_3 is irreducible over $\mathbf{Q}$, since its Alexander polynomial ϕ_3 is irreducible. On the other hand, the Alexander polynomial of M_4 is equal to $\phi_4 \phi_2$, which is not irreducible. Let us decompose M_4 into the irreducible factors over $\mathbf{Q}$.

By some computations, we see the following:

$$\begin{aligned}
M_4 &= \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \\
M_4^{-1} &= \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \\
S_4 &= M_4 + M_4^T = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}, \\
T_4 &= -M_4(M_4^{-1})^T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{pmatrix}, \\
H_4 &= (T_4)^T = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}.
\end{aligned}$$

Note that

$$\begin{aligned}
(H_4)^T M_4 H_4 &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} = M_4.
\end{aligned}$$

Let us first decompose M_4 over $\mathbf{C}$. The eigenvalues of H_4 are $-1, \pm\sqrt{-1}$, where -1 is the root of $\phi_2(t)$ and $\pm\sqrt{-1}$ are the roots of $\phi_4(t)$. Eigenvectors corresponding to the eigenvalues $-1, \sqrt{-1}$ and $-\sqrt{-1}$ are given by

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 - \sqrt{-1} \\ -\sqrt{-1} \end{pmatrix}, \begin{pmatrix} 1 \\ 1 + \sqrt{-1} \\ \sqrt{-1} \end{pmatrix},$$

respectively. Therefore, the ϕ_2 -primary component is generated by

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},$$

and the ϕ_4 -primary component is generated by

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

(For this, consider the real and the imaginary parts of the corresponding eigenvectors.) Set

$$P = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

Then, we have

$$P^{-1} = \frac{1}{2} \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{pmatrix}$$

and

$$P^{-1}H_4P = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

So, we have verified that P gives the correct decomposition of the monodromy into the irreducible components.

REMARK 5.1. We can show that we cannot choose an integral unimodular matrix as P as follows. If we choose

$$a \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \text{ and } a' \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + b' \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

as bases for the ϕ_4 -primary component for some integers a, b, a', b' , then we can show that the determinant of the corresponding (3×3) -matrix is an even integer.

Then, we have

$$P^T M_4 P = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix}.$$

So, over $\mathbf{Q}$, the bilinear form M_4 is isomorphic to $(2) \oplus R$, where

$$R = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Then, we have, over $\mathbf{Q}$,

$$\begin{aligned} L &= M_3 \otimes M_4 \otimes M_4 \\ &\cong M_3 \otimes ((2) \oplus R) \otimes ((2) \oplus R) \\ &\cong M_3 \otimes ((4) \oplus 2R \oplus 2R \oplus (R \otimes R)) \\ &\cong 4M_3 \oplus (2M_3 \otimes R) \oplus (2M_3 \otimes R) \oplus (M_3 \otimes R \otimes R). \end{aligned}$$

The characteristic polynomials corresponding to the four factors are given by

$$\phi_3(t), \phi_{12}(t), \phi_{12}(t) \text{ and } \phi_3(t)^2 \phi_6(t)^2,$$

respectively. So, in order to analyze the ϕ_6 -primary component of L , we still need to decompose $M_3 \otimes R \otimes R$, which is an (8×8) -matrix.

Note that by (5.1), the monodromy matrix H_3 corresponding to M_3 is given by

$$H_3 = -M_3^{-1}M_3 = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}.$$

We have

$$R^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

and the monodromy matrix H_R corresponding to R is given by

$$H_R = -R^{-1}R^T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Set $\omega = \exp(2\pi\sqrt{-1}/3)$. Eigenvectors of H_3 corresponding to the eigenvalues ω and $\bar{\omega}$ are given by

$$u_1 = \begin{pmatrix} 1 \\ -\omega \end{pmatrix} \text{ and } u_2 = \begin{pmatrix} 1 \\ -\bar{\omega} \end{pmatrix},$$

respectively. Eigenvectors of H_R corresponding to the eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$ are given by

$$v_1 = \begin{pmatrix} 1 \\ -\sqrt{-1} \end{pmatrix} \text{ and } v_2 = \begin{pmatrix} 1 \\ \sqrt{-1} \end{pmatrix},$$

respectively. Therefore, the monodromy matrix $H_{3,R,R}$ associated with $M_3 \otimes R \otimes R$ is diagonalized by the (8×8) -matrix Q consisting of the 8 column vectors

$$u_i \otimes v_j \otimes v_k,$$

$i, j, k = 1, 2$, in such a way that

$$Q^{-1}H_{3,R,R}Q = \begin{pmatrix} \omega & 0 \\ 0 & \bar{\omega} \end{pmatrix} \otimes \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix} \otimes \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}.$$

Therefore, the ϕ_6 -primary component is generated by $u_1 \otimes v_1 \otimes v_1$, $u_1 \otimes v_2 \otimes v_2$, $u_2 \otimes v_1 \otimes v_1$ and $u_2 \otimes v_2 \otimes v_2$ over $\mathbf{C}$. Note that

$$\begin{aligned} u_1 \otimes v_1 \otimes v_1 &= (1, -\sqrt{-1}, -\sqrt{-1}, -1, -\omega, \omega\sqrt{-1}, \omega\sqrt{-1}, \omega)^T, \\ u_1 \otimes v_2 \otimes v_2 &= (1, \sqrt{-1}, \sqrt{-1}, -1, -\omega, -\omega\sqrt{-1}, -\omega\sqrt{-1}, \omega)^T, \\ u_2 \otimes v_1 \otimes v_1 &= (1, -\sqrt{-1}, -\sqrt{-1}, -1, -\bar{\omega}, \bar{\omega}\sqrt{-1}, \bar{\omega}\sqrt{-1}, \bar{\omega})^T, \\ u_2 \otimes v_2 \otimes v_2 &= (1, \sqrt{-1}, \sqrt{-1}, -1, -\bar{\omega}, -\bar{\omega}\sqrt{-1}, -\bar{\omega}\sqrt{-1}, \bar{\omega})^T. \end{aligned}$$

By considering the real and imaginary parts, we see that the ϕ_6 -primary component is generated by

$$\begin{aligned} w_1 &= (1, 0, 0, -1, 1/2, -\sqrt{3}/2, -\sqrt{3}/2, -1/2)^T, \\ w_2 &= (0, -1, -1, 0, -\sqrt{3}/2, -1/2, -1/2, \sqrt{3}/2)^T, \\ w_3 &= (1, 0, 0, -1, 1/2, \sqrt{3}/2, \sqrt{3}/2, -1/2)^T, \\ w_4 &= (0, 1, 1, 0, -\sqrt{3}/2, 1/2, 1/2, \sqrt{3}/2)^T \end{aligned}$$

over $\mathbf{R}$. Then, we have

$$\begin{aligned} w_1 + w_3 &= (2, 0, 0, -2, 1, 0, 0, -1)^T, \\ (w_1 - w_3)/\sqrt{3} &= (0, 0, 0, 0, 0, -1, -1, 0)^T, \\ (w_2 + w_4)/\sqrt{3} &= (0, 0, 0, 0, -1, 0, 0, 1)^T, \\ -(w_2 - w_4) &= (0, 2, 2, 0, 0, 1, -, 0)^T. \end{aligned}$$

Note that these four vectors can be written as

$$\begin{aligned} r_1 &= (2, 1)^T \otimes (1, 0, 0, -1)^T, \\ r_2 &= (0, -1)^T \otimes (0, 1, 1, 0)^T, \\ r_3 &= (0, -1)^T \otimes (1, 0, 0, -1)^T, \\ r_4 &= (2, 1)^T \otimes (0, 1, 1, 0)^T, \end{aligned}$$

respectively. Then, by calculating

$$r_i^T (M_3 \otimes R \otimes R) r_j,$$

$i, j = 1, 2, 3, 4$, we see that the ϕ_6 -primary component of the bilinear form $M_3 \otimes R \otimes R$ is isomorphic over $\mathbf{Q}$ to

$$\begin{pmatrix} 0 & -4 & 0 & -12 \\ -4 & 0 & 4 & 0 \\ 0 & -4 & 0 & 4 \\ 12 & 0 & 4 & 0 \end{pmatrix},$$

which, in turn, is isomorphic to

$$\begin{pmatrix} 0 & 0 & -1 & -3 \\ 0 & 0 & -1 & 1 \\ -1 & 1 & 0 & 0 \\ 3 & 1 & 0 & 0 \end{pmatrix}$$

over $\mathbf{Q}$.

Consequently, we see that the ϕ_6 -primary component of $M_3 \otimes R \otimes R$, hence that of L , is algebraically null-cobordant, since it admits a metabolizer (see [6, §3.1], for example).

Let p be a positive integer relatively prime to 2 and 3, and consider the Brieskorn polynomial

$$g(z_1, z_2, z_3, z_4) = z_1^3 + z_2^4 + z_3^4 + z_4^p.$$

Then, the algebraic knot K_g associated with g is spherical, i.e. K_g is homeomorphic to the sphere S^7 (see Theorem 3.18). As the Seifert matrix L_g of K_g is given by the tensor product of L and M_p (see [26]), we see that a certain direct summand of L_g is null-cobordant over $\mathbf{Q}$ in the algebraic cobordism group.

The above explicit example shows that if we consider the image of the cobordism class of a spherical algebraic knot associated with a Brieskorn polynomial in $\oplus_\delta G_{\mathbf{Q}}^\delta$, then there might be a direct summand which vanishes in $G_{\mathbf{Q}}^\delta$ for some irreducible factor δ of the Alexander polynomial. In other words, the cobordism class represented by a spherical algebraic knot associated with a Brieskorn polynomial may not be able to recover some irreducible factors of the Alexander polynomial. This means that even if two algebraic knots are cobordant, the irreducible factors of their Alexander polynomials might be different, although they share at least one irreducible factor according to Proposition 3.17.

This suggests a major difficulty in proving the cobordism invariance of the exponents for Brieskorn polynomials. Note that in [31], topological invariance of exponents for Brieskorn polynomials was proved using the topological invariance of the Alexander polynomial. Such an approach seems not to work for the study of cobordisms.

6. CYCLIC SUSPENSION

In this section, we explore cyclic suspensions of simple fibered knots and algebraic knots, and also their properties concerning cobordisms.

Let $K \subset S^{2n+1}$ be a $(2n-1)$ -knot. Then, we can move the standard sphere $S^{2n+1} \subset S^{2n+3}$ ambient isotopically to get S' such that S' intersects S^{2n+1} transversely along K . For a positive integer d , we consider the d -fold cyclic branched covering $\tilde{S}$ of S^{2n+3} branched along S^{2n+1} , which is diffeomorphic to S^{2n+3} . Then the pull-back K_d of S' by the branched covering map in $\tilde{S}$ is called the d -fold cyclic suspension of K . Furthermore, we call the positive integer d the *suspension degree*. Note that K_d itself is diffeomorphic to the d -fold cyclic branched covering of S^{2n+1} branched along K , and that it is considered to be a $(2n+1)$ -knot in S^{2n+3} . This notion has been introduced by Kauffman [12] and Neumann [22] (see also [13]). Note that if K is a simple fibered knot, then so is K_d .

In this section, we consider the following problem.

PROBLEM 6.1. For a common integer d , let $(K_i)_d$ be the d -fold cyclic suspensions of two knots K_i , $i = 1, 2$. Furthermore, for another common integer e , let $(K_i)_{d,e}$ be the e -fold cyclic suspensions of $(K_i)_d$, $i = 1, 2$. Is it possible to construct examples such that K_i are not cobordant, that $(K_i)_d$ are cobordant and that $(K_i)_{d,e}$ are not cobordant?

If the answer is affirmative, then it would show that the cyclic suspensions do not preserve cobordisms in general.

Note that the algebraic knot associated with a Brieskorn polynomial $z_1^{a_1} + z_2^{a_2} + \cdots + z_{n+1}^{a_{n+1}}$ is the iterated cyclic suspension of the (a_1, a_2) -torus link in S^3 (see [22]). The above problem is closely related to the study of cobordisms of such knots.

In this section, we will often use the following (for details, see the argument just after [2, Remark 5.9]).

LEMMA 6.2. *Let L be a Seifert matrix associated with an $(n-1)$ -connected fiber F of a simple fibered $(2n-1)$ -knot K , and set $S = L + (-1)^n L^T$. Then, we have the exact sequence*

$$0 \rightarrow H_n(K; \mathbf{Z}) \rightarrow H_n(F; \mathbf{Z}) \xrightarrow{S^T} H_n(F, K; \mathbf{Z}) \rightarrow H_{n-1}(K; \mathbf{Z}) \rightarrow 0,$$

where S^T denotes the homomorphism naturally induced by the transpose of S .

Let $n \geq 3$ be an integer. For the moment, we will assume that n is odd. Consider the matrices

$$A_1 = \begin{pmatrix} B & C \\ -C^T & \mathbf{0} \end{pmatrix}$$

and

$$A_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

where B is a 2×2 integer matrix with $\det(B + B^T) = \pm 1$, C is a 2×2 integer matrix with $\det C = \pm 1$, and $\mathbf{0}$ denotes the 2×2 zero matrix. So, A_1 is a unimodular (4×4) -matrix and A_2 is a unimodular (2×2) -matrix. Let K_1 and K_2 be the simple fibered $(2n-1)$ -knots in S^{2n+1} whose Seifert matrices are given by A_1 and A_2 , respectively. Such simple fibered knots exist uniquely up to isotopy by [8, 11].

For positive integers a and b , let $(K_i)_a$ be the a -fold cyclic suspension of the knot K_i , and $(K_i)_{a,b}$ be the b -fold cyclic suspension of $(K_i)_a$, $i = 1, 2$. Then, their Seifert matrices $(A_i)_a$ and $(A_i)_{a,b}$, respectively, are given by

$$(A_i)_a = A_i \otimes M_a \text{ and } (A_i)_{a,b} = A_i \otimes M_a \otimes M_b$$

(see [12, 13, 23]).

Let us consider the 2-fold cyclic suspensions $(K_i)_2$, $i = 1, 2$. As M_2 is the (1×1) -matrix (1) , we can identify their Seifert matrices with those of K_i , $i = 1, 2$. As we have

$$(S_1)_2 = A_1 + A_1^T = \begin{pmatrix} B + B^T & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \quad (S_2)_2 = A_2 + A_2^T = \mathbf{0},$$

we see that

$$H_n((K_1)_2; \mathbf{Z}) \cong H_n((K_2)_2; \mathbf{Z}) \cong \mathbf{Z} \oplus \mathbf{Z} \cong H_{n-1}((K_1)_2; \mathbf{Z}) \cong H_{n-1}((K_2)_2; \mathbf{Z})$$

by Lemma 6.2. Furthermore, as A_1 and A_2 both have metabolizers, so does $A_1 \oplus (-A_2)$.

However, $(K_1)_2$ and $(K_2)_2$ are not cobordant, since the Seifert forms restricted to

$$H_n((K_i)_2; \mathbf{Z}) = \text{Ker } S_i, \quad i = 1, 2,$$

are not isomorphic (see [1]). Note that these knots are not spherical.

Let us now consider the 3-fold cyclic suspensions $(K_1)_3$ and $(K_2)_3$, respectively. Then, their Seifert matrices are given by

$$(A_1)_3 = A_1 \otimes M_3 = \begin{pmatrix} B \otimes M_3 & C \otimes M_3 \\ -C^T \otimes M_3 & \mathbf{0} \end{pmatrix}$$

and

$$(A_2)_3 = A_2 \otimes M_3 = \begin{pmatrix} \mathbf{0} & M_3 \\ -M_3 & \mathbf{0} \end{pmatrix},$$

respectively. Then, the intersection matrices of their fibers are given by

$$\begin{aligned} (S_1)_3 &= (A_1)_3 + (A_1)_3^T = \begin{pmatrix} B \otimes M_3 + B^T \otimes M_3^T & C \otimes M_3 - C \otimes M_3^T \\ C^T \otimes M_3^T - C^T \otimes M_3 & \mathbf{0} \end{pmatrix} \\ &= \begin{pmatrix} B \otimes M_3 + B^T \otimes M_3^T & C \otimes (M_3 - M_3^T) \\ -C^T \otimes (M_3 - M_3^T) & \mathbf{0} \end{pmatrix} \end{aligned}$$

and

$$(S_2)_3 = (A_2)_3 + (A_2)_3^T = \begin{pmatrix} \mathbf{0} & M_3 - M_3^T \\ -(M_3 - M_3^T) & \mathbf{0} \end{pmatrix},$$

respectively. As we have

$$\det(M_3 - M_3^T) = \det \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = 1,$$

we see that both $(S_1)_3$ and $(S_2)_3$ are unimodular. Therefore, the fibered knots $(K_1)_3$ and $(K_2)_3$ are spherical by Lemma 6.2. As their Seifert matrices are obviously algebraically null-cobordant, the knots are, in fact, null-cobordant, and in particular they are cobordant.

We can also show that K_1 and K_2 are not diffeomorphic to each other for an appropriate choice of C . For example, consider

$$C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

In this case, the intersection matrices are

$$S_1 = A_1 - A_1^T = \begin{pmatrix} B - B^T & C + C^T \\ -(C + C^T) & \mathbf{0} \end{pmatrix} = \begin{pmatrix} B - B^T & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$$

and

$$S_2 = A_2 - A_2^T = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}.$$

Therefore, the rank of $H_{n-1}(K_1; \mathbf{Z})$ is greater than or equal to 2, while $H_{n-1}(K_2; \mathbf{Z})$ is finite of order 4. So, K_1 and K_2 are not diffeomorphic and hence are not cobordant.

Let us now consider $(K_1)_{2,3} = (K_1)_{3,2}$ and $(K_2)_{2,3} = (K_2)_{3,2}$. Their Seifert forms are given by

$$(A_1)_{3,2} = A_1 \otimes M_3 \otimes M_2 = \begin{pmatrix} B \otimes M_3 & C \otimes M_3 \\ -C^T \otimes M_3 & \mathbf{0} \end{pmatrix}$$

and

$$(A_2)_{3,2} = A_2 \otimes M_3 \otimes M_2 = \begin{pmatrix} \mathbf{0} & M_3 \\ -M_3 & \mathbf{0} \end{pmatrix},$$

respectively. Then, their intersection matrices are

$$(S_1)_{3,2} = (A_1)_{3,2} - (A_1)_{3,2}^T = \begin{pmatrix} B \otimes M_3 - B^T \otimes M_3^T & C \otimes M_3 + C \otimes M_3^T \\ -C^T \otimes M_3 - C^T \otimes M_3^T & \mathbf{0} \end{pmatrix}$$

and

$$(S_2)_{3,2} = (A_2)_{3,2} - (A_2)_{3,2}^T = \begin{pmatrix} \mathbf{0} & M_3 + M_3^T \\ -M_3 - M_3^T & \mathbf{0} \end{pmatrix},$$

respectively. For C as above, we see that

$$|\det(S_1)_{3,2}| = 3^4, \quad |\det(S_2)_{3,2}| = 3^2,$$

and hence $(K_1)_{3,2}$ and $(K_2)_{3,2}$ are not diffeomorphic to each other and are not cobordant.

Summarizing, we have the following.

- (1) K_1 and K_2 are not diffeomorphic to each other and are not cobordant.
- (2) $(K_1)_2$ and $(K_2)_2$ are diffeomorphic to each other, but are not cobordant.
- (3) $(K_1)_3$ and $(K_2)_3$ are spherical and null-cobordant, so they are cobordant to each other.
- (4) $(K_1)_{3,2} = (K_1)_{2,3}$ and $(K_2)_{3,2} = (K_2)_{2,3}$ are not diffeomorphic to each other and are not cobordant.

So, this answers Problem 6.1 affirmatively.

REMARK 6.3. In general, if K_1 and K_2 are spherical knots which are cobordant, then $(K_1)_{2,2}$ and $(K_2)_{2,2}$ are also cobordant. See [13, §8].

Now, let us consider examples of algebraic knots. In [7], Du Bois and Michel constructed two polynomials

$$f = h_{r,s,p,q}(z_1, z_2, \dots, z_{n+1}) \text{ and } g = h_{s-8,r+8,p,q}(z_1, z_2, \dots, z_{n+1})$$

with isolated critical points at the origin such that K_f and K_g are cobordant, although they are not isotopic. Here, note that f and g are not Brieskorn polynomials. Let k be a positive integer called an *exponent* in the sense of [7] for both of f and g : i.e., $(t_f^k - 1)^2$ and $(t_g^k - 1)^2$ both vanish, where t_f and t_g are homological monodromies for the Milnor fibrations for f and g , respectively, and “1” denotes the identity homomorphism. Let us consider the algebraic knots $K_{\tilde{f}}$ and $K_{\tilde{g}}$ associated with

$$\tilde{f}(z_1, z_2, \dots, z_{n+2}) = f(z_1, z_2, \dots, z_{n+1}) + z_{n+2}^k$$

and

$$\tilde{g}(z_1, z_2, \dots, z_{n+2}) = g(z_1, z_2, \dots, z_{n+1}) + z_{n+2}^k,$$

respectively. Note that they are k -fold cyclic suspensions of K_f and K_g , respectively (see [22, Theorem 1]).

LEMMA 6.4. *The homology groups $H_n(K_{\tilde{f}}; \mathbf{Z})$ and $H_n(K_{\tilde{g}}; \mathbf{Z})$ have non-isomorphic torsions.*

Proof. Recall that $K_{\tilde{f}}$ (resp. $K_{\tilde{g}}$) is the k -fold cyclic branched cover of S^{2n+1} branched along K_f (resp. K_g). This implies that $K_{\tilde{f}}$ admits an open book structure with page diffeomorphic to F_f and with algebraic monodromy $t = t_f^k$.

Let $B \subset K_{\tilde{f}}$ be the branched locus and let E be the complement of an open tubular neighborhood of B in $K_{\tilde{f}}$. Thus, E is the total space of a fiber bundle over S^1 with fiber F_f and with algebraic monodromy $t = t_f^k$. Then, we have the following Wang exact sequence of homology [30] (see also [19, Lemma 8.4]):

$$H_n(F_f; \mathbf{Z}) \xrightarrow{t-1} H_n(F_f; \mathbf{Z}) \rightarrow H_n(E; \mathbf{Z}) \rightarrow H_{n-1}(F_f; \mathbf{Z}).$$

Since F_f is $(n-1)$ -connected [19], we have $H_{n-1}(F_f; \mathbf{Z}) = 0$ so that we have

$$H_n(E; \mathbf{Z}) \cong H_n(F_f; \mathbf{Z}) / \text{Im}(t-1).$$

Then, by the Meyer–Vietoris exact sequence for the pair $(E, N(B))$, where $N(B)$ is the closed tubular neighborhood of B in $K_{\tilde{f}}$, we have that

$$H_n(\partial N(B); \mathbf{Z}) \rightarrow H_n(N(B); \mathbf{Z}) \oplus H_n(E; \mathbf{Z}) \rightarrow H_n(K_{\tilde{f}}; \mathbf{Z}) \rightarrow H_{n-1}(\partial N(B); \mathbf{Z})$$

is exact. As $N(B) \cong K_f \times D^2$ and K_f is homeomorphic to S^{2n-1} with $n \geq 3$, we see that

$$H_n(\partial N(B); \mathbf{Z}), H_n(N(B); \mathbf{Z}), \text{ and } H_{n-1}(\partial N(B); \mathbf{Z})$$

all vanish. Therefore, we have $H_n(K_{\tilde{f}}; \mathbf{Z}) \cong H_n(E; \mathbf{Z})$, and hence they are isomorphic to the quotient $H_n(F_f; \mathbf{Z}) / (t_f^k - 1)H_n(F_f; \mathbf{Z})$.

On the other hand, $\text{Ker}(t_f^k - 1)$ is a pure submodule of the free abelian group $H_n(F_f; \mathbf{Z})$ of finite rank, where a submodule of a free abelian group is said to be *pure* if it is a direct summand. Therefore, there exists a free abelian subgroup H_f of $H_n(F_f; \mathbf{Z})$ such that

$$H_n(F_f; \mathbf{Z}) = H_f \oplus \text{Ker}(t_f^k - 1).$$

As $\text{Im}(t_f^k - 1)$ is contained in $\text{Ker}(t_f^k - 1)$, we see that $H_n(K_{\tilde{f}}; \mathbf{Z}) \cong H_n(F_f; \mathbf{Z}) / (t_f^k - 1)H_n(F_f; \mathbf{Z})$ is isomorphic to $H_f \oplus (\text{Ker}(t_f^k - 1) / \text{Im}(t_f^k - 1))$. Note that a similar isomorphism holds for $H_n(K_{\tilde{g}}; \mathbf{Z})$ as well.

Since the twist groups, which are the torsion subgroups of

$$\text{Ker}(t_f^k - 1) / \text{Im}(t_f^k - 1) \quad \text{and} \quad \text{Ker}(t_g^k - 1) / \text{Im}(t_g^k - 1),$$

are not isomorphic to each other according to [7], we see that the torsion subgroups of $H_n(K_{\tilde{f}}; \mathbf{Z})$ and $H_n(K_{\tilde{g}}; \mathbf{Z})$ are not isomorphic. $\square$

The above lemma implies that although K_f and K_g are cobordant, their cyclic suspensions $K_{\tilde{f}}$ and $K_{\tilde{g}}$ are not, since they are not diffeomorphic.

Let us take further iterated cyclic suspensions appropriately, say $K_{\hat{f}}$ and $K_{\hat{g}}$, where

$$\hat{f}(z_1, z_2, \dots, z_{n+3}, z_{n+4}) = \tilde{f}(z_1, z_2, \dots, z_{n+2}) + z_{n+3}^v + z_{n+4}^w$$

and

$$\hat{g}(z_1, z_2, \dots, z_{n+3}, z_{n+4}) = \tilde{g}(z_1, z_2, \dots, z_{n+2}) + z_{n+3}^v + z_{n+4}^w$$

for some distinct prime numbers v and w . The following lemma can be proved by an argument similar to that in [7, §1].

LEMMA 6.5. *For some distinct prime numbers v and w , the algebraic knots $K_{\hat{f}}$ and $K_{\hat{g}}$ are spherical.*

Proof. It is known that the roots of the characteristic polynomials for $K_{\tilde{f}}$ and $K_{\tilde{g}}$ are roots of unity [5]. Therefore, they are m -th roots of unity for a common positive integer m . If v and w are distinct prime numbers prime to m , then the roots of the characteristic polynomials for $K_{\hat{f}}$ and $K_{\hat{g}}$ are products of some m -th roots of unity and primitive vw -th roots of unity (see [28] or [7, §1]). This implies that if a cyclotomic polynomial $\phi_\ell(t)$ is a divisor of the characteristic polynomial of $K_{\hat{f}}$ or $K_{\hat{g}}$, then ℓ is divisible by v and w . Recall that if ℓ is divisible by two distinct primes, then the ℓ -th cyclotomic polynomial evaluated at 1 is equal to ± 1 , which is a standard property of cyclotomic polynomials. Therefore, we see that $\Delta_{K_{\hat{f}}}(1) = \pm 1$ and $\Delta_{K_{\hat{g}}}(1) = \pm 1$. Hence, $K_{\hat{f}}$ and $K_{\hat{g}}$ are spherical (for example, see [19, Theorem 8.5]). This completes the proof. $\square$

Since the Seifert forms of $K_{\tilde{f}}$ and $K_{\tilde{g}}$ are algebraically cobordant, so are the Seifert forms of $K_{\hat{f}}$ and $K_{\hat{g}}$. Since they are spherical, they are cobordant (see [14] or Remark 6.6 (1) below).

Summarizing, we have the following.

- (1) The algebraic knots K_f and K_g are cobordant, but are not isotopic.
- (2) Their k -fold cyclic suspensions $K_{\tilde{f}}$ and $K_{\tilde{g}}$ are not diffeomorphic to each other and are not cobordant.
- (3) The iterated cyclic suspensions $K_{\hat{f}}$ and $K_{\hat{g}}$ of $K_{\tilde{f}}$ and $K_{\tilde{g}}$, respectively, are cobordant.

This is yet another example that shows that cyclic suspensions (with a fixed suspension degree) do not behave well with respect to cobordisms. This time, the example shows this phenomenon for algebraic knots.

REMARK 6.6. (1) If K_0 and K_1 are cobordant knots, then if their cyclic suspensions $\tilde{K}_0$ and $\tilde{K}_1$, respectively, of the same degree are spherical of dimension greater than or equal to 3, then they are cobordant. This is because the Seifert matrices of $\tilde{K}_i$ are tensor products of those of K_i , which are (algebraically) cobordant, and the same matrix, and hence they are algebraically cobordant. For spherical higher dimensional knots, this implies that they are cobordant (see [14]).

(2) Similarly, if K is a spherical knot which has finite order in the knot cobordism group, then if its cyclic suspension $\tilde{K}$ is spherical, then $\tilde{K}$ also has finite order in the knot cobordism group. This is because, since the Seifert form of K is Witt equivalent to 0 over the real numbers, so is that of $\tilde{K}$.

(3) Suppose that K is a spherical knot and that its d -fold cyclic suspension $\tilde{K}$ is also spherical. Let us suppose that $\tilde{K}$ is null-cobordant. Then, we do not know if K is also null-cobordant or not.

Similarly, suppose that K_0 and K_1 are spherical knots and that their d -fold cyclic suspensions $\tilde{K}_0$ and $\tilde{K}_1$, respectively, are also spherical. Let us suppose that $\tilde{K}_0$ and $\tilde{K}_1$ are cobordant. Then, we do not know if K_0 and K_1 are also cobordant or not, except for the case $d = 2$.

Since the algebraic knots associated with Brieskorn polynomials are iterated cyclic suspensions of torus knots, the observations in this section may show that by adding extra variables we may encounter a pair of algebraic knots associated with Brieskorn polynomials which are cobordant but which have distinct exponents.

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