SUPPLEMENT TO: ARCS ON RATIONAL DOUBLE POINTS IN ARBITRARY CHARACTERISTIC

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ABSTRACT. This is a supplement to [dFW25] where we provide proofs for the D_{2n+1}, E_6, and E_7 cases as well as some of the computations omitted in the last step of the proof for the E_8 case.

1. D_{2n+1} SINGULARITIES

Here we present the proof of [dFW25, Theorem 2.1] for D_{2n+1} singularities. The proof is similar to the one given there for D_{2n} singularities.

Let $(X,0)$ be a D_{2n+1} singularity, with $n \geq 2$. We may assume that X given in $\mathbb{A}^3$ by

$$f(x,y,z) = z^2 + x^2y + y^nz + g(x,y,z) = 0,$$

where $g \in \{0, xy^{n-r}z \mid 1 \leq r \leq n-2\}$ if the characteristic is 2, and $g = 0$ otherwise.

Step 1. This step is very similar to the first step in the proof of the D_{2n} case, except this time we let $i = 1, \dots n-1$. We set $V_i := V(x_{\leq i-1}, y_0, z_{\leq i-1}) \subset \mathbb{A}^3_\infty$, observe that $V_i \subset V(f_0, \dots, f_{2i})$, and define $W_i := V_i \cap V(f_{2i}, f_{2i+1})$. We compute

$$\begin{cases} f_{2i} \equiv z_i^2 & \mod (x_{\leq i-1}, y_0, z_{\leq i-1}), \\ f_{2i+1} \equiv x_i^2 y_1 & \mod (x_{\leq i-1}, y_0, z_{\leq i}). \end{cases}$$

This implies that $W_i = V_i \cap V(z_i, x_iy_i)$. On $W_i \cap D(x_i)$, we have $y_1 = 0$. We eliminate the variables y_{m-2i}, for $m \geq 2i+2$, using the remaining euqations $f_m = 0$. This gives

$$C_i^\circ = V(x_{\leq i-1}, y_{\leq 1}, z_{\leq i})_{x_i}|_{y_{m-2i} = (x_i^2)^{-1}P_{i,m},\, m \geq 2i+2}$$

and leaves us with

$$W_i \cap V(x_i) = V(x_{\leq i}, y_0, z_{\leq i}).$$

If $i < n-1$, this is V_{i+1} and we repeat the process.

Step 2. The previous step produced the sets $C_1^\circ, \dots, C_{n-1}^\circ$, and left us with

$$V_n := V(x_{\leq n-1}, y_0, z_{\leq n-1}).$$

In the following computations, we use

$$\begin{cases} f_{2n} \equiv z_n^2 + y_1^n z_n & \mod (x_{\leq n-1}, y_0, z_{\leq n-1}), \\ f_{2n+1} \equiv x_n^2 y_1 + y_1^n z_{n+1} & \mod (x_{\leq n-1}, y_0, z_{\leq n}). \end{cases}$$

First we consider

$$W_n' := V_n \cap V(f_{2n}).$$

We stratify $W'_n = (W'_n \cap D(z_n)) \sqcup (W'_n \cap V(z_n))$. On $W'_n \cap D(z_n)$, we have $z_n + y_1^n = 0$, and we can solve for z_{m-n} for all $m \geq 2n+1$ using the remaining equations $f_m = 0$. This gives

$$C'^{\circ}_n = V(x_{\leq n-1}, y_0, z_{\leq n-1}, z_n + y_1^n)_{z_n}|_{z_{m-n}=(z_n)^{-1}P'_{n,m},\ m\geq 2n+1}$$

Next, we look at $W'_n \cap V(z_n) = V(x_{\leq n-2}, y_0, z_{\leq n-2})$. We let

$$W''_n := W'_n \cap V(z_n, f_{2n+1}).$$

We stratify $W''_n = (W''_n \cap D(y_1)) \sqcup (W''_n \cap V(y_1))$. On $W''_n \cap D(y_1)$, we solve again for z_{m-n} for all $m \geq 2n+2$ using the remaining equations $f_m = 0$. This gives

$$C''^{\circ}_n = V(x_{\leq n-1}, y_0, z_{\leq n})_{y_1}|_{z_{m-n}=(y_1^n)^{-1}P''_{n,m},\ m\geq 2n+2}$$

We are left with

$$W''_n \cap V(y_1) = V(x_{\leq n-1}, y_{\leq 1}, z_{\leq n}).$$

Step 3. For $i = n, \dots 2n-2$, consider the set

$$V_i := V(x_{\leq i-1}, y_{\leq 1}, z_{\leq i}).$$

Note that $V_n = V(x_{\leq n-1}, y_{\leq 1}, z_{\leq n})$, the set left to analyze from the previous step. Inductively, we see that $V_i \subset V(f_0, \dots, f_{2i})$. In this step, we rely on

$$\begin{cases} f_{2i+2} \equiv z_{i+1}^2 + x_i^2 y_2 & \mod (x_{\leq i-1}, y_{\leq 1}, z_{\leq i}), \\ f_{2i+3} \equiv 0 & \mod (x_{\leq i}, y_{\leq 1}, z_{\leq i+1}). \end{cases}$$

Let

$$W_i := V_i \cap V(f_{2i+2}).$$

We stratify $W_i = (W_i \cap D(x_i)) \sqcup (W_i \cap V(x_i))$. On $W_i \cap D(x_i)$, we use the equations $f_m =$ to eliminate the variables y_{m-2i} for $m \geq 2i+2$, and get

$$C_i^{\circ} = V(x_{\leq i-1}, y_{\leq 1}, z_{\leq i})_{x_i}|_{y_{m-2i}=(x_i^2)^{-1}P_{i,m},\ m\geq 2i+2}$$

On $W_i \cap V(x_i)$, we have $z_{i+1} = 0$, hence f_{2i+3} vanishes identically on $W_i \cap V(x_i)$. So, we are left with

$$W_i \cap V(x_i, f_{2i+3}) = V(x_{\leq i}, y_{\leq 1}, z_{\leq i+1}).$$

If $i < 2n-2$, then this is precisely V_{i+1}.

Step 4. The previous steps have produced the strata $C_1^{\circ}, \dots, C_{2n-2}^{\circ}, C'^{\circ}_n, C''^{\circ}_n$, and we are left with analyzing the set

$$V_{2n-1} := W_{2n-2} \cap V(x_{2n-2}, f_{4n-1}) = V(x_{\leq 2n-2}, y_{\leq 1}, z_{\leq 2n-1}).$$

So far, we have used the equations $f_m = 0$ for $m \leq 4n-1$. We define $W_{2n-1} := V_{2n-1} \cap V(f_{4n})$. We have

$$f_{4n} \equiv z_{2n}^2 + x_{2n-1}^2 y_2 + y_2^n z_{2n} \qquad \mod (x_{\leq 2n-3}, y_{\leq 1}, z_{\leq 2n-2}).$$

Covering W_{2n-1} with $D(x_{2n-1}) \cup D(y_2) \cup V(x_{2n-1}, y_2)$, we check that $W_{2n-1} \cap V(f_m \mid m \geq 4n+1)$ is irreducible This realizes the component

$$C_{2n-1}^{\circ} = V(x_{\leq 2n-2}, y_{\leq 1}, z_{\leq 2n-1})_{x_{2n-1}}|_{y_{m-4n+2}=(x_{2n-1}^2)^{-1}P_{i,m},\ m\geq 4n}$$

Step 5. Let C_i, C'_n, C''_n denote the closure of the sets $C_i^\circ, C_n'^\circ, C_n''^\circ$ constructed above. It is clear by construction that these sets are all irreducible and fill up X_∞^0. The last step is to show that there are no inclusions among them, hence X_∞^0 has $2n+1$ irreducible components.

By construction, we clearly have that $C_i \not\subset C_j$ whenever $i < j$, and $C_i \not\subset C'_n \cup C''_n$ for $i \leq n-1$. Observing that $z_n \not\equiv 0$ and $z_n + y_1^n \equiv 0$ on C'_n while $z_n \equiv 0$ and $z_n + y_1^n \not\equiv 0$ on C''_n, we also see that there are no inclusions among these two sets. Furthermore, we have that $C'_{n-1}, C''_{n-1} \not\subset C_i$ for all i since y_1 is identically zero on C_i and does not vanish identically on either C'_{n-1} or C''_{n-1}.

It remains to show that $C_j \not\subset C_i$ for $j > i$ and $C_i \not\subset C'_n \cup C''_n$ for $i \geq n$. Regarding the first, observe that

$$\begin{cases} f_{2i+2} \equiv x_i^2 y_2 & \mod (x_{\leq j-1} \setminus x_i, y_{\leq 1}, z_{\leq j}), \\ f_{2j+2} \equiv x_j^2 y_2 + z_{j+1}^2 & \mod (x_{\leq j-1}, y_{\leq 1}, z_{\leq j}). \end{cases}$$

Keeping in mind that x_i is a unit on C_i°, we see from the vanishing of f_{2i+2} that $y_2 \equiv 0$ on $C_i \cap V(x_{\leq j-1} \setminus x_i, y_{\leq 1}, z_{\leq j})$ On the other hand, using that x_j is a unit on C_j° and z_{j+1} is a free variable on this set, we see from the the vanishing of f_{2j+2} that $y_2 \not\equiv 0$ on C_j. As $C_j \subset V(x_{\leq j-1} \setminus x_i, y_{\leq 1}, z_{\leq j})$, we conclude that $C_j \not\subset C_i$ whenever $j > i$.

Finally, fix any C_i with $i \geq n$. Note that

$$f_{2i+1} \equiv x_i^2 y_1 \mod (x_{\leq i-1}, y_0, z_{\leq i}).$$

As y_1 restricts to a unit on both $C_n'^\circ$ and $C_n''^\circ$, we deduce that $x_i \equiv 0$ on $(C'_n \cup C''_n) \cap V(x_{\leq i-1}, y_0, z_{\leq i})$. On the other hand, $x_i \not\equiv 0$ on C_i, and since $C_i \subset V(x_{\leq i-1}, y_0, z_{\leq i})$, we conclude that $C_i \not\subset C'_n \cup C''_n$ for $i \geq n$.

2. E_6 singularities

Here we give the proof of [dFW25, Theorem 2.1] for E_6 singularities. The proof is similar to the one given there for E_8 singularities.

Let $0 \in X \subset \mathbb{A}^3$ be an E_6 singularity given by

$$f(x,y,z) = z^2 + x^3 + y^2 z + g(x,y,z),$$

where $g \in \{0, xyz\}$ if $\operatorname{char} k = 2$, $g \in \{0, x^2y^2\}$ if $\operatorname{char} k = 3$, and $g = 0$ otherwise.

Step 1. We set $V_1 := V(x_0, y_0, z_0)$. We have $f_0 \equiv f_1 \equiv 0$ modulo (x_0, y_0, z_0). The next few equations give

$$\begin{cases} f_2 \equiv z_1^2 & \mod (x_0, y_0, z_0), \\ f_3 \equiv x_1^3 & \mod (x_0, y_0, z_{\leq 1}), \\ f_4 \equiv z_2^2 + y_1^2 z_2 & \mod (x_{\leq 1}, y_0, z_{\leq 1}). \end{cases}$$

Let $W_1 := V_1 \cap V(f_2, f_3, f_4)$. We stratify this set using $D(z_2) \sqcup V(z_2)$. Note that, on $W_1 \cap D(z_2)$, we have $z_2 + y_1^2 = 0$. We eliminate the variables z_{m-2} by dividing by z_2, using the equations $f_m = 0$ for $m \geq 5$. This gives

$$C_1^\circ = V(x_{\leq 1}, y_0, z_{\leq 1}, z_2 + y_1^2)_{z_2}|_{z_{m-2} = (z_2)^{-1} Q_{1,m},\, m \geq 5}$$

This leaves us with

$$V_2 := W_1 \cap V(z_2) = V(x_{\leq 1}, y_0, z_{\leq 2}).$$

Step 2. The next equation gives

$$f_5 \equiv y_1^2 z_3 \qquad \mod (x_{\leq 1}, y_0, z_{\leq 2}).$$

We let $W_2 := V_2 \cap V(f_5)$. We stratify it using $D(y_1) \sqcup V(y_1)$. On $W_2 \cap D(y_1)$, we get $z_3 = 0$ and solve for z_{m-2} for $m \geq 6$ using the equations $f_m = 0$. This gives

$$C_2^\circ = V(x_{\leq 1}, y_0, z_{\leq 3})_{y_1}|_{z_{m-2}=(y_1^2)^{-1}P_{2,m},\, m\geq 6}$$

and leaves us with

$$V_3 := W_2 \cap V(y_1) = V(x_{\leq 1}, y_{\leq 1}, z_{\leq 2}).$$

Step 3. We have

$$f_6 \equiv z_3^2 + x_2^3 \qquad \mod (x_{\leq 1}, y_{\leq 1}, z_{\leq 2}).$$

We stratify $W_3 := V_2 \cap V(f_6)$ using $D(x_2) \sqcup V(x_2)$. On $W_3 \cap D(x_2)$, both x_2 and z_3 is a unit and we use the equations $f_m = 0$ for $m \geq 7$ to solve for z_{m-3} if $\operatorname{char} k \neq 2$ and for x_{m-4} if $\operatorname{char} k \neq 3$. This gives

$$C_3^\circ = \begin{cases} V(x_{\leq 1}, y_{\leq 1}, z_{\leq 2}, z_3^2 + x_2^3)_{x_2}|_{z_{m-3}=(2z_3)^{-1}P_{3,m},\, m\geq 7} & (\operatorname{char} k \neq 2) \\ V(x_{\leq 1}, y_{\leq 1}, z_{\leq 2}, z_3^2 + x_2^3)_{x_2}|_{x_{m-4}=(3x_2^2)^{-1}Q_{3,m},\, m\geq 7} & (\operatorname{char} k \neq 3) \end{cases}$$

This leaves us with

$$V_4 := W_3 \cap V(x_2) = V(x_{\leq 2}, y_{\leq 1}, z_{\leq 3}).$$

Step 4. The next two equations give

$$\begin{cases} f_7 \equiv 0 & \mod (x_{\leq 2}, y_{\leq 1}, z_{\leq 3}), \\ f_8 \equiv z_4^2 + y_2^2 z_4 & \mod (x_{\leq 2}, y_{\leq 1}, z_{\leq 3}). \end{cases}$$

Let $W_4 := V_4 \cap V(f_7, f_8)$. Stratifying using $D(z_4) \sqcup V(z_4)$, on $W_4 \cap D(z_4)$ we have $z_4 + y_2^2 = 0$, and we can solve for z_{m-4} for $m \geq 9$ using $f_m = 0$, obtaining

$$C_4^\circ = V(x_{\leq 2}, y_{\leq 1}, z_{\leq 3}, z_4 + y_2^2)_{z_4}|_{z_{m-4}=(z_4)^{-1}P_{4,m},\, m\geq 9}.$$

This leave us with

$$V_5 := W_4 \cap V(z_4) = V(x_{\leq 2}, y_{\leq 1}, z_{\leq 4}).$$

Step 5. The next equation is

$$f_9 \equiv x_3^3 + y_2^2 z_5 \mod (x_{\leq 2}, y_{\leq 1}, z_{\leq 4}).$$

Let $W_5 := V_5 \cap V(f_9)$. We stratify it using $D(y_2) \sqcup V(y_2)$. On $W_5 \cap D(y_2)$ we solve for z_{m-4} for $m \geq 10$ using $f_m = 0$. This gives

$$C_5^\circ = V(x_{\leq 2}, y_{\leq 1}, z_{\leq 4}, x_3^3 + y_2^2 z_5)_{y_2}|_{z_{m-4}=(y_2^2)^{-1}P_{5,m},\, m\geq 10}.$$

On $W_5 \cap V(y_2)$, we have $x_3 = 0$. We are left to analyze

$$V_6 := W_5 \cap V(y_2) = V(x_{\leq 3}, y_{\leq 2}, z_{\leq 4}).$$

Step 6. Using

$$\begin{cases} f_{10} \equiv z_5^2 & \mod (x_{\leq 3}, y_{\leq 2}, z_{\leq 4}), \\ f_{11} \equiv 0 & \mod (x_{\leq 3}, y_{\leq 2}, z_{\leq 5}), \\ f_{12} \equiv z_6^2 + x_4^3 + y_3^2 z_6 & \mod (x_{\leq 3}, y_{\leq 2}, z_{\leq 5}). \end{cases}$$

From here we compute the last set by solve for z_{m-7} for $m \geq 13$ using $f_m = 0$, obtaining

$$C_6^\circ = V(x_{\leq 3}, y_{\leq 2}, z_{\leq 5}, z_6^2 + x_4^3 + y_3^2 z_6)_{y_3}|_{z_{m-6}=(y_3^2)^{-1}P_{6,m},\, m\geq 13}.$$

Step 7. By taking the closures of the sets C_i° constructed in the previous steps, we obtain closed irreducible sets $C_1, \ldots, C_6 \subset X_\infty$. These sets fill up the whole fiber over the origin X_∞^0. All we are left to show is that there are no inclusions among them.

It is clear from the construction that $C_i \not\subset C_j$ whenever $i < j$. Similarly, we claim that $C_i \not\supset C_j$ for all $i < j$, hence there are no inclusions among any two of these sets. This is proved by following a similar argument as that given in [dFW25] to show that $C_1 \not\supset C_2$ in the E_8 case. Specifically, to show that $C_i \not\supset C_j$ whenever $i < j$, for each pair i, j we find a set $V_{i,j}$ and a variable that vanishes on $C_i \cap V_{i,j}$ but not on $C_j \cap V_{i,j}$.

The following table, where the C_i are listed in the first column and the C_j in the top row, lists the variables vanishing identically on $C_i \cap V_{i,j}$ but not on $C_j \cap V_{i,j}$. If a different variable is needed in characteristic 2, the we write it in round brackets. The existence of such variables imply that $C_i \not\supset C_j$ for all $i < j$.

	C_2	C_3	C_4	C_5	C_6
C_1	z_4	z_4	z_4	z_5	z_6
C_2		z_4	z_4	z_5	z_6
C_3			z_4 (x_3)	z_5 (x_3)	z_6 (x_4)
C_4				z_5	z_6
C_5					z_6

3. E_7 SINGULARITIES

Here we give the proof of [dFW25, Theorem 2.1] for E_7 singularities. The proof is similar to the one given there for E_8 singularities.

Let $0 \in X \subset \mathbb{A}^3$ be the E_7 singularity given by

$$f(x, y, z) = z^2 + x^3 + xy^3 + g(x, y, z),$$

where $g \in \{0, x^2yz, y^3z, xyz\}$ if $\operatorname{char} k = 2$, $g \in \{0, x^2y^2\}$ if $\operatorname{char} k = 3$, and $g = 0$ otherwise.

Step 1. We set $V_1 := V(x_0, y_0, z_0)$. The first nontrivial equations are

$$\begin{cases} f_2 \equiv z_1^2 & \mod (x_0, y_0, z_0), \\ f_3 \equiv x_1^3 & \mod (x_0, y_0, z_{\leq 1}), \\ f_4 \equiv z_2^2 & \mod (x_{\leq 1}, y_0, z_{\leq 1}), \\ f_5 \equiv x_2 y_1^3 & \mod (x_{\leq 1}, y_0, z_{\leq 2}). \end{cases}$$

Let $W_1 := V_1 \cap V(f_2, f_3, f_4, f_5)$. We stratify W_1. On $W_1 \cap D(y_1)$, we have $x_2 = 0$ and we eliminate the variables x_{m-3} for $m \geq 6$ using the remaining equations $f_m = 0$ and dividing by y_1^3. This gives

$$C_1^\circ = V(x_{\leq 2}, y_0, z_{\leq 2})_{y_1}|_{x_{m-3}=(y_1^3)^{-1}P_{1,m},\, m\geq 6}$$

and leaves us with

$$V_2 := W_1 \cap V(y_1) = V(x_{\leq 1}, y_{\leq 1}, z_{\leq 2}).$$

Step 2. The next equation gives

$$f_6 \equiv z_3^2 + x_2^3 \qquad \mod (x_{\leq 1}, y_{\leq 1}, z_{\leq 2}).$$

Let $W_2 := V_2 \cap V(f_6)$. On $W_2 \cap D(x_2)$, we have $z_3^2 + x_2^3 = 0$, so z_3 is also a unit. If $\operatorname{char} k \neq 2$, then we solve for z_{m-3} for $m \geq 7$ using the remaining equations $f_m = 0$ and dividing by $2z_3$. If $\operatorname{char} k \neq 3$, then we solve for x_{m-4} by dividing by $3x_2^2$. This gives

$$C_2^\circ = \begin{cases} V(x_{\leq 1}, y_{\leq 1}, z_{\leq 2}, z_3^2 + x_2^3)_{x_2}|_{z_{m-3}=(2z_3)^{-1}P_{2,m},\, m\geq 7} & (\operatorname{char} k \neq 2) \\ V(x_{\leq 1}, y_{\leq 1}, z_{\leq 2}, z_3^2 + x_2^3)_{x_2}|_{x_{m-4}=(3x_2^2)^{-1}Q_{2,m},\, m\geq 7} & (\operatorname{char} k \neq 3) \end{cases}$$

On $W_2 \cap V(x_2)$, we have $z_3 = 0$, hence

$$V_3 := W_2 \cap V(x_2) = V(x_{\leq 2}, y_{\leq 1}, z_{\leq 3}).$$

Step 3. The next three equations give

$$\begin{cases} f_7 \equiv 0 & \mod (x_{\leq 2}, y_{\leq 1}, z_{\leq 3}), \\ f_8 \equiv z_4^2 & \mod (x_{\leq 2}, y_{\leq 1}, z_{\leq 3}), \\ f_9 \equiv x_3^3 + x_3 y_2^3 & \mod (x_{\leq 2}, y_{\leq 1}, z_{\leq 4}). \end{cases}$$

Let $W_3 := V_3 \cap V(f_7, f_8, f_9)$. $W_3 \cap D(x_3)$ we have $x_3^2 + y_2^3 = 0$, hence y_2 is also a unit. If $\operatorname{char} k \neq 2$, then solve for x_{m-6} using the equations $f_m = 0$ for $m \geq 10$, by dividing by $2x_3^2$. If $\operatorname{char} k \neq 3$, then we solve for y_{m-7} by dividing by $3x_3y_2^2$. We obtain

$$C_3^\circ = \begin{cases} V(x_{\leq 2}, y_{\leq 1}, z_{\leq 4}, x_3^2 + y_2^3)_{x_3}|_{x_{m-6}=(2x_3^2)^{-1}P_{3,m},\, m\geq 10} & (\operatorname{char} k \neq 2) \\ V(x_{\leq 2}, y_{\leq 1}, z_{\leq 4}, x_3^2 + y_2^3)_{x_3}|_{y_{m-7}=(3x_3y_2^2)^{-1}Q_{3,m},\, m\geq 10} & (\operatorname{char} k \neq 3) \end{cases}$$

This leave us with

$$V_4 := W_3 \cap V(x_3) = V(x_{\leq 3}, y_{\leq 1}, z_{\leq 4}).$$

Step 4. The next equation is

$$f_{10} \equiv z_5^2 + x_4 y_2^3 \mod (x_{\leq 3}, y_{\leq 1}, z_{\leq 4}).$$

Let $W_4 := V_4 \cap V(f_{10})$. On $W_4 \cap D(y_2)$ we solve for x_{m-7} for $m \geq 11$ using $f_m = 0$. This gives

$$C_4^\circ = V(x_{\leq 3}, y_{\leq 1}, z_{\leq 4}, z_5^2 + x_4 y_2^3)_{y_2}|_{x_{m-7}=(y_2^3)^{-1}P_{4,m},\, m\geq 11}.$$

On $W_4 \cap V(y_2)$, we have $z_5 = 0$, and we are left with

$$V_5 := W_4 \cap V(y_2) = V(x_{\leq 3}, y_{\leq 2}, z_{\leq 5}).$$

Step 5. Next, we look at the equations give

$$\begin{cases} f_{11} \equiv 0 & \mod (x_{\leq 3}, y_{\leq 2}, z_{\leq 5}), \\ f_{12} \equiv z_6^2 + x_4^3 & \mod (x_{\leq 3}, y_{\leq 2}, z_{\leq 5}). \end{cases}$$

Let $W_5 := V_5 \cap V(f_{11}, f_{12})$. On $W_5 \cap D(z_6)$, we have $z_6^2 + x_4^3 = 0$, so both z_6 and x_4 are units. If $\operatorname{char} k \neq 2$, then we solve for z_{m-6} for $m \geq 13$ using $f_m = 0$. If $\operatorname{char} k \neq 3$, then we solve for x_{m-8}. We get

$$C_5^\circ = \begin{cases} V(x_{\leq 3}, y_{\leq 2}, z_{\leq 5}, z_6^2 + x_4^3)_{z_6}|_{z_{m-6}=(2z_6)^{-1}P_{5,m},\, m\geq 13} & (\operatorname{char} k \neq 2) \\ V(x_{\leq 3}, y_{\leq 2}, z_{\leq 5}, z_6^2 + x_4^3)_{z_6}|_{x_{m-8}=(3x_4^2)^{-1}P_{5,m},\, m\geq 13} & (\operatorname{char} k \neq 3) \end{cases}$$

On $W_5 \cap V(z_6)$, we have $x_4 = 0$, and we are left with

$$V_6 := W_5 \cap V(z_6) = V(x_{\leq 4}, y_{\leq 2}, z_{\leq 6}).$$

Step 6. The next two equations give

$$\begin{cases} f_{13} \equiv 0 & \mod (x_{\leq 4}, y_{\leq 2}, z_{\leq 6}), \\ f_{14} \equiv z_7^2 + x_5 y_3^3 & \mod (x_{\leq 4}, y_{\leq 2}, z_{\leq 6}). \end{cases}$$

Let $W_6 := V_6 \cap V(f_{13}, f_{14})$. On $W_6 \cap D(y_3)$, we solve for x_{m-9} for $m \geq 15$ using $f_m = 0$, which gives

$$C_6^\circ = V(x_{\leq 4}, y_{\leq 2}, z_{\leq 6}, z_7^2 + x_5 y_3^3)_{y_3}|_{x_{m-9}=(y_3^3)^{-1}P_{6,m},\, m\geq 15}.$$

On $W_6 \cap V(y_3)$, we have $z_7 = 0$, and we are left with

$$V_7 := W_6 \cap V(y_3) = V(x_{\leq 4}, y_{\leq 3}, z_{\leq 7}).$$

Step 7. Finally, we look at

$$\begin{cases} f_{15} \equiv x_5^5 & \mod (x_{\leq 4}, y_{\leq 3}, z_{\leq 7}), \\ f_{16} \equiv z_8 & \mod (x_{\leq 5}, y_{\leq 3}, z_{\leq 7}), \\ f_{17} \equiv 0 & \mod (x_{\leq 5}, y_{\leq 3}, z_{\leq 8}), \\ f_{18} \equiv z_9^2 + x_6^3 + x_6 y_4^3 & \mod (x_{\leq 5}, y_{\leq 3}, z_{\leq 8}). \end{cases}$$

From here we compute the last set

$$C_7^\circ = V(x_{\leq 5}, y_{\leq 3}, z_{\leq 8}, z_9^2 + x_6^3 + x_6 y_4^3)_{y_4}|_{x_{m-12}=(y_4^3)^{-1}P_{7,m},\, m\geq 19}.$$

Step 8. By taking the closures of the sets C_i° constructed in the previous steps, we obtain closed irreducible sets $C_1, \ldots, C_6 \subset X_\infty$. These sets fill up the whole fiber over the origin X_∞^0. All we are left to show is that there are no inclusions among them.

It is clear from the construction that $C_i \not\subset C_j$ whenever $i < j$. Similarly, we claim that $C_i \not\supset C_j$ for all $i < j$, hence there are no inclusions among any two of these sets. This is proved by following a similar argument as that given in [dFW25] to show that $C_1 \not\supset C_2$ in the E_8 case. Specifically, to show that $C_i \not\supset C_j$ whenever $i < j$, for each pair i, j we find a set $V_{i,j}$ and a variable that vanishes on $C_i \cap V_{i,j}$ but not on $C_j \cap V_{i,j}$.

The following table, where the C_i are listed in the first column and the C_j in the top row, lists the variables vanishing identically on $C_i \cap V_{i,j}$ but not on $C_j \cap V_{i,j}$. If a different variable is needed in characteristic 2, the we write it in round brackets. The existence of such variables imply that $C_i \not\supset C_j$ for all $i < j$.

	C_2	C_3	C_4	C_5	C_6	C_7
C_1	x_2	x_3	x_4	x_4	x_5	x_6
C_2		z_5 (x_3)	z_5 (x_4)	z_6 (x_4)	z_7 (x_5)	z_9 (x_6)
C_3			x_4 (y_3)	x_4 (y_3)	x_5 (y_3)	x_6 (y_4)
C_4				x_4	x_5	x_6
C_5					z_7 (x_6)	z_9 (x_6)
C_6						x_6

4. E_8 singularities

The proof for E_8 singularities is given in [dFW25, Theorem 2.1], but some of the computations showing in the last step of the proof that there are no inclusions among the sets C_i are omitted there.

As explained there, it is clear from the construction that $C_i \not\subset C_j$ whenever $i < j$, and the proof that $C_1 \not\supset C_2$ is given in details in [dFW25] by showing that, if $\operatorname{char} k \neq 2$,

then the variable z_5 vanishes identically on $C_1 \cap V_{1,2}$ but not on $C_2 \cap V_{1,2}$, where $V_{1,2} = V(x_{\leq 3}, y_{\leq 1}, z_{\leq 4} \setminus z_3)$. When $\operatorname{char} k = 2$, then one looks at x_4, which vanishes identically on $C_1 \cap V'_{1,2}$ but not on $C_2 \cap V'_{1,2}$ where $V'_{1,2} = V(x_{\leq 3} \setminus x_2, y_{\leq 1}, z_{\leq 4})$. The proof that $C_i \not\supset C_j$ for all other $i < j$ follows a similar computation, which produces a specific variable vanishing on $C_i \cap V_{i,j}$ but not on $C_j \cap V_{i,j}$ for a suitable set $V_{i,j}$.

The following table, where the C_i are listed in the first column and the C_j in the top row, lists the variables vanishing identically on $C_i \cap V_{i,j}$ but not on $C_j \cap V_{i,j}$. If a different variable is needed in characteristic 2, the we write it in round brackets. If a different variable is needed in characteristic 3, we write it in square brackets. The existence of such variables imply that $C_i \not\supset C_j$.

	C_2	C_3	C_4	C_5	C_6	C_7	C_8
C_1	z_5 (x_4)	z_6 (x_4)	z_8 (x_5)	z_9 (x_6)	z_{10} (x_7)	z_{12} (x_8)	z_{15} (x_{10})
C_2		z_6 (y_3)	z_8 (y_3)	z_9 (y_4)	z_{10} (y_4)	z_{12} (y_5)	z_{15} (y_6)
C_3			z_8 (x_5)	z_9 (x_6)	z_{10} (x_7)	z_{12} (x_8)	z_{15} (x_{10})
C_4				x_6 $[y_4]$	x_7 $[y_4]$	x_8 $[y_5]$	x_{10} $[y_6]$
C_5					z_{10} (x_7)	z_{12} (x_8)	z_{15} (x_{10})
C_6						z_{12} (y_5)	z_{15} (y_6)
C_7							z_{15} (x_{10})

References

[dFW25] Tommaso de Fernex and Shih-Hsin Wang, *Arcs on rational double points in arbitrary characteristic* (2025). Preprint. ↑1, 3, 5, 7

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