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ARCS ON RATIONAL DOUBLE POINTS IN ARBITRARY CHARACTERISTIC

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Abstract. We settle the Nash problem for two-dimensional rational double points in all characteristics. The proof is based on a direct computation of the families of arcs through these singularities.

1. Introduction

In the late sixties, Nash uncovered a connection between certain information coming from resolution of singularities of complex varieties and the topology of the space of formal arcs through their singularities [Nas95]. The connection takes a particularly simple form in dimension two, where a naturally defined injective map, called the *Nash map*, is established from the set of irreducible components of the space of arcs of the singularity to the set of exceptional divisors in the minimal resolution. This map can be interpreted as an inclusion between certain sets of divisorial valuations, and Nash conjectured that the map is surjective, namely, the two sets of valuations are the same. This conjecture became known as the *Nash problem*. Nash also proposed a conjectural picture for higher dimensional singularities which turned out to hold in some cases but fail in general [IK03, dF13]. The Nash problem in higher dimension eventually evolved into the question about determining the image of the Nash map (see, e.g., [LJR12, dFD16, dFW24, BLM25] for recent works in this direction), a question which still remains largely open.

Pioneer work towards the Nash problem was done in [LJ80] where an approach via *wedges* (namely, arcs on arc spaces) to detect adjacencies in the space of arcs was first introduced, though a rigorous proof of the existence of such wedges, based on a result known as the *curve selection lemma*, was only established years later in [Reg06]. Several special cases of the Nash problem in dimension two were solved throughout the years in the works [Nas95, Reg95, LJRL99, IK03, Reg04, PPP06, Plé08, LA11, PS12, PP13] before the problem was completely solved in [FdBPP12]. An independent proof for rational surface singularities was given in [Reg12]. Surveys can be found in [PS15, dF18].

People have looked at the Nash problem for normal surfaces in positive characteristics as well, see for instance [Reg95, IK03, Reg06, Reg12, Nob18]. Some cases such as minimal surface singularities (i.e., two-dimensional rational singularities whose fundamental cycle in the minimal resolution is reduced) and toric singularities (in all dimensions) were settled, but beyond that, little is know about the Nash problem in positive characteristics. It should be noted that, except for A_n singularities, rational double points are not covered by those results. These can be considered in many regards as the simplest singularities. They were classically studied, with

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their classification tracing through [DV34, Art66, Lip69, GK90], and appear in many different contexts.

The purpose of this paper is to settle the Nash problem for rational double points in positive characteristics.

Theorem 1.1. *The Nash problem holds for two-dimensional rational double points over algebraically closed fields of arbitrary characteristic.*

It is instructive to look at the history of the Nash problem for rational double points in characteristic zero. While the case of A_n singularities can be checked with a simple computation and was already well understood by Nash, it took several decades before the remaining rational double points were settled in characteristic zero. The cases of D_n and E_6 singularities were treated in [Plé08, PS12], and a new proof for D_n and E_6 was given [LA11] where the E_7 singularity case was settled as well. However, the remaining case of the E_8 singularity proved to be too hard to be dealt with the methods introduced in those works. For this last case, a completely different route was taken in [PP13], where the Nash problem was solved for all quotient surface singularities after translating it into a topological one (cf. [FdB12]); notably, a similar approach was later applied to tackle the general case in [FdBPP12]. In all of the above works the ground field is assumed to have characteristic zero. This assumption is especially relevant for the proof of the E_8 case which uses transcendental methods in an essential way.

A purely algebraic proof of the Nash problem for surfaces was given in [dFD16]. Both proofs from [FdBPP12, dFD16] rely on wedges and the curve selection lemma. In fact, as remarked by Lejeune-Jalabert and Reguera, the proof of [dFD16] solves an *a priori* harder problem, known as the *wedge lifting problem*, which goes back to [LJ80] and is known to imply the Nash problem [Reg04]. By contrast, it was shown in [Reg12] that the wedge problem fails in positive characteristics; the examples are given by an E_8 singularity in characteristics $2, 3, 5$. Reguera's counterexamples to the wedge lifting problem easily generalizes to all characteritics (see Appendix A).

With Theorem 1.1 now settling the Nash problem for E_8 in all characteristics, we conclude that these two problems—the Nash problem and the wedge lifting problem—are not equivalent in positive characteristic.

The proof in [dFD16] breaks down in positive characteristics due to the possible occurrence of wild ramification, which invalidate a key estimate in the proof. With the approach via wedges failing in positive characteristics, we address Theorem 1.1 via a direct computation from the given equations. The idea is to construct the irreducible components in the space of arcs from the irreducible components at the finite jet level, by going up one level at a time. The proof is inspired by the computations of the irreducible components of the jets spaces through rational double points carried out in characteristic zero in [Mou14], though there are some differences in the way the computations are organized here.

In [dFW24], we constructed a map $\Psi_{X,m}$, defined for $m \gg 1$, from the set of irreducible components of the space of arcs through a singularity and the set of irreducible components of the space of m-jets through the singularity. There, we used results on the Nash problem to show that this map is surjective for a certain class of singularities of characteristic zero that generalize the notion of Du Val singularities in higher dimensions. Here, looking at two-dimensional rational double points of arbitrary characteristic, we follow an opposite approach, where we first compute the irreducible components at the jet level, and then show that map $\Psi_{X,m}$ is surjective, deducing from there the validity of the Nash problem.

A crucial part of the proof is to distinguish the components at the arc level (which is equivalent, in this setting, to showing that $\Psi_{X,m}$ is surjective). This in principle requires controlling infinitely many equations. We make this possible by carefully organizing our computations, taking radicals at each step to simplify the equations of the sets.

Our computations apply mostly uniformly to all characteristics, though some minor adjustments are needed in small characteristics (between characteristics 2 and 3 for E_6 and E_7, and between characteristics 2, 3 and 5 for E_8). The fact that the computations are sensitive to small characteristics can be seen as a manifestation of the subtlety of the problem in that setting, as one should expect in view of Reguera's aforementioned counterexamples to the wedge lifting problem. In the cases where different isomorphism types exist in small characteristics, the same computations carry through for all isomorphism types.

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2. Jet and arc components through singularities

Let k be an algebraically closed field. For a k-algebra R, let R_m be the universal Hasse–Schmidt k-algebra of order m of R (cf. [Voj07]). For any $f \in R$ and $i \leq m$, we denote by $f_i \in R_m$ the i-th Hasse–Schmidt derivative of f. For instance, if $R = k[x,y,z]$, then $R_m = k[x_i, y_i, z_i \mid 0 \leq i \leq m]$, and if $R = k[x,y,z]/(f)$ for some $f \in k[x,y,z]$, then

$$R_m = k[x_i, y_i, z_i \mid 0 \leq i \leq m]/(f_j \mid 0 \leq j \leq m).$$

Let X be a k-scheme and $m \in \mathbb{N}$. We denote by X_m and X_∞ the m-jet scheme and arc space of X, respectively (cf. [EM09]). These are schemes over k whose K-valued points, for any field extension K/k, are jets $\gamma\colon \operatorname{Spec} K[t]/(t^{m+1}) \to X$ and arcs $\alpha\colon \operatorname{Spec} K[[t]] \to X$ with coefficients in K. If $X = \operatorname{Spec} R$ for a k-algebra R, then we have $X_m = \operatorname{Spec} R_m$. We denote by $\pi_m = \pi_{X,m}\colon X_m \to X$ and $\pi = \pi_X\colon X_\infty \to X$ the natural projections mapping a jet $\gamma(t)$ and an arc $\alpha(t)$ to their base point $\gamma(0), \alpha(0) \in X$, respectively. For a closed set $\Sigma \in X$, we denote by $X_m^\Sigma := \pi_m^{-1}(\Sigma)_{\mathrm{red}}$ and $X_\infty^\Sigma := \pi^{-1}(\Sigma)_{\mathrm{red}}$ the sets of m-jets and of arcs through $\Sigma \subset X$.

Assume now that X is a variety and $\Sigma = \operatorname{Sing} X$ is its singular locus. Then X_m is a scheme of finite type over k, and therefore X_m^Σ, as a closed subscheme of a scheme of finite type, has finitely many irreducible components. We call them the *m-jet components* through $\Sigma \subset X$. In general, however, X_∞ is not of finite type, and in fact it is not Noetherian. Nonetheless, it is still the case that X_∞^Σ has finitely many irreducible components as long as X admits a resolution of singularities [dF18, Corollary 8.8]. We call them the *arc components* through $\Sigma \subset X$.

If either $\operatorname{char} k = 0$ or if X has isolated singularities and admits a resolution of singularities $f\colon Y \to X$, then every irreducible component C of X_∞^Σ is equal to the closure of a set of the form $f_\infty(\pi_Y^{-1}(E))$ for some prime divisor E on Y with $f(E) \subset \Sigma$; here $f_\infty\colon Y_\infty \to X_\infty$ is the natural map defined by $f_\infty(\alpha) := f \circ \alpha$. In general, the same is true if we restrict our attention to arc components that are not fully contained in Σ_∞; we refer to these as the *non-degenerate* components. This correspondence associating ord_E to C defines an injective map

$$\mathcal{N}_X\colon \{\text{non-degenerate arc components}\} \to \{\text{essential divisorial valuations}\},$$

called the *Nash map*, from the set of non-degenerate arc components through the singularities of X to the set of a special class of valuations on X, called *essential divisorial valuations* [Nas95, IK03]. When X is a surface, these are precisely the valuations defined by the exceptional divisors on the minimal resolution.

At its core, the Nash problem aims to understand the image of this map. If $(X,0)$ is a surface singularity, then the question is whether the image of the Nash map $\mathcal{N}_X$ is the set of divisorial valuations defined by the exceptional divisors on the minimal resolution of singularities of $(X,0)$. As we already discussed in the introduction, [FdBPP12] solves this problem in characteristic zero.

Arc components also relate to jet components. This was studied in [dFW24], where an injective map

$$\Psi_{X,m}\colon \{\text{non-degenerate arc components}\} \to \{m\text{-jet components}\}$$

from the set of non-degenerate arc components through the singular locus Σ of X to the set of m-jet components through Σ is defined for all $m \gg 1$. This map sends a non-degenerate arc component C to the unique m-jet component D containing the image of C under the truncation map $X_\infty \to X_m$ [dFW24, Theorem 4.1]. The characteristic is there assumed to be zero, but the construction of $\Psi_{X,m}$ works in all characteristics. The definition of $\Psi_{X,m}$ extends to all m as a multivalued function.

The function $\Psi_{X,m}$ is related to the directed graph Γ_X determined by all jet components as m varies. This graph was studied in [Mou11, Mou14, Mou17, CM21], and is defined as follows: vertices v correspond to jet components through $\Sigma \subset X$ and are labelled by the order m of the jets, and an edge is drawn from a vertex v of order m to a vertex v' of order $m+1$ whenever the component corresponding to v' maps into the one corresponding to v under the truncation map $X_{m+1} \to X_m$. We refer to Γ_X as the *graph of jet components* through the singular locus Σ of X.

It is natural to connect the Nash problem to properties of the graph Γ_X. With this in mind, we introduce some terminology. A *branch* of Γ_X stemming from a vertex v is the directed subgraph of Γ_X obtained by removing all vertices that are not reachable by v. A *directed chain* in Γ_X is a connected directed subgraph of Γ_X such that every vertex in it has at most one edge stemming from it. A branch is said to be *simple* if it is a directed chain in Γ_X. A subgraph of Γ_X is *unbounded* if it contains vertices of arbitrarily large order. We regard unbounded subgraphs of Γ_X up to equivalence, where any two such subgraphs are said to be equivalent if they agree in large enough order (i.e., after removing all vertices of oder $< m$ for some m).

In view of the injectivity and well-definedness of $\Psi_{X,m}$ for $m \gg 1$, each arc component through $\Sigma \subset X$ determines a unique (up to equivalence) unbounded directed chain in Γ_X, hence we have an injective map

$$\Psi_X \colon \{\text{non-degenerate arc components}\} \to \{\text{unbounded directed chains}\}$$

which sends an arc component C to the unbounded directed chain in Γ_X whose vertex of order m, for $m \gg 1$, corresponds to the m-jet component given by $\Psi_{X,m}(C)$.

In some cases (e.g., when $\Psi_{X,m}$ is surjective for $m \gg 1$), Ψ_X is surjective and every unbounded directed chain in Γ_X is equivalent to an unbounded simple branch, but the latter property fails in general (e.g., for non-canonical toric surface singularities [Mou17]) and it is unclear whether the first should hold in general.

These properties are closely related to the Nash problem. Knowing the image of the Nash map can help determine when these properties hold (see, e.g., [dFW24]). Conversely, computing jet components and understanding the image of Ψ_X provides a way to determine the image of the Nash map. The latter is our approach to prove Theorem 1.1. We summarize our main result in the next statement, which includes Theorem 1.1.

Theorem 2.1. *Let $(X,0)$ be a two-dimensional rational double point defined over an algebraically closed field k. Then:*

(1) *The map $\mathcal{N}_X$ from the set of non-degenerate arc components through $0 \in X$ to the set of valuations defined by the exceptional divisors on the minimal resolution of X is surjective, hence a bijection.*
(2) *The map $\Psi_{X,m}$ from the set of arc-components to the set of m-jet components through $0 \in X$ is surjective, hence a bijection, for every $m \gg 1$ (and for every m as a multivalued function).*
(3) *The map Ψ_X from the set of arc-components through $0 \in X$ to the set of equivalent classes of unbounded directed chains in Γ_X is surjective, hence a bijection, for every m. Furthermore, every unbounded directed chain is equivalent to an unbounded simple branch of Γ_X.*

The proof of Theorem 2.1 is contained in the following sections, which are organized depending on the type of singularity. We summarize in the next section some general principle that will guide our computations. We are hoping that a similar approach may be applied in the future to study the Nash maps in other situations as well, including higher dimensional cases.

Notation 2.2. The following notation will be used in the proof of Theorem 2.1. Let I and J be ordered index sets. For a set of variables $\{u_i \mid i \in I\}$ and $m \in I$, we use the short-hand notation $u_{\leq m}$ to denote the set $\{u_i \mid i \leq m\}$ and $u_{>m}$ to denote $\{u_i \mid i > m\}$. For any $j \in I$ with $j < m$, we denote the set $\{u_i \mid i \leq m, i \neq j\}$ by $u_{\leq m} \setminus u_j$. In a polynomial ring $R = A[u_i, v, w_j \mid (i,j) \in I \times J]$, given $(m,n) \in I \times J$ we write $V(u_{\leq m})_v \,|_{w_j = P_j,\ j \geq n}$ to denote a subset of $\operatorname{Spec} R$ obtained by first setting $u_i = 0$ for all $i \leq m$, then localizing at v, and finally eliminating the variables w_j, for $j \geq n$, using equations $w_j = P_j$ where $P_j \in A[u_{>m}, v, w_{>j}]_v$. Finally, given a nonzero divisor f and an ideal $\mathfrak{a}$ in a ring R, we denote by $D(f) := \operatorname{Spec} R_f$ the localization at f, and use the symbols $Z(\mathfrak{a})$ and $V(\mathfrak{a})$ to denote the subscheme and the (reduced) zero locus of $\mathfrak{a}$ in $\operatorname{Spec} R$.

3. Computational strategy

Let $f \in k[x,y,z]$ be a polynomial, and let f_i denote its i-th Hasse–Schmidt derivatives. We summarize two key observations that are implicitly used in our analysis of the fiber X_∞^0 when $X = \{f = 0\} \subset \mathbb{A}^3$ is a surface with a rational double point at 0. Similar properties hold for polynomials in any number of variables.

Proposition 3.1. *Let*

$$f = \sum_{(a,b,c) \in I} r_{a,b,c} \cdot x^a y^b z^c$$

with $r_{a,b,c} \in k$, and assume that the characteristic is either zero or larger than any of the exponents a, b, c appearing in f. Suppose that $f_0, f_1, \ldots, f_{n-1} \in (x_{\leq i-1}, y_{\leq j-1}, z_{\leq h-1})$. Then the next derivative satisfies

$$f_n \equiv f|_J(x_i, y_j, z_h) \mod (x_{\leq i-1}, y_{\leq j-1}, z_{\leq h-1}),$$

for some (possibly empty) $J \subset I$, where

$$f|_J := \sum_{(a,b,c) \in J} r_{a,b,c} \cdot x^a y^b z^c.$$

That is, f_n, modulo lower-order derivatives, is congruent to $f|_J$ evaluated at the variables x_i, y_j, z_h. Importantly, each variable appears only in one specific order of its derivative; for instance, only x_i appears and not $x_{i+1}, x_{i+2}, \ldots, x_n$.

Proof. Suppose, for contradiction, that after reducing modulo $(x_{\leq i-1}, y_{\leq j-1}, z_{\leq h-1})$, the derivative f_n remains nonzero and contains a monomial involving a variable of strictly higher order than those in $\{x_i, y_j, z_h\}$. Without loss of generality, assume there is a variable $x_{i+\ell}$ for some $\ell \geq 1$, appearing in a monomial of f_n of the form

$$u \cdot x_{i+\ell}^{\alpha} \cdot x_i^{\beta} \cdot M,$$

where $u \in k$, $\alpha > 0$, $\beta \geq 0$, and M is a monomial in $k[x_{i'}, y_{j'}, z_{h'} \mid i' \geq i + \ell + 1, j' \geq j, h' \geq h]$. In other words, ℓ is chosen to be the smallest positive integer such that $x_{i+\ell}$ appears in this monomial of f_n. Then $f_{n-\alpha\ell}$ contains a monomial of the form

$$v \cdot x_i^{\alpha+\beta} \cdot M,$$

where $v \in k$ differs from u only by factors given by binomial coefficients. Specifically, we have $u = v \cdot \binom{\alpha+\beta}{\alpha}$. It follows that if $u \neq 0$ then $v \neq 0$. This gives a contradiction to our assumption that $f_{n-\alpha\ell} \in (x_{\leq i-1}, y_{\leq j-1}, z_{\leq h-1})$. □

Proposition 3.2. *With the same assumptions and notation as in Proposition 3.1, suppose that $f|_J \neq 0$ and let $g := f|_J(x_i, y_j, z_h)$. Consider the scheme $Z := Z(x_{\leq i-1}, y_{\leq j-1}, z_{\leq h-1}, g)$, and let Z_{sm} denote the smooth locus of Z, i.e.,*

$$Z_{\mathrm{sm}} = Z \cap \Big(D\Big(\frac{\partial g}{\partial x_i}\Big) \cup D\Big(\frac{\partial g}{\partial y_j}\Big) \cup D\Big(\frac{\partial g}{\partial z_k}\Big)\Big).$$

Then on each of the open sets $Z \cap D\big(\frac{\partial g}{\partial x_i}\big)$, $Z \cap D\big(\frac{\partial g}{\partial y_j}\big)$, and $Z \cap D\big(\frac{\partial g}{\partial z_k}\big)$, we can use the remaining equations $f_{n+\ell} = 0$, for $\ell \geq 0$, to eliminate the variables $x_{i+\ell}$, $y_{j+\ell}$, or $z_{h+\ell}$, respectively.

Proof. For any $\ell \geq 1$, we observe that, modulo $(x_{\leq i-1}, y_{\leq j-1}, z_{\leq h-1})$, the $(n+\ell)$-th Hasse–Schmidt derivative of f has the form

$$f_{n+\ell} \equiv \frac{\partial g}{\partial x_i}(x_i, y_j, z_h) \cdot x_{i+\ell} + \frac{\partial g}{\partial y_j}(x_i, y_j, z_h) \cdot y_{j+\ell} + \frac{\partial g}{\partial z_k}(x_i, y_j, z_h) \cdot z_{h+\ell} + G_{n,\ell}(x_{i'}, y_{j'}, z_{h'}) \mod (x_{\leq i-1}, y_{\leq j-1}, z_{\leq k-1}),$$

where $G_{n,\ell} \in k[x_{i'}, y_{j'}, z_{h'} \mid i \leq i' < i+\ell, j \leq j' < j+\ell, k \leq k' < k+\ell]$. The result easily follows from this. □

Remark 3.3. Implicit in Proposition 3.2 is the assumption that Z_{sm} is nonempty, as otherwise the statement is vacuous. If Z is generically smooth, then Proposition 3.2 provides a way of computing the inverse image of its smooth locus in X_∞. The proposition does not say anything, however, if these schemes are non-reduced. In our setting, when this occurs in the proof of Theorem 2.1, then g typically consists of a single monomial. Those situations are treated by setting each variable appearing in the monomial equal to zero, and go one step up in the jet schemes to continue the computations.

4. A_n SINGULARITIES

In this section we verify Theorem 2.1 for A_n singularities. The computation of jet and arc components in this case is straightforward and completely characteristic free. We include it here for completeness and to gives an idea in this simple case of how the argument will be organized in the other cases.

Let $(X, 0)$ be an A_n singularity, with $n \geq 1$. We may assume that X is defined in $\mathbb{A}^3$ by

$$f(x, y, z) = z^{n+1} + xy.$$

Step 1. For $i = 1, \dots n-1$, we set $V_i := V(x_{\leq i-1}, y_0, z_0) \subset \mathbb{A}^3_\infty$. Inductively, we see that $V_i \subset V(f_0, \dots, f_i)$. We define $W_i := V_i \cap V(f_{i+1})$. Note that

$$f_{i+1} \equiv x_i y_1 \mod (x_{\leq i-1}, y_0, z_0).$$

We stratify $W_i = (W_i \cap D(x_i)) \sqcup (W_i \cap V(x_i))$. On $W_i \cap D(x_i)$, we have $y_1 = 0$, and we use the equations $f_m = 0$, for $m \geq i+1$, and the fact that x_i is a unit on this set, to eliminate the variables y_{m-i}. The first $n-i$ of these equations give $y_1 = \cdots = y_{n-i} = 0$, and we obtain the set

$$C_i^\circ = V(x_{\leq i-1}, y_{\leq n-i}, z_0)_{x_i}|_{y_{m-i} = (x_i)^{-1} P_{i,m},\ m \geq n+1}$$

where the polynomials $P_{i,m}$ do not depend on y_j for any $j \geq m-i$. We denote by $C_i \subset X_\infty$ the closure of C_i°.

We are left with $W_i \cap V(x_i) = V(x_{\leq i}, y_0, z_0)$. If $i < n-1$, then this is precisely V_{i+1}, and we apply induction. If $i = n-1$, then we end up with $V_n := W_{n-1} \cap V(x_{n-1}) = V(x_{\leq n-1}, y_0, z_0)$.

Let $W_n := V_n \cap V(f_{n+1})$. We have

$$f_{n+1} \equiv z_1^{n+1} + x_n y_1 \mod (x_{\leq n-1}, y_0, z_0).$$

We cover W_n with $D(x_n) \cup D(y_1) \cup V(x_n, y_1)$. On $W_n \cap D(x_n)$, we use the equations $f_{n+j} = 0$ for $j \geq 1$ to eliminate y_j. This produces C_n°. On $W_n \cap D(y_1)$, we use the remaining equations

to solve for x_j for $j \geq n$. The resulting set is irreducible and agree with C_n° on $D(x_n) \cap D(y_1)$. Finally, on $W_n \cap V(x_n, y_1)$ we automatically have $z_1 = 0$, and we end up with a set contained in the closure of C_n°. This shows that $W_n \cap V(f_m \mid m \geq n+2)$ is irreducible and agrees with the closure C_n of C_n°.

Step 2. To conclude the proof, it suffices to note that the sets $C_1, \ldots, C_n$ are irreducible, they fill up the fiber X_∞^0, and there are no containments among any two of them, namely, $C_i \not\subset C_j$ whenever $i \neq j$. The first two assertions are clear from the construction, and the last assertion follows immediately by comparing the defining equations of the sets C_i°.

5. D_n SINGULARITIES

In this section we look at D_n singularities. We give the proof of Theorem 2.1 in the even case D_{2n}. The proof for D_{2n+1} singularities is similar and is omitted.

Let $(X, 0)$ be a D_{2n} singularity, with $n \geq 2$. We may assume that X given in $\mathbb{A}^3$ by

$$f(x, y, z) = z^2 + x^2 y + xy^n + h(x, y, z) = 0,$$

where $h \in \{0, xy^{n-r}z \mid 1 \leq r \leq n-2\}$ if the characteristic is 2, and $h = 0$ otherwise.

The proof goes by constructing the locally closed sets $C_1^\circ, \ldots, C_{2n-2}^\circ, C_{n-1}'^\circ, C_{n-1}''^\circ$ using the equations of X_∞^0, and prove that their closures $C_1, \ldots, C_{2n-2}, C_{n-1}', C_{n-1}''$ are the irreducible components of X_∞^0.

The construction breaks into four steps. First, we compute the first $n-2$ sets $C_1^\circ, \ldots, C_{n-2}^\circ$ by looking at the jet schemes up to order $2n-3$. Next we compute C_{n-1}°, $C_{n-1}'^\circ$ and $C_{n-1}''^\circ$ from the jet scheme at level $2n-1$. We then compute $C_n^\circ, \ldots, C_{2n-3}^\circ$ going up on higher levels, until we reach level $4n-2$, which produces the last component C_{2n-2}. By construction, these sets are irreducible and cover all of X_∞^0. The last step of the proof will be to prove that there are no inclusions among these sets.

Step 1. For $i = 1, \ldots n-2$, we set

$$V_i := V(x_{\leq i-1}, y_0, z_{\leq i-1}) \subset \mathbb{A}_\infty^3.$$

Inductively, we see that $V_i \subset V(f_0, \ldots, f_{2i-1})$. We cut V_i with the next equations

$$f_{2i} = f_{2i+1} = 0,$$

and define

$$W_i := V_i \cap V(f_{2i}, f_{2i+1}).$$

We compute

$$\begin{cases} f_{2i} \equiv z_i^2 & \mod (x_{\leq i-1}, y_0, z_{\leq i-1}), \\ f_{2i+1} \equiv x_i^2 y_1 & \mod (x_{\leq i-1}, y_0, z_{\leq i}). \end{cases}$$

This implies that $W_i = V_i \cap V(z_i, x_i y_1)$. We stratify $W_i = (W_i \cap D(x_i)) \sqcup (W_i \cap V(x_i))$. On $W_i \cap D(x_i)$, we have $y_1 = 0$. For $m \geq 2i+2$, we have

$$f_m \equiv x_1^2 y_{m-2} + G_{i,m} \mod (x_{\leq i-1}, y_{\leq 1}, z_{\leq i})$$

where $G_{i,m}$ does not depend on y_{j-2} for any $j \geq m$. Therefore, we use the remaining equations $f_m = 0$, for $m \geq 2i+2$, and the fact that x_i is a unit on this set, to eliminate the variables y_{m-2i}. This gives the set

$$C_i^\circ = V(x_{\leq i-1}, y_{\leq 1}, z_{\leq i})_{x_i}|_{y_{m-2i} = (x_i^2)^{-1} P_{i,m},\ m \geq 2i+2}$$

where $P_{i,m} = -g_{i,m}$. We are left with

$$W_i \cap V(x_i) = V(x_{\leq i}, y_0, z_{\leq i}).$$

If $i < n-2$, then this is precisely V_{i+1}, and we apply induction.

Step 2. The previous step gives us the sets $C_1^\circ, \dots, C_{n-2}^\circ$, and leaves us in the end with the set

$$V_{n-1} := V(x_{\leq n-2}, y_0, z_{\leq n-2}).$$

Observing that $V_{n-1} \subset V(f_0, \dots, f_{2n-3})$, we define

$$W_{n-1} := V_{n-1} \cap V(f_{2n-2}, f_{2n-1}).$$

We have

$$\begin{cases} f_{2n-2} \equiv z_{n-1}^2 & \mod (x_{\leq n-2}, y_0, z_{\leq n-2}), \\ f_{2n-1} \equiv x_{n-1}^2 y_1 + x_{n-1} y_1^n & \mod (x_{\leq n-2}, y_0, z_{\leq n-1}). \end{cases}$$

hence $W_{n-1} = V_{n-1} \cap V(z_{n-1}, x_{n-1}^2 y_1 + x_{n-1} y_1^n)$. We stratify W_{n-1} using the stratification

$$D(x_{n-1}y_1) \sqcup (D(x_{n-1}) \cap V(y_1)) \sqcup (V(x_{n-1}) \cap D(y_1)) \sqcup V(x_{n-1}, y_1).$$

This stratification reflects a decomposition of W_{n-1} into three irreducible components, cut out, respectively, by the three equations $x_{n-1} = 0$, $y_1 = 0$, and $x_{n-1} + y_1^n = 0$, which we analyze next.

We first look at $W_{n-1} \cap D(x_{n-1}) \cap V(y_1)$. For $m \geq 2n$, we have

$$f_m \equiv x_{n-1}^2 y_{m-2n+2} + g_{n-1,m} \mod (x_{\leq n-2}, y_{\leq 1}, z_{\leq n-1}),$$

where $g_{n-1,m}$ does not depend on y_{j-2n+2} for any $j \geq m$. Thus we can solve for y_{m-2n+2} using the remaining equations $f_m = 0$ for $m \geq 2n$. This produces

$$C_{n-1}^\circ = V(x_{\leq n-2}, y_{\leq 1}, z_{\leq n-1})_{x_{n-1}}|_{y_{m-2n+2} = (x_{n-1}^2)^{-1} P_{n-1,m},\, m \geq 2n}$$

where $P_{n-1,m} = -g_{n-1,m}$.

Next, we look at $W_{n-1} \cap D(y_1) \cap V(x_{n-1})$. For $m \geq 2n$, we have

$$f_m \equiv x_{m-n} y_1^n + G'_{n-1,m} \mod (x_{\leq n-1}, y_{\leq 0}, z_{\leq n-1}),$$

where $G'_{n-1,m}$ does not depend on x_{j-n} for any $j \geq m$. Thus we can solve for x_{m-n} using the remaining equations $f_m = 0$ for $m \geq 2n$, obtaining

$$C_{n-1}'^\circ = V(x_{\leq n-1}, y_0, z_{\leq n-1})_{y_1}|_{x_{m-n} = (y_1^n)^{-1} P'_{n-1,m},\, m \geq 2n},$$

where $P'_{n-1,m} = -G'_{n-1,m}$.

We now analyze at the stratum $W_{n-1} \cap D(x_{n-1}y_1)$. Note that on this set we have $x_{n-1} + y_1^{n-1} \equiv 0$, a relation that is used without warning in the next computation. For $m \geq 2n$, we have

$$f_m \equiv x_{n-1} x_{m-n} y_1 + G''_{n-1,m} \mod (x_{\leq n-2}, y_{\leq 0}, z_{\leq n-1})$$

where $G''_{n-1,m}$ does not depend on x_{j-n} for any $j \geq m$. We use the equations $f_m = 0$ to solve for x_{m-n} for $m \geq 2n$, by inverting $2x_{n-1}y_1$. This yields

$$C_{n-1}''^\circ = V(x_{\leq n-2}, y_0, z_{\leq n-1}, x_{n-1} + y_1^{n-1})_{x_{n-1}y_1}|_{x_{m-n} = (x_{n-1}y_1)^{-1} P''_{n-1,m},\, m \geq 2n}$$

where $P''_{n-1,m} = -G''_{n-1.m}$.

We are left with $W_{n-1} \cap V(x_{n-1}, y_1)$. Note that this is equal to $V(x_{\leq n-1}, y_{\leq 1}, z_{\leq n-1})$. We cut with with the next equation $f_{2n} = 0$, and define

$$V_n := W_{n-1} \cap V(x_{n-1}, y_1, f_{2n}).$$

Observing that

$$f_{2n} \equiv z_n^2 \mod (x_{\leq n-1}, y_0, z_{\leq n-1}),$$

we compute $V_n = V(x_{\leq n-1}, y_{\leq 1}, z_{\leq n})$.

Step 3. For $i = n, \dots 2n-3$, consider the set

$$V_i := V(x_{\leq i-1}, y_{\leq 1}, z_{\leq i}).$$

Note that, for $i = n$, the definition of this set agrees with the set V_n defined at the end of the previous step. Inductively, we see that $V_i \subset V(f_0, \dots, f_{2i})$. We define

$$W_i := V_i \cap V(f_{2i+1}, f_{2i+2}).$$

Note that

$$\begin{cases} f_{2i+1} \equiv 0 & \mod (x_{\leq i-1}, y_{\leq 1}, z_{\leq i}), \\ f_{2i+2} \equiv z_{i+1}^2 + x_i^2 y_2 & \mod (x_{\leq i-1}, y_{\leq 1}, z_{\leq i}). \end{cases}$$

Therefore $W_i = V_i \cap V(z_{i+1}^2 + x_i^2 y_2)$. We stratify $W_i = (W_i \cap D(x_i)) \sqcup (W_i \cap V(x_i))$. On $W_i \cap D(x_i)$, we have

$$f_m \equiv x_i^2 y_{m-2i} + G_{i,m} \mod (x_{\leq i-1}, y_{\leq 1}, z_{\leq i})$$

for $m \geq 2i+2$, where $G_{i,m}$ does not depend on y_{j-2i} for any $j \geq m$. We can eliminate the variables y_{m-2i}, obtaining

$$C_i^\circ = V(x_{\leq i-1}, y_{\leq 1}, z_{\leq i})_{x_i}|_{y_{m-2i} = (x_i^2)^{-1} P_{i,m},\ m \geq 2i+2}$$

where $P_{i,m} = -G_{i,m}$. We are left with $W_i \cap V(x_i)$. On this set, we have $z_{i+1} = 0$, hence

$$W_i \cap V(x_i) = V(x_{\leq i}, y_{\leq 1}, z_{\leq i+1}).$$

If $i < 2n-3$, then this is precisely V_{i+1}, and we apply induction.

Step 4. The previous steps have produced the strata $C_1^\circ, \dots, C_{2n-3}^\circ, C_{n-1}'^\circ, C_{n-1}''^\circ$, and we are left with analyzing the set

$$V_{2n-2} := W_{2n-3} \cap V(x_{2n-3}, f_{4n-4}) = V(x_{\leq 2n-3}, y_{\leq 1}, z_{\leq 2n-2}).$$

So far, we have used the equations $f_m = 0$ for $m \leq 4n-4$. We define

$$W_{2n-2} := V_{2n-2} \cap V(f_{4n-3}, f_{4n-2}).$$

We have

$$\begin{cases} f_{4n-3} \equiv 0 & \mod (x_{\leq 2n-3}, y_{\leq 1}, z_{\leq 2n-2}), \\ f_{4n-2} \equiv z_{2n-1}^2 + x_{2n-2}^2 y_2 + x_{2n-2} y_2^n & \mod (x_{\leq 2n-3}, y_{\leq 1}, z_{\leq 2n-2}). \end{cases}$$

On $W_{2n-2} \cap D(x_{2n-2})$, we have

$$f_m \equiv x_1^2 y_{m-2} + x_{m-2n-1} y_1^n + G_{2n-3,m} \mod (x_{\leq 2n-3}, y_{\leq 1}, z_{\leq 2n-2})$$

for $m \geq 4n-1$, where $G_{2n-3,m}$ does not depend on y_{j-2} nor x_{j-2n-1} for any $j \geq m$.

We cover W_{2n-2} with $D(x_{2n-2}) \cup D(y_2) \cup V(x_{2n-2}, y_2)$. On $W_{2n-2} \cap D(x_{2n-2})$, we use the equations $f_m = 0$ to eliminate the variables y_{m-2} for $m \geq 4m-1$, obtaining

$$C_{2n-2}^\circ = V(x_{\leq 2n-3}, y_{\leq 1}, z_{\leq 2n-2})_{x_{2n-2}}|_{y_{m-2} = (y_{m-2})^{-1} P_{2n-3,m},\ m \geq 2i+2}$$

where $P_{2n-3,m} = -x_{m-2n-1} y_1^n - G_{2n-3,m}$. On $W_{2n-2} \cap D(y_2)$, we eliminate the variables x_{m-2n-1} using the same equations $f_m = 0$, but dividing this time by y_2^n; the resulting set agrees with C_{2n-2}° on $D(x_{2n-2}) \cap D(y_2)$. Finally, on $W_n \cap V(x_{2n-2}, y_2)$ we end up with something contained in the closure of C_{2n-2}°.

This shows that $W_{2n-2} \cap V(f_m \mid m \geq 4n-2)$ is irreducible and agrees with the closure of C_{2n-2}°. This realizes the component C_{2n-2}.

Step 5. Let C_i, C'_{n-1}, C''_{n-1} denote the closure of the sets $C_i^\circ, C'^\circ_{n-1}, C''^\circ_{n-1}$ constructed above. It is clear by construction that these sets are all irreducible and fill up X^0_∞. The last step is to show that there are no inclusions among them, hence X^0_∞ has $2n$ irreducible components.

By construction, we clearly have that $C_i \not\subset C_j$ whenever $i < j$, and $C_i \not\subset C'_{n-1} \cup C''_{n-1}$ for $i \leq n-2$. Similarly, $C_{n-1} \not\subset C'_{n-1} \cup C''_{n-1}$, since $x_{n-1} \not\equiv 0$ on C_{n-1} (showing that $C_{n-1} \not\subset C'_{n-1}$) and $x_{n-1} + y_1^{n-1} \not\equiv 0$ on C_{n-1} (showing that $C_{n-1} \not\subset C''_{n-1}$). Furthermore, we have $x_{n-1} \equiv 0$ and $x_{n-1} + y_1^{n-1} \not\equiv 0$ on C'_{n-1} while $x_{n-1} \not\equiv 0$ and $x_{n-1} + y_1^{n-1} \equiv 0$ on C''_{n-1}, hence there are no inclusions among these two sets. Note also that $C'_{n-1}, C''_{n-1} \not\subset C_i$ for all i since y_1 is identically zero on C_i and does not vanish identically on either C'_{n-1} or C''_{n-1}.

It remains to show that $C_j \not\subset C_i$ for $j > i$ and $C_i \not\subset C'_{n-1} \cup C''_{n-1}$ for $i \geq n$. Regarding the first, observe that

$$\begin{cases} f_{2i+2} \equiv x_i^2 y_2 & \mod (x_{\leq j-1} \setminus x_i, y_{\leq 1}, z_{\leq j}), \\ f_{2j+2} \equiv x_j^2 y_2 + z_{j+1}^2 & \mod (x_{\leq j-1}, y_{\leq 1}, z_{\leq j}). \end{cases}$$

Keeping in mind that x_i is a unit on C_i°, we see from the vanishing of f_{2i+2} that $y_2 \equiv 0$ on $C_i \cap V(x_{\leq j-1} \setminus x_i, y_{\leq 1}, z_{\leq j})$ On the other hand, using that x_j is a unit on C_j° and z_{j+1} is a free variable on this set, we see from the the vanishing of f_{2j+2} that $y_2 \not\equiv 0$ on C_j. As $C_j \subset V(x_{\leq j-1} \setminus x_i, y_{\leq 1}, z_{\leq j})$, we conclude that $C_j \not\subset C_i$ whenever $j > i$.

Finally, fix any C_i with $i \geq n$. Note that

$$f_{2i+1} \equiv x_i^2 y_1 \mod (x_{\leq i-1}, y_0, z_{\leq i}).$$

As y_1 restricts to a unit on both C'°_{n-1} and C''°_{n-1}, we deduce that $x_i \equiv 0$ on

$$(C'_{n-1} \cup C''_{n-1}) \cap V(x_{\leq i-1}, y_0, z_{\leq i}).$$

On the other hand, $x_i \not\equiv 0$ on C_i, and since $C_i \subset V(x_{\leq i-1}, y_0, z_{\leq i})$, we conclude that $C_i \not\subset C'_{n-1} \cup C''_{n-1}$ for $i \geq n$.

6. E_6, E_7, and E_8 singularities

In this section we look at the remaining exceptional cases. We present the proof of Theorem 2.1 for E_8. The proofs of the other two cases are similar and are left to the reader.

We therefore assume that X is given in $\mathbb{A}^3$ by

$$f(x, y, z) = z^2 + x^3 + y^5 + h(x, y, z),$$

where $h \in \{0, xy^3z, xy^2z, y^3z, xyz\}$ if $\operatorname{char} k = 2$, $h \in \{0, x^2y^3, x^2y^2\}$ if $\operatorname{char} k = 3$, $h \in \{0, xy^4\}$ if $\operatorname{char} k = 5$, and $h = 0$ otherwise.

We will construct locally closed sets $C_1^\circ, \ldots, C_8^\circ$ whose closures $C_1, \ldots, C_8$ are irreducible and fill up X^0_∞. The last step of the proof will be to prove that there are no inclusions among these sets.

Step 1. We set $V_1 := V(x_0, y_0, z_0)$. The first nontrivial equations are

$$\begin{cases} f_2 \equiv z_1^2 & \mod (x_0, y_0, z_0), \\ f_3 \equiv x_1^3 & \mod (x_0, y_0, z_{\leq 1}), \\ f_4 \equiv z_2^2 & \mod (x_{\leq 1}, y_0, z_{\leq 1}), \\ f_5 \equiv y_1^5 & \mod (x_{\leq 1}, y_0, z_{\leq 2}) \\ f_6 \equiv z_3^2 + x_2^3 & \mod (x_{\leq 1}, y_{\leq 1}, z_{\leq 2}). \end{cases}$$

Let $W_1 := V_1 \cap V(f_2, \ldots, f_6)$. On $W_1 \cap D(x_2)$, both x_2 and z_3 are units (the fact that z_3 is a unit on $W_1 \cap D(x_2)$ follows from the equation $z_3^2 + x_2^3 = 0$, which holds on this set). For $m \geq 7$, we can write

$$f_m \equiv 2z_3 z_{m-3} + 3x_2^2 x_{m-4} + G_{1,m} \mod (x_{\leq 1}, y_{\leq 1}, z_{\leq 2}),$$

where $G_{1,m}$ does not depend on z_{j-3} or x_{j-4} for any $j \geq m$.

If the characteristic is different from 2, we can divide by $2z_3$ to solve for z_{m-3} using the equation $f_m = 0$ for $m \geq 7$. If the characteristic is different from 3, we can divide by $3x_2^2$ and use the same equations to solve for x_{m-4} for $m \geq 7$. Either way, we obtain the irreducible locally closed set

$$C_1^\circ = \begin{cases} V(x_{\leq 1}, y_{\leq 1}, z_{\leq 2}, z_3^2 + x_2^3)_{x_2}|_{z_{m-3}=(2z_3)^{-1}P_{1,m},\ m\geq 7} & (\operatorname{char} k \neq 2) \\ V(x_{\leq 1}, y_{\leq 1}, z_{\leq 2}, z_3^2 + x_2^3)_{x_2}|_{x_{m-4}=(3x_2^2)^{-1}Q_{1,m},\ m\geq 7} & (\operatorname{char} k \neq 3), \end{cases}$$

where $P_{1,m} = -3x_2^2 x_{m-4} - G_{1,m}$ and $Q_{1,m} = -2z_3 z_{m-3} - G_{1,m}$.

Note that the two definitions of C_1° given for $\operatorname{char} k \geq 5$ agree. The closure of C_1° defines an irreducible set $C_1 \subset X_\infty^0$.

Step 2. On $W_1 \cap V(x_2)$ we have $z_2 = 0$, and this leaves us with

$$V_2 := W_1 \cap V(x_2) = V(x_{\leq 2}, y_{\leq 1}, z_{\leq 3}).$$

Next, we have

$$\begin{cases} f_7 \equiv 0 & \mod (x_{\leq 2}, y_{\leq 1}, z_{\leq 3}), \\ f_8 \equiv z_4^2 & \mod (x_{\leq 2}, y_{\leq 1}, z_{\leq 3}), \\ f_9 \equiv x_3^3 & \mod (x_{\leq 2}, y_{\leq 1}, z_{\leq 4}), \\ f_{10} \equiv z_5^2 + y_2^5 & \mod (x_{\leq 3}, y_{\leq 1}, z_{\leq 4}). \end{cases}$$

Let $W_2 := V_2 \cap V(f_7, \ldots, f_{10})$. On $W_2 \cap D(y_2)$, both y_2 and z_5 are units. For $m \geq 11$, we write

$$f_m \equiv 2z_5 z_{m-5} + 5y_2^4 y_{m-8} + G_{2,m} \mod (x_{\leq 3}, y_{\leq 1}, z_{\leq 4}),$$

where $G_{2,m}$ does not depend on z_{j-5} or y_{j-8} for any $j \geq m$.

If $\operatorname{char} k \neq 2$, we use the equations $f_m = 0$ for $m \geq 11$ to solve for z_{m-5}, and if $\operatorname{char} k \neq 5$, we use the same equations to solve for y_{m-8}. We obtain the set

$$C_2^\circ = \begin{cases} V(x_{\leq 3}, y_{\leq 1}, z_{\leq 4}, z_5^2 + y_2^5)_{y_2}|_{z_{m-5}=(2z_5)^{-1}P_{2,m},\ m\geq 11} & (\operatorname{char} k \neq 2) \\ V(x_{\leq 3}, y_{\leq 1}, z_{\leq 4}, z_5^2 + y_2^5)_{y_2}|_{y_{m-8}=(5y_2^4)^{-1}R_{2,m},\ m\geq 11} & (\operatorname{char} k \neq 5) \end{cases}$$

where $P_{2,m} = -5y_2^4 y_{m-8} - G_{2,m}$ and $R_{2,m} = -2z_5 z_{m-5} - G_{2,m}$. The closure of C_2° defines an irreducible set $C_2 \subset X_\infty^0$.

Step 3. On $W_2 \cap V(y_2)$ we have $z_5 = 0$, and this leaves us with

$$V_3 := W_2 \cap V(y_2) = V(x_{\leq 3}, y_{\leq 2}, z_{\leq 5}).$$

Next, we look at

$$\begin{cases} f_{11} \equiv 0 & \mod (x_{\leq 3}, y_{\leq 2}, z_{\leq 5}), \\ f_{12} \equiv z_6^2 + x_4^3 & \mod (x_{\leq 3}, y_{\leq 2}, z_{\leq 5}). \end{cases}$$

Let $W_3 := V_3 \cap V(f_{11}, f_{12})$. On $W_3 \cap D(x_4)$, both x_4 and z_6 are units. For $m \geq 13$, we write

$$f_m \equiv 2z_6 z_{m-6} + 3x_4^2 x_{m-8} + G_{3,m} \mod (x_{\leq 3}, y_{\leq 2}, z_{\leq 5}),$$

where $G_{3,m}$ does not depend on z_{j-6} or x_{j-8} for any $j \geq m$. We use the equations $f_m = 0$, for $m \geq 13$, to solve for z_{m-6} if $\operatorname{char} k \neq 2$, and to solve for x_{m-8} if $\operatorname{char} k \neq 3$. We obtain the set

$$C_3^\circ = \begin{cases} V(x_{\leq 3}, y_{\leq 2}, z_{\leq 5}, z_6^2 + x_4^3)_{x_4}|_{z_{m-6}=(2z_6)^{-1}P_{3,m},\ m\geq 13} & (\operatorname{char} k \neq 2) \\ V(x_{\leq 3}, y_{\leq 2}, z_{\leq 5}, z_6^2 + x_4^3)_{x_4}|_{x_{m-8}=(3x_4^2)^{-1}Q_{3,m},\ m\geq 13} & (\operatorname{char} k \neq 3) \end{cases}$$

where $P_{3,m} = -3x_4^2 x_{m-8} - G_{3,m}$ and $Q_{3,m} = -2z_6 z_{m-6} G_{3,m}$. Let $C_3 \subset X_\infty^0$ be the closure of C_3°.

Step 4. On $W_3 \cap V(x_4)$ we have $z_6 = 0$, and this leaves us with

$$V_4 := W_3 \cap V(x_4) = V(x_{\leq 4}, y_{\leq 2}, z_{\leq 6}).$$

Next, we look at

$$\begin{cases} f_{13} \equiv 0 & \mod (x_{\leq 4}, y_{\leq 2}, z_{\leq 6}), \\ f_{14} \equiv z_7^2 & \mod (x_{\leq 4}, y_{\leq 2}, z_{\leq 6}), \\ f_{15} \equiv x_5^3 + y_3^5 & \mod (x_{\leq 4}, y_{\leq 2}, z_{\leq 7}). \end{cases}$$

Let $W_4 := V_4 \cap V(f_{13}, f_{14}, f_{15})$. On $W_4 \cap D(x_5)$, both x_5 and y_3 are units. For $m \geq 16$, we write

$$f_m \equiv 3x_5^2 x_{m-10} + 5y_3^4 y_{m-12} + G_{4,m} \mod (x_{\leq 4}, y_{\leq 2}, z_{\leq 7}),$$

where $G_{4,m}$ does not depend on x_{j-10} or y_{j-12} for $j \geq m$. We use the equations $f_m = 0$, for $m \geq 16$, to solve for x_{m-10} if $\operatorname{char} k \neq 3$, and to solve for y_{m-12} if $\operatorname{char} k \neq 5$. We obtain the set

$$C_4^\circ = \begin{cases} V(x_{\leq 4}, y_{\leq 2}, z_{\leq 7}, x_5^3 + y_3^5)_{x_5}|_{x_{m-10} = (3x_5^2)^{-1} Q_{4,m},\ m \geq 16} & (\operatorname{char} k \neq 3) \\ V(x_{\leq 4}, y_{\leq 2}, z_{\leq 7}, x_5^3 + y_3^5)_{x_5}|_{y_{m-12} = (5y_3^4)^{-1} R_{4,m},\ m \geq 16} & (\operatorname{char} k \neq 5) \end{cases}$$

where $Q_{4,m} = -5y_3^4 y_{m-12} - G_{4,m}$ and $R_{4,m} = -3x_5^2 x_{m-10} - G_{4,m}$. Let $C_4 \subset X_\infty^0$ be the closure of C_4°.

Step 5. On $W_4 \cap V(x_5)$ we have $y_3 = 0$, and this leaves us with

$$V_5 := W_4 \cap V(x_5) = V(x_{\leq 5}, y_{\leq 3}, z_{\leq 7}).$$

Next, we look at

$$\begin{cases} f_{16} \equiv z_8^2 & \mod (x_{\leq 5}, y_{\leq 3}, z_{\leq 7}), \\ f_{17} \equiv 0 & \mod (x_{\leq 5}, y_{\leq 3}, z_{\leq 8}), \\ f_{18} \equiv z_9^2 + x_6^3 & \mod (x_{\leq 5}, y_{\leq 3}, z_{\leq 8}). \end{cases}$$

Let $W_5 := V_5 \cap V(f_{16}, f_{17}, f_{18})$. On $W_5 \cap D(x_6)$, we have units z_9 and x_6. For $m \geq 19$, we write

$$f_m \equiv 2z_9 z_{m-9} + 3x_6^2 x_{m-12} + G_{5,m} \mod (x_{\leq 4}, y_{\leq 2}, z_{\leq 7}),$$

where $G_{5,m}$ does not depend on z_{j-9} or x_{j-12} for any $j \geq m$. We use the equations $f_m = 0$, for $m \geq 19$, to solve for z_{m-9} if $\operatorname{char} k \neq 2$ and for x_{m-12} if $\operatorname{char} k \neq 3$. We obtain the set

$$C_5^\circ = \begin{cases} V(x_{\leq 5}, y_{\leq 3}, z_{\leq 8}, z_9^2 + x_6^3)_{x_6}|_{z_{m-9} = (2z_9)^{-1} P_{5,m},\ m \geq 19} & (\operatorname{char} k \neq 2) \\ V(x_{\leq 5}, y_{\leq 3}, z_{\leq 8}, z_9^2 + x_6^3)_{x_6}|_{x_{m-12} = (3x_6^2)^{-1} Q_{5,m},\ m \geq 19} & (\operatorname{char} k \neq 3) \end{cases}$$

where $P_{5,m} = -3x_6^2 x_{m-12} - G_{5,m}$ and $Q_{5,m} = -2z_9 z_{m-9} - G_{5,m}$. Let $C_5 \subset X_\infty^0$ be the closure of C_5°.

Step 6. On $W_5 \cap V(x_6)$ we have $z_9 = 0$, and this leaves us with

$$V_6 := W_5 \cap V(x_6) = V(x_{\leq 6}, y_{\leq 3}, z_{\leq 9}).$$

Next, we look at

$$\begin{cases} f_{19} \equiv 0 & \mod (x_{\leq 6}, y_{\leq 3}, z_{\leq 9}), \\ f_{20} \equiv z_{10}^2 + y_4^5 & \mod (x_{\leq 6}, y_{\leq 3}, z_{\leq 9}). \end{cases}$$

Let $W_6 := V_6 \cap V(f_{19}, f_{20})$. On $W_6 \cap D(y_4)$, the units are z_{10} and y_4. For $m \geq 21$, we write

$$f_m \equiv 2z_{10} z_{m-10} + 5y_4^4 y_{m-16} + G_{6,m} \mod (x_{\leq 6}, y_{\leq 3}, z_{\leq 9}),$$

where $G_{6,m}$ does not depend on z_{j-10} or y_{j-16} for any $j \geq m$. We use the equations $f_m = 0$, for $m \geq 21$, to solve for z_{m-10} if $\operatorname{char} k \neq 2$ and for y_{m-16} if $\operatorname{char} k \neq 5$. We obtain the set

$$C_6^\circ = \begin{cases} V(x_{\leq 6}, y_{\leq 3}, z_{\leq 9}, z_{10}^2 + y_4^5)_{y_4}|_{z_{m-10} = (2z_{10})^{-1} P_{6,m},\ m \geq 21} & (\operatorname{char} k \neq 2) \\ V(x_{\leq 6}, y_{\leq 3}, z_{\leq 9}, z_{10}^2 + y_4^5)_{y_4}|_{y_{m-16} = (5y_4^4)^{-1} R_{6,m},\ m \geq 21} & (\operatorname{char} k \neq 5), \end{cases}$$

where $P_{6,m} = -5y_4^4 y_{m-16} - G_{6,m}$ and $R_{6,m} = -2z_{10}z_{m-10} - G_{6,m}$. Let $C_6 \subset X_\infty^0$ be the closure of C_6°.

Step 7. On $W_6 \cap V(y_4)$ we have $z_{10} = 0$, and this leaves us with

$$V_7 := W_6 \cap V(y_4) = V(x_{\le 6}, y_{\le 4}, z_{\le 10}).$$

Next, we look at

$$\begin{cases} f_{21} \equiv x_7^3 & \mod (x_{\le 6}, y_{\le 4}, z_{\le 10}), \\ f_{22} \equiv z_{11}^2 & \mod (x_{\le 7}, y_{\le 4}, z_{\le 10}), \\ f_{23} \equiv 0 & \mod (x_{\le 7}, y_{\le 4}, z_{\le 11}), \\ f_{24} \equiv z_{12}^2 + x_8^3 & \mod (x_{\le 7}, y_{\le 4}, z_{\le 11}). \end{cases}$$

Let $W_7 := V_7 \cap V(f_{21}, \dots, f_{24})$. On $W_7 \cap D(x_8)$, both z_{12} and x_8 are units. For $m \ge 25$, we write

$$f_m \equiv 2z_{12}z_{m-12} + 3x_8^2 x_{m-16} + G_{7,m} \mod (x_{\le 7}, y_{\le 4}, z_{\le 11}),$$

where $G_{7,m}$ does not depend on z_{j-12} or x_{j-16} for any $j \ge m$. We use the equations $f_m = 0$, for $m \ge 25$, to solve for z_{m-12} if $\operatorname{char} k \ne 2$, and to solve for x_{m-16} if $\operatorname{char} k \ne 3$. We obtain the set

$$C_7^\circ = \begin{cases} V(x_{\le 7}, y_{\le 4}, z_{\le 11}, z_{12}^2 + x_8^3)_{x_8}|_{z_{m-12} = (2z_{12})^{-1} P_{7,m},\ m \ge 25} & (\operatorname{char} k \ne 2) \\ V(x_{\le 7}, y_{\le 4}, z_{\le 11}, z_{12}^2 + x_8^3)_{x_8}|_{x_{m-16} = (3x_8^2)^{-1} Q_{7,m},\ m \ge 25} & (\operatorname{char} k \ne 3) \end{cases}$$

where $P_{7,m} = -3x_8^2 x_{m-16} - G_{7,m}$ and $Q_{7,m} = -2z_{12}z_{m-12} - G_{7,m}$. Let $C_7 \subset X_\infty^0$ be the closure of C_7°.

Step 8. On $W_7 \cap V(x_8)$ we have $z_{12} = 0$, and this leaves us with

$$V_8 := W_7 \cap V(x_8) = V(x_{\le 8}, y_{\le 4}, z_{\le 12}).$$

Finally, we look at

$$\begin{cases} f_{25} \equiv y_5^5 & \mod (x_{\le 8}, y_{\le 4}, z_{\le 12}), \\ f_{26} \equiv z_{13}^2 & \mod (x_{\le 8}, y_{\le 5}, z_{\le 12}), \\ f_{27} \equiv x_9^3 & \mod (x_{\le 8}, y_{\le 5}, z_{\le 13}), \\ f_{28} \equiv z_{14}^2 & \mod (x_{\le 9}, y_{\le 5}, z_{\le 13}), \\ f_{29} \equiv 0 & \mod (x_{\le 9}, y_{\le 5}, z_{\le 14}), \\ f_{30} \equiv z_{15}^2 + x_{10}^3 + y_6^5 & \mod (x_{\le 9}, y_{\le 5}, z_{\le 14}). \end{cases}$$

Let $W_8 := V_8 \cap V(f_{25}, \dots, f_{30})$. This time we localize at a product to make all variables involved in the last equation units: on $W_8 \cap D(z_{15}x_{10})$, all three variables z_{15}, x_{10}, and y_6 are units. For $m \ge 31$, we write

$$f_m \equiv 2z_{15}z_{m-15} + 3x_{10}^2 x_{m-20} + 5y_6^4 y_{m-24} + G_{8,m} \mod (x_{\le 9}, y_{\le 5}, z_{\le 14}),$$

where $G_{8,m}$ does not depend on z_{j-15}, x_{j-20} or y_{j-24} for any $j \ge m$. We use the equations $f_m = 0$, for $m \ge 31$, to solve for z_{m-15} if $\operatorname{char} k \ne 2$, and to solve for x_{m-20} if $\operatorname{char} k \ne 3$. We obtain the set

$$C_8^\circ = \begin{cases} V(x_{\le 9}, y_{\le 5}, z_{\le 14}, z_{15}^2 + x_{10}^3 + y_6^5)_{z_{15}}|_{z_{m-15} = (2z_{15})^{-1} P_{8,m},\ m \ge 31} & (\operatorname{char} k \ne 2) \\ V(x_{\le 9}, y_{\le 5}, z_{\le 14}, z_{15}^2 + x_{10}^3 + y_6^5)_{x_{10}}|_{x_{m-20} = (3x_{10}^2)^{-1} Q_{8,m},\ m \ge 31} & (\operatorname{char} k \ne 3), \end{cases}$$

where $P_{8,m} = -3x_{10}^2 x_{m-20} - 5y_6^4 y_{m-24} - G_{8,m}$ and $Q_{8,m} = -2z_{15}z_{m-15} - 5y_6^4 y_{m-24} - G_{8,m}$. Let $C_8 \subset X_\infty^0$ be the closure of C_8°.

Step 9. In the above steps, we have constructed closed irreducible sets $C_1, \dots, C_8 \subset X_\infty$. These sets fill up the whole fiber over the origin, hence

$$X_\infty^0 = C_1 \cup \cdots \cup C_8.$$

All we are left to show is that each C_i is an irreducible component of X_∞^0. Equivalently, we need to show that there are no inclusions among any two of them.

It is clear from the construction (where we constructed the sets C_i° by a series of localizations and subsequent restrictions to the complements) that $C_i \not\subset C_j$ whenever $i < j$. The delicate part is to show that, similarly, $C_i \not\supset C_j$ for all $i < j$, hence there are no inclusions among any two of these sets.

Here we explain why $C_1 \not\supset C_2$. The other cases are treated similarly and are left to the reader.

First, assume that $\operatorname{char} k \neq 2$. We look at one particular defining equation of C_1° in $D(x_2)$, namely, $z_5 = (2z_3)^{-1}P_{1,8}$. The reason we want to focus on this equation is that $z_5 \not\equiv 0$ on C_2. Rewriting the equation as $z_3 z_5 = \frac{1}{2}P_{1,8}$, we can extend it beyond $D(x_2)$. We look at this equation on the set $V(x_{\leq 3}, y_{\leq 1}, z_{\leq 4} \setminus z_3)$. The polynomial $P_{1,8}$ vanishes identically over this set, thus the given equation reduces to $z_3 z_5 = 0$. Note that $C_1 \cap V(x_{\leq 3}, y_{\leq 1}, z_{\leq 4} \setminus z_3)$ is an irreducible set and, by what we just said, $z_3 z_5$ vanishes identically on this set. As $z_3 \not\equiv 0$ on $C_1 \cap V(x_{\leq 3}, y_{\leq 1}, z_{\leq 4} \setminus z_3)$, we conclude that $z_5 \equiv 0$ over there. However, $z_5 \not\equiv 0$ over C_2, since the latter was constructed as the closure of the set C_2°, which is contained in $D(z_5)$. Noting that $C_2 \subset V(x_{\leq 3}, y_{\leq 1}, z_{\leq 4} \setminus z_3)$, we conclude that $C_2 \not\subset C_1$.

If $\operatorname{char} k = 2$, then we do not have the equation $z_5 = (2z_3)^{-1}P_{1,8}$, but we can use instead the equation $x_4 = (3x_2^2)^{-1}Q_{1,8}$, which also comes from setting $f_8 = 0$. This time we look at the equation on the set $V(x_{\leq 3} \setminus x_2, y_{\leq 1}, z_{\leq 4})$. The polynomial $Q_{1,8}$ vanishes identically on this set, hence the equation reduces to $x_2^2 x_4 = 0$. By the same argument as in the previous case, we conclude that x_4 vanishes along $C_1 \cap V(x_{\leq 3} \setminus x_2, y_{\leq 1}, z_{\leq 4})$. On the other hand, a quick inspection of the equations of C_2° in $D(y_2)$ show that $x_4 \not\equiv 0$ on C_2, and since C_2 is contained $V(x_{\leq 3} \setminus x_2, y_{\leq 1}, z_{\leq 4})$, we conclude that $C_2 \not\subset C_1$ in characteristic 2 as well.

Appendix A. On the wedge lifting problem

Appendix by Tommaso de Fernex, Roi Docampo, and Gebhard Martin

Let X be a surface defined over an uncountable algebraically closed field k. A *k-wedge* on X is a morphism $\Phi\colon \operatorname{Spec} k[[s,t]] \to X$. Denoting by 0 and η the closed point and generic point of $\operatorname{Spec} k[[s]]$, one can view Φ as defining an arc $\operatorname{Spec} k[[s]] \to X_\infty$ on the arc space of X giving a deformation of the *special arc* $\varphi_0 := \Phi(0,t)$ of the wedge to its *generic arc* $\varphi_\eta := \Phi(\eta, t)$.

Let $f\colon Y \to X$ denote the minimal resolution. As long as they are not fully contained in the indeterminacy locus of f^{-1}, the arcs φ_0 and φ_η lift individually to arcs $\widetilde{\varphi}_0$ and $\widetilde{\varphi}_\eta$ on Y. The question is whether the wedge itself lifts to Y. More precisely, let E be an irreducible component of the exceptional locus of f, and denote by V the set of k-rational points $P \in E$ with the property that every k-wedge Φ on X such that $\widetilde{\varphi}_0$ intersects E transversally at P automatically factors through the minimal resolution. Then the *k-wedge lifting problem* asks whether, for every E, the set V has a nonempty intersection with $\bigcap_{n\in\mathbb{N}} U_n$ for every family $\{U_n\}_{n\in\mathbb{N}}$ of dense open subsets of E [LJ80, LJR12].

The proof of the two-dimensional Nash problem given in [dFD16] solves affermatively the two-dimensional k-wedge lifting problem in characteristic zero. By contrast, Reguera showed that the k-wedge lifting problem fails for the E_8 singularity $x^2 + y^3 + z^5 = 0$ in characteristics 2, 3, and 5 [Reg12]. Here we extend Reguera's characteristic 2 example to arbitrary characteristics.

Proposition A.1. *Let p be a prime, and let $X \subset \mathbb{A}^3$ be the surface singularity defined by $x^p + y^{p+1} + z^{2p+1} = 0$ over an algebraically closed field k of characteristic p. For every $\lambda \in k \setminus \{0, -1\}$, the wedge $\Phi\colon \operatorname{Spec} k[[s,t]] \to X$ given by*

$$\begin{cases} x = -t^{2p+1}(s+\lambda t)^{2p^2+p-1}(s+(1+\lambda)t) \\ y = t^{2p}(s+\lambda t)^{2p^2-p} \\ z = t^p(s+\lambda t)^{p^2} \end{cases}$$

does not lift to the minimal resolution of X. By varying λ, one obtains a k-rational wedge whose special arc $\widetilde{\varphi}_0$ intersects E at any given k-rational point away from its intersection with other exceptional components.

For every pairwise coprime integers $a, b, c \geq 2$, a log resolution $f\colon Y \to X$ of the Brieskorn singularity $X = \{x^a + y^b + z^c = 0\} \subset \mathbb{A}^3$ over $\mathbb{C}$ was computed in [HJ69, OW71]. The dual graph of the exceptional divisor of this resolution is a star shaped tree with three branches. The central component is extracted by the normalized weighted blow-up with weights (bc, ac, ab) in the coordinates (x, y, z). This is the singular Seifert fibration of [OW71]; in modern terminology it is the one-step embedded $\mathbb{Q}$-resolution by weighted blow-up [MM13]. Resolving its three cyclic quotient singularities gives the Hirzebruch–Jung arms of the resolution graph. While the classical references rely on topological methods, the resolution can be computed purely algebraically and is defined over $\mathbb{Z}$ (see, e.g., [LS23, AGMM24]). The central component E on the resolution Y is a rational curve with self intersection -1 or -2, depending on the exponents; the precise arithmetic conditions determining which case occurs for each triple (a, b, c) are explicitly given in [HNK71].

We now specialize to the case $(a, b, c) = (p, p+1, 2p+1)$, where p is a prime. In this case we have $E^2 = -2$, and f is the minimal resolution of X. By adjoining the variable

$$u := \frac{z^{2p+1}}{y^{p+1}}$$

and suitably localizing, we obtain a chart on the weighted blow-up of X given by $\operatorname{Spec} S$, where

$$S := \frac{k[x, y, z, u, u^{-1}, (1+u)^{-1}]}{(x^p + (1+u^{-1})z^{2p+1}, uy^{p+1} - z^{2p+1})}.$$

The normalization $\widetilde{S}$ of S gives a chart $\operatorname{Spec} \widetilde{S}$ on Y intersecting E.

Lemma A.2. *The normalization $\widetilde{S}$ of S is a regular ring and is given by $\widetilde{S} = S[v]$ where*

$$v := \frac{xz^{2p-3}}{y^p}.$$

For every $\lambda \neq 0, -1$, the pair $(u - \lambda, v)$ gives a regular system of parameters in $\widetilde{S}$ and E is defined by $v = 0$ in this chart.

Proof. Observing that

$$1 + u = \frac{y^{p+1} + z^{2p+1}}{y^{p+1}} = -\frac{x^p}{y^{p+1}},$$

one checks that

$$\begin{aligned} v^{2p^2+3p+1} &= -u^{2p^2-p-3}(1+u)^{2p+3}x \\ v^{2p^2+p} &= -u^{2p^2-3p}(1+u)^{2p+1}y \\ v^{p^2+p} &= u^{p^2-p-1}(1+u)^{p+1}z. \end{aligned}$$

Any one of these equations shows that v is integral over S. Solving for (x, y, z), we obtain an isomorphism

$$S[v] \simeq k[u, u^{-1}, (1+u)^{-1}, v].$$

Since $k[u, u^{-1}, (1+u)^{-1}, v]$ is a regular ring, we conclude that $S[v]$ is regular and is equal to $\widetilde{S}$. The last assertion is clear in view of the above isomorphism and the fact that $\operatorname{val}_E(v) = 1$. □

Proof of Proposition A.1. Using that we are in characteristic p, it is immediate to check that the image of Φ is indeed contained in X.

As we shall see in a moment, the special arc $\varphi_0 := \Phi(0, t)$ of the wedge lifts to an arc $\widetilde{\varphi}_0$ on Y which intersects transversally the divisor E at a point p that does not belong to any other

exceptional divisor. The generic arc $\varphi_\eta := \Phi(\eta, t)$ of the wedge, on the other hand, lifts to an arc $\widetilde{\varphi}_\eta$ on Y which intersects some exceptional divisor F other than E.

We work in the chart $\operatorname{Spec} \widetilde{S} \subset Y$ and consider the elements

$$u = \frac{z^{2p+1}}{y^{p+1}}, \quad v = \frac{xz^{2p-3}}{y^p}.$$

The wedge lifts to the map

$$\widetilde{\Phi}\colon \operatorname{Spec} k[[st^{-1}, t]] \to V$$

given by

$$\begin{cases} u = \dfrac{(s+\lambda t)^p}{t^p} \\ v = -\dfrac{(s+\lambda t)^{p-1}(s+(1+\lambda)t)}{t^{p-1}}, \end{cases}$$

where we regard these rational functions as polynomials in s/t and t. From the equations, we see that Φ does not factor through f, hence it does not lift to the minimal resolution. Geometrically, for $\lambda \in k \setminus \{0, -1\}$, the arc $\widetilde{\varphi}_0$ intersects E with order of contact 1 at the point $(u, v) = (\lambda^p, 0)$, hitting any k-point of E away from the other exceptional components, and $\widetilde{\varphi}_\eta$ intersects a different component. □

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